

IDENTITIES IN ITERATED RASCAL TRIANGLES

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ABSTRACT. In this manuscript, we show new binomial identities in iterated rascal triangles, revealing a connection between the Vandermonde convolution and iterated rascal numbers. We also present Vandermonde-like binomial identities. Furthermore, we establish a relation between iterated rascal triangle and $(1, q)$ -binomial coefficients.

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1. INTRODUCTION

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were

Date: July 6, 2024.

2010 Mathematics Subject Classification. 11B25, 11B99.

Key words and phrases. Pascal's triangle, Rascal triangle, Binomial coefficients, Binomial identities, Binomial theorem, Generalized Rascal triangles, Iterated rascal triangles, Iterated rascal numbers, Vandermonde identity, Vandermonde convolution .

Sources: <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>

challenged to provide the next row for the following number triangle

$$\begin{array}{ccccccc}
 & & & & 1 & & & \\
 & & & & & & 1 & \\
 & & & 1 & & 2 & & 1 \\
 & & 1 & & 3 & & 3 & & 1 \\
 & & & & & & \dots & &
 \end{array}$$

Teacher's expected answer was the one that matches Pascal's triangle, e.g. "1 4 6 4 1". However, Anggoro, Liu, and Tulloch suggested "1 4 5 4 1" instead. They devised this new row via what they called diamond formula

$$\text{South} = \frac{\text{East} \cdot \text{West} + 1}{\text{North}}$$

So they obtained the following triangle

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	5	4	1			
5	1	5	7	7	5	1		
6	1	6	9	10	9	6	1	
7	1	7	11	13	13	11	7	1

Table 1. Rascal triangle. See the OEIS sequence [A077028](#) [3].

Indeed, the forth row is "1 4 5 4 1" because $4 = \frac{1 \cdot 3 + 1}{1}$ and $5 = \frac{3 \cdot 3 + 1}{2}$.

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [4], in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [5, 6]. In particular, the

concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

2. BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number

Definition 2.1. *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} \quad (2.1)$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [7]

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

Implies

$$\binom{n}{k} = \sum_{m=0}^k \binom{n-k}{m} \binom{k}{k-m}$$

Thus,

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \quad (2.2)$$

Meaning that iterated rascal number is partial case of Vandermonde convolution of $\binom{n}{k}$ with the upper summation bound equals to i . Without further hesitation consider our findings.

Proposition 2.2. *Iterated rascal triangle equals to Pascal's triangle up to i -th column. For every $k \leq i$*

$$\binom{n}{k}_i = \binom{n}{k} \quad (2.3)$$

Then binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying binomial coefficients symmetry principle we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

Proof. Proof of proposition (2.2). Consider the following relation between binomial coefficients and iterated rascal numbers, for every $0 \leq k \leq i$

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=0}^k \binom{n-k}{m} \binom{k}{k-m} - \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} = 0$$

Yields

$$\sum_{m=k+1}^i \binom{n-k}{m} \binom{k}{m} = 0$$

It is indeed true, because binomial coefficients $\binom{k}{m}$ are zero for each $m \geq k+1$. So that for every $k \leq i$

$$\binom{n}{k} - \binom{n}{k}_i = 0$$

Therefore, the proposition (2.2) is true. \square

Proposition 2.3. *Iterated rascal triangle equals to Pascal's triangle up to $2i+1$ -th row. For every $n \leq 2i+1$*

$$\binom{n}{k}_i = \binom{n}{k}$$

Therefore, for every $i \geq 0$ and $n \geq 0$

$$\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k} \quad (2.4)$$

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over k for each i , so that it is true for all cases in i, k : $i < k$, $i = k$ and $k > i$. In particular, equation (2.4) implies the row sums identity in iterated rascal triangles

$$\sum_{k=0}^{\infty} \binom{2i+1-n}{k}_i = 2^{2i+1-n}$$

Given $n = 0$ we obtain

$$\sum_{k=0}^{\infty} \binom{2i+1}{k}_i = 2^{2i+1}$$

and so on. Taking $t = 2i + 1$ in (2.4) yields

$$\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$$

Proof. Proof of proposition (2.3). We have to prove that for every i, k

$$\sum_{m=0}^k \binom{2i+1-n-k}{m} \binom{k}{m} - \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

For the case $k < i$ proof is the same as proof of proposition (2.2). For the case $k = i$ proof is trivial. Thus, the remaining case is $k > i$ yields

$$\sum_{m=i+1}^k \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

Introducing sum in k to above equation

$$\sum_{m=i+1}^k \sum_k \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

Implies

$$\sum_{m=i+1}^k \binom{2i+2-n}{2m+1} = 0$$

because $\sum_{k=0}^l \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1}$, see equation (5.26) in [15]. Substituting $m = i + 1 + m$ we get

$$\sum_{m=0}^k \binom{2i+2-n}{2(i+1+m)+1} = \sum_{m=0}^k \binom{2i+2-n}{2i+3+2m} = 0$$

Which is indeed true because $\binom{2i+2-n}{2i+3+2m} = 0$ for every $m, n \geq 0$. □

Moreover, equation (2.4) gives Vandermonde-like identity

Proposition 2.4. (*Vandermonde-like identity.*)

$$\binom{2i+1-n}{k} = \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m}$$

$$\begin{aligned}\binom{2i+1}{k} &= \sum_{m=0}^i \binom{2i+1-k}{m} \binom{k}{m} \\ \binom{2i}{k} &= \sum_{m=0}^i \binom{2i-k}{m} \binom{k}{m}\end{aligned}$$

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences $\binom{n}{k} - \binom{n}{k}_3$



We can spot that having $i = 3$ the $k = 4$ -th column gives binomial coefficient $\binom{n}{4}$. Indeed, this rule is true for every i .

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

Proof. We have previously stated that iterated rascal numbers are closely related to Vandermonde convolution (2.2). Thus, proposition (2.5) can be rewritten as

$$\sum_{m=0}^i \binom{n+i}{m} \binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m} \binom{i}{i-m} = \binom{n+i}{i} \binom{i}{0}$$

Therefore, $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$ is indeed true. \square

Proposition (2.5) yields to few more identities. Applying binomial coefficients symmetry

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking $j = n + i$ gives

$$\begin{aligned} \binom{j+i}{j} - \binom{j+i}{j}_{i-1} &= \binom{j}{j-i} \\ \binom{j+i}{i} - \binom{j+i}{i}_{i-1} &= \binom{j}{i} \end{aligned}$$

Proposition (2.5) can be generalized even further, for every fixed $i < k$ and $i > k$.

Proposition 2.6. *(Finite difference of binomial coefficients and iterated rascal numbers for $i < k$.) For every fixed $i < k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^k \binom{n-k}{m} \binom{k}{k-m}$$

Proof. It is true by means of Vandermonde convolution. \square

Proposition 2.7. *(Finite difference of binomial coefficients and iterated rascal numbers for $i > k$.) For every fixed $i > k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=k+1}^i \binom{n-k}{m} \binom{k}{k-m}$$

Proof. It is true by means of Vandermonde convolution. \square

3. Q-BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.

```
In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame -> All]
```

0																				
0	0																			
0	0	0																		
0	0	0	0																	
0	0	0	0	0																
0	0	0	0	0	0															
0	0	0	0	0	0	0														
0	0	0	0	0	0	0	0													
0	0	0	0	0	0	0	0	0												
0	0	0	0	0	0	0	0	0	0											
0	0	0	0	0	0	0	0	0	0	0										
0	0	0	0	0	0	0	0	0	0	0	0									
0	0	0	0	0	0	0	0	0	0	0	0	0								
0	0	0	0	0	0	0	0	0	0	0	0	0	0							
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0						
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0					
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 2. Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $(1, 5)$ -binomial coefficient $\binom{n}{k}^5$. Sequence **A096943** in the OEIS [9].

The $(1, q)$ -binomial coefficients $\binom{n}{k}^q$ are special kind of binomial coefficients defined by

Definition 3.1. $(1, q)$ -Binomial coefficient

$$\binom{n}{k}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \binom{n-1}{k}^q + \binom{n-1}{k-1}^q & \end{cases} \quad (3.1)$$

Indeed, the relation shown in Figure (2) is true for every i , so that it establishes a relation between $(1, q)$ -binomial coefficients and iterated rascal numbers.

Proposition 3.2. *(Relation between iterated rascal numbers and $(1, q)$ -binomial coefficients.)*

For every fixed $i \geq 0$

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \left[\begin{matrix} i+2+j \\ i+2 \end{matrix} \right]^{i+2}$$

Taking $t = i + 2$ in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[\begin{matrix} t+j \\ t \end{matrix} \right]^t$$

In particular, having $i = 1$ proposition (3.2) gives the OEIS sequence [A006503](#) [10] such that third column of $(1, 3)$ -Pascal triangle [A095660](#) [11].

Having $i = 3$ proposition (3.2) gives the OEIS sequence [A096943](#) [9] such that third column of $(1, 5)$ -Pascal triangle [A096940](#) [12].

For $i = 5$, the proposition (3.2) yields the OEIS sequence [A097297](#) [13] such that seventh column of $(1, 6)$ -Pascal triangle [A096940](#) [14].

4. ROW SUMS CONJECTURE

In [6] the authors propose the following conjecture for row sums of iterated rascal triangles.

Conjecture 4.1. *(Conjecture 7.5 in [6].) For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

Proof. Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^i \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^i \sum_k \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2} \quad (4.1)$$

In Concrete mathematics [[15], p. 169, eq (5.26)], Knuth et al. provide the identity for the column sum of binomial coefficients multiplication

$$\sum_{k=0}^l \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1} \quad (4.2)$$

We can observe this pattern in the equation (4.1), thus the sum $\sum_k \binom{4i+3-k}{m} \binom{k}{m}$ equals to

$$\sum_k \binom{4i+3-k}{m} \binom{k}{m} = \binom{4i+4}{2m+1}$$

Therefore, conjecture (4.1) is equivalent to

$$\sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+3}$$

So that

$$\frac{1}{2} \sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

This completes the proof. \square

Proposition 4.2. *For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

In particular, equation (4.2) assumes the following identity in row sums of iterated rascal triangles

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^i \binom{n+1}{2m+1}$$

5. CONCLUSIONS

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.5), (2.6), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.4). Furthermore, we establish a relation between iterated rascal triangles and $(1, q)$ -binomial coefficients (3.2). All the results can be validated using supplementary Mathematica scripts at [16].

6. ACKNOWLEDGEMENTS

Author is grateful to Oleksandr Kulkov, Markus Scheuer, Amelia Gibbs for their valuable feedback and suggestions regarding the conjecture (4.1) at MathStackExchange discussion [17].

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Version: Local-0.1.0

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