DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

ABSTRACT. We study an identity expressing odd powers n^{2m+1} as a double sum of the form

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r,$$

and focus on determining the real coefficients $\mathbf{A}_{m,r}$. By utilizing classical tools such as the Binomial theorem and Faulhaber's formula, we obtain a recurrence for the coefficients

$$\mathbf{A}_{m,r} = (2r+1) \binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

from which all values $\mathbf{A}_{m,r}$ can be computed.

1. INTRODUCTION AND MAIN RESULTS

Assume that the following odd power identity holds

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

Our main goal is to identify the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that identity above is true.

Although, the recurrence relation is already given at [1], a few key points in proof are worth to discuss additionally.

The main idea of Alekseyev's approach was to utilize a recurrence relation to evaluate the set of coefficients $\mathbf{A}_{m,r}$ starting from the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ recursively, continuing similarly up to $\mathbf{A}_{m,0}$.

We utilize Binomial theorem $(n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t$ and specific version of Faulhaber's formula [2]

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} = \frac{1}{p+1} \left[\sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1}$$

$$= \frac{1}{p+1} \left[\sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1}$$

The reason we use modified version of Faulhaber's formula is because we want to omit summation bounds, for simplicity. This would help us to collapse the common terms across complex sums, see also [3, p. 2]. Therefore, we expand the sum $\sum_{k=1}^{n} k^r (n-k)^r$ using Binomial theorem

$$\sum_{k=1}^{n} k^r (n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r}$$

By applying Faulhaber's formula above, we obtain

$$\sum_{k=1}^{n} k^r (n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \left[\left(\frac{1}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{t+r+1-j} \right) - \frac{B_{t+r+1}}{t+r+1} \right]$$

By moving the common term $\frac{(-1)^t}{t+r+1}$ out of brackets

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \left[\sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t} \right]$$

By expanding the brackets

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} \right]$$

$$- \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

By moving the sum in j and omitting summation bounds in t

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_{j,t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

By rearranging the sums we obtain

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_j B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] \qquad (2)$$

$$- \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

We can notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j = 0 \\ \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1} & \text{if } j > 0 \end{cases} \qquad (3)$$

An elegant proof of the binomial identity (3) is done by Markus Scheuer in [4].

In particular, the equation (3) is zero for $0 < t \leq j$.

To utilize the equation (3), we have to move $j = 0$ out of summation in (2) to avoid

division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j=1}^{\infty} B_j n^{2r+1-j} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right]$$

$$- \left[\sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

Hence, we simplify the equation above by using (3) so that

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j=1}^{\infty} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1} \right]}_{(\star)}$$

$$- \underbrace{\left[\sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)}$$

By introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms

across two sums

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right]$$

$$- \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right]$$

$$= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by the odd-power identity (1), we obtain the following relation

for polynomials in n

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^{m} \sum_{\text{odd } \ell} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by $\ell = 2k + 1$ we get

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^{m} \sum_{k} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1}$$

By simplifying the term 2

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{r=0}^{m} \sum_{k} \mathbf{A}_{m,r} \frac{(-1)^r}{r-k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1} \qquad (4)$$

Basically, the relation (4) is the generating function we utilize to evaluate the values of

$\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$. We now fix the unused values of $\mathbf{A}_{m,r}$ so that $\mathbf{A}_{m,r} = 0$ for every

$r < 0$ or $r > m$.

Taking the coefficient of n^{2m+1} in (4) yields

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \qquad (5)$$

because $\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1$.

That's may not be immediately clear why the coefficient of n^{2m+1} in (4) is $(2m+1)\binom{2m}{m}$.

To extract the coefficient of n^{2m+1} from the expression (4), we isolate the relevant terms by

setting $r = m$ in the first sum, and $k = m$ in the second sum. This gives

$$[n^{2m+1}] \left(\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{r=0}^{m} \sum_{k=0}^{\infty} \mathbf{A}_{m,r} \frac{(-1)^r}{r-k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} - n^{2m+1} \right)$$

$$= \mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} + \sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{(-1)^r}{r-m} \binom{r}{2m+1} B_{2r-2m} - 1$$

We observe that the sum

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{(-1)^r}{r-m} \binom{r}{2m+1} B_{2r-2m}$$

does not contribute to the determination of the coefficients because the binomial coefficient

$\binom{r}{2m+1}$ vanishes for all $r \leq m$. Consequently, all terms in the sum are zero. Thus,

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0 \implies \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d \leq m-1$ in (4) gives

$$[n^{2d+1}]\left(\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{r=0}^{m}\sum_{k} \mathbf{A}_{m,r} \frac{(-1)^r}{r-k}\binom{r}{2k+1} B_{2r-2k} n^{2k+1} - n^{2m+1}\right)$$

$$= \mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + \sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{(-1)^r}{r-d}\binom{r}{2d+1} B_{2r-2d}.$$

For every $\frac{m}{2} \leq d$, the binomial coefficient $\binom{r}{2d+1}$ vanishes, because for all $r \leq m$ holds

$r < 2d+1$. As a particular example, when $r = m$ and $d = \frac{m}{2}$, we have

$$\binom{m}{m+1} = 0.$$

Therefore, the entire sum involving $\binom{r}{2d+1}$ vanishes, and we conclude

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} = 0 \implies \mathbf{A}_{m,d} = 0.$$

Hence, for all integers d such that $\frac{m}{2} \leq d \leq m-1$, the coefficient $\mathbf{A}_{m,d} = 0$.

In contrast, for values $d \leq \frac{m}{2} - 1$, the binomial coefficient $\binom{r}{2d+1}$ can be nonzero; for instance, if $r = m$ and $d = \frac{m}{2} - 1$, then

$$\binom{m}{m-1} \neq 0,$$

allowing the corresponding terms to contribute to the determination of $\mathbf{A}_{m,d}$.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ in (4), we obtain

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0. \tag{6}$$

Solving for $\mathbf{A}_{m,d}$ yields

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!\,d!\,m!\,(m-2d-1)!} \cdot \frac{1}{m-d}B_{2m-2d}.$$

Proceeding recursively, we can compute each coefficient $\mathbf{A}_{m,r}$ for integers r in the ranges

$$\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}, \quad \text{for } s = 1, 2, \ldots$$

by using previously computed values $\mathbf{A}_{m,d}$ for $d > r$, via the relation

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}.$$

Finally, we define the following recurrence relation for coefficients $\mathbf{A}_{m,r}$

Definition 1.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\[2ex] (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \leq r < m \\[2ex] 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{7}$$

where B_t are Bernoulli numbers [5]. It is assumed that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [6, 7].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$

- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$

- $\mathbf{A}_{m,r} = 0$ for $r < 0$

- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$

- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$

- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$

- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

2. Conclusions

We have investigated a class of polynomial identities representing odd powers n^{2m+1} as double sums of the form

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r,$$

and focused on explicitly determining the real coefficients $\mathbf{A}_{m,r}$. By employing the Binomial theorem and Faulhaber's formula, we derived a recurrence relation

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r},$$

which allows for computation of all coefficients $\mathbf{A}_{m,r}$.

References

[1] Alekseyev, Max. MathOverflow answer 297916/113033, 2018. https://mathoverflow.net/a/297916/
113033.

[2] Alan F. Beardon. Sums of powers of integers. *The American mathematical monthly*, 103(3):201–213, 1996.

https://doi.org/10.1080/00029890.1996.12004725.

[3] Knuth, Donald E. Two notes on notation. *The American Mathematical monthly*, 99(5):403–422, 1992.

https://arxiv.org/abs/math/9205211.

[4] Scheuer, Markus. MathStackExchange answer 4724343/463487, 2023. https://math.stackexchange.

com/a/4724343/463487.

[5] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book com-

pany, 1953.

[6] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences, 2018. https://oeis.

org/A302971.

[7] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences, 2018. https://oeis.

org/A304042.

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3. Questions

Question 3.1. *Although, a proof of combinatorial identity* (3) *is already present, it is good*

to point out literature or more context on it. Reference to a book or article with deeper

discussion.

Question 3.2. *Are these coefficients* $\mathbf{A}_{m,r}$ *appear in widely-known mathematical literature?*

Question 3.3. *I have struggle to understand the equation* (5), *it takes the coefficient of*

n^{2m+1} *meaning that we substitute* $r = m$ *into* (4) *evaluating it, if I understand it properly.*

So that coefficient of n^{2m+1} *is*

$$\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} + \sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{(-1)^r}{r-m}\binom{r}{2m+1}B_{2r-2m} - 1$$

It implies that following sum is zero

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{(-1)^r}{r-m}\binom{r}{2m+1}B_{2r-2m} = 0$$

So that

$$\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} = 1; \qquad \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Which is indeed true because $\binom{r}{2m+1} = 0$ *as* r *runs over* $0 \le r \le m$.

Question 3.4. *Almost the same problem with equation* (3.3), *taking the coefficient of* n^{2d+1}

for an integer d *in the range* $\frac{m}{2} \le d \le m-1$, *we get*

$$\mathbf{A}_{m,d} = 0$$

Let be $r = d$ *and* $k = d$ *in* (4), *then the coefficient of* n^{2d+1} *is*

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + \sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{(-1)^r}{r-d}\binom{r}{2d+1}B_{2r-2d} - 0$$

The sum

$$\sum_r \mathbf{A}_{m,r} \frac{(-1)^r}{r-d} \binom{r}{2d+1} B_{2r-2d} = 0$$

because $\binom{r}{2d+1} = 0$ for all r such that $0 \le r \le m$ and d such that $\frac{m}{2} \le d \le m-1$.

To summarize, the value of d should be in the range $d \le \frac{m}{2} - 1$ so that binomial coefficient

$\binom{r}{2d+1}$ is non-zero. For example, let be $r = m$ and $d = \frac{m}{2} - 1$ then $\binom{r}{2d+1} = \binom{m}{m-1} \ne 0$ and

so on for each $d \le \frac{m}{2} - 1$.

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