

# POLYNOMIAL IDENTITIES INVOLVING RASCAL TRIANGLE

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ABSTRACT. Abstract

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## 1. DEFINITIONS

Definition of generalized Rascal triangle

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \quad (1.1)$$

$$= \binom{n-k}{0} \binom{k}{0} + \binom{n-k}{1} \binom{k}{1} + \binom{n-k}{2} \binom{k}{2} + \dots + \binom{n-k}{i} \binom{k}{i} \quad (1.2)$$

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Definition of  $(1, q)$ -Pascal triangle

$$\begin{bmatrix} n \\ k \end{bmatrix}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \begin{bmatrix} n-1 \\ k \end{bmatrix}^q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^q & \end{cases}$$

Pascals triangle as polynomial

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{1}{k!} n(n-1)(n-2) \cdots (n-(k-1)) = \prod_{i=1}^k \frac{n-i+1}{i} \quad (1.3)$$

## 2. FORMULAE

**2.1. Claim 0. Vandermonde convolution.** Generalized rascal number is partial case of Vandermonde convolution Consider Vandermonde convolution

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

Then generalized rascal number is partial case of Vandermonde convolution with upper summation bound equals  $i$

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m}$$

**2.2. Claim 1. I-th column identity.** Generalized rascal triangle equals to Pascal's triangle up to  $i$ -th column

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq k \leq i \quad (2.1)$$

$$\binom{n}{i-j}_i = \binom{n}{i-j}, \quad \text{ColumnIdentity1} \quad (2.2)$$

$$\binom{n}{n-i+j}_i = \binom{n}{n-i+j}, \quad \text{ColumnIdentity2} \quad (2.3)$$

**2.3. Claim 2. 2i+1 row identity.** Generalized rascal triangle equals to Pascal's triangle up to  $2i + 1$ -th row

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq n \leq 2i + 1 \quad (2.4)$$

For every fixed  $i \geq 0$

$$\binom{2i+1-j}{k}_i = \binom{2i+1-j}{k} \quad \text{RowIdentity1} \quad (2.5)$$

$$\binom{(2i+1)-j}{k}_{(2i+1)-i-1} = \binom{(2i+1)-j}{k} \quad (2.6)$$

For every fixed  $i \geq 0$  and  $t \geq 2i + 1$

$$\binom{t-j}{k}_{t-i-1} = \binom{t-j}{k} \quad \text{RowIdentity2} \quad (2.7)$$

For  $k = j$

$$\binom{2i+1-j}{j}_i = \binom{2i+1-j}{j}, \quad 0 \leq j \leq i \quad \text{RowIdentity3}$$

$$\binom{2i+1-j}{2i+1-2j}_i = \binom{2i+1-j}{2i+1-2j}$$

$$\binom{(2i+1)-j}{(2i+1)-2j}_{(2i+1)-i-1} = \binom{(2i+1)-j}{(2i+1)-2j}$$

$$\binom{t-j}{t-2j}_{t-i-1} = \binom{t-j}{t-2j}, \quad t \geq 2i + 1, \quad 0 \leq j \leq t - i - 1, \quad \text{RowIdentity4}$$

**2.4. Proof of 2i+1 row identity.** Let be definition of rascal number

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m}$$

Then for every  $0 \leq n \leq 2i + 1$  holds

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq n \leq 2i + 1 \quad (2.8)$$

Let be Vandermonde formula

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

By Vandermonde formula

$$\binom{n}{k} = \sum_{m=0}^k \binom{n-k}{m} \binom{k}{k-m}$$

Then for every  $0 \leq n \leq 2i+1$  and  $0 \leq k \leq 2i+1-n$

$$\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k}$$

Thus we have to prove that

$$\begin{aligned} \binom{2i+1-n}{k}_i &= \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{k-m} = \binom{2i+1-n}{k} \\ \binom{2i+1-n}{k} &= \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{k-m} = \sum_{m=0}^k \binom{2i+1-n-k}{m} \binom{k}{k-m} \end{aligned}$$

Rewrite explicitly gives for  $i$

$$\begin{aligned} \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{k-m} &= \binom{2i+1-n-k}{0} \binom{k}{k} + \binom{2i+1-n-k}{1} \binom{k}{k-1} \\ &\quad + \binom{2i+1-n-k}{2} \binom{k}{k-2} + \binom{2i+1-n-k}{3} \binom{k}{k-3} \\ &\quad + \dots + \binom{2i+1-n-k}{i} \binom{k}{k-i} \end{aligned}$$

Rewrite explicitly gives for  $k$

$$\begin{aligned} \sum_{m=0}^k \binom{2i+1-n-k}{m} \binom{k}{k-m} &= \binom{2i+1-n-k}{0} \binom{k}{k} + \binom{2i+1-n-k}{1} \binom{k}{k-1} \\ &\quad + \binom{2i+1-n-k}{2} \binom{k}{k-2} + \binom{2i+1-n-k}{3} \binom{k}{k-3} \\ &\quad + \dots + \binom{2i+1-n-k}{k} \binom{k}{k-k} \end{aligned}$$

We have three cases in general  $k < i$ ,  $k = i$ ,  $k > i$ , so we have to prove that

$$\sum_{m=0}^k \binom{2i+1-n-k}{m} \binom{k}{k-m} - \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{k-m} = 0$$

For the case  $k < i$  proof in Jenna et al paper. For the case  $k = i$  proof is trivial. Thus,

$$\sum_{m=i+1}^k \binom{2i+1-n-k}{m} \binom{k}{k-m} = 0$$

Considering the constraints,

$$\begin{cases} n \geq 0 \\ k \geq i + 1 \\ 2i + 1 - n - k \leq i - n \\ m \geq i + 1 \end{cases}$$

Thus,

$$\sum_{m=i+1}^k \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

because binomial coefficients  $\binom{i-n-s}{i+1+s}$  are zero for each  $i, n, s \geq 0$ .

**2.5. Claim 3. Row-column difference binomial identity.** Row-column difference identity. Proof via Vandermonde's identity. For every fixed  $i \geq 1$

$$\begin{aligned} \binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} &= \binom{n+i}{i} \quad \text{RowColumnDifferenceIdentity1} \\ \binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} &= \binom{n+i}{n} \\ \binom{(n+i)+i}{(n+i)} - \binom{(n+i)+i}{(n+i)}_{i-1} &= \binom{(n+i)}{(n+i)-i} \\ \binom{j+i}{j} - \binom{j+i}{j}_{i-1} &= \binom{j}{j-i}, \quad \text{RowColumnDifferenceIdentity2} \end{aligned}$$

**2.6. Proof of Row-column difference binomial identity.** Let be Vandermonde formula

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

Let be definition of rascal number

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m}$$

Then

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

$$\sum_{m=0}^i \binom{n+i}{m} \binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m} \binom{i}{m} = \binom{n+i}{i}$$

*Generalized relation of difference between rascal numbers and binomial coefficients?*

2.7. **Claim 4.** Relation between  $(1, q)$ -Pascal's triangle

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \left[ \begin{matrix} i+2+j \\ i+2 \end{matrix} \right]^{i+2}, \quad \text{OneQPascalIdentity1}$$

$$\binom{2(i+2)-1+j}{i+2} - \binom{2(i+2)-1+j}{i+2}_i = \left[ \begin{matrix} i+2+j \\ i+2 \end{matrix} \right]^{i+2}$$

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[ \begin{matrix} t+j \\ t \end{matrix} \right]^t, \quad \text{OneQPascalIdentity2}$$

### 3. SIDES OF WORLD

$$\mathbf{North} = \binom{n-2}{k-1}_i$$

$$\mathbf{South} = \binom{n}{k}_i$$

$$\mathbf{West} = \binom{n-1}{k-1}_i$$

$$\mathbf{East} = \binom{n-1}{k}_i$$

Identity see Hotchkiss

$$\mathbf{South} = \frac{\mathbf{East} \cdot \mathbf{West} + 1}{\mathbf{North}} \quad (3.1)$$

$$\binom{n}{k}_i = \frac{\binom{n-1}{k}_i \binom{n-1}{k-1}_i + 1}{\binom{n-2}{k-1}_i} \quad (3.2)$$

Identity see Hotchkiss, for all inner  $k > 0$  and  $k < n$

$$\mathbf{South} = \mathbf{East} + \mathbf{West} - \mathbf{North} + 1 \quad (3.3)$$

$$\binom{n}{k}_i = \binom{n-1}{k}_i + \binom{n-1}{k-1}_i - \binom{n-2}{k-1}_i + 1 \quad (3.4)$$

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