AN EFFICIENT METHOD OF SPLINE APPROXIMATION FOR POWER FUNCTION

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Abstract. Let $P(m, X, N)$ be an m-degree polynomials in $X \in \mathbb{R}$ having fixed non-negative integers m and N. In this manuscript an efficient method of spline approximation for power function is shown and discussed. Approximation technique is based on the fact that polynomial $P(m, X, N)$ approximates odd-power function X^{2m+1} for a in some neighborhood of fixed N.

Contents

1. Introduction

Consider the m-degree polynomial $P(m, X, N)$ having fixed non-negative integers m and N

$$P(m, X, N) = \sum_{r=0}^{m} \sum_{k=1}^{N} \mathbf{A}_{m,r} k^r (X-k)^r$$

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Sources: https://github.com/kolosovpetro/AnEfficientMethodOfSplineApproximation

For example

$$P(2,X,0)=0$$

$$P(2,X,1)=30X^2-60X+31$$

$$P(2,X,2)=150X^2-540X+512$$

$$P(2,X,3)=420X^2-2160X+2943$$

$$P(2,X,4)=900X^2-6000X+10624$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively, see [1, 2, 3, 4]. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [5, 6].

Essentially, the polynomial $P(m,X,N)$ is a result of rearrangement inside Faulhaber's formula. It was inspired by Knuth's *Johann Faulhaber and sums of powers*, see [7]. In particular, the polynomial $P(m,X,N)$ yields an identity for odd powers

$$P(m,X,X)=X^{2m+1}$$

In extended form

$$X^{2m+1}=\sum_{r=0}^{m}\sum_{k=1}^{X}\mathbf{A}_{m,r}k^r(X-k)^r$$

Precisely, the relation between Faulhaber's formula and $P(m,X,N)$ is shown by [8].

However, apart polynomial identity for odd powers, I've spotted several approximation properties of $P(m, X, N)$. Therefore, in this manuscript we discuss approximation properties of polynomial $P(m, X, N)$. I use a few well-known criteria to measure and estimate error of approximation: Absolute error, Relative error and Percentage error. Assume that function $f_2(x)$ approximates the function $f_1(x)$ then the errors are

$$\text{Absolute Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|}$$
$$\text{Relative Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|}$$
$$\text{Percentage Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|} \times 100\%$$

Diving straight to the point, we switch our focus to already mentioned polynomial $P(2, X, 4) = 900X^2 - 6000X + 10624$ to show the first example of how it approximates the odd power function X^5. In fact, we approximate the polynomial X^{2m+1} by lower degree polynomial X^m as the following image presents

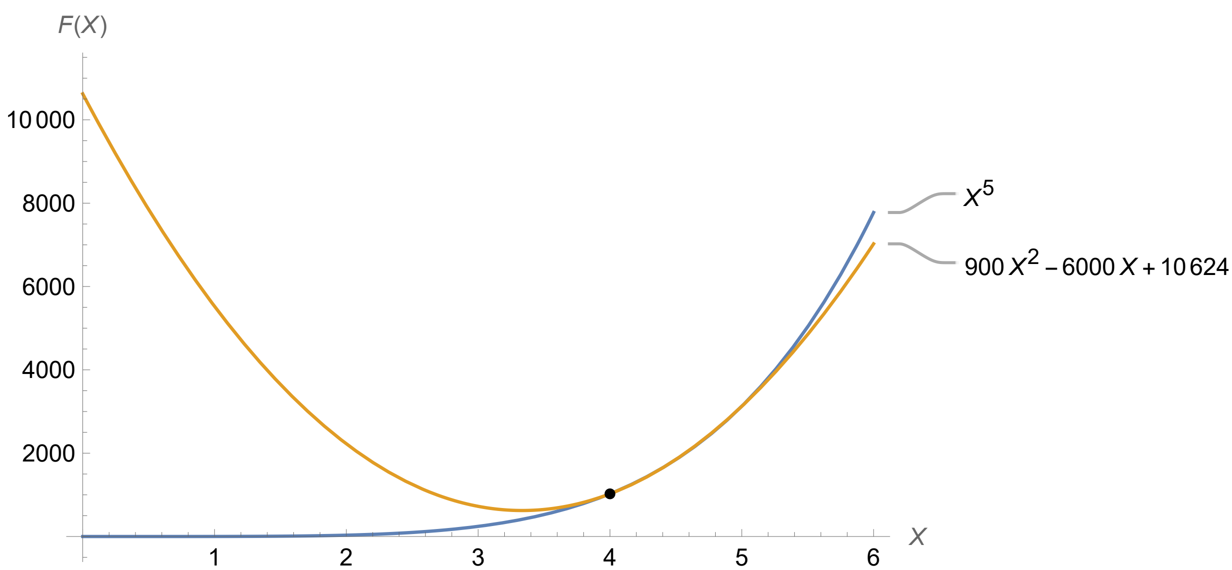

Figure 1. Polynomial plot $P(2, X, 4)$ with fifth power X^5. Points of intersection $X = 4$, $X = 4.42472$, $X = 4.99181$. Interval of convergence: $3.9 \leq X \leq 5.3$ with percentage error $E < 2\%$.

As we see, the interval $3.9 \leq X \leq 5.3$ has the percentage error lesser than 2% which is quite impressive. Therefore, having fixed $N = 4$ the polynomial $P(2, X, 4)$ approximates odd power in neighborhood of $N = 4$ which is $3.9 \leq X \leq 5.3$. To showcase the concrete values of absolute, relative and percentage errors of approximation above, I attach a separate table in addenda.

One more interesting observation can be done by increasing the value of N in $P(m, X, N)$ having fixed m, it follows that by increasing N the length of interval of convergence with odd-power X^{2m+1} increasing as well. For instance,

- Having $P(2, X, 4)$ the interval of convergence with percentage error lesser than 1% is $4.0 \leq X \leq 5.1$
- Having $P(2, X, 20)$ the interval of convergence with percentage error lesser than 1% is $18.7 \leq X \leq 22.9$
- Having $P(2, X, 120)$ the interval of convergence with percentage error lesser than 1% is $110.0 \leq X \leq 134.7$

The reason why the lenght of convergence interval rises as N rise lays beneath the implicit form of polynomial $P(m, X, N)$ meaning that

$$P(m, X, N) = \sum_{r=0}^{m} (-1)^{m-r} U(m, N, r) \cdot X^r$$

where $U(m, N, r)$ is a polynomial defined as follows

$$U(m, N, r) = (-1)^m \sum_{k=1}^{N} \sum_{j=r}^{m} \binom{j}{r} \mathbf{A}_{m,j} k^{2j-r} (-1)^j$$

which rises as N rise.

To wrap up the current state of the manuscript, refresh the key facts and finding we got so far, therefore, the polynomial $P(m, X, N)$ is an m-degree polynomial in $X \in \mathbb{R}$, having fixed non-negative integers m and N. It approximates odd power function in some neighborhood of fixed N. The length of interval of convergence between X^{2m+1} and $P(m, X, N)$ rises as N rise.

For the sake of clear and definite results verification I attach mathematica programs to generate plots and data tables so that reader is able to verify the main results of current part of manuscript, see the link.

So far we have discussed approximation of odd power function, now we focus on its even case, which is quite straightforward. Considering the same example $P(2, X, 4)$ we reach the approximation of even power X^6 by means of multiplication, which graphically looks as follows

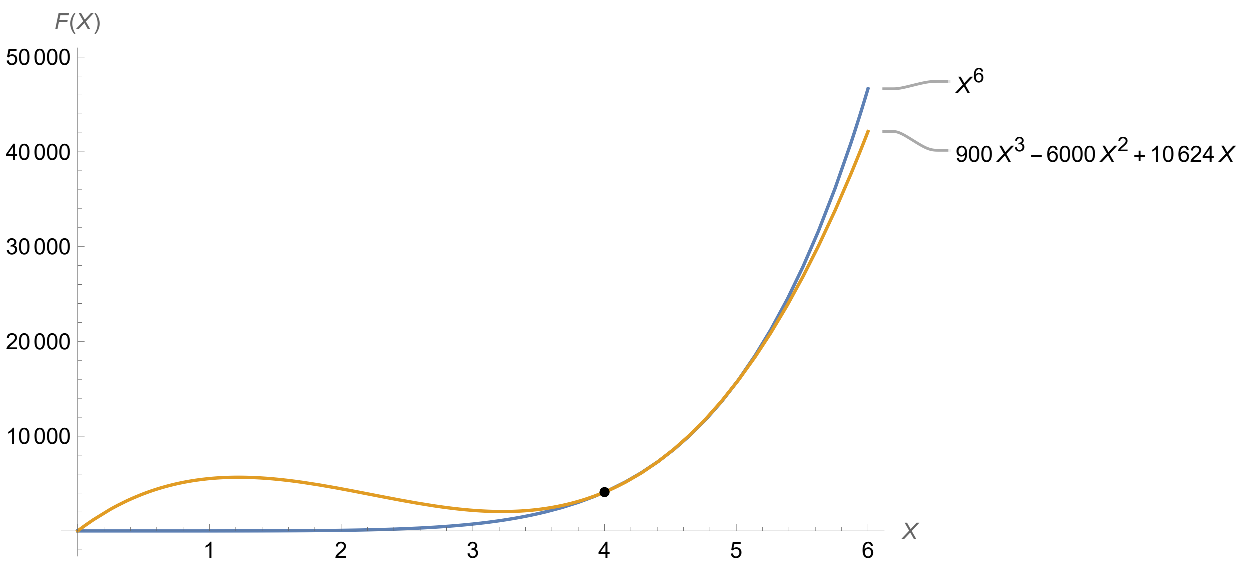

Figure 2. Polynomial plot $P(2, X, 4) \cdot X$ with sixth power X^6. Interval of convergence: $3.9 \leq X \leq 5.3$ with percentage error $E < 2\%$.

Therefore, we have reached the statement that the polynomial $P(m, X, N)$ is an m-degree polynomial in X, having fixed non-negative integers m and N. It approximates power function in some neighborhood of fixed N. The length of interval of convergence between power function and $P(m, X, N)$ or $P(m, X, N) \cdot X$ rises as N rise.

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2. Addendum

Table 2. Comparison of X^5 and $P(2, X, 4) = 900X^2 - 6000X + 10624$

X	X^5	$900X^2 - 6000X + 10624$	**ABS**	**Relative**	**% Error**
3.8	792.352	820.0	27.6483	0.034894	3.4894
3.9	902.242	913.0	10.758	0.0119236	1.19236
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1153.0	5.56201	0.00480079	0.480079
4.2	1306.91	1300.0	6.91232	0.00528905	0.528905
4.3	1470.08	1465.0	5.08443	0.0034586	0.34586
4.4	1649.16	1648.0	1.16224	0.000704746	0.0704746
4.5	1845.28	1849.0	3.71875	0.00201528	0.201528
4.6	2059.63	2068.0	8.37024	0.00406395	0.406395
4.7	2293.45	2305.0	11.5499	0.00503605	0.503605
4.8	2548.04	2560.0	11.9603	0.00469393	0.469393
4.9	2824.75	2833.0	8.24751	0.00291973	0.291973
5.0	3125.0	3124.0	1.0	0.00032	0.032
5.1	3450.25	3433.0	17.2525	0.00500036	0.500036
5.2	3802.04	3760.0	42.0403	0.0110573	1.10573
5.3	4181.95	4105.0	76.9549	0.0184017	1.84017
5.4	4591.65	4468.0	123.65	0.0269294	2.69294
5.5	5032.84	4849.0	183.844	0.0365288	3.65288
5.6	5507.32	5248.0	259.318	0.047086	4.7086
5.7	6016.92	5665.0	351.921	0.0584885	5.84885
5.8	6563.57	6100.0	463.568	0.0706274	7.06274
5.9	7149.24	6553.0	596.243	0.0833995	8.33995