

# IDENTITIES IN ITERATED RASCAL TRIANGLES

PETRO KOLOSOV

ABSTRACT. In this manuscript, we show new binomial identities in iterated rascal triangles, revealing a connection between the Vandermonde convolution and iterated rascal numbers. We also present Vandermonde-like binomial identities. Furthermore, we establish a relation between iterated rascal triangle and  $(1, q)$ -binomial coefficients.

## CONTENTS

1. Introduction	1
2. Binomial identities in Iterated Rascal Triangles	3
3. Q-Binomial identities in Iterated Rascal Triangles	7
4. Row sums conjecture	9
5. Conclusions	10
6. Acknowledgements	11
References	11

## 1. INTRODUCTION

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were

---

*Date:* July 6, 2024.

*2010 Mathematics Subject Classification.* 11B25, 11B99.

*Key words and phrases.* Pascal's triangle, Rascal triangle, Binomial coefficients, Binomial identities, Binomial theorem, Generalized Rascal triangles, Iterated rascal triangles, Iterated rascal numbers, Vandermonde identity, Vandermonde convolution .

Sources: <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>

challenged to provide the next row for the following number triangle

$$\begin{array}{ccccccc}
 & & & & 1 & & & \\
 & & & & & & 1 & \\
 & & & 1 & & 2 & & 1 \\
 & & 1 & & 3 & & 3 & & 1 \\
 & & & & & & \dots & & 
 \end{array}$$

Teacher's expected answer was the one that matches Pascal's triangle, e.g "1 4 6 4 1". However, Anggoro, Liu, and Tulloch suggested "1 4 5 4 1" instead. They devised this new row via what they called diamond formula

$$\text{South} = \frac{\text{East} \cdot \text{West} + 1}{\text{North}}$$

So they obtained the following triangle

$n/k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	5	4	1			
5	1	5	7	7	5	1		
6	1	6	9	10	9	6	1	
7	1	7	11	13	13	11	7	1

**Table 1.** Rascal triangle. See the OEIS sequence [A077028](#) [3].

Indeed, the forth row is "1 4 5 4 1" because  $4 = \frac{1 \cdot 3 + 1}{1}$  and  $5 = \frac{3 \cdot 3 + 1}{2}$ .

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [4], in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [5, 6]. In particular, the

concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

## 2. BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number

**Definition 2.1.** *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} \quad (2.1)$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [7]

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}; \quad \binom{n}{k} = \sum_{m=0}^k \binom{n-k}{m} \binom{k}{k-m}$$

Thus,

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \quad (2.2)$$

Meaning that iterated rascal number is partial case of Vandermonde convolution of  $\binom{n}{k}$  with the upper summation bound equals to  $i$ . Without further hesitation consider our findings.

**Proposition 2.2.** *Iterated rascal triangle equals to Pascal's triangle up to  $i$ -th column.*

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq k \leq i \quad (2.3)$$

Then binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying binomial coefficients symmetry principle we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

*Proof.* Proof of proposition (2.2). Consider the following relation between binomial coefficients and iterated rascal numbers, for every  $0 \leq k \leq i$

$$\begin{aligned} \binom{n}{k} - \binom{n}{k}_i &= \sum_{m=0}^k \binom{n-k}{m} \binom{k}{k-m} - \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \\ &= \sum_{m=k+1}^i \binom{n-k}{m} \binom{k}{m} = 0 \end{aligned}$$

It is indeed true, because binomial coefficients  $\binom{k}{m}$  are zero for each  $m \geq k+1$ . So that for every  $0 \leq k \leq i$

$$\binom{n}{k} - \binom{n}{k}_i = 0; \quad \binom{n}{k} = \binom{n}{k}_i$$

Therefore, the proposition (2.2) is true.  $\square$

**Proposition 2.3.** *Iterated rascal triangle equals to Pascal's triangle up to  $2i+1$ -th row*

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq n \leq 2i+1$$

Therefore, for every fixed  $i \geq 0$  and  $n \geq 0$

$$\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k} \quad (2.4)$$

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over  $k$  for each  $i$ , so that it is true for all cases in  $i, k$ :  $i < k$ ,  $i = k$  and  $k > i$ . In particular, equation (2.4) implies the row sums identity in iterated rascal triangles

$$\sum_{k=0}^{\infty} \binom{2i+1-n}{k}_i = 2^{2i+1-n}$$

Given  $n = 0$  we obtain

$$\sum_{k=0}^{\infty} \binom{2i+1}{k}_i = 2^{2i+1}$$

and so on. Taking  $t = 2i+1$  in (2.4) yields

$$\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$$

Moreover, equation (2.4) gives Vandermonde-like identity

**Proposition 2.4.** (*Vandermonde-like identity.*)

$$\binom{2i+1-n}{k} = \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m}$$

In particular, given  $n = 0, 1$  proposition (2.4) gives the following binomial identities

$$\begin{aligned} \binom{2i+1}{k} &= \sum_{m=0}^i \binom{2i+1-k}{m} \binom{k}{m} \\ \binom{2i}{k} &= \sum_{m=0}^i \binom{2i-k}{m} \binom{k}{m} \end{aligned}$$

*Proof.* Proof of proposition (2.3). We have to prove that for every  $i, k$

$$\sum_{m=0}^k \binom{2i+1-n-k}{m} \binom{k}{m} - \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

For the case  $k < i$  proof is the same as proof of proposition (2.2). For the case  $k = i$  proof is trivial. Thus, the remaining case is  $k > i$  yields

$$\sum_{m=i+1}^k \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

Introducing sum in  $k$  to above equation

$$\sum_{m=i+1}^k \sum_k \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

Implies

$$\sum_{m=i+1}^k \binom{2i+2-n}{2m+1} = 0$$

because  $\sum_{k=0}^l \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1}$ , see equation (5.26) in [15]. Substituting  $m = i+1+m$  we get

$$\sum_{m=0}^k \binom{2i+2-n}{2(i+1+m)+1} = \sum_{m=0}^k \binom{2i+2-n}{2i+3+2m} = 0$$

Which is indeed true for every  $m, n \geq 0$ . This completes the proof.  $\square$

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences  $\binom{n}{k} - \binom{n}{k}_3$

0				k=4																	
0	0																				
0	0	0																			
0	0	0	0																		
0	0	0	0	0				A000332													
0	0	0	0	0	0																
0	0	0	0	0	0	0															
0	0	0	0	0	0	0	0														
0	0	0	0	0	0	0	0	0													
Out[67]=	0	0	0	0	1	0	0	0	0	0								n=8			
	0	0	0	0	5	5	0	0	0	0											
	0	0	0	0	15	26	15	0	0	0	0										
	0	0	0	0	35	81	81	35	0	0	0	0									
	0	0	0	0	70	196	262	196	70	0	0	0	0								
	0	0	0	0	126	406	658	658	406	126	0	0	0	0							
	0	0	0	0	210	756	1414	1716	1414	756	210	0	0	0	0						
	0	0	0	0	330	1302	2730	3830	3830	2730	1302	330	0	0	0	0					
	0	0	0	0	495	2112	4872	7680	8885	7680	4872	2112	495	0	0	0	0				
	0	0	0	0	715	3267	8184	14232	18525	18525	14232	8184	3267	715	0	0	0	0			
	0	0	0	0	1001	4862	13101	24816	35697	40186	35697	24816	13101	4862	1001	0	0	0	0		
	0	0	0	0	1365	7007	20163	41217	64713	80587	80587	64713	41217	20163	7007	1365	0	0	0	0	
	0	0	0	0	1820	9828	30030	65780	111705	152020	168230	152020	111705	65780	30030	9828	1820	0	0	0	0

We can spot that having  $i = 3$  the  $k = 4$ -th column gives binomial coefficient  $\binom{n}{4}$ . Indeed, this rule is true for every  $i$ .

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$
$$\sum_{m=0}^i \binom{n+i}{m} \binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m} \binom{i}{i-m} = \binom{n+i}{i} \binom{i}{0}$$
☐
$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking  $j = n + i$  gives

$$\begin{aligned} \binom{j+i}{j} - \binom{j+i}{j}_{i-1} &= \binom{j}{j-i} \\ \binom{j+i}{i} - \binom{j+i}{i}_{i-1} &= \binom{j}{i} \end{aligned}$$

Proposition (2.5) can be generalized even further, for every fixed  $i < k$  and  $i > k$ .

**Proposition 2.6.** *(Finite difference of binomial coefficients and iterated rascal numbers for  $i < k$ .) For every fixed  $i < k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^k \binom{n-k}{m} \binom{k}{k-m}$$

*Proof.* It is true by means of Vandermonde convolution. □

**Proposition 2.7.** *(Finite difference of binomial coefficients and iterated rascal numbers for  $i > k$ .) For every fixed  $i > k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=k+1}^i \binom{n-k}{m} \binom{k}{k-m}$$

*Proof.* It is true by means of Vandermonde convolution. □

### 3. Q-BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.

```
In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame -> All]
```

0																				
0	0																			
0	0	0																		
0	0	0	0																	
0	0	0	0	0																
0	0	0	0	0	0															
0	0	0	0	0	0	0														
0	0	0	0	0	0	0	0													
0	0	0	0	0	1	0	0	0	0											
0	0	0	0	0	5	5	0	0	0	0										
0	0	0	0	0	15	26	15	0	0	0	0									
0	0	0	0	0	35	81	81	35	0	0	0	0								
0	0	0	0	0	70	196	262	196	70	0	0	0	0							
0	0	0	0	0	126	406	658	658	406	126	0	0	0	0						
0	0	0	0	0	210	756	1414	1716	1414	756	210	0	0	0	0					
0	0	0	0	0	330	1302	2730	3830	3830	2730	1302	330	0	0	0	0				
0	0	0	0	0	495	2112	4872	7680	8885	7680	4872	2112	495	0	0	0	0			
0	0	0	0	0	715	3267	8184	14232	18525	18525	14232	8184	3267	715	0	0	0	0		
0	0	0	0	0	1001	4862	13101	24816	35697	40186	35697	24816	13101	4862	1001	0	0	0	0	
0	0	0	0	0	1365	7007	20163	41217	64713	80587	80587	64713	41217	20163	7007	1365	0	0	0	0
0	0	0	0	0	1820	9828	30030	65780	111705	152020	168230	152020	111705	65780	30030	9828	1820	0	0	0

**Figure 2.** Difference  $\binom{n}{k} - \binom{n}{k}_3$ . Highlighted column is  $(1, 5)$ -binomial coefficient  $\begin{bmatrix} n \\ k \end{bmatrix}^5$ . Sequence **A096943** in the OEIS [9].

The  $(1, q)$ -binomial coefficients  $\begin{bmatrix} n \\ k \end{bmatrix}^q$  are special kind of binomial coefficients defined by

**Definition 3.1.**  $(1, q)$ -Binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \begin{bmatrix} n-1 \\ k \end{bmatrix}^q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^q & \end{cases} \quad (3.1)$$

Indeed, the relation shown in Figure (2) is true for every  $i$ , so that it establishes a relation between  $(1, q)$ -binomial coefficients and iterated rascal numbers.

**Proposition 3.2.** (Relation between iterated rascal numbers and  $(1, q)$ -binomial coefficients.)

For every fixed  $i \geq 0$

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \begin{bmatrix} i+2+j \\ i+2 \end{bmatrix}^{i+2}$$



Taking  $t = i + 2$  in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[ \begin{matrix} t+j \\ t \end{matrix} \right]^t$$

In particular, having  $i = 1$  proposition (3.2) gives the OEIS sequence [A006503](#) [10] such that third column of (1, 3)-Pascal triangle [A095660](#) [11].

Having  $i = 3$  proposition (3.2) gives the OEIS sequence [A096943](#) [9] such that third column of (1, 5)-Pascal triangle [A096940](#) [12].

For  $i = 5$ , the proposition (3.2) yields the OEIS sequence [A097297](#) [13] such that seventh column of (1, 6)-Pascal triangle [A096940](#) [14].

#### 4. ROW SUMS CONJECTURE

In [6] the authors propose the following conjecture for row sums of iterated rascal triangles.

**Conjecture 4.1.** (*Conjecture 7.5 in [6].*) For every  $i$

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

*Proof.* Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^i \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^i \sum_k \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2} \quad (4.1)$$

In Concrete mathematics [[15], p. 169, eq (5.26)], Knuth et al. provide the identity for the column sum of binomial coefficients multiplication

$$\sum_{k=0}^l \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1} \quad (4.2)$$

We can observe this pattern in the equation (4.1), thus the sum  $\sum_k \binom{4i+3-k}{m} \binom{k}{m}$  equals to

$$\sum_k \binom{4i+3-k}{m} \binom{k}{m} = \binom{4i+4}{2m+1}$$

Therefore, conjecture (4.1) is equivalent to

$$\sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+3}$$

So that

$$\frac{1}{2} \sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

This completes the proof. □

**Proposition 4.2.** *For every  $i$*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

In particular, equation (4.2) assumes the following identity in row sums of iterated rascal triangles

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^i \binom{n+1}{2m+1}$$

## 5. CONCLUSIONS

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.5), (2.6), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.4). Furthermore, we establish a relation between iterated rascal triangles and  $(1, q)$ -binomial coefficients (3.2). All the results can be validated using supplementary Mathematica scripts at [16].

## 6. ACKNOWLEDGEMENTS

Author is grateful to Oleksandr Kulkov, Markus Scheuer, Amelia Gibbs for their valuable feedback and suggestions regarding the conjecture (4.1) at MathStackExchange discussion [17].

## REFERENCES

- [1] Anggoro, Alif and Liu, Eddy and Tulloch, Angus. The Rascal Triangle. *The College Mathematics Journal*, 41(5):393–395, 2010. <https://doi.org/10.4169/074683410X521991>.
- [2] Sloane, N. J. A. Pascal’s triangle read by rows. Entry A007318 in The On-Line Encyclopedia of Integer Sequences, 1994. <https://oeis.org/A007318>.
- [3] Sloane, N. J. A. The Rascal triangle read by rows. Entry A077028 in The On-Line Encyclopedia of Integer Sequences, 2002. <https://oeis.org/A077028>.
- [4] Amelia Gibbs and Brian K. Miceli. Two Combinatorial Interpretations of Rascal Numbers. *arXiv preprint arXiv:2405.11045*, 2024. <https://arxiv.org/abs/2405.11045>.
- [5] Hotchkiss, Philip K. Generalized Rascal Triangles. *arXiv preprint arXiv:1907.11159*, 2019. <https://arxiv.org/abs/1907.11159>.
- [6] Gregory, Jena and Kronholm, Brandt and White, Jacob. Iterated rascal triangles. *Aequationes mathematicae*, pages 1–18, 2023. <https://doi.org/10.1007/s00010-023-00987-6>.
- [7] Andrews, George E. and Askey, Richard and Roy, Ranjan and Roy, Ranjan and Askey, Richard. *Special functions*, volume 71. Cambridge university press Cambridge, 2000.
- [8] Sloane, N. J. A. Binomial coefficient  $\text{binomial}(n,4)$ . Entry A000332 in The On-Line Encyclopedia of Integer Sequences, 2009. <https://oeis.org/A000332>.
- [9] Sloane, N. J. A. Sixth column of (1,5)-Pascal triangle. Entry A096943 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A096943>.
- [10] Sloane, N. J. A. Entry A006503 in The On-Line Encyclopedia of Integer Sequences, 1995. <https://oeis.org/A006503>.
- [11] Sloane, N. J. A. Pascal (1,3) triangle. Entry A095660 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A095660>.
- [12] Sloane, N. J. A. Pascal (1,5) triangle. Entry A096940 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A096940>.

- [13] Sloane, N. J. A. Seventh column ( $m=6$ ) of  $(1,6)$ -Pascal triangle. Entry A097297 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A097297>.
- [14] Sloane, N. J. A. Pascal  $(1,6)$  triangle. Entry A096940 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A096940>.
- [15] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: a foundation for computer science*. Pearson Education India, 1994.
- [16] Kolosov, Petro. Identities in Iterated Rascal Triangles Mathematica programs, 2024. <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>.
- [17] Kolosov, Petro. "Iterated rascal triangle row sums" on MathStackExchange, 2024. <https://math.stackexchange.com/q/4941336/463487>.

**Version:** Local-0.1.0

SOFTWARE DEVELOPER, DEVOPS ENGINEER

*Email address:* kolosovp94@gmail.com

*URL:* <https://kolosovpetro.github.io>