DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

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Abstract. https://mathoverflow.net/a/297916/113033

1. Introduction

Assuming that following odd power identity holds

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

Our main goal is to identify the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that identity above is true.

Although, the recurrence relation is already given at [1], a few key points in proof are worth to discuss additionally.

The main idea of Alekseyev's approach was to utilize a recurrence relation to evaluate the set of coefficients $\mathbf{A}_{m,r}$, starting from the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ by using backtracking, continuing similarly up to $\mathbf{A}_{m,0}$.

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By applying Binomial theorem $(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}k^t$ and Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}$, we get

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r}$$
$$= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right]$$
$$= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right]$$
$$= \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$
$$= \left[\sum_{j,t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

By rearranging the sums we obtain

$$= \left[\sum_j B_j n^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] - \left[\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \tag{2}$$

We can notice that

$$\sum_t\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j=0 \\ \frac{(-1)^r}{j}\binom{r}{2r-j+1} & \text{if } j>0 \end{cases} \tag{3}$$

An elegant proof of the binomial identity (3) is done by Markus Scheuer in [2].

In particular, the equation (3) is zero for $0 < t \leq j$.

To utilize the equation (3), we have to move $j = 0$ out of summation in (2) to avoid division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j\geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right]$$
$$- \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

Now we do not care about division by zero in $\frac{(-1)^r}{j}$, hence by simplifying the equation above we get

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j\geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1}\right]}_{(\star)}$$
$$- \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]}_{(\diamond)}$$

By introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms across two sums

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right]$$
$$- \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right]$$
$$= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by the odd-power identity (1), we obtain the following relation for polynomials in n

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r,\text{ odd } \ell} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by k we get

$$T = \sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r,\, k} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1} \tag{4}$$

Taking the coefficient of n^{2m+1} in (4) yields

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \tag{5}$$

because $\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1$.

That's may not be immediately clear why the coefficient of n^{2m+1} in (4) is $(2m+1)\binom{2m}{m}$.

$$[n^{2m+1}]T = \mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} + 2 \sum_r \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2m} \binom{r}{2m+1} B_{2r-2m} \equiv 1$$

The sum $2 \sum_r \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2m} \binom{r}{2m+1} B_{2r-2m}$ collapses because r runs over the interval $0 \le r \le m$ making the coefficient $\binom{r}{2m+1} = 0$ for every value of r.

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \le d < m$, we get

$$[n^{2d+1}]T = \mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2 \sum_r \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} \equiv 0$$

$$\mathbf{A}_{m,d} = 0 \tag{6}$$

because we focus on sum $2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}$, in particular on n^{2k+1} and binomial coefficient $\binom{r}{2k+1}$.

For instance, if we have to get coefficient of n^{2d+1} in range $\frac{m}{2}\le d<m$, we set $d=m-1$, thus we have to get coefficient of $m-1$ in $2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}$.

Therefore, we set $k=m-1$ and $r=m-1$ which implies that $\binom{r}{2k+1}=\binom{m-1}{2m-1}=0$, hence $\mathbf{A}_{m,m-1}\frac{1}{(2m-1)\binom{2m-2}{m-1}}n^{2m-1}=0$.

The same principle applies for every d in the range $\frac{m}{2}\le d<m$, because $r=\frac{m}{2}$ and $k=\frac{m}{2}$ means that $\binom{r}{2k+1}=\binom{\frac{m}{2}}{m+1}=0$.

To summarize, the value of k should be in range $k\le\frac{d-1}{2}$ so that binomial coefficient $\binom{d}{2k+1}$ is non-zero.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4}\le d<\frac{m}{2}$ we get

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}}+2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d}=0 \tag{7}$$

i.e

$$\mathbf{A}_{m,d}=(-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $\frac{m}{2^{s+1}}\le r<\frac{m}{2^s}$, iterating consecutively over $s=1,2,\dots$ by using previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r}=(2r+1)\binom{2r}{r}\sum_{d\ge 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient $\mathbf{A}_{m,r}$

Definition 1.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r}\sum_{d\geq 2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{8}$$

where B_t are Bernoulli numbers [3]. It is assumed that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [4, 5].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

2. Questions

Question 2.1. *Although, a proof of combinatorial identity (3) is already present, it is good to point out literature or more context on it. Reference to a book or article with deeper discussion.*

Question 2.2. *I have struggle to understand the equation (5), it takes the coefficient of n^{2m+1} meaning that we substitute $r = m$ into (4) evaluating it, if I understand it properly. So that coefficient of n^{2m+1} is*

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} n^{2m+1} + 2\sum_{k} \mathbf{A}_{m,m} \frac{(-1)^m}{2m-2k} \binom{m}{2k+1} B_{2m-2k} n^{2k+1} = 1$$

It implies that coefficient of n^{2m+1} in following sum is zero

$$2\sum_{k} \mathbf{A}_{m,m} \frac{(-1)^m}{2m-2k} \binom{m}{2k+1} B_{2m-2k} n^{2k+1} = 0$$

So that

$$\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} = 1; \qquad \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Which is indeed true because $\binom{m}{2k+1} = 0$ *as* $k = m$.

Question 2.3. *Almost the same problem with equation (6), taking the coefficient of* n^{2d+1} *for an integer* d *in the range* $\frac{m}{2} \leq d < m$, *we get*

$$\mathbf{A}_{m,d} = 0$$

Let be $r = d$ *in (4)*

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}}n^{2d+1} + 2\sum_{k}\mathbf{A}_{m,d}\frac{(-1)^d}{2d-2k}\binom{d}{2k+1}B_{2d-2k}n^{2k+1} = 0$$

Let be $d = m - 1$ *then again same principle*

$$2\sum_{k}\mathbf{A}_{m,d}\frac{(-1)^d}{2d-2k}\binom{d}{2k+1}B_{2d-2k}n^{2k+1} = 0$$

because $\binom{m-1}{2k+1} = 0$ *as* $k = m - 1$.

To summarize, the value of k should be in range $k \leq \frac{d-1}{2}$ so that binomial coefficient $\binom{d}{2k+1}$ is non-zero.

References

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