

# AN EFFICIENT METHOD OF SPLINE APPROXIMATION FOR POWER FUNCTION

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ABSTRACT. Let  $P(m, X, N)$  be an  $m$ -degree polynomials in  $X \in \mathbb{R}$  having fixed non-negative integers  $m$  and  $N$ . Essentially, the polynomial  $P(m, X, N)$  is a result of a rearrangement inside Faulhaber's formula in the context of Knuth's work *Johann Faulhaber and sums of powers*.

In this manuscript we discuss the approximation properties of polynomial  $P(m, X, N)$ . In particular, the polynomial  $P(m, X, N)$  approximates the odd power function  $X^{2m+1}$  in a certain neighborhood of a fixed non-negative integer  $N$  with a percentage error less than 1%.

By increasing the value of  $N$  the length of convergence interval with odd-power  $X^{2m+1}$  also increases.

Furthermore, this approximation technique is generalized for arbitrary non-negative exponent of the power function  $X^j$  by using splines.

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## 1. INTRODUCTION

Consider the  $m$ -degree polynomial  $P(m, X, N)$  having fixed non-negative integers  $m$  and  $N$

$$P(m, X, N) = \sum_{r=0}^m \sum_{k=1}^N \mathbf{A}_{m,r} k^r (X - k)^r$$

For example

$$P(2, X, 0) = 0$$

$$P(2, X, 1) = 30X^2 - 60X + 31$$

$$P(2, X, 2) = 150X^2 - 540X + 512$$

$$P(2, X, 3) = 420X^2 - 2160X + 2943$$

$$P(2, X, 4) = 900X^2 - 6000X + 10624$$

where  $\mathbf{A}_{m,r}$  is a real coefficient defined recursively, see [1, 2, 3, 4]. For example,

$m/r$	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

**Table 1.** Coefficients  $\mathbf{A}_{m,r}$ . See the OEIS sequences [5, 6].

Essentially, the polynomial  $P(m, X, N)$  is derived from a rearrangement of Faulhaber's formula. It was inspired by Knuth's *Johann Faulhaber and sums of powers*, see [7]. In particular, the polynomial  $P(m, X, N)$  yields an identity for odd powers

$$P(m, X, X) = X^{2m+1}$$

In its extended form

$$X^{2m+1} = \sum_{r=0}^m \sum_{k=1}^X \mathbf{A}_{m,r} k^r (X - k)^r$$

The exact relation between Faulhaber's formula and  $P(m, X, N)$  is shown by [8].

However, apart from the polynomial identity for odd powers, I've discovered several approximation properties of  $P(m, X, N)$ . Therefore, in this manuscript we explore the approximation properties of the polynomial  $P(m, X, N)$ . I use a few well-known criteria to measure and estimate error of approximation: Absolute error, Relative error and Percentage error. Assume that the function  $f_2(x)$  approximates the function  $f_1(x)$ , then errors are given by

$$\text{Absolute Error} = |f_1(x) - f_2(x)|$$

$$\text{Relative Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|}$$

$$\text{Percentage Error} = \frac{|f_1(x) - f_2(x)|}{|f_1(x)|} \times 100\%$$

Diving straight into the point, we switch our focus to the previously mentioned polynomial  $P(2, X, 4) = 900X^2 - 6000X + 10624$  to show the first example of how it approximates the odd power function  $X^5$ . In fact, we approximate the polynomial  $X^{2m+1}$  using a lower-degree polynomial of degree  $m$ , as shown in the following image



**Figure 1.** Approximation of fifth power  $X^5$  by  $P(2, X, 4)$ . Points of intersection  $X = 4$ ,  $X = 4.42472$ ,  $X = 4.99181$ . Convergence interval is  $4.0 \leq X \leq 5.1$  with percentage error  $E < 1\%$ .

As observed, the polynomial  $P(2, X, 4)$  approximates  $X^5$  in the neighborhood of  $N = 4$  with the convergence interval  $4.0 \leq X \leq 5.1$  where the percentage error is less than 1% which is quite remarkable. The following table presents specific values of absolute, relative, and percentage errors for this approximation

<b>X</b>	$X^5$	$900X^2 - 6000X + 10624$	<b>ABS</b>	<b>Relative</b>	<b>% Error</b>
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1153.0	5.56201	0.00480079	0.480079
4.2	1306.91	1300.0	6.91232	0.00528905	0.528905
4.3	1470.08	1465.0	5.08443	0.0034586	0.34586
4.4	1649.16	1648.0	1.16224	0.000704746	0.0704746
4.5	1845.28	1849.0	3.71875	0.00201528	0.201528
4.6	2059.63	2068.0	8.37024	0.00406395	0.406395
4.7	2293.45	2305.0	11.5499	0.00503605	0.503605
4.8	2548.04	2560.0	11.9603	0.00469393	0.469393
4.9	2824.75	2833.0	8.24751	0.00291973	0.291973
5.0	3125.0	3124.0	1.0	0.00032	0.032
5.1	3450.25	3433.0	17.2525	0.00500036	0.500036

**Table 2.** Comparison of  $X^5$  and  $P(2, X, 4) = 900X^2 - 6000X + 10624$

One more interesting observation arises by increasing the value of  $N$  in  $P(m, X, N)$  while keeping  $m$  fixed. As  $N$  increases, the length of the convergence interval with the odd-power  $X^{2m+1}$  also increases. For instance,

- For  $P(2, X, 4)$  and  $X^5$ , the convergence interval with a percentage error less than 1% is  $4.0 \leq X \leq 5.1$ , with a length  $L = 1.1$
- For  $P(2, X, 20)$  and  $X^5$ , the convergence interval with a percentage error less than 1% is  $18.7 \leq X \leq 22.9$ , with a length  $L = 4.2$
- For  $P(2, X, 120)$  and  $X^5$ , the convergence interval with a percentage error less than 1% is  $110.0 \leq X \leq 134.7$ , with a length  $L = 24.7$

The reason behind this behavior lies in the implicit form of the polynomial  $P(m, X, N)$ , meaning that

$$P(m, X, N) = \sum_{r=0}^m (-1)^{m-r} U(m, N, r) \cdot X^r$$

where  $U(m, N, r)$  is a polynomial defined as follows

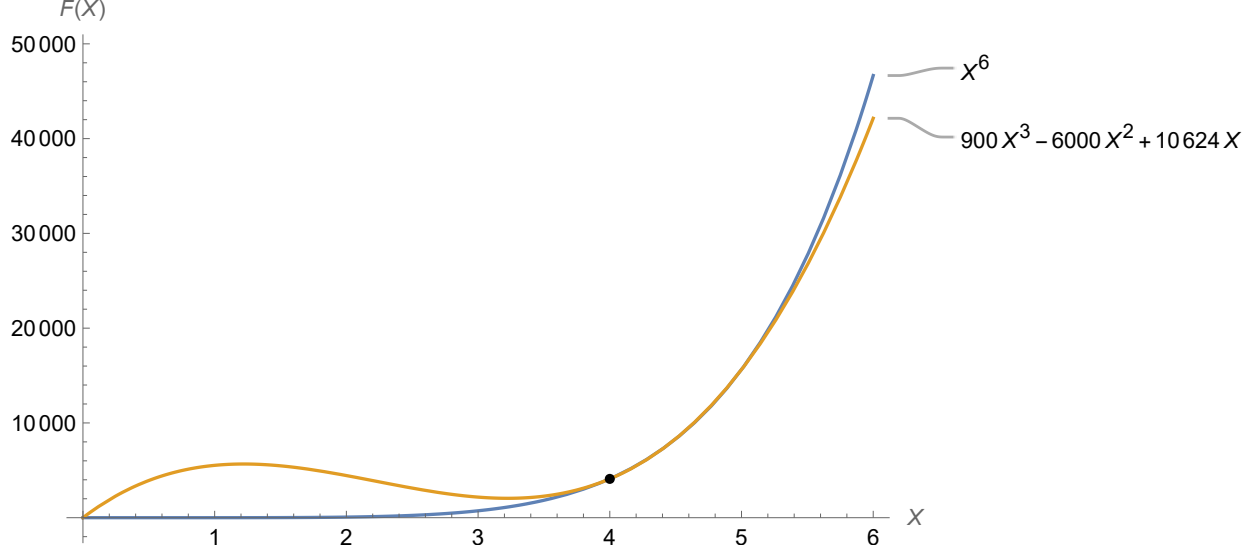
$$U(m, N, r) = (-1)^m \sum_{k=1}^N \sum_{j=r}^m \binom{j}{r} \mathbf{A}_{m,j} k^{2j-r} (-1)^j$$

which grows as  $N$  increases. Few cases of coefficients  $U(m, N, r)$  are registered as OEIS sequences [9, 10, 11].

To summarize, let us recap the key findings so far. The polynomial  $P(m, X, N)$  is an  $m$ -degree polynomial in  $X \in \mathbb{R}$  with fixed non-negative integers  $m$  and  $N$ . It approximates the odd power function  $X^{2m+1}$  within a specific neighborhood of  $N$ . The length  $L$  of the convergence interval between  $X^{2m+1}$  and  $P(m, X, N)$  increases as  $N$  grows.

For the sake of clear and precise verification of results, I attach mathematica programs to generate plots and data tables, so that reader is able to verify the main results of current part of manuscript, see the [link](#).

So far we have discussed approximation of odd power function  $X^{2m+1}$ , now we focus on its even case  $X^{2m+2}$  which is quite straightforward. Considering the same example  $P(2, X, 4)$  we reach the approximation of even power  $X^6$  by means of  $K$ -times multiplication by  $X$ , with graphic representation as follows



**Figure 2.** Approximation of sixth power  $X^6$  by  $P(2, X, 4) \cdot X$ . Convergence interval is  $3.9 \leq X \leq 5.1$  with percentage error  $E < 3\%$ .

Therefore, we have reached the statement that the polynomial  $P(m, X, N)$  is an  $m$ -degree polynomial in  $X$ , having fixed non-negative integers  $m$  and  $N$ . It approximates the power function  $X^j$  in some neighborhood of fixed  $N$ . The length of convergence interval between power function  $X^j$  and  $P(m, X, N) \cdot X^K$  rises as  $N$  rise.

## 2. GENERALIZATIONS

Previously, we have discussed that polynomial  $P(m, X, N)$  approximates power function  $X^j$  in some neighborhood of fixed non-negative integer  $N$ . Approximation by  $P(m, X, N)$  can be adjusted by  $X^k$  multiplication for even exponents of power function. In general, it is safe to say that power function  $X^j$  is approximated by  $P(m, X, N) \cdot X^k$  where  $k = 0$  for odd exponent  $j$  and  $k$  is either  $k = 1$  or  $k = -1$  for an even exponent  $j$ . Therefore, for arbitrary exponent  $j$  in  $X^j$  we have

$$X^j \approx \begin{cases} P(m, X, N) & j = 2m + 1 \\ P(m, X, N) \cdot X & j = 2m + 2 \\ P(m, X, N) \cdot X^{-1} & j = 2m \end{cases}$$

Of course, there are other variations of the value of  $k$ , but we will stick to the simple case for the moment.

As we also discussed, the length  $L$  of the convergence interval between  $X^j$  and its approximation by  $P(m, X, N)$  increases as  $N$  grow. However, the convergence interval is still bounded, which could not satisfy certain approximation scenarios. Depending on the approximation requirements in terms of convergence interval length  $L$  a single polynomial  $P(m, X, N)$  with fixed  $m$  and  $N$  may be unsuitable. Here is the place where spline approximation comes into play. The spline  $S(x)$  is piecewise defined function over the interval  $(x_0, \dots, x_n)$

$$S(x) = \begin{cases} f_1(x), & x_0 \leq x < x_1 \\ f_2(x), & x_1 \leq x < x_2 \\ \vdots & \vdots \\ f_n(x), & x_{n-1} \leq x \leq x_n \end{cases}$$

The given points  $x_k$  are called *knots*.

Assume that the approximation requirement in terms of convergence length  $L$  is to approximate the power function  $X^j$  bounded by real points  $A$  and  $B$  such that  $A < B$ . Splines perfectly fit the need to match an arbitrary convergence range for the power function  $X^j$  using the approximation by  $P(m, X, N)$ . Formally,

$$X^j \approx \begin{cases} P(m, X, N + t_1) \cdot X^k & x_0 \leq x < x_1 \\ P(m, X, N + t_2) \cdot X^k & x_1 \leq x < x_2 \\ \vdots & \vdots \\ P(m, X, N + t_{n-1}) \cdot X^k & x_{n-1} \leq x < x_n \end{cases}$$

The values of  $t_r$  to be adjusted according to approximation requirements in terms of accuracy.



### 3. USE CASES

Use case scenarios of the approximation technique we discuss have their own constraints and limitations. For instance, approximation requirements should have precisely specified exponent  $j$  in  $X^j$  because for each  $j$  there is a matching polynomial  $P(m, X, N)$ . Perfectly, there should be a set precompiled polynomials  $P(m, X, N)$  matching precise exponent  $j$  in  $X^j$  over precisely defined approximation range with required error of approximation  $E$  constraint. Generally, approximation of power function  $X^j$  by  $P(m, X, N)$  can be split by the following steps

- (1) Define exponent  $j$  in  $X^j$
- (2) Define required error threshold  $E$
- (3) Define required interval of approximation  $I$
- (4) Choose and precompile polynomials  $P(m, X, N)$  such that required interval of approximation and error threshold  $E$  are satisfied
- (5) Define a set of knots so that error threshold  $E$  and interval of approximation  $I$  requirements are satisfied

Defining set of spline knots essentially requires inspection of convergence intervals of between  $X^j$  and  $P(m, X, N + t_k)$  by choosing knots such that interval of approximation and error threshold are satisfied. Consider an example. Let be the following approximation requirement

- (1) Exponent  $j = 3$
- (2) Percentage error threshold  $E \leq 1\%$
- (3) Interval of approximation  $10 \leq X \leq 15$

Now we have to choose a set of polynomials  $P(m, X, N + t_k)$  based on which we adjust a set of spline knots. We can safely choose integers  $t_k$  in range  $10 \leq t_k \leq 15$  because of the

following properties of  $P(m, X, N)$ .

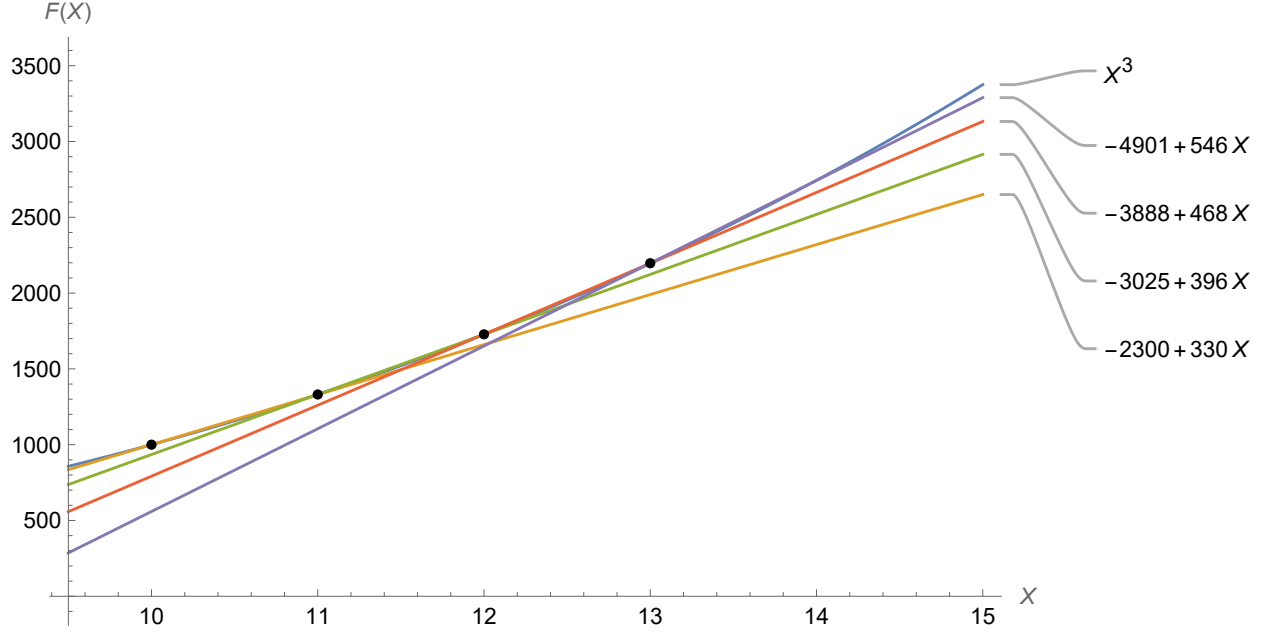
$$P(m, X, X) = X^{2m+1}$$

$$P(m, X, X+1) = (X+1)^{2m+1} - 1$$

Therefore, for each two consequential points  $N = X, N = X + 1$  the absolute difference is 1, making that range at least 1% percentage error for  $X^j \leq 100$ . I have chosen the approximation range  $10 \leq X \leq 15$  and  $j = 3$  intentionally to show spline approximation with percentage error lesser than 1%. Therefore, to approximate  $X^3$  in range  $10 \leq X \leq 15$ , we have to following spline function

$$X^3 \approx \begin{cases} P(1, X, 10) = -2300 + 330X, & 10 \leq X < 11 \\ P(1, X, 11) = -3025 + 396X, & 11 \leq X < 12 \\ P(1, X, 12) = -3888 + 468X, & 12 \leq X < 13 \\ P(1, X, 13) = -4901 + 546X, & 13 \leq X < 14 \\ P(1, X, 14) = -6076 + 630X, & 14 \leq X \leq 15 \end{cases} \quad (1)$$

Graphically this linear approximation of cubes looks as follows



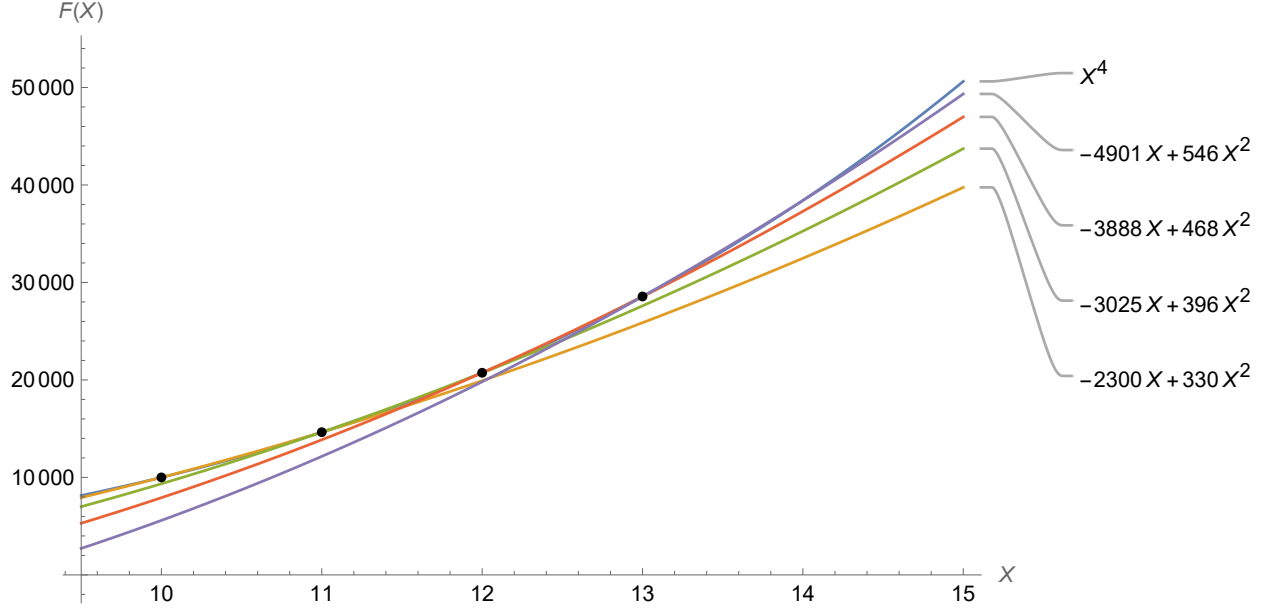
**Figure 3.** Approximation of cubes  $X^3$  by splines (1). Convergence interval is  $10 \leq X \leq 15$  with percentage error  $E \leq 1\%$ .

where spline knots are integers in range  $10 \leq N \leq 14$ .

The same principle applies for even exponent  $j = 4$  in  $X^j$  with the same convergence interval  $10 \leq X \leq 15$  and approximation error under 1% constraints

$$X^4 \approx \begin{cases} P(1, X, 10) \cdot X = -2300X + 330X^2, & 10 \leq X < 11 \\ P(1, X, 10) \cdot X = -3025X + 396X^2, & 11 \leq X < 12 \\ P(1, X, 10) \cdot X = -3888X + 468X^2, & 12 \leq X < 13 \\ P(1, X, 10) \cdot X = -4901X + 546X^2, & 13 \leq X < 14 \\ P(1, X, 10) \cdot X = -6076X + 630X^2, & 14 \leq X \leq 15 \end{cases} \quad (2)$$

Which graphically looks as follows



**Figure 4.** Approximation of cubes  $X^4$  by splines (2). Convergence interval is  $10 \leq X \leq 15$  with percentage error  $E \leq 1\%$ .

In general, for each variation of  $X^j$  such that  $X^j \geq 100$  the approximation can be done using splines in  $P(m, X, N)$  over the interval  $A \leq X \leq B$  with spline knot vector be the integers in range from  $A$  to  $B$  so that knots vector is  $\{A, A + 1, A + 2, \dots, B\}$ . Because,

$$P(m, X, X) = X^{2m+1}$$

$$P(m, X, X + 1) = (X + 1)^{2m+1} - 1$$

This furthermore can be optimized depending on value of  $N$  in  $P(m, X, N)$  because convergence interval with power function  $X^j$  increases as  $N$  rises.

#### 4. CONCLUSIONS

We have established that  $P(m, X, N)$  is an  $m$ -degree polynomial in  $X \in \mathbb{R}$  having fixed non-negative integers  $m$  and  $N$ .

The polynomial  $P(m, X, N)$  is a result of rearrangement inside Faulhaber's formula that was inspired by Knuth's *Johann Faulhaber and sums of powers*.

In this manuscript we have discussed approximation properties of polynomial  $P(m, X, N)$ .

In particular, the polynomial  $P(m, X, N)$  approximates odd power function  $X^{2m+1}$  in certain neighborhood of fixed non-negative integer  $N$  with percentage error lesser than 1%.

By increasing the value of  $N$  the length of convergence interval with odd-power  $X^{2m+1}$  increasing as well.

Furthermore, this approximation technique is generalized for arbitrary non-negative exponent power function  $X^j$  by using splines.

In general, for each variation of  $X^j$  such that  $X^j \geq 100$  the approximation can be done using splines in  $P(m, X, N)$  over the interval  $A \leq X \leq B$  with spline knot vector be the integers in range from  $A$  to  $B$  so that knots vector is  $\{A, A + 1, A + 2, \dots, B\}$ . Because,

$$P(m, X, X) = X^{2m+1}$$

$$P(m, X, X + 1) = (X + 1)^{2m+1} - 1$$

This can be further optimized depending on value of  $N$  in  $P(m, X, N)$  because convergence interval with power function  $X^j$  increases as  $N$  rises.

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