

FINITE DIFFERENCE OF CONSECUTIVE ODD POWERS MINUS 1 IS ALWAYS DIVISIBLE BY 6

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MAIN RESULTS

We show that for every odd power P and non-negative integer N holds

$$(N + 1)^P - N^P - 1 \equiv 0 \pmod{6}$$

To prove the statement we have to show that for every non-negative integer N and odd power P

$$(N + 1)^P - N^P - 1$$

is always divisible by both 2 and 3. For every non-negative integer N , either N is even, and $N + 1$ is odd, and vice versa. It means that $(N + 1)^P - N^P$ is always a positive odd integer. Therefore,

$$(N + 1)^P - N^P - 1 \equiv 0 \pmod{2}$$

Let be $F(N) = (N + 1)^P - N^P - 1$, then we want to show that for every non-negative N

$$F(N) \equiv 0 \pmod{3}$$

Since that every integer N is congruent to either 0, 1, or 2 modulo 3, we check each case.

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Consider the case $N \equiv 0 \pmod{3}$

$$N \equiv 0 \pmod{3}$$

$$N + 1 \equiv 1 \pmod{3}$$

$$N^p \equiv 0^p = 0 \pmod{3}$$

$$(N + 1)^p \equiv 1^p = 1 \pmod{3}$$

Therefore,

$$f(N) = (N + 1)^p - N^p - 1 \equiv 1 - 0 - 1 = 0 \pmod{3}$$

Consider the case $N \equiv 1 \pmod{3}$

$$N \equiv 1 \pmod{3}$$

$$N + 1 \equiv 2 \pmod{3}$$

$$N^p \equiv 1^p = 1 \pmod{3}$$

$$(N + 1)^p \equiv 2^p \pmod{3}$$

Since p is odd, then $2^p \equiv 2 \pmod{3}$, hence

$$f(N) \equiv 2 - 1 - 1 = 0 \pmod{3}$$

Consider the case $N \equiv 2 \pmod{3}$

$$N \equiv 2 \pmod{3}$$

$$N + 1 \equiv 0 \pmod{3}$$

$$N^p \equiv 2^p \equiv 2 \pmod{3}$$

$$(N + 1)^p \equiv 0^p = 0 \pmod{3}$$

Hence,

$$f(N) \equiv 0 - 2 - 1 = -3 \equiv 0 \pmod{3}$$

Therefore, for every odd power P and non-negative integer N holds

$$(N + 1)^P - N^P - 1 \equiv 0 \pmod{6}$$