AN EFFICIENT METHOD OF SPLINE APPROXIMATION FOR POWER FUNCTION

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Abstract. Let $P(m, X, N)$ be an m-degree polynomials in $X \in \mathbb{R}$ having fixed non-negative integers m and N. In this manuscript an efficient method of spline approximation for power function is shown and discussed. Approximation technique is based on the fact that polynomial $P(m, X, N)$ approximates odd-power function X^{2m+1} for a in some neighborhood of fixed N.

Contents

1. Introduction

Consider the m-degree polynomial $P(m, X, N)$ having fixed non-negative integers m and N

$$P(m, X, N) = \sum_{r=0}^{m} \sum_{k=1}^{N} \mathbf{A}_{m,r} k^r (X-k)^r$$

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Sources: https://github.com/kolosovpetro/AnEfficientMethodOfSplineApproximation

For example

$$P(2, X, 0) = 0$$

$$P(2, X, 1) = 30X^2 - 60X + 31$$

$$P(2, X, 2) = 150X^2 - 540X + 512$$

$$P(2, X, 3) = 420X^2 - 2160X + 2943$$

$$P(2, X, 4) = 900X^2 - 6000X + 10624$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively, see [1, 2, 3, 4]. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [5, 6].

In this manuscript we discuss approximation properties of polynomial $P(m, X, N)$. I use a few well-known criteria to measure and estimate error of approximation: Absolute error, Relative error and Percentage error. Assume that function $f_2(x)$ approximates the function $f_1(x)$ then the errors are

$$\text{Absolute Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|}$$

$$\text{Relative Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|}$$

$$\text{Percentage Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|} \times 100\%$$

Diving straight to the point, we switch our focus to already mentioned polynomial $P(2, X, 4) = 900X^2 - 6000X + 10624$ to show the first example of how it approximates the odd power function X^5. In fact, we approximate the polynomial X^{2m+1} by lower degree polynomial X^m as the following image presents

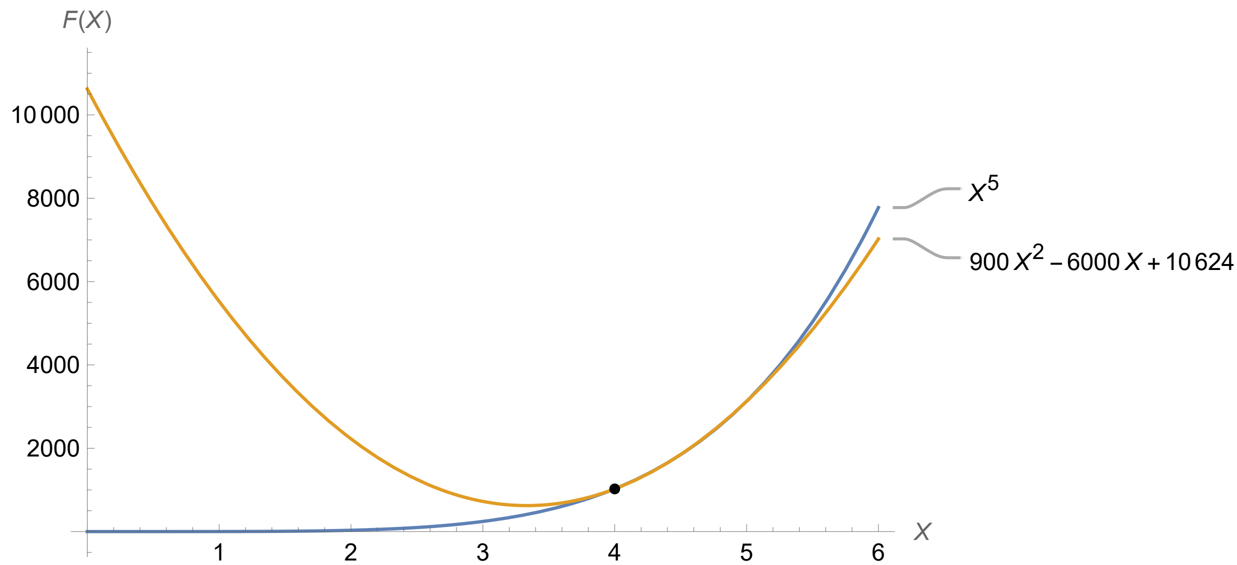

Figure 1. Polynomial plot $P(2, X, 4)$ with fifth power X^5. Points of intersection $X = 4$, $X = 4.42472$, $X = 4.99181$. Interval of convergence: $3.9 \leq X \leq 5.3$ with $E < 2\%$.

As we see, the interval $3.9 \leq X \leq 5.3$ has the percentage error lesser than 2% which is quite impressive. Therefore, having fixed $N = 4$ the polynomial $P(2, X, 4)$ approximates odd power in neighborhood of $N = 4$ which is $3.9 \leq X \leq 5.3$. To showcase the concrete values of absolute, relative and percentage errors of approximation above, I attach a separate table in addenda.

One more interesting observation can be done by increasing the value of N in $P(m, X, N)$ having fixed m, it follows that by increasing N the interval of convergence with odd-power X^{2m+1} increasing as well. For instance,

- Having $P(2, X, 4)$ the interval of convergence with percentage error lesser than 1% is $4.0 \leq X \leq 5.1$

- Having $P(2, X, 20)$ the interval of convergence with percentage error lesser than 1% is $18.7 \leq X \leq 22.9$
- Having $P(2, X, 120)$ the interval of convergence with percentage error lesser than 1% is $110.0 \leq X \leq 134.7$

The reason why convergence interval rises as N rise lays beneath the implicit form of polynomial $P(m, X, N)$ meaning that

$$P(m, X, N) = \sum_{r=0}^{m} (-1)^{m-r} U(m, N, r) \cdot X^r$$

where $U(m, N, r)$ is a polynomial defined as follows

$$U(m, N, r) = (-1)^m \sum_{k=1}^{N} \sum_{j=r}^{m} \binom{j}{r} \mathbf{A}_{m,j} k^{2j-r} (-1)^j$$

which rises as N rise.

2. Conclusions

References

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3. Addendum

Table 2. Comparison of X^5, $2X4 = 900X^2 - 6000X + 10624$

X	X^5	$900X^2 - 6000X + 10624$	**ABS**	**Relative**	**% Error**
3.8	792.352	820.0	27.6483	0.034894	3.4894
3.9	902.242	913.0	10.758	0.0119236	1.19236
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1153.0	5.56201	0.00480079	0.480079
4.2	1306.91	1300.0	6.91232	0.00528905	0.528905
4.3	1470.08	1465.0	5.08443	0.0034586	0.34586
4.4	1649.16	1648.0	1.16224	0.000704746	0.0704746
4.5	1845.28	1849.0	3.71875	0.00201528	0.201528
4.6	2059.63	2068.0	8.37024	0.00406395	0.406395
4.7	2293.45	2305.0	11.5499	0.00503605	0.503605
4.8	2548.04	2560.0	11.9603	0.00469393	0.469393
4.9	2824.75	2833.0	8.24751	0.00291973	0.291973
5.0	3125.0	3124.0	1.0	0.00032	0.032
5.1	3450.25	3433.0	17.2525	0.00500036	0.500036
5.2	3802.04	3760.0	42.0403	0.0110573	1.10573
5.3	4181.95	4105.0	76.9549	0.0184017	1.84017
5.4	4591.65	4468.0	123.65	0.0269294	2.69294
5.5	5032.84	4849.0	183.844	0.0365288	3.65288
5.6	5507.32	5248.0	259.318	0.047086	4.7086
5.7	6016.92	5665.0	351.921	0.0584885	5.84885
5.8	6563.57	6100.0	463.568	0.0706274	7.06274
5.9	7149.24	6553.0	596.243	0.0833995	8.33995