AN EFFICIENT METHOD OF SPLINE APPROXIMATION FOR POWER FUNCTION

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Abstract. Let $P(m, X, N)$ be an m-degree polynomials in $X \in \mathbb{R}$ having fixed non-negative integers m and N. In this manuscript we discuss approximation properties of polynomial $P(m, X, N)$. In particular, the polynomial $P(m, X, N)$ approximates odd power function X^{2m+1} in some neighborhood of fixed non-negative integer N with percentage error lesser than 1%. Percentage error is free for adjustments, depending on required approximation accuracy. By increasing the value of N the length of convergence interval with odd-power X^{2m+1} increasing as well. Furthermore, above approximation property is generalized for arbitrary non-negative exponent power function, using splines.

Contents

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Sources: https://github.com/kolosovpetro/AnEfficientMethodOfSplineApproximation

1. Introduction

Consider the m-degree polynomial $P(m, X, N)$ having fixed non-negative integers m and N

$$P(m, X, N) = \sum_{r=0}^{m} \sum_{k=1}^{N} \mathbf{A}_{m,r} k^r (X-k)^r$$

For example

$$P(2, X, 0) = 0$$

$$P(2, X, 1) = 30X^2 - 60X + 31$$

$$P(2, X, 2) = 150X^2 - 540X + 512$$

$$P(2, X, 3) = 420X^2 - 2160X + 2943$$

$$P(2, X, 4) = 900X^2 - 6000X + 10624$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively, see [1, 2, 3, 4]. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [5, 6].

Essentially, the polynomial $P(m, X, N)$ is a result of rearrangement inside Faulhaber's formula. It was inspired by Knuth's *Johann Faulhaber and sums of powers*, see [7]. In particular, the polynomial $P(m, X, N)$ yields an identity for odd powers

$$P(m, X, X) = X^{2m+1}$$

In extended form

$$X^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{X} \mathbf{A}_{m,r} k^r (X-k)^r$$

Precisely, the relation between Faulhaber's formula and $P(m, X, N)$ is shown by [8].

However, apart polynomial identity for odd powers, I've spotted several approximation properties of $P(m, X, N)$. Therefore, in this manuscript we discuss approximation properties of polynomial $P(m, X, N)$. I use a few well-known criteria to measure and estimate error of approximation: Absolute error, Relative error and Percentage error. Assume that function $f_2(x)$ approximates the function $f_1(x)$ then the errors are

$$\text{Absolute Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|}$$

$$\text{Relative Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|}$$

$$\text{Percentage Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|} \times 100\%$$

Diving straight to the point, we switch our focus to already mentioned polynomial $P(2, X, 4) = 900X^2 - 6000X + 10624$ to show the first example of how it approximates the odd power function X^5. In fact, we approximate the polynomial X^{2m+1} by lower degree polynomial X^m as the following image presents

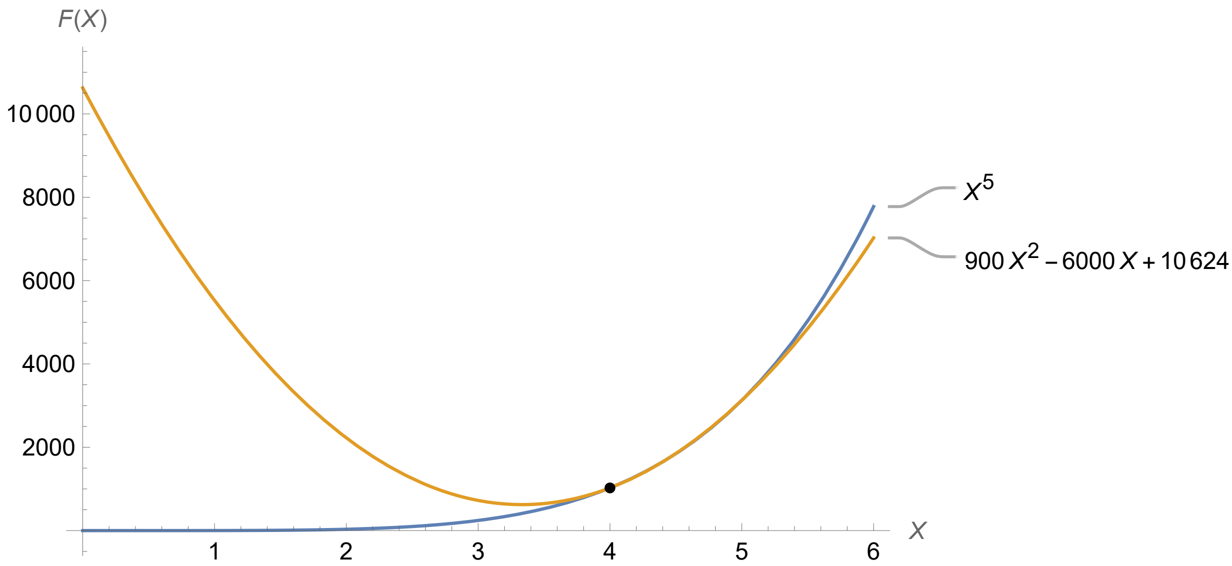

Figure 1. Polynomial plot $P(2, X, 4)$ with fifth power X^5. Points of intersection $X = 4$, $X = 4.42472$, $X = 4.99181$. Convergence interval: $4.0 \leq X \leq 5.1$ with percentage error $E < 1\%$.

As we see, polynomial $P(2, X, 4)$ approximates X^5 in a neighborhood of $N = 4$ with the convergence interval $4.0 \leq X \leq 5.1$ that has percentage error lesser than 1% which is quite impressive. To showcase the concrete values of absolute, relative and percentage errors of this approximation, I attach a separate table to addendum.

One more interesting observation can be done by increasing the value of N in $P(m, X, N)$ having fixed m, it follows that by increasing N the length of convergence interval with odd-power X^{2m+1} increasing as well. For instance,

- Having $P(2, X, 4)$ and X^5 the convergence interval with percentage error lesser than 1% is $4.0 \leq X \leq 5.1$ with lenght of interval $L = 1.1$
- Having $P(2, X, 20)$ and X^5 the convergence interval with percentage error lesser than 1% is $18.7 \leq X \leq 22.9$ with lenght of interval $L = 4.2$
- Having $P(2, X, 120)$ and X^5 the convergence interval with percentage error lesser than 1% is $110.0 \leq X \leq 134.7$ with lenght of interval $L = 24.7$

The reason why the lenght of convergence interval rises as N rise lays beneath the implicit form of polynomial $P(m,X,N)$ meaning that

$$P(m,X,N)=\sum_{r=0}^{m}(-1)^{m-r}U(m,N,r)\cdot X^{r}$$

where $U(m,N,r)$ is a polynomial defined as follows

$$U(m,N,r)=(-1)^{m}\sum_{k=1}^{N}\sum_{j=r}^{m}\binom{j}{r}\mathbf{A}_{m,j}k^{2j-r}(-1)^{j}$$

which rises as N rise.

To wrap up the current state of the manuscript, refresh the key facts and finding we got so far. Therefore, the polynomial $P(m,X,N)$ is an m-degree polynomial in $X\in\mathbb{R}$, having fixed non-negative integers m and N. It approximates odd power function X^{2m+1} in some neighborhood of fixed N. The length L of convergence interval between X^{2m+1} and $P(m,X,N)$ rises as N rise.

For the sake of clear and precise verification of results, I attach mathematica programs to generate plots and data tables, so that reader is able to verify the main results of current part of manuscript, see the `link`.

So far we have discussed approximation of odd power function X^{2m+1}, now we focus on its even case X^{2m+2} which is quite straightforward. Considering the same example $P(2,X,4)$ we reach the approximation of even power X^6 by means of K-times multiplication by X, with graphic representation as follows

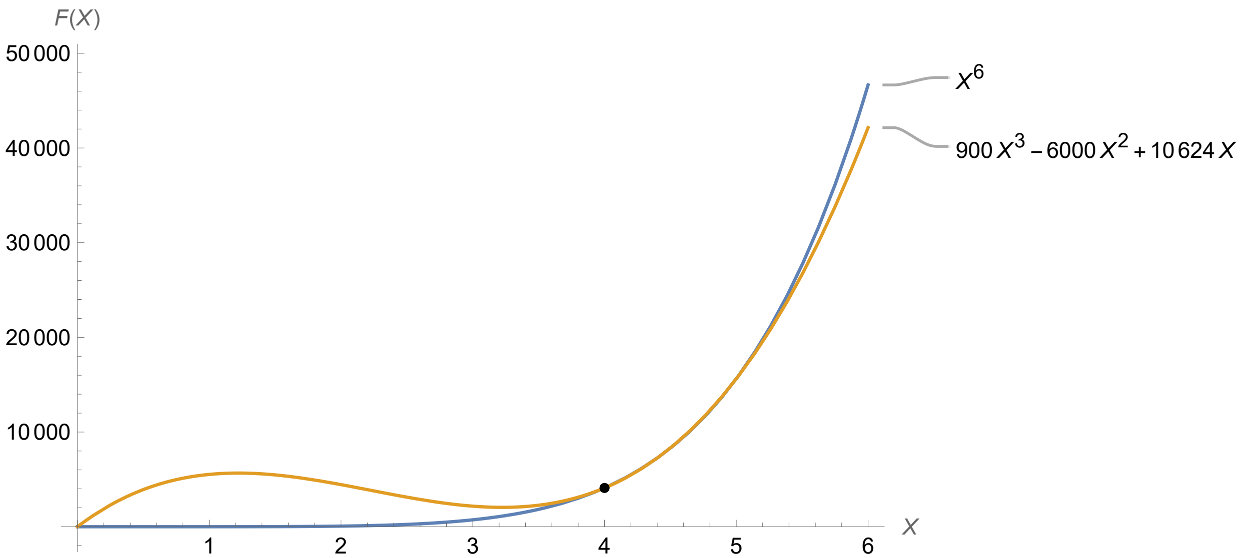

Figure 2. Polynomial plot $P(2, X, 4) \cdot X$ with sixth power X^6. Convergence interval: $3.9 \leq X \leq 5.3$ with percentage error $E < 2\%$.

Therefore, we have reached the statement that the polynomial $P(m, X, N)$ is an m-degree polynomial in X, having fixed non-negative integers m and N. It approximates the power function X^j in some neighborhood of fixed N. The length of convergence interval between power function and $P(m, X, N)$ or $P(m, X, N) \cdot X^K$ rises as N rise.

2. Generalizations

Previously, we have discussed that polynomial $P(m, X, N)$ approximates power function X^j in some neighborhood of fixed non-negative integer N. Approximation by $P(m, X, N)$ can be adjusted by K-times multiplication by X for even exponents of power function. In general, it is safe to say that power function X^j is approximated by $P(m, X, N) \cdot X^k$, where $k = 0$ for odd exponent j and k either $k = 1$ or $k = -1$ for even exponent j

$$X^j \approx \begin{cases} P(m, X, N) & j = 2m + 1 \\ P(m, X, N) \cdot X & j = 2m + 2 \\ P(m, X, N) \cdot X^{-1} & j = 2m \end{cases}$$

which covers arbitrary exponent j.

As we also discussed, the lenght L of convergence interval between X^j and approximation by $P(m, X, N)$ rises as N rise. However, the convergence interval is still bounded, which could not satisfy certain approximation scenarios. Depending on approximation requirements in terms of convergence interval length L a single polynomial $P(m, X, N)$ with fixed m and N can be unfit. Here is the place where spline approximation comes to play. The spline $S(x)$ is piecewise defined function over the interval $(x_0, \dots x_n)$

$$S(x) = \begin{cases} f_1(x), & x_0 \leq x < x_1 \\ f_2(x), & x_0 \leq x < x_1 \\ \vdots & \vdots \\ f_n(x), & x_{n-1} \leq x \leq x_n \end{cases}$$

The given points x_k are called *knots*.

Assume that approximation requirement in terms of convergence length L is to approximate the power function X^j bounded by real points A and B such that $A < B$. Splines perfectly fits the need to match arbitrary convergence range of power function's X^j approximation by $P(m, X, N)$. Formally,

$$X^j \approx \begin{cases} P(m, X, N + t_1) \cdot X^k & x_0 \leq x < x_1 \\ P(m, X, N + t_2) \cdot X^k & x_0 \leq x < x_1 \\ \vdots & \vdots \\ P(m, X, N + t_{n-1}) \cdot X^k & x_{n-1} \leq x < x_n \end{cases}$$

The values of t_r to be adjusted according to approximation requirements in terms of accuracy.

3. Use cases

Use case scenarios of the approximation technique we discuss have their own constraints and limitations. For instance, approximation requirements should have precisely specified exponent j in X^j because for each j there is a matching polynomial $P(m, X, N)$. Perfectly, there should be a set precompiled polynomials $P(m, X, N)$ matching precise exponent j in X^j over precisely defined approximation range with required error of approximation E constraint.

4. Conclusions

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5. Addendum

Table 2. Comparison of X^5 and $P(2, X, 4) = 900X^2 - 6000X + 10624$

X	X^5	$900X^2 - 6000X + 10624$	**ABS**	**Relative**	**% Error**
3.8	792.352	820.0	27.6483	0.034894	3.4894
3.9	902.242	913.0	10.758	0.0119236	1.19236
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1153.0	5.56201	0.00480079	0.480079
4.2	1306.91	1300.0	6.91232	0.00528905	0.528905
4.3	1470.08	1465.0	5.08443	0.0034586	0.34586
4.4	1649.16	1648.0	1.16224	0.000704746	0.0704746
4.5	1845.28	1849.0	3.71875	0.00201528	0.201528
4.6	2059.63	2068.0	8.37024	0.00406395	0.406395
4.7	2293.45	2305.0	11.5499	0.00503605	0.503605
4.8	2548.04	2560.0	11.9603	0.00469393	0.469393
4.9	2824.75	2833.0	8.24751	0.00291973	0.291973
5.0	3125.0	3124.0	1.0	0.00032	0.032
5.1	3450.25	3433.0	17.2525	0.00500036	0.500036
5.2	3802.04	3760.0	42.0403	0.0110573	1.10573
5.3	4181.95	4105.0	76.9549	0.0184017	1.84017
5.4	4591.65	4468.0	123.65	0.0269294	2.69294
5.5	5032.84	4849.0	183.844	0.0365288	3.65288
5.6	5507.32	5248.0	259.318	0.047086	4.7086
5.7	6016.92	5665.0	351.921	0.0584885	5.84885
5.8	6563.57	6100.0	463.568	0.0706274	7.06274
5.9	7149.24	6553.0	596.243	0.0833995	8.33995

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