## IDENTITIES IN ITERATED RASCAL TRIANGLES

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ABSTRACT. In this manuscript, we explore new binomial identities within iterated Rascal triangles, revealing a connection between Vandermonde convolution and iterated Rascal numbers. We also provide new identities in terms of finite differences of iterated Rascal numbers and Binomial coefficients. Apart that, the manuscript provides a proof of the row sums conjecture in iterated Rascal triangles. Also, we establish and explore a connection between iterated Rascal triangles and (1,q)-Binomial coefficients, showing related OEIS sequences. All result can be validated through the supplementary Mathematica programs.

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### 1. Introduction

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were challenged to provide the next row for the following number triangle

The teacher anticipated that the next row would match Pascal's triangle, such as "1 4 6 4 1", by applying the binomial coefficient recurrence rule South = East + West. However, Anggoro, Liu, and Tulloch proposed that the next row should be "1 4 5 4 1". Instead of using Pascal's triangle rule South = East + West, they derived this new row using a relation they termed the diamond formula

$$South = \frac{East \cdot West + 1}{North} \tag{1.1}$$

By applying the recurrence relation from equation (1.1), the students successfully generated an entirely new triangular sequence, now referred to as the Rascal triangle.

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	5	4	1					
5	1	5	7	7	5	1				
6	1	6	9	10	9	6	1			
7	1	7	11	13	13	11	7	1		
8	1	8	13	16	17	16	13	8	1	
9	1	9	15	19	21	21	19	15	9	1

Table 1. Rascal triangle. Sequence A077028 in OEIS [2].

For example, the fourth row is "1 4 5 4 1" because  $4 = \frac{1 \cdot 3 + 1}{1}$  and  $5 = \frac{3 \cdot 3 + 1}{2}$ . Moreover, the Rascal triangle, as presented in table (1), represents the first and foundational instance of a new family of Pascal-like triangles. This family, known as *iterated Rascal triangles*, was first introduced by J. Gregory in her master's thesis [3].

We define the k-th element in the n-th row of an iterated Rascal triangle as  $\binom{n}{k}_i$ , where i represents the number of iterations. The integer sequence produced by  $\binom{n}{k}_i$  is referred to as an iterated Rascal triangle Ri, and each  $\binom{n}{k}_i$  is termed an iterated Rascal number. Therefore, the Rascal triangle shown in table (1) corresponds to the iterated Rascal triangle R1, generated by the formula  $\binom{n}{k}_1 = k(n-k) + 1$ . While the iterated Rascal number  $\binom{n}{k}_i$  is defined by the diamond rule (1.1), which differs from the standard binomial coefficient recurrence, it still maintains a significant connection with the binomial coefficients  $\binom{n}{k}$ , as demonstrated by

$$\binom{n}{k}_{i} = \sum_{m=0}^{i} \binom{n-k}{m} \binom{k}{m} \tag{1.2}$$

For example,  $\binom{7}{4}_3 = 35$ ,  $\binom{12}{7}_5 = 792$ ,  $\binom{11}{5}_5 = 462$ .

**Example 1.1.** Rascal triangle R2 generated by  $\binom{n}{k}_2$ 

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	19	15	6	1			
7	1	7	21	31	31	21	7	1		
8	1	8	28	46	53	46	28	8	1	
9	1	9	36	64	81	81	64	36	9	1

Table 2. Rascal triangle R2. Sequence A374378 in OEIS [4].

**Example 1.2.** Rascal triangle R3 generated by  $\binom{n}{k}_3$ 

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	69	56	28	8	1	
9	1	9	36	84	121	121	84	36	9	1

Table 3. Rascal triangle R3. Sequence A374452 in OEIS [5].

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [6], in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [7, 8]. In particular, the

concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

### 2. Binomial identities in Iterated Rascal Triangles

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number [3, eq. 3.2]

# **Definition 2.1.** Iterated rascal number

$$\binom{n}{k}_{i} = \sum_{m=0}^{i} \binom{n-k}{m} \binom{k}{m} \tag{2.1}$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [9]

$$\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m} \binom{b}{r-m}$$

**Implies** 

$$\binom{n}{k} = \sum_{m=0}^{k} \binom{n-k}{m} \binom{k}{k-m}$$

Thus,

$$\binom{n}{k}_{i} = \sum_{m=0}^{i} \binom{n-k}{m} \binom{k}{k-m} \tag{2.2}$$

Meaning that iterated rascal number is partial case of Vandermonde convolution of  $\binom{n}{k}$  with the upper summation bound equals to i. Without further hesitation consider our findings.

**Proposition 2.2.** Iterated rascal triangle equals to Pascal's triangle up to i-th column. For every  $k \leq i$ 

$$\binom{n}{k}_i = \binom{n}{k} \tag{2.3}$$

Then binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying binomial coefficients symmetry principle we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

*Proof.* Proof of proposition (2.2) is given by [8, proposition 6.0.1].

**Proposition 2.3.** Iterated rascal triangle equals to Pascal's triangle up to 2i+1-th row. For every  $n \le 2i+1$ 

$$\binom{n}{k}_i = \binom{n}{k}$$

Therefore, for every  $i \geq 0$  and  $n \geq 0$ 

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over k for each i, so that it is true for all cases in i, k: i < k, i = k and k > i. In particular, equation (2.4) implies the row sums identity in iterated rascal triangles

$$\sum_{k=0}^{\infty} \binom{2i+1-n}{k}_{i} = 2^{2i+1-n}$$

Given n = 0 we obtain

$$\sum_{k=0}^{\infty} \binom{2i+1}{k}_i = 2^{2i+1}$$

and so on. Taking t = 2i + 1 in (2.4) yields

$$\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$$

Moreover, equation (2.4) gives Vandermonde-like identity

$$\binom{2i+1-n}{k} = \sum_{m=0}^{i} \binom{2i+1-n-k}{m} \binom{k}{m}$$
 (2.5)

In particular, given n = 0, 1 equation (2.5) yields the following Vandermonde-like identities

$$\binom{2i+1}{k} = \sum_{m=0}^{i} \binom{2i+1-k}{m} \binom{k}{m}$$

$$\binom{2i}{k} = \sum_{m=0}^{i} \binom{2i-k}{m} \binom{k}{m}$$

*Proof.* Proof of proposition (2.3). We have to prove that for every i, k

$$\sum_{m=0}^{k} {2i+1-n-k \choose m} {k \choose m} - \sum_{m=0}^{i} {2i+1-n-k \choose m} {k \choose m} = 0$$

For the case k < i proof is given by [8, proposition 6.0.1]. For the case k = i proof is trivial. Thus, the remaining case is k > i yields

$$\sum_{m=i+1}^{k} {2i+1-n-k \choose m} {k \choose m} = 0$$
 (2.6)

If (2.6) is true for each k > i, then its sum over k should be zero as well. Introducing sum in k to (2.6) we get

$$\sum_{k} \sum_{m=i+1}^{k} {2i+1-n-k \choose m} {k \choose m} = 0$$

The sum  $\sum_{k} {2i+1-n-k \choose m} {k \choose m}$  appears to match the equation (5.26) in Concrete mathematics [10, eq. 5.26]

$$\sum_{k=0}^{\ell} {\ell-k \choose m} {q+k \choose n} = {\ell+q+1 \choose m+n+1}$$
 (2.7)

Therefore,

$$\sum_{k} {2i+1-n-k \choose m} {k \choose m} = {2i+2-n \choose 2m+1}$$

Thus, our main assumption is equivalent to

$$\sum_{m=i+1}^{k} \sum_{k} {2i+1-n-k \choose m} {k \choose m} \equiv \sum_{m=i+1}^{k} {2i+2-n \choose 2m+1}$$

Hence, we have to prove that

$$\sum_{m=i+1}^{k} \binom{2i+2-n}{2m+1} = 0 \tag{2.8}$$

Substituting m = i + 1 + m into (2.8), we get

$$\sum_{m=0}^{k} {2i+2-n \choose 2(i+1+m)+1} = \sum_{m=0}^{k} {2i+2-n \choose 2i+3+2m} = 0$$

Which is indeed true because  $\binom{2i+2-n}{2i+3+2m} = 0$  for every  $m, n \ge 0$ . Thus, the proposition (2.4) is true.

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences  $\binom{n}{k} - \binom{n}{k}_3$ 

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	0 6	—	$\mathbf{H}$		406	658	658	406	126	0	0	0	0						$\perp$	
	0 6	0	0		756	1414	1716	1414	756	210	0	0	0	0						
	0 6	9 0	0	330	1302	2730	3830	3830	2730	1302	330	0	0	0	0				$\perp$	
	0 6	_	$\mathbf{H}$		2112	4872	7680	8885	7680	4872	2112	495	0	0	0	0			$\perp$	
	0 6				3267	8184	14 232	18 525	18525	14 232	8184	3267	715	0	0	0	0	$\perp$	$\perp$	
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**Figure 1.** Difference  $\binom{n}{k} - \binom{n}{k}_3$ . Highlighted column is  $\binom{n}{4}$ . Sequence A000332 in the OEIS [11].

We can spot that having i = 3 the k = 4-th column gives binomial coefficient  $\binom{n}{4}$ . Indeed, this rule is true for every i.

**Proposition 2.4.** (Row-column difference.) For every  $i \geq 0$ 

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

*Proof.* We have previously stated that iterated rascal numbers are closely related to Vandermonde convolution (2.2). Thus, proposition (2.4) can be rewritten as

$$\sum_{m=0}^{i} \binom{n+i}{m} \binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m} \binom{i}{i-m} = \binom{n+i}{i} \binom{i}{0}$$

Therefore,  $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$  is indeed true.

Proposition (2.4) yields to few more identities. Applying binomial coefficients symmetry

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking j = n + i gives

$$\binom{j+i}{j} - \binom{j+i}{j}_{i-1} = \binom{j}{j-i}$$

By symmetry

$$\binom{j+i}{i} - \binom{j+i}{i}_{i-1} = \binom{j}{i}$$

Proposition (2.4) can be generalized even further, for every i < k and i > k.

**Proposition 2.5.** (Finite difference of binomial coefficients and iterated rascal numbers for i < k.) For every i < k

$$\binom{n}{k} - \binom{n}{k}_{i} = \sum_{m=i+1}^{k} \binom{n-k}{m} \binom{k}{k-m}$$

*Proof.* It is true by means of Vandermonde convolution.

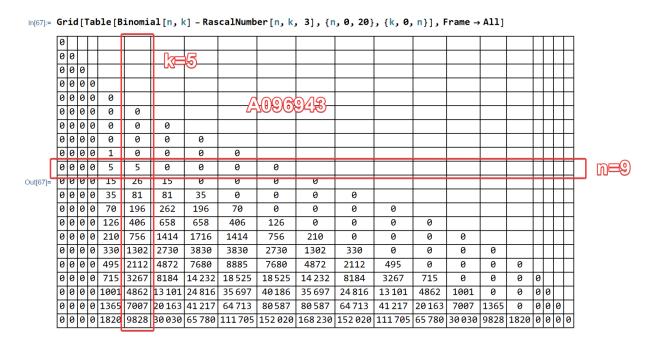
**Proposition 2.6.** (Finite difference of binomial coefficients and iterated rascal numbers for i > k.) For every i > k

$$\binom{n}{k} - \binom{n}{k}_{i} = \sum_{m=k+1}^{i} \binom{n-k}{m} \binom{k}{k-m}$$

*Proof.* It is true by means of Vandermonde convolution.

## 3. Q-Binomial identities in Iterated Rascal Triangles

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.



**Figure 2.** Difference  $\binom{n}{k} - \binom{n}{k}_3$ . Highlighted column is (1,5)-binomial coefficient  $\binom{n}{k}^5$ . Sequence A096943 in the OEIS [12].

The (1,q)-binomial coefficients  $\binom{n}{k}^q$  are special kind of binomial coefficients defined by

**Definition 3.1.** (1,q)-Binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}^{q} = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \end{cases}$$

$$\begin{bmatrix} \binom{n-1}{k} \rceil^{q} + \binom{n-1}{k-1} \rceil^{q} \\ \end{cases}$$
(3.1)

Indeed, the relation shown in Figure (2) is true for every i, so that it establishes a relation between (1, q)-binomial coefficients and iterated rascal numbers.

**Proposition 3.2.** (Relation between iterated rascal numbers and (1, q)-binomial coefficients.) For every  $i \ge 0$ 

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \begin{bmatrix} i+2+j \\ i+2 \end{bmatrix}^{i+2}_i$$

Taking t = i + 2 in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \begin{bmatrix} t+j \\ t \end{bmatrix}^t$$

In particular, having i = 1 proposition (3.2) gives the OEIS sequence A006503 [13] such that third column of (1,3)-Pascal triangle A095660 [14].

Having i=3 proposition (3.2) gives the OEIS sequence A096943 [12] such that third column of (1,5)-Pascal triangle A096940 [15].

For i = 5, the proposition (3.2) yields the OEIS sequence A097297 [16] such that seventh column of (1, 6)-Pascal triangle A096940 [17].

## 4. Row sums conjecture

In [8] the authors propose the following conjecture for row sums of iterated rascal triangles.

Conjecture 4.1. (Conjecture 7.5 in [8].) For every i

$$\sum_{k=0}^{4i+3} {4i+3 \choose k}_i = 2^{4i+2}$$

*Proof.* Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^{i} {4i+3-k \choose m} {k \choose m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^{i} \sum_{k} {4i+3-k \choose m} {k \choose m} = 2^{4i+2}$$
(4.1)

We can observe the pattern (2.7) in equation (4.1). Thus, the sum  $\sum_{k} {4i+3-k \choose m} {k \choose m}$  equals to

$$\sum_{k} {4i+3-k \choose m} {k \choose m} = {4i+4 \choose 2m+1}$$

Therefore, conjecture (4.1) is equivalent to

$$\sum_{m=0}^{i} \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} {4i+4 \choose 2m+1} = \sum_{m=0}^{2i+1} \left[ {4i+3 \choose 2m+1} + {4i+3 \choose 2m} \right] = 2^{4i+3}$$

So that

$$\frac{1}{2} \sum_{m=0}^{2i+1} {4i+4 \choose 2m+1} = \sum_{m=0}^{i} {4i+4 \choose 2m+1} = 2^{4i+2}$$

This completes the proof.

Proposition 4.2. For every i

$$\sum_{k=0}^{4i+3} {4i+3 \choose k}_i = 2^{4i+2}$$

In particular, equation (2.7) assumes the following identity in row sums of iterated rascal triangles

$$\sum_{k=0}^{n} \binom{n}{k}_{i} = \sum_{m=0}^{i} \binom{n+1}{2m+1}$$

Decomposing  $\binom{n+1}{2m+1}$  in above equation yields

**Proposition 4.3.** (Iterated rascal triangles row sums.) For every i

$$\sum_{k=0}^{n} \binom{n}{k}_{i} = \sum_{m=0}^{2i+1} \binom{n}{m}$$

Proof.

$$\sum_{k=0}^{n} \binom{n}{k}_{i} = \sum_{m=0}^{i} \binom{n+1}{2m+1} = \sum_{m=0}^{i} \binom{n}{2m} + \binom{n}{2m+1} = \sum_{m=0}^{2i+1} \binom{n}{m}$$

## 5. Conclusions

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.4), (2.5), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.5). Furthermore, we establish a relation between iterated rascal triangles and (1,q)-binomial coefficients (3.2). All the results can be validated using supplementary Mathematica scripts at [18].

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