DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

1. Introduction

2. Recurrence relation

In 2018, the recurrence relation [1] for the coefficients $\mathbf{A}_{m,r}$ was provided by Dr. Max Alekseyev, George Washington University. The main idea of Alekseyev's approach was to utilize a generating function to evaluate the set of coefficients $\mathbf{A}_{m,r}$ starting from the base case $\mathbf{A}_{m,m}$, then to evaluate previous coefficient $\mathbf{A}_{m,m-1}$ recursively, similarly up to $\mathbf{A}_{m,0}$. We utilize Binomial theorem and a specific version of Faulhaber's formula [2] with upper summation bound set to $p+1$

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} = \frac{1}{p+1} \left[\sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1}$$

The reason we use the Faulhaber's formula above is because we tend to omit summation bounds, for simplicity. This helps us to collapse the common terms across complex sums, because now we can let the sum run over all integers j, while only finitely many terms $\binom{p+1}{j}$ are non-zero, see also [3]. Hence,

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \left[\sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1} \tag{1}$$

Now we expand the sum $\sum_{k=1}^{n} k^r (n-k)^r$ using Binomial theorem

$$\sum_{k=1}^{n} k^r (n-k)^r = \sum_{k=1}^{n} k^r \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r}$$

By applying Faulhaber's formula (1) to $\sum_{k=1}^{n} k^{t+r}$, we get

$$\begin{aligned} \sum_{k=1}^{n} k^r (n-k)^r &= \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \left[\left(\frac{1}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{t+r+1-j} \right) - \frac{B_{t+r+1}}{t+r+1} \right] \\ &= \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \left[\left(\sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} \right) - B_{t+r+1} n^{r-t} \right] \end{aligned}$$

By expanding brackets

$$\sum_{k=1}^{n} k^r (n-k)^r = \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

By moving the sum in j and omitting summation bounds in t

$$\sum_{k=1}^{n} k^r (n-k)^r = \left[\sum_{j,t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[\sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

By rearranging the sums we obtain

$$\begin{aligned} \sum_{k=1}^{n} k^r (n-k)^r = &\left[\sum_{j} B_j n^{2r+1-j} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] \\ &- \left[\sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \end{aligned} \tag{2}$$

We can notice that

Lemma 2.1 (Altering Binomial identity)**.** *For integers r, j, we have*

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \dfrac{1}{(2r+1)\binom{2r}{r}} & \text{if } j = 0, \\ \dfrac{(-1)^r}{j} \dbinom{r}{2r-j+1} & \text{if } j > 0. \end{cases}$$

Proof. For $j = 0$ we have

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} = \sum_t \binom{r}{t} (-1)^t \int_0^1 z^{r+t}\, dz$$

Because $\frac{1}{r+t+1} = \int_0^1 z^{r+t}\, dz$.

$$\sum_t \binom{r}{t} (-1)^t \int_0^1 z^{r+t}\, dz = \int_0^1 z^r \left(\sum_t \binom{r}{t} (-1)^t z^t \right) dz = \int_0^1 z^r (1-z)^r\, dz$$

The work [?] provides the identity $\binom{n}{k}^{-1} = (n+1) \int_0^1 z^k (1-z)^{n-k}\, dz$. By setting $n = 2r$ and $k = r$ yields

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} = \int_0^1 z^r (1-z)^r\, dz = \binom{2r}{r}^{-1} \frac{1}{2r+1}$$

This completes the proof for $j = 0$.

For $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \sum_t \frac{(-1)^t}{j} \binom{r}{t} \binom{r+t}{j-1}$$

Because $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$. Now apply the coefficient extraction $[z^k]$ to represent the coefficient of z^k. For example: $[z^k](1+z)^r = \binom{r}{k}$. Therefore,

$$\sum_t \frac{(-1)^t}{j} \binom{r}{t} \binom{r+t}{j-1} = \sum_t \frac{(-1)^t}{j} \binom{r}{t} [z^{j-1}](1+z)^{r+t} = [z^{j-1}] \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^{r+t}$$

By factoring out $(1+z)^r$ from the sum

$$[z^{j-1}] \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^{r+t} = [z^{j-1}](1+z)^r \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^t$$

Now apply the binomial theorem to the inner sum

$$\sum_t \binom{r}{t} (-1)^t (1+z)^t = (1-(1+z))^r = (-z)^r = (-1)^r z^r$$

Hence, for $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \frac{(-1)^r}{j} [z^{j-1}](1+z)^r z^r$$

By applying the identity $[z^{p-q}]A(z) = [z^p]z^q A(z)$

$$\frac{(-1)^r}{j} [z^{j-1}](1+z)^r z^r = \frac{(-1)^r}{j} [z^{j-1-r}](1+z)^r = \frac{(-1)^r}{j} \binom{r}{j-1-r}$$

Finally, we use the symmetry $\binom{n}{k} = \binom{n}{n-k}$ to show that for $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \frac{(-1)^r}{j} \binom{r}{j-1-r} = \frac{(-1)^r}{j} \binom{r}{2r-j+1}$$

This completes the proof. $\square$

To simplify (2) using lemma (2.1), we have to move $j = 0$ out of the sum Σ in (2) to avoid division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j=1}^{\infty} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right] - \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

Hence, we simplify the equation (2) by using lemma (2.1)

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j=1}^{\infty} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1}\right] - \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

By setting $\ell = 2r - j + 1$ to the sum $\sum_{j=1}^{\infty}$, and $\ell = r - t$ to the sum $\sum_t$, we collapse common terms across two sums, thus

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_\ell \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^\ell\right] \\ &\quad - \left[\sum_\ell \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^\ell\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^\ell\end{aligned}$$

By replacing odd $\ell = 2k + 1$, and by simplifying 2's, we get

Proposition 2.2 (Bivariate Faulhaber's Formula)**.**

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{k=0}^{\infty} \frac{(-1)^r}{r-k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by the odd-power identity in conjecture (**??**), we obtain the following relation for polynomials in n

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{r=0}^{m} \sum_{k=0}^{\infty} \mathbf{A}_{m,r} \frac{(-1)^r}{r-k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} - n^{2m+1} = 0$$

We denote it as

$$F_m = \sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{r=0}^{m} \sum_{k=0}^{\infty} \mathbf{A}_{m,r} \frac{(-1)^r}{r-k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} - n^{2m+1} \tag{3}$$

Basically, the relation (3) is the generating function we utilize to evaluate the values of $\mathbf{A}_{m,m}, \mathbf{A}_{m,m-1}, \ldots, \mathbf{A}_{m,0}$. We now fix the unused values of $\mathbf{A}_{m,r}$ so that $\mathbf{A}_{m,r} = 0$ for every $r < 0$ or $r > m$.

Extracting the coefficient of n^{2m+1} in (3) yields

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

because $\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1$.

That's may not be immediately clear why the coefficient of n^{2m+1} is $(2m+1)\binom{2m}{m}$. To extract the coefficient of n^{2m+1} from the generating function (3), we isolate the relevant terms by setting $r = m$ in the first sum, and $k = m$ in the second sum. This gives

$$[n^{2m+1}]F_m = \mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} + \sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{(-1)^r}{r-m} \binom{r}{2m+1} B_{2r-2m} - 1$$

We observe that the sum

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{(-1)^r}{r-m} \binom{r}{2m+1} B_{2r-2m}$$

does not contribute to the determination of the coefficients $\mathbf{A}_{m,r}$, because the binomial coefficient $\binom{r}{2m+1}$ vanishes for all $r \leq m$. Consequently, all terms in the sum are zero. Thus,

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0 \implies \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d \leq m-1$ in (3) gives

$$[n^{2d+1}]F_m = \mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + \sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{(-1)^r}{r-d} \binom{r}{2d+1} B_{2r-2d}.$$

For every $\frac{m}{2} \leq d$, the binomial coefficient $\binom{r}{2d+1}$ vanishes, because for all $r \leq m$ holds $r < 2d+1$. As a particular example, when $r = m$ and $d = \frac{m}{2}$, we have

$$\binom{m}{m+1} = 0.$$

Therefore, the entire sum involving $\binom{r}{2d+1}$ vanishes, and we conclude that for all integers d such that $\frac{m}{2} \leq d \leq m-1$ the coefficients $\mathbf{A}_{m,d}$ are zeroes

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} = 0 \implies \mathbf{A}_{m,d} = 0.$$

In contrast, for values $d \leq \frac{m}{2} - 1$, the binomial coefficient $\binom{r}{2d+1}$ can be nonzero; for instance, if $r = m$ and $d = \frac{m}{2} - 1$, then

$$\binom{m}{m-1} \neq 0,$$

allowing the corresponding terms to contribute to the determination of $\mathbf{A}_{m,d}$.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ in (3), we obtain

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0.$$

Solving for $\mathbf{A}_{m,d}$ yields

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!\, d!\, m!\, (m-2d-1)!} \cdot \frac{1}{m-d} B_{2m-2d}.$$

Proceeding recursively, we can compute each coefficient $\mathbf{A}_{m,r}$ for integers r in the ranges $\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}$, for $s = 1, 2, \ldots$, by using previously computed values $\mathbf{A}_{m,d}$ for $d > r$, via the relation

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}.$$

Finally, we define the following recurrence relation for coefficients $\mathbf{A}_{m,r}$

Proposition 2.3. *For integers m and r*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

where B_t are Bernoulli numbers [4]. It is assumed that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [5, 6].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$.
- $\mathbf{A}_{m,r} = 0$ for $r < 0$ and $r > m$.
- $\mathbf{A}_{m,r} = 0$ for $m < 0$.
- $\mathbf{A}_{m,r} = 0$ for $\lfloor \frac{m}{2} \rfloor \leq r < m$.
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$.
- $\mathbf{A}_{m,r}$ are all integers up to row $m = 11$.
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$.

References

[1] Alekseyev, Max. MathOverflow answer 297916/113033, 2018. https://mathoverflow.net/a/297916/113033.

[2] Alan F. Beardon. Sums of powers of integers. *The American mathematical monthly*, 103(3):201–213, 1996. https://doi.org/10.1080/00029890.1996.12004725.

[3] Knuth, Donald E. Two notes on notation. *The American Mathematical monthly*, 99(5):403–422, 1992. https://arxiv.org/abs/math/9205211.

[4] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book company, 1953.

[5] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences, 2018. https://oeis.org/A302971.

[6] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences, 2018. https://oeis.org/A304042.