DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

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Abstract. https://mathoverflow.net/a/297916/113033

1. Introduction

Assuming that following odd power identity holds

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

Our main goal is to identify the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that identity above is true.

Although, the recurrence relation is already given at [1], a few key points in proof are worth to discuss additionally.

The main idea of Alekseyev's approach was to utilize a recurrence relation to evaluate the set of coefficients $\mathbf{A}_{m,r}$ starting from the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ by using backtracking, continuing similarly up to $\mathbf{A}_{m,0}$.

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By applying Binomial theorem $(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}k^t$ and Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}$, we get

$$\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r} \\
&= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right] \\
&= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right] \\
&= \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \\
&= \left[\sum_{j,t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]
\end{aligned}$$

By rearranging the sums we obtain

$$= \left[\sum_j B_j n^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] - \left[\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \tag{2}$$

We can notice that

$$\sum_t\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j=0 \\ \frac{(-1)^r}{j}\binom{r}{2r-j+1} & \text{if } j>0 \end{cases} \tag{3}$$

An elegant proof of the binomial identity (3) is done by Markus Scheuer in [2].

In particular, the equation (3) is zero for $0 < t \le j$.

To utilize the equation (3), we have to move $j = 0$ out of summation in (2) to avoid division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j\geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

Now we do not care about division by zero in $\frac{(-1)^r}{j}$. Hence, we simplify the equation above by using (3) so that

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j\geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1}\right]}_{(\star)} - \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]}_{(\diamond)}$$

By introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms across two sums

$$\begin{aligned} \sum_{k=1}^{n} k^r(n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by the odd-power identity (1), we obtain the following relation for polynomials in n

$$\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1}+2\sum_{r,\text{ odd }\ell}\mathbf{A}_{m,r}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^{\ell}\equiv n^{2m+1}$$

Replacing odd ℓ by k we get

$$\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1}+2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}\equiv n^{2m+1} \tag{4}$$

$$\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1}+2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}-n^{2m+1}=0$$

Taking the coefficient of n^{2m+1} in (4) yields

$$\mathbf{A}_{m,m}=(2m+1)\binom{2m}{m} \tag{5}$$

because $\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}}=1$.

That's may not be immediately clear why the coefficient of n^{2m+1} in (4) is $(2m+1)\binom{2m}{m}$.

To take the coefficient of n^{2m+1} we fix $r=m$ and $k=m$ in (4)

$$[n^{2m+1}]\left(\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1}+2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}-n^{2m+1}\right)$$

$$=\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}}+2\sum_{r=m}^{m}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2m}\binom{r}{2m+1}B_{2r-2m}-1$$

The sum $2\sum_r \mathbf{A}_{m,r}\frac{(-1)^r}{2r-2m}\binom{r}{2m+1}B_{2r-2m}$ collapses because r runs over the interval $0 \leq r \leq m$ making the coefficient $\binom{r}{2m+1} = 0$ for every value of r. Thus

$$\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0$$

Taking the coefficient of n^{2d+1} in (4) for an integer d in the range $\frac{m}{2} \leq d \leq m-1$, we get

$$[n^{2d+1}]\left(\sum_r \mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1} - n^{2m+1}\right)$$

$$= \mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2\sum_{r=d}^{m}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2d}\binom{r}{2d+1}B_{2r-2d} - 0$$

For every integer d in the range $\frac{m}{2} \leq d \leq m-1$, the binomial coefficient $\binom{r}{2d+1} = 0$ because r runs over $0 \leq r \leq m$. Thus, the sum $2\sum_r \mathbf{A}_{m,r}\frac{(-1)^r}{2r-2d}\binom{r}{2d+1}B_{2r-2d}$ collapses. Consider the corner case, we fix $r = m$ and $d = \frac{m}{2}$ then

$$\binom{r}{2d+1} = \binom{m}{m+1} = 0$$

Therefore, for every integer d in the range $\frac{m}{2} \leq d \leq m-1$

$$\mathbf{A}_{m,d} = 0 \tag{6}$$

To summarize, the value of d should be in the range $d \leq \frac{m}{2} - 1$ so that binomial coefficient $\binom{r}{2d+1}$ is non-zero. For example, let be $r = m$ and $d = \frac{m}{2} - 1$ then $\binom{r}{2d+1} = \binom{m}{m-1} \neq 0$ and so on for each $d \leq \frac{m}{2} - 1$.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0 \tag{7}$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}$, iterating consecutively over $s = 1, 2, \ldots$ by using previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient $\mathbf{A}_{m,r}$

Definition 1.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r}\sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{8}$$

where B_t are Bernoulli numbers [3]. It is assumed that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [4, 5].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

2. Questions

Question 2.1. *Although, a proof of combinatorial identity (3) is already present, it is good to point out literature or more context on it. Reference to a book or article with deeper discussion.*

Question 2.2. *Are these coefficients $\mathbf{A}_{m,r}$ appear in widely-known mathematical literature?*

Question 2.3. *I have struggle to understand the equation (5), it takes the coefficient of n^{2m+1} meaning that we substitute $r = m$ into (4) evaluating it, if I understand it properly. So that coefficient of n^{2m+1} is*

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} + 2 \sum_{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2m} \binom{r}{2m+1} B_{2r-2m} - 1$$

It implies that following sum is zero

$$2 \sum_{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2m} \binom{r}{2m+1} B_{2r-2m} = 0$$

So that

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1; \qquad \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Which is indeed true because $\binom{r}{2m+1} = 0$ as r runs over $0 \le r \le m$.

Question 2.4. *Almost the same problem with equation (6), taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d \leq m-1$, we get*

$$\mathbf{A}_{m,d} = 0$$

Let be $r = d$ and $k = d$ in (4), then the coefficient of n^{2d+1} is

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2 \sum_{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} - 0$$

The sum

$$2 \sum_{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} = 0$$

because $\binom{r}{2d+1} = 0$ for all r such that $0 \leq r \leq m$ and d such that $\frac{m}{2} \leq d \leq m-1$.

To summarize, the value of d should be in the range $d \leq \frac{m}{2} - 1$ so that binomial coefficient $\binom{r}{2d+1}$ is non-zero. For example, let be $r = m$ and $d = \frac{m}{2} - 1$ then $\binom{r}{2d+1} = \binom{m}{m-1} \neq 0$ and so on for each $d \leq \frac{m}{2} - 1$.

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