

POLYNOMIAL IDENTITIES INVOLVING RASCAL TRIANGLE

PETRO KOLOSOV

ABSTRACT. Abstract

1. DEFINITIONS

Definition of generalized Rascal triangle

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \quad (1.1)$$

$$= \binom{n-k}{0} \binom{k}{0} + \binom{n-k}{1} \binom{k}{1} + \binom{n-k}{2} \binom{k}{2} + \dots + \binom{n-k}{i} \binom{k}{i} \quad (1.2)$$

Definition of $(1, q)$ -Pascal triangle

$$\begin{bmatrix} n \\ k \end{bmatrix}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \begin{bmatrix} n-1 \\ k \end{bmatrix}^q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^q & \end{cases}$$

Pascals triangle as polynomial

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{1}{k!} n(n-1)(n-2) \dots (n-(k-1)) = \prod_{i=1}^k \frac{n-i+1}{i} \quad (1.3)$$

Date: June 30, 2024.

2010 Mathematics Subject Classification. 26E70, 05A30.

Key words and phrases. Keyword1, Keyword2 .

2. SIDES OF WORLD

$$\mathbf{North} = \binom{n-2}{k-1}_i$$

$$\mathbf{South} = \binom{n}{k}_i$$

$$\mathbf{West} = \binom{n-1}{k-1}_i$$

$$\mathbf{East} = \binom{n-1}{k}_i$$

Identity see Hotchkiss

$$\mathbf{South} = \frac{\mathbf{East} \cdot \mathbf{West} + 1}{\mathbf{North}} \quad (2.1)$$

$$\binom{n}{k}_i = \frac{\binom{n-1}{k}_i \binom{n-1}{k-1}_i + 1}{\binom{n-2}{k-1}_i} \quad (2.2)$$

Identity see Hotchkiss, for all inner $k > 0$ and $k < n$

$$\mathbf{South} = \mathbf{East} + \mathbf{West} - \mathbf{North} + 1 \quad (2.3)$$

$$\binom{n}{k}_i = \binom{n-1}{k}_i + \binom{n-1}{k-1}_i - \binom{n-2}{k-1}_i + 1 \quad (2.4)$$

3. FORMULAE

3.1. **Claim 0.** Generalized rascal triangle is partial case of Chu-Vandermonde convolution

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

3.2. **Claim 1.** Generalized rascal triangle equals to Pascal's triangle up to i -th column

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq k \leq i \quad (3.1)$$

$$\binom{n}{i-j}_i = \binom{n}{i-j}, \quad \text{ColumnIdentity1} \quad (3.2)$$

$$\binom{n}{n-i+j}_i = \binom{n}{n-i+j}, \quad \text{ColumnIdentity2} \quad (3.3)$$

$$(3.4)$$

3.3. **Claim 2.** Generalized rascal triangle equals to Pascal's triangle up to $2i+1$ -th row

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq n \leq 2i+1 \quad (3.5)$$

For every fixed $i \geq 0$

$$\binom{2i+1-j}{k}_i = \binom{2i+1-j}{k} \quad \text{RowIdentity1} \quad (3.6)$$

$$\binom{(2i+1)-j}{k}_{(2i+1)-i-1} = \binom{(2i+1)-j}{k} \quad (3.7)$$

For every fixed $i \geq 0$ and $t \geq 2i+1$

$$\binom{t-j}{k}_{t-i-1} = \binom{t-j}{k} \quad \text{RowIdentity2} \quad (3.8)$$

For $k = j$

$$\binom{2i+1-j}{j}_i = \binom{2i+1-j}{j}, \quad 0 \leq j \leq i \quad \text{RowIdentity3}$$

$$\binom{2i+1-j}{2i+1-2j}_i = \binom{2i+1-j}{2i+1-2j}$$

$$\binom{(2i+1)-j}{(2i+1)-2j}_{(2i+1)-i-1} = \binom{(2i+1)-j}{(2i+1)-2j}$$

$$\binom{t-j}{t-2j}_{t-i-1} = \binom{t-j}{t-2j}, \quad t \geq 2i+1, \quad 0 \leq j \leq t-i-1, \quad \text{RowIdentity4}$$

Proof.

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

$$\binom{2i+1-j}{k}_i = \binom{2i+1-j}{k}$$

$$\binom{2i+1-j}{k}_i = \sum_{m=0}^i \binom{2i+1-j-k}{m} \binom{k}{m} = \binom{2i+1-j}{k}$$

$$\binom{2i+1-j}{k} = \sum_{m=0}^i \binom{2i+1-j-k}{m} \binom{k}{m}$$

$$\binom{2i}{k} = \sum_{m=0}^i \binom{2i-k}{m} \binom{k}{m}$$

3.4. **Claim 3.** Row-column difference identity. Proof via Vandermonde's identity. For every fixed $i \geq 1$

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i} \quad \text{RowColumnDifferenceIdentity1}$$

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

$$\binom{(n+i)+i}{(n+i)} - \binom{(n+i)+i}{(n+i)}_{i-1} = \binom{(n+i)}{(n+i)-i}$$

$$\binom{j+i}{j} - \binom{j+i}{j}_{i-1} = \binom{j}{j-i}, \quad \text{RowColumnDifferenceIdentity2}$$

3.5. **Claim 4.** Relation between $(1, q)$ -Pascal's triangle

$$\begin{aligned} \binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i &= \left[\begin{matrix} i+2+j \\ i+2 \end{matrix} \right]^{i+2}, \quad \text{OneQPascalIdentity1} \\ \binom{2(i+2)-1+j}{i+2} - \binom{2(i+2)-1+j}{i+2}_i &= \left[\begin{matrix} i+2+j \\ i+2 \end{matrix} \right]^{i+2} \\ \binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} &= \left[\begin{matrix} t+j \\ t \end{matrix} \right]^t, \quad \text{OneQPascalIdentity2} \end{aligned}$$

SOFTWARE DEVELOPER, DEVOPS ENGINEER

Email address: kolosovp94@gmail.com

URL: <https://kolosovpetro.github.io>