IDENTITIES IN ITERATED RASCAL TRIANGLES

PETRO KOLOSOV

Abstract. In this manuscript, we show new binomial identities in iterated rascal triangles, revealing a connection between the Vandermonde convolution and iterated rascal numbers. We also present Vandermonde-like binomial identities. Furthermore, we establish a relation between iterated rascal triangle and $(1, q)$-binomial coefficients.

Contents

1. Introduction

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were

Date: July 6, 2024.

2010 *Mathematics Subject Classification.* 11B25, 11B99.

Key words and phrases. Pascal's triangle, Rascal triangle, Binomial coefficients, Binomial identities, Binomial theorem, Generalized Rascal triangles, Iterated rascal triangles, Iterated rascal numbers, Vandermonde identity, Vandermonde convolution .

Sources: https://github.com/kolosovpetro/IdentitiesInRascalTriangle challenged to provide the next row for the following number triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & & & \cdots & & & \end{array}$$

Teacher's expected answer was the one that matches Pascal's triangle, e.g "1 4 6 4 1". However, Anggoro, Liu, and Tulloch suggested "1 4 5 4 1" instead. They devised this new row via what they called diamond formula

$$\mathbf{South} = \frac{\mathbf{East} \cdot \mathbf{West} + 1}{\mathbf{North}}$$

So they obtained the following triangle

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	5	4	1			
5	1	5	7	7	5	1		
6	1	6	9	10	9	6	1	
7	1	7	11	13	13	11	7	1

Table 1. Rascal triangle. See the OEIS sequence A077028 [3].

Indeed, the forth row is "1 4 5 4 1" because $4 = \frac{1 \cdot 3+1}{1}$ and $5 = \frac{3 \cdot 3+1}{2}$.

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [4], in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [5, 6]. In particular, the concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

2. Binomial identities in Iterated Rascal Triangles

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number

Definition 2.1. *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} \tag{2.1}$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [7]

$$\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m}\binom{b}{r-m}; \quad \binom{n}{k} = \sum_{m=0}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

Thus,

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m} \tag{2.2}$$

Meaning that iterated rascal number is partial case of Vandermonde convolution of $\binom{n}{k}$ with the upper summation bound equals to i. Without further hesitation consider our findings.

Proposition 2.2. *Iterated rascal triangle equals to Pascal's triangle up to i-th column.*

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq k \leq i \tag{2.3}$$

Then binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying binomial coefficients symmetry principle we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

Proof. Proof of proposition (2.2). Consider the following relation between binomial coefficients and iterated rascal numbers, for every $0 \leq k \leq i$

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=0}^{k} \binom{n-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m}$$
$$= \sum_{m=k+1}^{i} \binom{n-k}{m}\binom{k}{m} = 0$$

It is indeed true, because binomial coefficients $\binom{k}{m}$ are zero for each $m \geq k+1$. So that for every $0 \leq k \leq i$

$$\binom{n}{k} - \binom{n}{k}_i = 0; \quad \binom{n}{k} = \binom{n}{k}_i$$

Therefore, the proposition (2.2) is true. □

Proposition 2.3. *Iterated rascal triangle equals to Pascal's triangle up to $2i+1$-th row*

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq n \leq 2i+1$$

Therefore, for every fixed $i \geq 0$ and $n \geq 0$

$$\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k} \tag{2.4}$$

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over k for each i, so that it is true for all cases in i, k: $i < k$, $i = k$ and $k > i$. In particular, equation (2.4) implies the row sums identity in iterated rascal triangles

$$\sum_{k=0}^{\infty} \binom{2i+1-n}{k}_i = 2^{2i+1-n}$$

Given $n = 0$ we obtain

$$\sum_{k=0}^{\infty} \binom{2i+1}{k}_i = 2^{2i+1}$$

and so on. Taking $t = 2i+1$ in (2.4) yields

$$\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$$

Moreover, equation (2.4) gives Vandermonde-like identity

Proposition 2.4. *(Vandermonde-like identity.)*

$$\binom{2i+1-n}{k} = \sum_{m=0}^{i} \binom{2i+1-n-k}{m}\binom{k}{m}$$

In particular, given $n = 0, 1$ proposition (2.4) yields

$$\binom{2i+1}{k} = \sum_{m=0}^{i} \binom{2i+1-k}{m}\binom{k}{m}$$

$$\binom{2i}{k} = \sum_{m=0}^{i} \binom{2i-k}{m}\binom{k}{m}$$

Proof. Proof of proposition (2.3). We have three possible relations between i, k: $k < i$, $k = i$, $k > i$. So we have to prove that for every i, k

$$\sum_{m=0}^{k} \binom{2i+1-n-k}{m}\binom{k}{m} - \sum_{m=0}^{i} \binom{2i+1-n-k}{m}\binom{k}{m} = 0$$

For the case $k < i$ proof is the same as proof of proposition (2.2). For the case $k = i$ proof is trivial. Thus, the remaining case is $k > i$ yields

$$\sum_{m=i+1}^{k} \binom{2i+1-n-k}{m}\binom{k}{m} = 0$$

Introducing sum in k to above equation

$$\sum_{m=i+1}^{k} \sum_{k} \binom{2i+1-n-k}{m}\binom{k}{m} = 0$$

Implies

$$\sum_{m=i+1}^{k} \binom{2i+2-n}{2m+1} = 0$$

because $\sum_{k=0}^{l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}$, see equation (5.26) in [15]. Substituting $m = i + 1 + m$ we get

$$\sum_{m=0}^{k} \binom{2i+2-n}{2(i+1+m)+1} = \sum_{m=0}^{k} \binom{2i+2-n}{2i+2m+3} = 0$$

Which is indeed true for every $m, n \geq 0$. This completes the proof. $\square$

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences $\binom{n}{k} - \binom{n}{k}_3$

```
In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame → All]
```

k=4 A000332 n=8

Out[67]=

0																				
0	0																			
0	0	0																		
0	0	0	0																	
0	0	0	0	0																
0	0	0	0	0	0															
0	0	0	0	0	0	0														
0	0	0	0	0	0	0	0													
0	0	0	0	1	0	0	0	0												
0	0	0	0	5	5	0	0	0	0											
0	0	0	0	15	26	15	0	0	0	0										
0	0	0	0	35	81	81	35	0	0	0	0									
0	0	0	0	70	196	262	196	70	0	0	0	0								
0	0	0	0	126	406	658	658	406	126	0	0	0	0							
0	0	0	0	210	756	1414	1716	1414	756	210	0	0	0	0						
0	0	0	0	330	1302	2730	3830	3830	2730	1302	330	0	0	0	0					
0	0	0	0	495	2112	4872	7680	8885	7680	4872	2112	495	0	0	0	0				
0	0	0	0	715	3267	8184	14 232	18 525	18 525	14 232	8184	3267	715	0	0	0	0			
0	0	0	0	1001	4862	13 101	24 816	35 697	40 186	35 697	24 816	13 101	4862	1001	0	0	0	0		
0	0	0	0	1365	7007	20 163	41 217	64 713	80 587	80 587	64 713	41 217	20 163	7007	1365	0	0	0	0	
0	0	0	0	1820	9828	30 030	65 780	111 705	152 020	168 230	152 020	111 705	65 780	30 030	9828	1820	0	0	0	0

Figure 1. Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $\binom{n}{4}$. Sequence A000332 in the OEIS [8].

We can spot that having $i = 3$ the $k = 4$-th column gives binomial coefficient $\binom{n}{4}$. Indeed, this rule is true for every i.

Proposition 2.5. *(Row-column difference.) For every fixed $i \geq 0$*

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

Proof. We have previously stated that iterated rascal numbers are closely related to Vandermonde convolution (2.2). Thus, proposition (2.5) can be rewritten as

$$\sum_{m=0}^{i} \binom{n+i}{m}\binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m}\binom{i}{m}$$

Therefore, $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$ is indeed true. □

Proposition (2.5) yields to few more identities. Applying binomial coefficients symmetry

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking $j = n + i$ gives

$$\binom{j+i}{j} - \binom{j+i}{j}_{i-1} = \binom{j}{j-i}$$

$$\binom{j+i}{i} - \binom{j+i}{i}_{i-1} = \binom{j}{i}$$

Proposition (2.5) can be generalized even further, for every fixed $i < k$ and $i > k$.

Proposition 2.6. *(Finite difference of binomial coefficients and iterated rascal numbers for $i < k$.) For every fixed $i < k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

Proof. It is true by means of Vandermonde convolution. □

Proposition 2.7. *(Finite difference of binomial coefficients and iterated rascal numbers for $i > k$.) For every fixed $i > k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=k+1}^{i} \binom{n-k}{m}\binom{k}{k-m}$$

Proof. It is true by means of Vandermonde convolution. □

3. Q-Binomial identities in Iterated Rascal Triangles

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.

```
In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame → All]
```

k=5

A096943

n=9

Out[67]=

0																				
0	0																			
0	0	0																		
0	0	0	0																	
0	0	0	0	0																
0	0	0	0	0	0															
0	0	0	0	0	0	0														
0	0	0	0	0	0	0	0													
0	0	0	0	1	0	0	0	0												
0	0	0	0	5	5	0	0	0	0											
0	0	0	0	15	26	15	0	0	0	0										
0	0	0	0	35	81	81	35	0	0	0	0									
0	0	0	0	70	196	262	196	70	0	0	0	0								
0	0	0	0	126	406	658	658	406	126	0	0	0	0							
0	0	0	0	210	756	1414	1716	1414	756	210	0	0	0	0						
0	0	0	0	330	1302	2730	3830	3830	2730	1302	330	0	0	0	0					
0	0	0	0	495	2112	4872	7680	8885	7680	4872	2112	495	0	0	0	0				
0	0	0	0	715	3267	8184	14 232	18 525	18 525	14 232	8184	3267	715	0	0	0	0			
0	0	0	0	1001	4862	13 101	24 816	35 697	40 186	35 697	24 816	13 101	4862	1001	0	0	0	0		
0	0	0	0	1365	7007	20 163	41 217	64 713	80 587	80 587	64 713	41 217	20 163	7007	1365	0	0	0	0	
0	0	0	0	1820	9828	30 030	65 780	111 705	152 020	168 230	152 020	111 705	65 780	30 030	9828	1820	0	0	0	0

Figure 2. Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $(1,5)$-binomial coefficient $\left[{n \atop k}\right]^5$. Sequence A096943 in the OEIS [9].

The $(1,q)$-binomial coefficients $\left[{n \atop k}\right]^q$ are special kind of binomial coefficients defined by

Definition 3.1. $(1,q)$-*Binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \left[{n-1 \atop k}\right]^q + \left[{n-1 \atop k-1}\right]^q & \end{cases} \tag{3.1}$$

Indeed, the relation shown in Figure (2) is true for every i, so that it establishes a relation between $(1,q)$-binomial coefficients and iterated rascal numbers.

Proposition 3.2. *(Relation between iterated rascal numbers and* $(1,q)$*-binomial coefficients.) For every fixed* $i \geq 0$

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \begin{bmatrix} i+2+j \\ i+2 \end{bmatrix}^{i+2}$$

Taking $t = i + 2$ in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \begin{bmatrix} t+j \\ t \end{bmatrix}^t$$

In particular, having $i = 1$ proposition (3.2) gives the OEIS sequence A006503 [10] such that third column of $(1, 3)$-Pascal triangle A095660 [11].

Having $i = 3$ proposition (3.2) gives the OEIS sequence A096943 [9] such that third column of $(1, 5)$-Pascal triangle A096940 [12].

For $i = 5$, the proposition (3.2) yields the OEIS sequence A097297 [13] such that seventh column of $(1, 6)$-Pascal triangle A096940 [14].

4. Row sums conjecture

In [6] the authors propose the following conjecture for row sums of iterated rascal triangles.

Conjecture 4.1. *(Conjecture 7.5 in [6].) For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

Proof. Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^{i} \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^{i} \sum_{k} \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2} \tag{4.1}$$

In Concrete mathematics [[15], p. 169, eq (5.26)], Knuth et al. provide the identity for the column sum of binomial coefficients multiplication

$$\sum_{k=0}^{l} \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1} \tag{4.2}$$

We can observe this pattern in the equation (4.1), thus the sum $\sum_k \binom{4i+3-k}{m}\binom{k}{m}$ equals to

$$\sum_{k} \binom{4i+3-k}{m} \binom{k}{m} = \binom{4i+4}{2m+1}$$

Therefore, conjecture (4.1) is equivalent to

$$\sum_{m=0}^{i} \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+3}$$

So that

$$\frac{1}{2}\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^{i} \binom{4i+4}{2m+1} = 2^{4i+2}$$

This completes the proof. $\square$

Proposition 4.2. *For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

In particular, equation (4.2) assumes the following identity in row sums of iterated rascal triangles

$$\sum_{k=0}^{n} \binom{n}{k}_i = \sum_{m=0}^{i} \binom{n+1}{2m+1}$$

5. Conclusions

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.5), (2.6), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.4). Furthermore, we establish a relation between iterated rascal triangles and $(1, q)$-binomial coefficients (3.2). All the results can be validated using supplementary Mathematica scripts at [16].

6. Acknowledgements

Author is grateful to Oleksandr Kulkov, Markus Scheuer, Amelia Gibbs for their valuable feedback and suggestions regarding the conjecture (4.1) at MathStackExchange discussion [17].

References

[1] Anggoro, Alif and Liu, Eddy and Tulloch, Angus. The Rascal Triangle. *The College Mathematics Journal*, 41(5):393–395, 2010. https://doi.org/10.4169/074683410X521991.

[2] Sloane, N. J. A. Pascal's triangle read by rows. Entry A007318 in The On-Line Encyclopedia of Integer Sequences, 1994. https://oeis.org/A007318.

[3] Sloane, N. J. A. The Rascal triangle read by rows. Entry A077028 in The On-Line Encyclopedia of Integer Sequences, 2002. https://oeis.org/A077028.

[4] Amelia Gibbs and Brian K. Miceli. Two Combinatorial Interpretations of Rascal Numbers. *arXiv preprint arXiv:2405.11045*, 2024. https://arxiv.org/abs/2405.11045.

[5] Hotchkiss, Philip K. Generalized Rascal Triangles. *arXiv preprint arXiv:1907.11159*, 2019. https://arxiv.org/abs/1907.11159.

[6] Gregory, Jena and Kronholm, Brandt and White, Jacob. Iterated rascal triangles. *Aequationes mathematicae*, pages 1–18, 2023. https://doi.org/10.1007/s00010-023-00987-6.

[7] Andrews, George E. and Askey, Richard and Roy, Ranjan and Roy, Ranjan and Askey, Richard. *Special functions*, volume 71. Cambridge university press Cambridge, 2000.

[8] Sloane, N. J. A. Binomial coefficient binomial(n,4). Entry A000332 in The On-Line Encyclopedia of Integer Sequences, 2009. https://oeis.org/A000332.

[9] Sloane, N. J. A. Sixth column of (1,5)-Pascal triangle. Entry A096943 in The On-Line Encyclopedia of Integer Sequences, 2004. https://oeis.org/A096943.

[10] Sloane, N. J. A. Entry A006503 in The On-Line Encyclopedia of Integer Sequences, 1995. https://oeis.org/A006503.

[11] Sloane, N. J. A. Pascal (1,3) triangle. Entry A095660 in The On-Line Encyclopedia of Integer Sequences, 2004. https://oeis.org/A095660.

[12] Sloane, N. J. A. Pascal (1,5) triangle. Entry A096940 in The On-Line Encyclopedia of Integer Sequences, 2004. https://oeis.org/A096940.

[13] Sloane, N. J. A. Seventh column (m=6) of (1,6)-Pascal triangle. Entry A097297 in The On-Line Encyclopedia of Integer Sequences, 2004. https://oeis.org/A097297.

[14] Sloane, N. J. A. Pascal (1,6) triangle. Entry A096940 in The On-Line Encyclopedia of Integer Sequences, 2004. https://oeis.org/A096940.

[15] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: a foundation for computer science*. Pearson Education India, 1994.

[16] Kolosov, Petro. Identities in Iterated Rascal Triangles Mathematica programs, 2024. https://github.com/kolosovpetro/IdentitiesInRascalTriangle.

[17] Kolosov, Petro. "Iterated rascal triangle row sums" on MathStackExchange, 2024. https://math.stackexchange.com/q/4941336/463487.

Version: `Local-0.1.0`

Software Developer, DevOps Engineer

Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io