AN EFFICIENT METHOD OF SPLINE APPROXIMATION FOR POWER FUNCTION

PETRO KOLOSOV

1. Introduction

Consider the m-degree polynomial $P(m, X, N)$ having fixed non-negative integers m and N

$$P(m, X, N) = \sum_{r=0}^{m} \sum_{k=1}^{N} \mathbf{A}_{m,r} k^r (X - k)^r$$

For example

$$\begin{aligned}
P(2, X, 0) &= 0 \\
P(2, X, 1) &= 30X^2 - 60X + 31 \\
P(2, X, 2) &= 150X^2 - 540X + 512 \\
P(2, X, 3) &= 420X^2 - 2160X + 2943 \\
P(2, X, 4) &= 900X^2 - 6000X + 10624
\end{aligned}$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively, see [1, 2]. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See the OEIS sequences [3, 4].

Essentially, the polynomial $P(m, X, N)$ is derived from a rearrangement of Faulhaber's formula. It was inspired by Knuth's *Johann Faulhaber and sums of powers*, see [5]. In particular, the polynomial $P(m, X, N)$

yields an identity for odd powers

$$P(m,X,X) = X^{2m+1}$$

In its extended form

$$X^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{X} \mathbf{A}_{m,r} k^r (X-k)^r$$

The exact relation between Faulhaber's formula and $P(m,X,N)$ is shown by [6].

However, apart from the polynomial identity for odd powers, I've discovered several approximation properties of $P(m,X,N)$. Therefore, in this manuscript we explore the approximation properties of the polynomial $P(m,X,N)$. I use a few well-known criteria to measure and estimate error of approximation: Absolute error, Relative error and Percentage error. Assume that the function $f_2(x)$ approximates the function $f_1(x)$, then errors are given by

$$\begin{aligned}
\text{Absolute Error} &= |f_1(x) - f_2(x)| \\
\text{Relative Error} &= \frac{|f_1(x) - f_2(x)|}{|f_1(x)|} \\
\text{Percentage Error} &= \frac{|f_1(x) - f_2(x)|}{|f_1(x)|} \times 100\%
\end{aligned}$$

Diving straight into the point, we switch our focus to the previously mentioned polynomial $P(2,X,4) = 900X^2 - 6000X + 10624$ to show the first example of how it approximates the odd power function X^5. In fact, we approximate the polynomial X^{2m+1} using a lower-degree polynomial of degree m, as shown in the following image

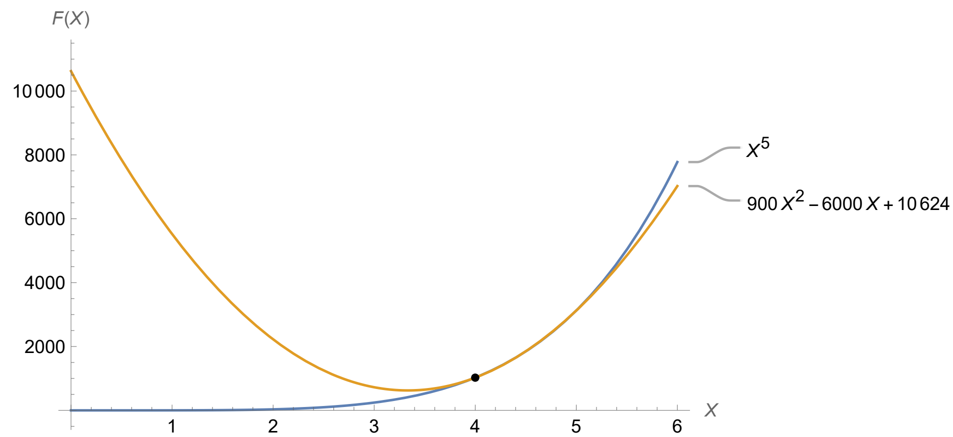

Figure 1. Approximation of fifth power X^5 by $P(2, X, 4)$. Points of intersection $X = 4$, $X = 4.42472$, $X = 4.99181$. Convergence interval is $4.0 \leq X \leq 5.1$ with percentage error $E < 1\%$.

As observed, the polynomial $P(2, X, 4)$ approximates X^5 in the neighborhood of $N = 4$ with the convergence interval $4.0 \leq X \leq 5.1$ where the percentage error is less than 1% which is quite remarkable. The following table presents specific values of absolute, relative, and percentage errors for this approximation

X	X^5	$900X^2 - 6000X + 10624$	**ABS**	**Relative**	**% Error**
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1153.0	5.56201	0.00480079	0.480079
4.2	1306.91	1300.0	6.91232	0.00528905	0.528905
4.3	1470.08	1465.0	5.08443	0.0034586	0.34586
4.4	1649.16	1648.0	1.16224	0.000704746	0.0704746
4.5	1845.28	1849.0	3.71875	0.00201528	0.201528
4.6	2059.63	2068.0	8.37024	0.00406395	0.406395
4.7	2293.45	2305.0	11.5499	0.00503605	0.503605
4.8	2548.04	2560.0	11.9603	0.00469393	0.469393
4.9	2824.75	2833.0	8.24751	0.00291973	0.291973
5.0	3125.0	3124.0	1.0	0.00032	0.032
5.1	3450.25	3433.0	17.2525	0.00500036	0.500036

Table 2. Comparison of X^5 and $P(2, X, 4) = 900X^2 - 6000X + 10624$

One more interesting observation arises by increasing the value of N in $P(m, X, N)$ while keeping m fixed. As N increases, the length of the convergence interval with the odd-power X^{2m+1} also increases. For instance,

- For $P(2, X, 4)$ and X^5, the convergence interval with a percentage error less than 1% is $4.0 \leq X \leq 5.1$, with a length $L = 1.1$
- For $P(2, X, 20)$ and X^5, the convergence interval with a percentage error less than 1% is $18.7 \leq X \leq 22.9$, with a length $L = 4.2$
- For $P(2, X, 120)$ and X^5, the convergence interval with a percentage error less than 1% is $110.0 \leq X \leq 134.7$, with a length $L = 24.7$

The reason behind this behavior lies in the implicit form of the polynomial $P(m, X, N)$, meaning that

$$P(m, X, N) = \sum_{r=0}^{m} (-1)^{m-r} U(m, N, r) \cdot X^r$$

where $U(m, N, r)$ is a polynomial defined as follows

$$U(m, N, r) = (-1)^m \sum_{k=1}^{N} \sum_{j=r}^{m} \binom{j}{r} \mathbf{A}_{m,j} k^{2j-r} (-1)^j$$

which grows as N increases. Few cases of coefficients $U(m, N, r)$ are registered as OEIS sequences [7, 8, 9].

To summarize, let us recap the key findings so far. The polynomial $P(m, X, N)$ is an m-degree polynomial in $X \in \mathbb{R}$ with fixed non-negative integers m and N. It approximates the odd power function X^{2m+1} within a specific neighborhood of N. The length L of the convergence interval between X^{2m+1} and $P(m, X, N)$ increases as N grows.

So far we have discussed approximation of odd power function X^{2m+1}, now we focus on its even case X^{2m+2} which is quite straightforward. Considering the same example $P(2, X, 4)$ we reach the approximation of even power X^6 by means of K-times multiplication by X, with graphic representation as follows

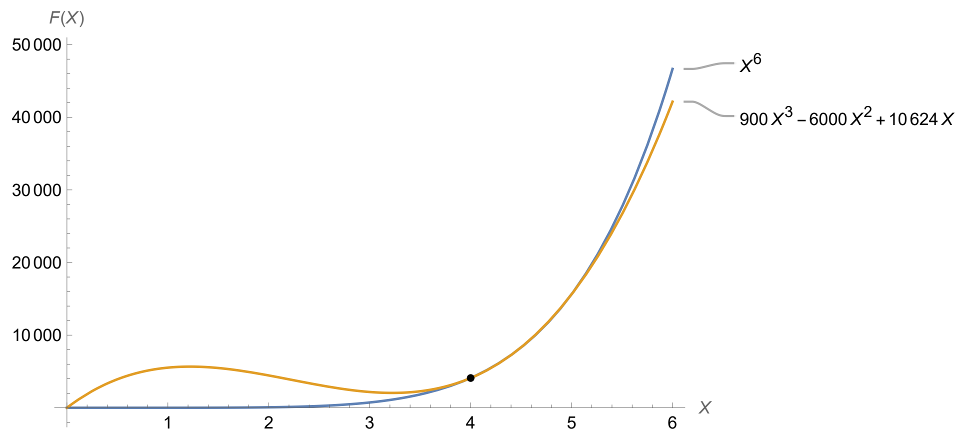

Figure 2. Approximation of sixth power X^6 by $P(2, X, 4) \cdot X$. Convergence interval is $3.9 \leq X \leq 5.1$ with percentage error $E < 3\%$.

Therefore, we have reached the statement that the polynomial $P(m, X, N)$ is an m-degree polynomial in X, having fixed non-negative integers m and N. It approximates the power function X^j in some neighborhood of fixed N. The length of convergence interval between power function X^j and $P(m, X, N) \cdot X^K$ rises as N rise.

2. Generalizations

Previously, we have discussed that polynomial $P(m, X, N)$ approximates power function X^j in some neighborhood of fixed non-negative integer N. Approximation by $P(m, X, N)$ can be adjusted by X^k multiplication for even exponents of power function. In general, it is safe to say that power function X^j is approximated by $P(m, X, N) \cdot X^k$ where $k = 0$ for odd exponent j and k is either $k = 1$ or $k = -1$ for an even exponent j. Therefore, for arbitrary exponent j in X^j we have

$$X^j \approx \begin{cases} P(m, X, N) & j = 2m + 1 \\ P(m, X, N) \cdot X & j = 2m + 2 \\ P(m, X, N) \cdot X^{-1} & j = 2m \end{cases}$$

Of course, there are other variations of the value of k, but we will stick to the simple case for the moment.

As we also discussed, the length L of the convergence interval between X^j and its approximation by $P(m, X, N)$ increases as N grow. However, the convergence interval is still bounded, which could not satisfy certain approximation scenarios. Depending on the approximation requirements in terms of convergence interval length L a single polynomial $P(m, X, N)$ with fixed m and N may be unsuitable. Here is the place where spline approximation comes into play. The spline $S(x)$ is piecewise defined function over the interval $(x_0, \dots x_n)$

$$S(x) = \begin{cases} f_1(x), & x_0 \le x < x_1 \\ f_2(x), & x_0 \le x < x_1 \\ \vdots & \vdots \\ f_n(x), & x_{n-1} \le x \le x_n \end{cases}$$

The given points x_k are called *knots*.

Assume that the approximation requirement in terms of convergence length L is to approximate the power function X^j bounded by real points A and B such that $A < B$. Splines perfectly fit the need to match an arbitrary convergence range for the power function X^j using the approximation by $P(m, X, N)$. Formally,

$$X^j \approx \begin{cases} P(m, X, N + t_1) \cdot X^k & x_0 \le x < x_1 \\ P(m, X, N + t_2) \cdot X^k & x_0 \le x < x_1 \\ \vdots & \vdots \\ P(m, X, N + t_{n-1}) \cdot X^k & x_{n-1} \le x < x_n \end{cases}$$

The values of t_r to be adjusted according to approximation requirements in terms of accuracy.

3. Use cases

Use case scenarios of the approximation technique we discuss have their own constraints and limitations. For instance, approximation requirements should have precisely specified exponent j in X^j because for each j there is a matching polynomial $P(m, X, N)$. Perfectly, there should be a set precompiled polynomials $P(m, X, N)$ matching precise exponent j in X^j over precisely defined approximation range with required error of approximation E as a constraint. Generally, the approximation of power function X^j by $P(m, X, N)$ can be broken down into the following steps

(1) Define the exponent j in X^j
(2) Define the required error threshold E
(3) Define the required interval of approximation I
(4) Choose and precompile polynomials $P(m, X, N)$ so that the required interval of approximation and error threshold E are satisfied
(5) Define a set of knots so that the error threshold E and the interval of approximation I requirements are satisfied

Defining set of spline knots essentially requires an inspection of the convergence intervals between X^j and $P(m, X, N + t_k)$ by choosing knots such that the interval of approximation and error threshold are satisfied. Consider an example. Let be the following approximation requirements

(1) Exponent $j = 3$
(2) Percentage error threshold $E \leq 1\%$
(3) Interval of approximation $10 \leq X \leq 15$

Now we have to choose a set of polynomials $P(m, X, N + t_k)$ based on which we adjust a set of spline knots. We can safely choose integers t_k in range $10 \leq t_k \leq 15$ because of the following properties of $P(m, X, N)$.

$$P(m, X, X) = X^{2m+1}$$
$$P(m, X, X + 1) = (X + 1)^{2m+1} - 1$$

Therefore, for each two consequential points $N = X, N = X + 1$ the absolute difference is 1, making that range at least 1% percentage error for $X^j \leq 100$. I have chosen the approximation range $10 \leq X \leq 15$ and $j = 3$ intentionally to show the spline approximation with percentage error threshold less than 1%. Therefore, to approximate X^3 in the range $10 \leq X \leq 15$, we use the following spline function

$$X^3 \approx \begin{cases} P(1, X, 10) = -2300 + 330X, & 10 \leq X < 11 \\ P(1, X, 11) = -3025 + 396X, & 11 \leq X < 12 \\ P(1, X, 12) = -3888 + 468X, & 12 \leq X < 13 \\ P(1, X, 13) = -4901 + 546X, & 13 \leq X < 14 \\ P(1, X, 14) = -6076 + 630X, & 14 \leq X \leq 15 \end{cases} \tag{1}$$

Graphically, this linear approximation of cubes looks as follows

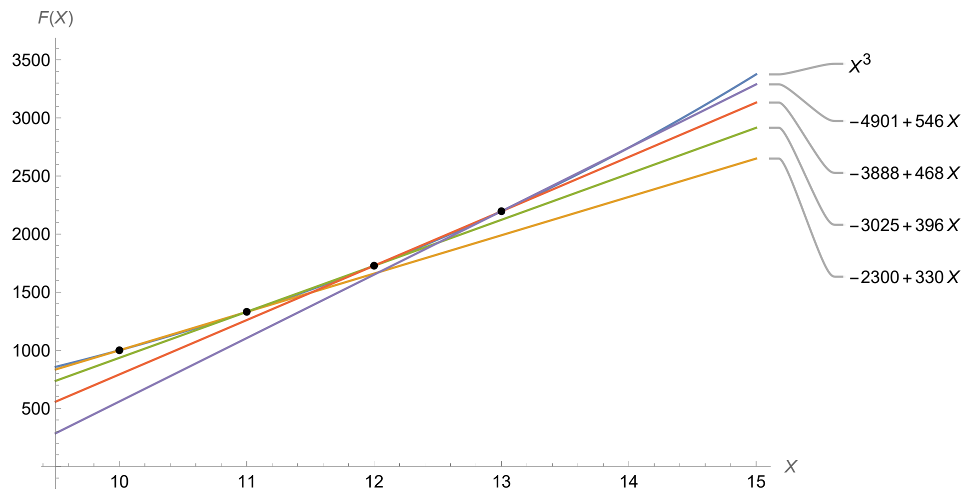

Figure 3. Approximation of cubes X^3 by splines (1). Convergence interval is $10 \leq X \leq 15$ with percentage error $E \leq 1\%$.

where the spline knots are integers in the range $10 \leq N \leq 14$.

The same principle applies for even exponent $j = 4$ in X^j with the same convergence interval $10 \leq X \leq 15$ and approximation error under 1% constraints

$$X^4 \approx \begin{cases} P(1,X,10)\cdot X = -2300X + 330X^2, & 10 \leq X < 11 \\ P(1,X,10)\cdot X = -3025X + 396X^2, & 11 \leq X < 12 \\ P(1,X,10)\cdot X = -3888X + 468X^2, & 12 \leq X < 13 \\ P(1,X,10)\cdot X = -4901X + 546X^2, & 13 \leq X < 14 \\ P(1,X,10)\cdot X = -6076X + 630X^2, & 14 \leq X \leq 15 \end{cases} \tag{2}$$

Which graphically looks as follows

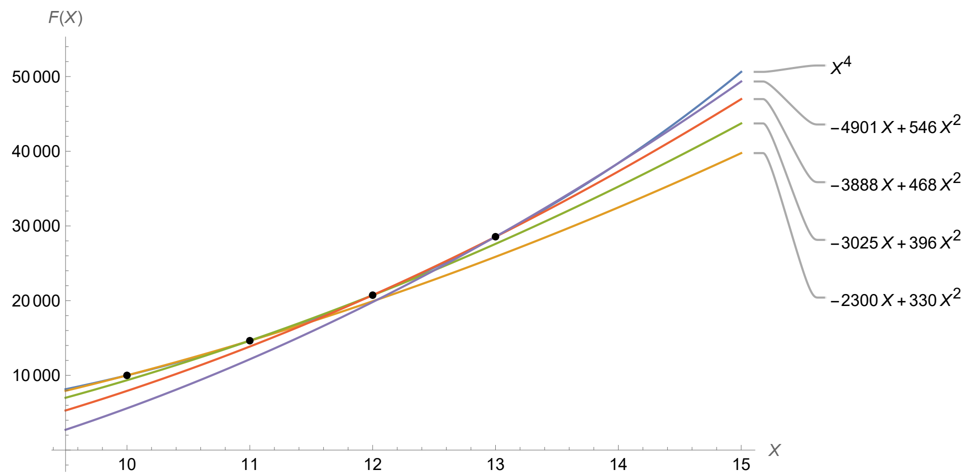

Figure 4. Approximation of X^4 by splines (2). Convergence interval is $10 \leq X \leq 15$ with percentage error $E \leq 1\%$.

In general, for each variation of X^j such that $X^j \geq 100$ the approximation can be done using splines in $P(m, X, N)$ over the interval $A \leq X \leq B$ with spline knot vector be the integers in range from A to B so that knots vector is $\{A, A+1, A+2, \dots, B\}$. Because,

$$P(m, X, X) = X^{2m+1}$$
$$P(m, X, X+1) = (X+1)^{2m+1} - 1$$

This can be further optimized depending on the value of N in $P(m, X, N)$ because the convergence interval with the power function X^j increases as N grows.

References

[1] Kolosov, Petro. On the link between binomial theorem and discrete convolution. *arXiv preprint arXiv:1603.02468*, 2016. https://arxiv.org/abs/1603.02468.

[2] Kolosov, Petro. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. https://doi.org/10.1017/mag.2022.129.

[3] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences, 2018. https://oeis.org/A302971.

[4] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences, 2018. https://oeis.org/A304042.

[5] Knuth, Donald E. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203):277–294, 1993. https://arxiv.org/abs/math/9207222.

[6] Petro Kolosov. Unexpected polynomial identity, 2025. https://kolosovpetro.github.io/pdf/UnexpectedPolynomialIdentity.pdf.

[7] Petro Kolosov. The coefficients U(m, l, k), m = 1 defined by the polynomial identity, 2018. https://oeis.org/A320047.

[8] Petro Kolosov. The coefficients U(m, l, k), m = 2 defined by the polynomial identity, 2018. https://oeis.org/A316349.
[9] Petro Kolosov. The coefficients U(m, l, k), m = 3 defined by the polynomial identity, 2018. https://oeis.org/A316387.