AN EFFICIENT METHOD OF SPLINE APPROXIMATION FOR POWER FUNCTION

PETRO KOLOSOV

Abstract. Let $P(m, X, N)$ be an m-degree polynomials in $X \in \mathbb{R}$ having fixed non-negative integers m and N. Essentially, the polynomial $P(m, X, N)$ is a result of rearrangement inside Faulhaber's formula in context of Knuth's work *Johann Faulhaber and sums of powers*.

In this manuscript we discuss approximation properties of polynomial $P(m, X, N)$. In particular, the polynomial $P(m, X, N)$ approximates odd power function X^{2m+1} in certain neighborhood of fixed non-negative integer N with percentage error lesser than 1%.

By increasing the value of N the length of convergence interval with odd-power X^{2m+1} increasing as well.

Furthermore, this approximation technique is generalized for arbitrary non-negative exponent power function X^j by using splines.

Contents

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Sources: https://github.com/kolosovpetro/AnEfficientMethodOfSplineApproximation

1. Introduction

Consider the m-degree polynomial $P(m,X,N)$ having fixed non-negative integers m and N

$$P(m,X,N)=\sum_{r=0}^{m}\sum_{k=1}^{N}\mathbf{A}_{m,r}k^r(X-k)^r$$

For example

$$P(2,X,0)=0$$

$$P(2,X,1)=30X^2-60X+31$$

$$P(2,X,2)=150X^2-540X+512$$

$$P(2,X,3)=420X^2-2160X+2943$$

$$P(2,X,4)=900X^2-6000X+10624$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively, see [1, 2, 3, 4]. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See the OEIS sequences [5, 6].

Essentially, the polynomial $P(m,X,N)$ is a result of rearrangement inside Faulhaber's formula. It was inspired by Knuth's *Johann Faulhaber and sums of powers*, see [7]. In particular, the polynomial $P(m,X,N)$ yields an identity for odd powers

$$P(m,X,X)=X^{2m+1}$$

In extended form

$$X^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{X} \mathbf{A}_{m,r} k^r (X-k)^r$$

Precisely, the relation between Faulhaber's formula and $P(m, X, N)$ is shown by [8].

However, apart polynomial identity for odd powers, I've spotted several approximation properties of $P(m, X, N)$. Therefore, in this manuscript we discuss approximation properties of polynomial $P(m, X, N)$. I use a few well-known criteria to measure and estimate error of approximation: Absolute error, Relative error and Percentage error. Assume that function $f_2(x)$ approximates the function $f_1(x)$ then the errors are

$$\text{Absolute Error} = |f_1(x) - f_2(x)|$$

$$\text{Relative Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|}$$

$$\text{Percentage Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|} \times 100\%$$

Diving straight to the point, we switch our focus to already mentioned polynomial $P(2, X, 4) = 900X^2 - 6000X + 10624$ to show the first example of how it approximates the odd power function X^5. In fact, we approximate the polynomial X^{2m+1} by lower degree polynomial X^m as the following image presents

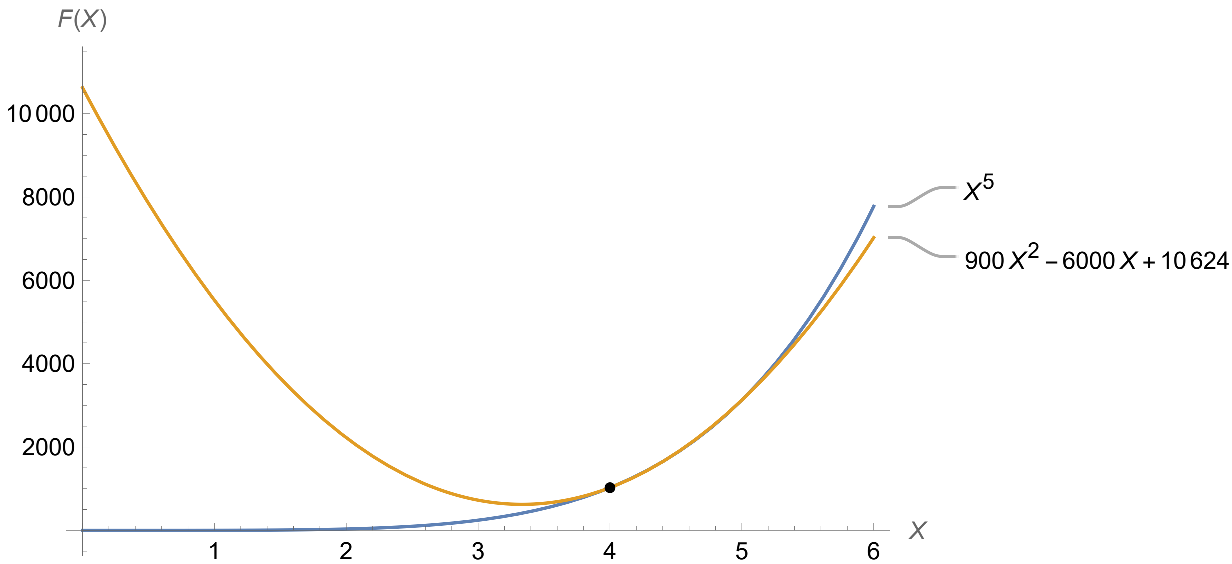

Figure 1. Polynomial plot $P(2, X, 4)$ with fifth power X^5. Points of intersection $X = 4$, $X = 4.42472$, $X = 4.99181$. Convergence interval: $4.0 \leq X \leq 5.1$ with percentage error $E < 1\%$.

As we see, polynomial $P(2, X, 4)$ approximates X^5 in a neighborhood of $N = 4$ with the convergence interval $4.0 \leq X \leq 5.1$ that has percentage error lesser than 1% which is quite impressive. Consider the table below to showcase the concrete values of absolute, relative and percentage errors of this approximation

X	X^5	$900X^2 - 6000X + 10624$	**ABS**	**Relative**	**% Error**
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1153.0	5.56201	0.00480079	0.480079
4.2	1306.91	1300.0	6.91232	0.00528905	0.528905
4.3	1470.08	1465.0	5.08443	0.0034586	0.34586
4.4	1649.16	1648.0	1.16224	0.000704746	0.0704746
4.5	1845.28	1849.0	3.71875	0.00201528	0.201528
4.6	2059.63	2068.0	8.37024	0.00406395	0.406395
4.7	2293.45	2305.0	11.5499	0.00503605	0.503605
4.8	2548.04	2560.0	11.9603	0.00469393	0.469393
4.9	2824.75	2833.0	8.24751	0.00291973	0.291973
5.0	3125.0	3124.0	1.0	0.00032	0.032
5.1	3450.25	3433.0	17.2525	0.00500036	0.500036

Table 2. Comparison of X^5 and $P(2, X, 4) = 900X^2 - 6000X + 10624$

One more interesting observation can be done by increasing the value of N in $P(m, X, N)$ having fixed m, it follows that by increasing N the length of convergence interval with odd-power X^{2m+1} increasing as well. For instance,

- Having $P(2, X, 4)$ and X^5 the convergence interval with percentage error lesser than 1% is $4.0 \le X \le 5.1$ with lenght of interval $L = 1.1$
- Having $P(2, X, 20)$ and X^5 the convergence interval with percentage error lesser than 1% is $18.7 \le X \le 22.9$ with lenght of interval $L = 4.2$
- Having $P(2, X, 120)$ and X^5 the convergence interval with percentage error lesser than 1% is $110.0 \le X \le 134.7$ with lenght of interval $L = 24.7$

The reason why the lenght of convergence interval rises as N rise lays beneath the implicit form of polynomial $P(m, X, N)$ meaning that

$$P(m, X, N) = \sum_{r=0}^{m} (-1)^{m-r} U(m, N, r) \cdot X^r$$

where $U(m, N, r)$ is a polynomial defined as follows

$$U(m, N, r) = (-1)^m \sum_{k=1}^{N} \sum_{j=r}^{m} \binom{j}{r} \mathbf{A}_{m,j} k^{2j-r} (-1)^j$$

which rises as N rise.

To wrap up the current state of the manuscript, refresh the key facts and finding we got so far. Therefore, the polynomial $P(m, X, N)$ is an m-degree polynomial in $X \in \mathbb{R}$, having fixed non-negative integers m and N. It approximates odd power function X^{2m+1} in some neighborhood of fixed N. The length L of convergence interval between X^{2m+1} and $P(m, X, N)$ rises as N rise.

For the sake of clear and precise verification of results, I attach mathematica programs to generate plots and data tables, so that reader is able to verify the main results of current part of manuscript, see the `link`.

So far we have discussed approximation of odd power function X^{2m+1}, now we focus on its even case X^{2m+2} which is quite straightforward. Considering the same example $P(2, X, 4)$ we reach the approximation of even power X^6 by means of K-times multiplication by X, with graphic representation as follows

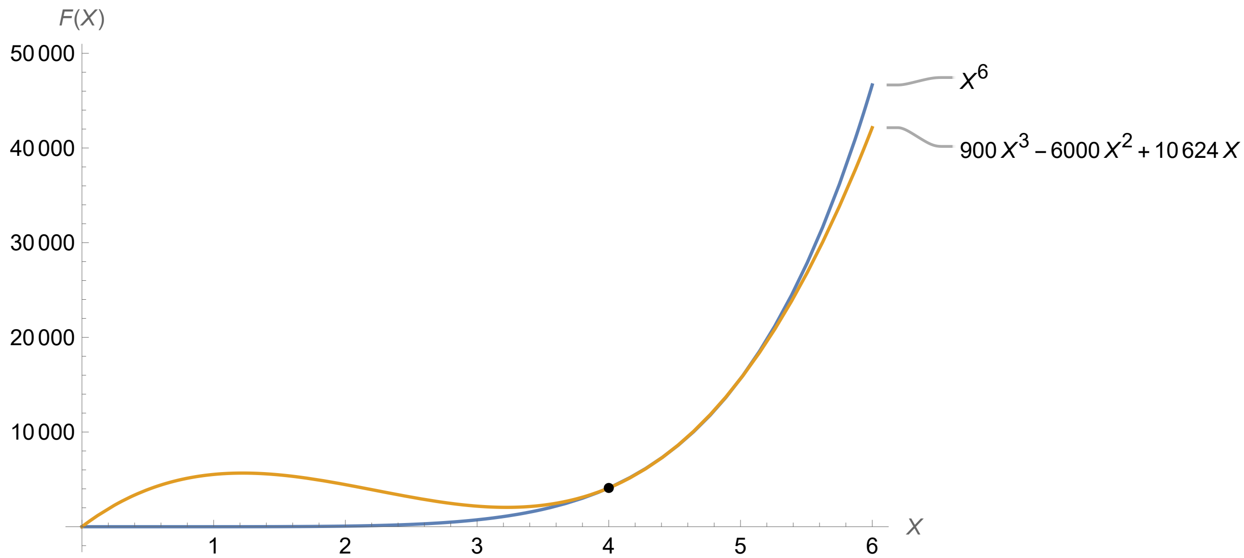

Figure 2. Polynomial plot $P(2, X, 4) \cdot X$ with sixth power X^6. Convergence interval: $3.9 \leq X \leq 5.1$ with percentage error $E < 3\%$.

Therefore, we have reached the statement that the polynomial $P(m, X, N)$ is an m-degree polynomial in X, having fixed non-negative integers m and N. It approximates the power function X^j in some neighborhood of fixed N. The length of convergence interval between power function X^j and $P(m, X, N) \cdot X^K$ rises as N rise.

2. Generalizations

Previously, we have discussed that polynomial $P(m, X, N)$ approximates power function X^j in some neighborhood of fixed non-negative integer N. Approximation by $P(m, X, N)$ can be adjusted by X^k multiplication for even exponents of power function. In general, it is safe to say that power function X^j is approximated by $P(m, X, N) \cdot X^k$ where $k = 0$ for odd exponent j and k either $k = 1$ or $k = -1$ for even exponent j. Therefore, for arbitrary exponent j in X^j we have

$$X^j \approx \begin{cases} P(m, X, N) & j = 2m + 1 \\ P(m, X, N) \cdot X & j = 2m + 2 \\ P(m, X, N) \cdot X^{-1} & j = 2m \end{cases}$$

Of course, there are other variations of the value of k, we stick to simple case for the moment.

As we also discussed, the lenght L of convergence interval between X^j and approximation by $P(m, X, N)$ rises as N rise. However, the convergence interval is still bounded, which could not satisfy certain approximation scenarios. Depending on approximation requirements in terms of convergence interval length L a single polynomial $P(m, X, N)$ with fixed m and N can be unfit. Here is the place where spline approximation comes to play. The spline $S(x)$ is piecewise defined function over the interval $(x_0, \ldots x_n)$

$$S(x) = \begin{cases} f_1(x), & x_0 \leq x < x_1 \\ f_2(x), & x_0 \leq x < x_1 \\ \vdots & \vdots \\ f_n(x), & x_{n-1} \leq x \leq x_n \end{cases}$$

The given points x_k are called *knots*.

Assume that approximation requirement in terms of convergence length L is to approximate the power function X^j bounded by real points A and B such that $A < B$. Splines perfectly fits the need to match arbitrary convergence range of power function's X^j approximation by $P(m, X, N)$. Formally,

$$X^j \approx \begin{cases} P(m, X, N + t_1) \cdot X^k & x_0 \leq x < x_1 \\ P(m, X, N + t_2) \cdot X^k & x_0 \leq x < x_1 \\ \vdots & \vdots \\ P(m, X, N + t_{n-1}) \cdot X^k & x_{n-1} \leq x < x_n \end{cases}$$

The values of t_r to be adjusted according to approximation requirements in terms of accuracy.

3. Use cases

Use case scenarios of the approximation technique we discuss have their own constraints and limitations. For instance, approximation requirements should have precisely specified exponent j in X^j because for each j there is a matching polynomial $P(m, X, N)$. Perfectly, there should be a set precompiled polynomials $P(m, X, N)$ matching precise exponent j in X^j over precisely defined approximation range with required error of approximation E constraint. Generally, approximation of power function X^j by $P(m, X, N)$ can be split by the following steps

(1) Define exponent j in X^j
(2) Define required error threshold E
(3) Define required interval of approximation I
(4) Choose and precompile polynomials $P(m, X, N)$ such that required interval of approximation and error threshold E are satisfied
(5) Define a set of knots so that error threshold E and interval of approximation I requirements are satisfied

Defining set of spline knots essentially requires inspection of convergence intervals of between X^j and $P(m, X, N + t_k)$ by choosing knots such that interval of approximation and error threshold are satisfied. Consider an example. Let be the following approximation requirement

(1) Exponent $j = 3$
(2) Percentage error threshold $E \leq 1\%$
(3) Interval of approximation $10 \leq X \leq 15$

Now we have to choose a set of polynomials $P(m, X, N + t_k)$ based on which we adjust a set of spline knots. We can safely choose integers t_k in range $10 \leq t_k \leq 15$ because of the following properties of $P(m, X, N)$.

$$P(m, X, X) = X^{2m+1}$$

$$P(m, X, X + 1) = (X + 1)^{2m+1} - 1$$

Therefore, for each two consequential points $N = X, N = X + 1$ the absolute difference is 1, making that range at least 1% percentage error for $X^j \leq 100$. I have chosen the approximation range $10 \leq X \leq 15$ and $j = 3$ intentionally to show spline approximation with percentage error lesser than 1%. Therefore, to approximate X^3 in range $10 \leq X \leq 15$, we have to following spline function

$$X^3 \approx \begin{cases} P(m, X, 10) = -2300 + 330X, & 10 \leq X < 11 \\ P(m, X, 11) = -3025 + 396X, & 11 \leq X < 12 \\ P(m, X, 12) = -3888 + 468X, & 12 \leq X < 13 \\ P(m, X, 13) = -4901 + 546X, & 13 \leq X < 14 \\ P(m, X, 14) = -6076 + 630X, & 14 \leq X \leq 15 \end{cases}$$

Graphically this linear approximation of cubes looks as follows

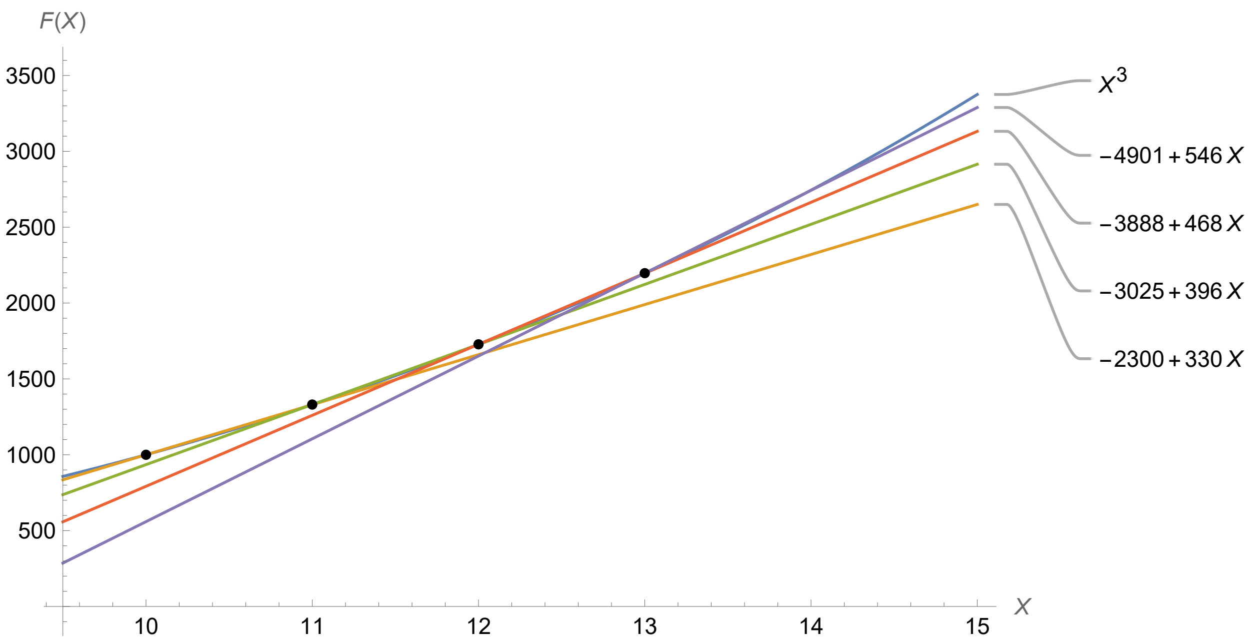

Figure 3. Approximation of cubes X^3 by splines of $P(m, X, N)$.

where spline knots are integers in range $10 \leq N \leq 14$.

The same principle applies for even exponent $j = 4$ in X^j with the same convergence interval $10 \le X \le 15$ and approximation error under 1% constraints

$$X^4 \approx \begin{cases} -2300X + 330X^2, & 10 \le X < 11 \\ -3025X + 396X^2, & 11 \le X < 12 \\ -3888X + 468X^2, & 12 \le X < 13 \\ -4901X + 546X^2, & 13 \le X < 14 \\ -6076X + 630X^2, & 14 \le X \le 15 \end{cases}$$

Which graphically looks as follows

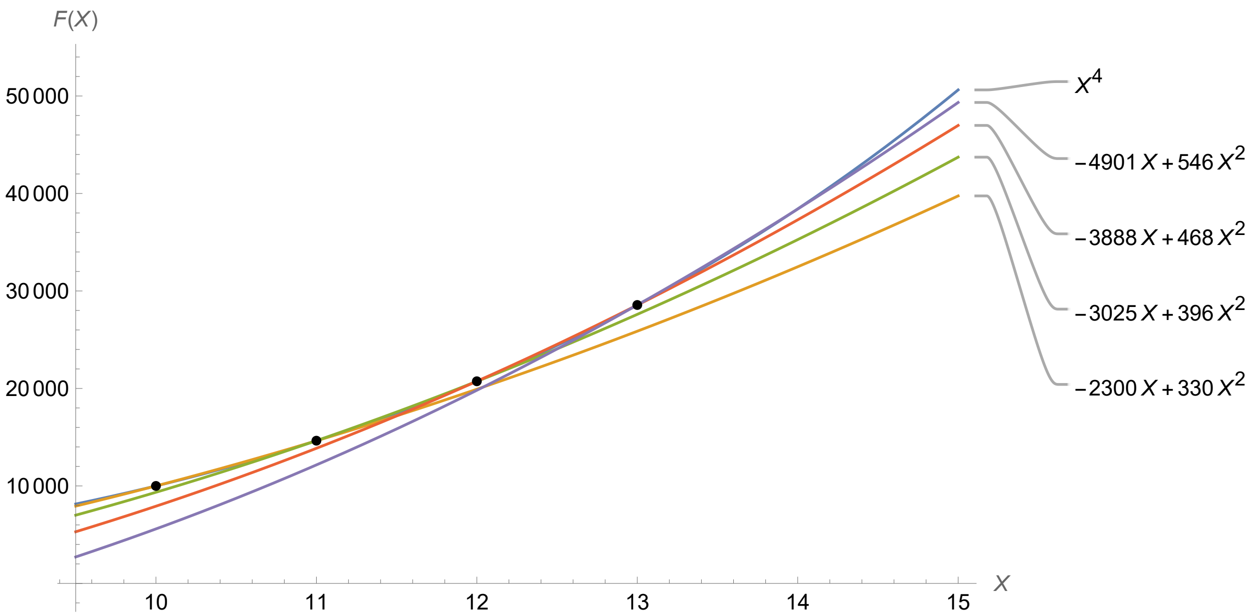

Figure 4. Approximation of cubes X^4 by splines of $P(m, X, N) \cdot X$.

In general, for each variation of X^j such that $X^j \ge 100$ the approximation can be done using splines in $P(m, X, N)$ over the interval $A \le X \le B$ with spline knot vector be the integers in range from A to B so that knots vector is $\{A, A+1, A+2, \ldots, B\}$. Because,

$$P(m, X, X) = X^{2m+1}$$

$$P(m, X, X+1) = (X+1)^{2m+1} - 1$$

This furthermore can be optimized depending on value of N in $P(m,X,N)$ because convergence interval with power function X^j increases as N rises.

4. Conclusions

We have established that $P(m,X,N)$ is an m-degree polynomial in $X \in \mathbb{R}$ having fixed non-negative integers m and N.

The polynomial $P(m,X,N)$ is a result of rearrangement inside Faulhaber's formula that was inspired by Knuth's *Johann Faulhaber and sums of powers*.

In this manuscript we have discussed approximation properties of polynomial $P(m,X,N)$.

In particular, the polynomial $P(m,X,N)$ approximates odd power function X^{2m+1} in certain neighborhood of fixed non-negative integer N with percentage error lesser than 1%.

By increasing the value of N the length of convergence interval with odd-power X^{2m+1} increasing as well.

Furthermore, this approximation technique is generalized for arbitrary non-negative exponent power function X^j by using splines.

In general, for each variation of X^j such that $X^j \geq 100$ the approximation can be done using splines in $P(m,X,N)$ over the interval $A \leq X \leq B$ with spline knot vector be the integers in range from A to B so that knots vector is $\{A, A+1, A+2, \dots, B\}$. Because,

$$P(m,X,X) = X^{2m+1}$$

$$P(m,X,X+1) = (X+1)^{2m+1} - 1$$

This can be further optimized depending on value of N in $P(m,X,N)$ because convergence interval with power function X^j increases as N rises.

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Software Developer, DevOps Engineer

Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io