DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

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Abstract. https://mathoverflow.net/a/297916/113033

1. Introduction

Assuming that following odd power identity holds

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=0}^{n-1} \mathbf{A}_{m,r} k^r (n-k)^r$$

Our main goal is to identify the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that identity above is true.

Although, the recurrence relation is already given at [1], a few key points in proof are worth to discuss additionally.

The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate the $\mathbf{A}_{m,r}$ recursively, taking the base case $\mathbf{A}_{m,m}$, then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ by using backtracking, continuing similarly up to $\mathbf{A}_{m,0}$.

Expanding $(n-k)^r$ using Faulhaber's formula yields

$$\sum_{k=0}^{n-1} k^j (n-k)^j = \sum_{k=0}^{n-1} \sum_{i} \binom{j}{i} n^{j-i} k^{i+j} \tag{1}$$

$$= \sum_{i} \binom{j}{i} n^{j-i} \frac{(-1)^i}{i+j+1} \left[\sum_{t} \binom{i+j+1}{t} B_t n^{i+j+1-t} - B_{i+j+1} \right] \tag{2}$$

$$= \sum_{i,t} \binom{j}{i} \frac{(-1)^i}{i+j+1} \binom{i+j+1}{t} B_t n^{2j+1-t} - \sum_{i} \binom{j}{i} \frac{(-1)^i}{i+j+1} B_{i+j+1} n^{j-i} \tag{3}$$

where B_t are Bernoulli numbers.

Comment: Now let's break down the steps we performed to get formulae above

- In (1) we expanded $(n-k)^j$ by using Binomial theorem

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- In (2) we expanded the sum $\sum_{k=0}^{n-1} k^{i+j}$ by using Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{t=0}^{p}\binom{p+1}{t}B_j n^{p+1-t}$. Note that the Faulhaber's formula above was modified in order to omit summation bounds, meaning that
$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{t=0}^{p}\binom{p+1}{t}B_j n^{p+1-t} = \left[\frac{1}{p+1}\sum_{t}\binom{p+1}{t}B_t n^{p+1-t}\right] - B_{p+1}$$
- In (3) we expand brackets by merging two sums

Now, we notice that
$$\sum_{i}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t} = \begin{cases}\frac{1}{(2j+1)\binom{2j}{j}}, & \text{if } t=0;\\ \frac{(-1)^j}{t}\binom{j}{2j-t+1}, & \text{if } t>0.\end{cases} \tag{4}$$

Comment: The source or proof of identity (4) is not clear yet. It would be nice to provide it.

Hence, introducing $\ell = 2j+1-t$ and $\ell = j-i$ respectively, we get
$$\sum_{k=0}^{n-1}(n-k)^j k^j = \frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + \sum_{\ell}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell}B_{2j+1-\ell}n^{\ell} - \sum_{\ell}\binom{j}{\ell}\frac{(-1)^{j-\ell}}{2j+1-\ell}B_{2j+1-\ell}n^{\ell} \tag{5}$$
$$= \frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + 2\sum_{\text{odd } \ell}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell}B_{2j+1-\ell}n^{\ell} \tag{6}$$

Comment: Let's break down what is going in (5) and (6)

- *Step 5.1*: Double sum $\sum_{i,t}$ simplified to single one $\sum_t$ by means of (4).
- *Step 5.2*: The term $\frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1}$ moved out of summation by means of (4). It is iteration step $t=0$.
- *Step 5.3*: Introducing $\ell = 2j+1-t$ and $\ell = j-i$ to sums $\sum_t$ and $\sum_i$ to collapse common terms in (5).
- *Step 5.4*: Expression with common terms collapsed (6).

Using the definition of $\mathbf{A}_{m,j}$, we obtain the following identity for polynomials in n
$$\sum_{j}\mathbf{A}_{m,j}\frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + 2\sum_{j,\text{ odd } \ell}\mathbf{A}_{m,j}\binom{j}{\ell}\frac{(-1)^j}{2j+1-\ell}B_{2j+1-\ell}n^{\ell} \equiv n^{2m+1} \tag{7}$$

Taking the coefficient of n^{2m+1} we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \tag{8}$$

Comment: It is not clear how exactly $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$ follows. As I understand, here is about *Taking coefficient of* in context of generating functions? Meaning that coefficient of $2m+1$ so that $j = m$ in $\sum_j \mathbf{A}_{m,j} \frac{1}{(2j+1)\binom{2j}{j}} n^{2j+1}$? Hence, coefficient of it to be $\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}}$. To evaluate $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$ it should be assumed that $\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1$, which is indeed true because of n^{2m+1} in RHS.

References

[1] Alekseyev, Max. MathOverflow answer 297916/113033, 2018. https://mathoverflow.net/a/297916/113033.

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