DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

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Abstract. https://mathoverflow.net/a/297916/113033

1. Introduction

Assume that the following odd power identity holds

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

Our main goal is to identify the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that identity above is true.

Although, the recurrence relation is already given at [1], a few key points in proof are worth to discuss additionally.

The main idea of Alekseyev's approach was to utilize a recurrence relation to evaluate the set of coefficients $\mathbf{A}_{m,r}$ starting from the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ by using backtracking, continuing similarly up to $\mathbf{A}_{m,0}$.

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We are going to utilize Binomial theorem $(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}k^t$ and specific version of Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} = \frac{1}{p+1}\left[\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - \frac{B_{p+1}}{p+1}$$
$$= \frac{1}{p+1}\left[\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - \frac{B_{p+1}}{p+1}$$

The reason we use modified version of Faulhaber's formula is because its summation bounds can be omitted, which helps to collapse common terms across complex sums. Therefore, we expand the sum $\sum_{k=1}^{n} k^r(n-k)^r$ as follows

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r}$$
$$= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\left(\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j}\right) - \frac{B_{t+r+1}}{t+r+1}\right]$$
$$= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\left[\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right]$$
$$= \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$
$$= \left[\sum_{j,t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

By rearranging the sums we obtain

$$= \left[\sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] - \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \qquad (2)$$

We can notice that

$$\sum_{t} \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j = 0 \\ \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1} & \text{if } j > 0 \end{cases} \tag{3}$$

An elegant proof of the binomial identity (3) is done by Markus Scheuer in [2].

In particular, the equation (3) is zero for $0 < t \leq j$.

To utilize the equation (3), we have to move $j = 0$ out of summation in (2) to avoid division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j=1} B_j n^{2r+1-j} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right]$$
$$- \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

Hence, we simplify the equation above by using (3) so that

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j=1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1} \right]}_{(\star)}$$
$$- \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)}$$

By introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms across two sums

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^{\ell}\right] \\ &- \left[\sum_{\ell}\binom{r}{\ell}\frac{(-1)^{r-\ell}}{2r+1-\ell}B_{2r+1-\ell}n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{\text{odd } \ell}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^{\ell}\end{aligned}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by the odd-power identity (1), we obtain the following relation for polynomials in n

$$\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\ \text{odd } \ell}\mathbf{A}_{m,r}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by $\ell = 2k+1$ we get

$$\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\ k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1} \equiv n^{2m+1} \tag{4}$$

$$\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\ k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1} - n^{2m+1} = 0$$

We now fix the unused values of $\mathbf{A}_{m,r}$ so that $\mathbf{A}_{m,r} = 0$ for every $r < 0$ or $r > m$.

Taking the coefficient of n^{2m+1} in (4) yields

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \tag{5}$$

because $\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} = 1$.

That's may not be immediately clear why the coefficient of n^{2m+1} in (4) is $(2m+1)\binom{2m}{m}$.

To take the coefficient of n^{2m+1} we fix $r = m$ and $k = m$ in (4)

$$[n^{2m+1}]\left(\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1} - n^{2m+1}\right)$$

$$= \mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} + 2\sum_{r}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2m}\binom{r}{2m+1}B_{2r-2m} - 1$$

The sum $2\sum_r \mathbf{A}_{m,r}\frac{(-1)^r}{2r-2m}\binom{r}{2m+1}B_{2r-2m}$ equals to zero because it contains only a single iteration step $r = m$ so that $\binom{m}{2m+1} = 0$. Thus

$$\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0$$

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \le d \le m-1$ in (4), we obtain

$$[n^{2d+1}]\left(\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1} - n^{2m+1}\right)$$

$$= \mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2\sum_{r}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2d}\binom{r}{2d+1}B_{2r-2d} - 0$$

For every integer d in the range $\frac{m}{2} \le d \le m-1$, the binomial coefficient $\binom{r}{2d+1} = 0$ because r runs over $d \le r \le m$. Thus, the sum $2\sum_r \mathbf{A}_{m,r}\frac{(-1)^r}{2r-2d}\binom{r}{2d+1}B_{2r-2d}$ collapses. Consider the corner case, we fix $r = m$ and $d = \frac{m}{2}$ then

$$\binom{r}{2d+1} = \binom{m}{m+1} = 0$$

Therefore, for every integer d in the range $\frac{m}{2} \leq d \leq m-1$

$$\mathbf{A}_{m,d} = 0 \tag{6}$$

To summarize, the value of d should be in the range $d \leq \frac{m}{2} - 1$ so that binomial coefficient $\binom{r}{2d+1}$ is non-zero. For example, let be $r = m$ and $d = \frac{m}{2} - 1$ then $\binom{r}{2d+1} = \binom{m}{m-1} \neq 0$ and so on for each $d \leq \frac{m}{2} - 1$.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d} B_{2m-2d} = 0 \tag{7}$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}$, iterating consecutively over $s = 1, 2, \ldots$ by using previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient $\mathbf{A}_{m,r}$

Definition 1.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{8}$$

where B_t are Bernoulli numbers [3]. It is assumed that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [4, 5].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$

- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

2. Questions

Question 2.1. *Although, a proof of combinatorial identity (3) is already present, it is good to point out literature or more context on it. Reference to a book or article with deeper discussion.*

Question 2.2. *Are these coefficients $\mathbf{A}_{m,r}$ appear in widely-known mathematical literature?*

Question 2.3. *I have struggle to understand the equation (5), it takes the coefficient of n^{2m+1} meaning that we substitute $r = m$ into (4) evaluating it, if I understand it properly. So that coefficient of n^{2m+1} is*

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} + 2 \sum_{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2m} \binom{r}{2m+1} B_{2r-2m} - 1$$

It implies that following sum is zero

$$2\sum_{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2m} \binom{r}{2m+1} B_{2r-2m} = 0$$

So that

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1; \qquad \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Which is indeed true because $\binom{r}{2m+1} = 0$ *as* r *runs over* $0 \leq r \leq m$.

Question 2.4. *Almost the same problem with equation (6), taking the coefficient of* n^{2d+1} *for an integer* d *in the range* $\frac{m}{2} \leq d \leq m-1$, *we get*

$$\mathbf{A}_{m,d} = 0$$

Let be $r = d$ *and* $k = d$ *in (4), then the coefficient of* n^{2d+1} *is*

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2\sum_{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} - 0$$

The sum

$$2\sum_{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} = 0$$

because $\binom{r}{2d+1} = 0$ *for all* r *such that* $0 \leq r \leq m$ *and* d *such that* $\frac{m}{2} \leq d \leq m-1$.

To summarize, the value of d should be in the range $d \leq \frac{m}{2} - 1$ so that binomial coefficient $\binom{r}{2d+1}$ is non-zero. For example, let be $r = m$ and $d = \frac{m}{2} - 1$ then $\binom{r}{2d+1} = \binom{m}{m-1} \neq 0$ and so on for each $d \leq \frac{m}{2} - 1$.

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