POLYNOMIAL IDENTITIES INVOLVING RASCAL TRIANGLE

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Abstract. Abstract

1. Definitions

Definition of generalized Rascal triangle

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m} \tag{1.1}$$

$$= \binom{n-k}{0}\binom{k}{0} + \binom{n-k}{1}\binom{k}{1} + \binom{n-k}{2}\binom{k}{2} + \ldots + \binom{n-k}{i}\binom{k}{i} \tag{1.2}$$

Definition of $(1, q)$-Pascal triangle

$$\begin{bmatrix} n \\ k \end{bmatrix}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \begin{bmatrix} n-1 \\ k \end{bmatrix}^q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^q \end{cases}$$

Pascals triangle as polynomial

$$\binom{n}{k} = \frac{(n)^{\underline{k}}}{k!} = \frac{1}{k!} n(n-1)(n-2)\cdots(n-(k-1)) = \prod_{i=1}^{k} \frac{n-i+1}{i} \tag{1.3}$$

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2. Sides of world

$$\mathbf{North} = \binom{n-2}{k-1}_i$$

$$\mathbf{South} = \binom{n}{k}_i$$

$$\mathbf{West} = \binom{n-1}{k-1}_i$$

$$\mathbf{East} = \binom{n-1}{k}_i$$

Identity see Hotchkiss

$$\mathbf{South} = \frac{\mathbf{East} \cdot \mathbf{West} + 1}{\mathbf{North}} \tag{2.1}$$

$$\binom{n}{k}_i = \frac{\binom{n-1}{k}_i \binom{n-1}{k-1}_i + 1}{\binom{n-2}{k-1}_i} \tag{2.2}$$

Identity see Hotchkiss, for all inner $k > 0$ and $k < n$

$$\mathbf{South} = \mathbf{East} + \mathbf{West} - \mathbf{North} + 1 \tag{2.3}$$

$$\binom{n}{k}_i = \binom{n-1}{k}_i + \binom{n-1}{k-1}_i - \binom{n-2}{k-1}_i + 1 \tag{2.4}$$

3. Formulae

3.1. **Claim 0.** Generalized rascal triangle is partial case of Chu-Vandermonde convolution

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

3.2. **Claim 1.** Generalized rascal triangle equals to Pascal's triangle up to i-th column

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \le k \le i \tag{3.1}$$

$$\binom{n}{i-j}_i = \binom{n}{i-j}, \quad ColumnIdentity1 \tag{3.2}$$

$$\binom{n}{n-i+j}_i = \binom{n}{n-i+j}, \quad ColumnIdentity2 \tag{3.3}$$

$$\tag{3.4}$$

3.3. **Claim 2.** Generalized rascal triangle equals to Pascal's triangle up to $2i+1$-th row

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \le n \le 2i+1 \tag{3.5}$$

For every fixed $i \ge 0$

$$\binom{2i+1-j}{k}_i = \binom{2i+1-j}{k} \quad RowIdentity1 \tag{3.6}$$

$$\binom{(2i+1)-j}{k}_{(2i+1)-i-1} = \binom{(2i+1)-j}{k} \tag{3.7}$$

For every fixed $i \ge 0$ and $t \ge 2i+1$

$$\binom{t-j}{k}_{t-i-1} = \binom{t-j}{k} \quad RowIdentity2 \tag{3.8}$$

For $k = j$

$$\binom{2i+1-j}{j}_i = \binom{2i+1-j}{j}, \quad 0 \le j \le i \quad RowIdentity3$$

$$\binom{2i+1-j}{2i+1-2j}_i = \binom{2i+1-j}{2i+1-2j}$$

$$\binom{(2i+1)-j}{(2i+1)-2j}_{(2i+1)-i-1} = \binom{(2i+1)-j}{(2i+1)-2j}$$

$$\binom{t-j}{t-2j}_{t-i-1} = \binom{t-j}{t-2j}, \quad t \ge 2i+1, \quad 0 \le j \le t-i-1, \quad RowIdentity4$$

Proof.

$$\binom{2i+1-j}{k}_i = \binom{2i+1-j}{k}$$

$$\binom{2i+1-j}{k}_i = \sum_{m=0}^{i} \binom{2i+1-j-k}{m}\binom{k}{m} = \binom{2i+1-j}{k}$$

3.4. **Claim 3.** Row-column difference identity. Proof via Vandermonde's identity. For every fixed $i \geq 1$

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i} \quad RowColumnDifferenceIdentity1$$

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

$$\binom{(n+i)+i}{(n+i)} - \binom{(n+i)+i}{(n+i)}_{i-1} = \binom{(n+i)}{(n+i)-i}$$

$$\binom{j+i}{j} - \binom{j+i}{j}_{i-1} = \binom{j}{j-i}, \quad RowColumnDifferenceIdentity2$$

3.5. **Claim 4.** Relation between $(1,q)$-Pascal's triangle

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \begin{bmatrix} i+2+j \\ i+2 \end{bmatrix}^{i+2}, \quad OneQPascalIdentity1$$

$$\binom{2(i+2)-1+j}{i+2} - \binom{2(i+2)-1+j}{i+2}_i = \begin{bmatrix} i+2+j \\ i+2 \end{bmatrix}^{i+2}$$

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \begin{bmatrix} t+j \\ t \end{bmatrix}^{t}, \quad OneQPascalIdentity2$$

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