

# IDENTITIES IN ITERATED RASCAL TRIANGLES

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ABSTRACT. In this manuscript, we show new binomial identities in iterated rascal triangles, revealing a connection between Vandermonde convolution and iterated rascal numbers. We also present Vandermonde-like binomial identities. Furthermore, we establish a relation between iterated rascal triangle and  $(1, q)$ -binomial coefficients.

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## 1. INTRODUCTION

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were

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Sources: <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>

challenged to provide the next row for the following number triangle

$$\begin{array}{ccccccc}
 & & & & 1 & & & \\
 & & & & & & 1 & \\
 & & & 1 & & 2 & & 1 \\
 & & 1 & & 3 & & 3 & & 1 \\
 & & & & & & \dots & & 
 \end{array}$$

Teacher's expected answer was the one that matches Pascal's triangle, e.g. "1 4 6 4 1". However, Anggoro, Liu, and Tulloch suggested "1 4 5 4 1" instead. They devised this new row via what they called diamond formula

$$South = \frac{East \cdot West + 1}{North}$$

So they obtained the following triangle

$n/k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	5	4	1			
5	1	5	7	7	5	1		
6	1	6	9	10	9	6	1	
7	1	7	11	13	13	11	7	1

**Table 1.** Rascal triangle. See the OEIS sequence [A077028](#) [2].

Indeed, the forth row is "1 4 5 4 1" because  $4 = \frac{1 \cdot 3 + 1}{1}$  and  $5 = \frac{3 \cdot 3 + 1}{2}$ .

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [3], in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [4, 5]. In particular, the

concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

## 2. BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number

**Definition 2.1.** *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} \quad (2.1)$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [6]

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

Implies

$$\binom{n}{k} = \sum_{m=0}^k \binom{n-k}{m} \binom{k}{k-m}$$

Thus,

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \quad (2.2)$$

Meaning that iterated rascal number is partial case of Vandermonde convolution of  $\binom{n}{k}$  with the upper summation bound equals to  $i$ . Without further hesitation consider our findings.

**Proposition 2.2.** *Iterated rascal triangle equals to Pascal's triangle up to  $i$ -th column. For every  $k \leq i$*

$$\binom{n}{k}_i = \binom{n}{k} \quad (2.3)$$

Then binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying binomial coefficients symmetry principle we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

*Proof.* Proof of proposition (2.2). Consider the following relation between binomial coefficients and iterated rascal numbers, for every  $k \leq i$

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=0}^k \binom{n-k}{m} \binom{k}{k-m} - \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} = 0$$

Yields

$$\sum_{m=k+1}^i \binom{n-k}{m} \binom{k}{m} = 0$$

It is indeed true, because binomial coefficients  $\binom{k}{m}$  are zero for each  $m \geq k+1$ . So that for every  $k \leq i$

$$\binom{n}{k} - \binom{n}{k}_i = 0$$

Therefore, the proposition (2.2) is true. □

**Proposition 2.3.** *Iterated rascal triangle equals to Pascal's triangle up to  $2i+1$ -th row. For every  $n \leq 2i+1$*

$$\binom{n}{k}_i = \binom{n}{k}$$

Therefore, for every  $i \geq 0$  and  $n \geq 0$

$$\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k} \tag{2.4}$$

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over  $k$  for each  $i$ , so that it is true for all cases in  $i, k$ :  $i < k$ ,  $i = k$  and  $k > i$ . In particular, equation (2.4) implies the row sums identity in iterated rascal triangles

$$\sum_{k=0}^{\infty} \binom{2i+1-n}{k}_i = 2^{2i+1-n}$$

Given  $n = 0$  we obtain

$$\sum_{k=0}^{\infty} \binom{2i+1}{k}_i = 2^{2i+1}$$

and so on. Taking  $t = 2i + 1$  in (2.4) yields

$$\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$$

*Proof.* Proof of proposition (2.3). We have to prove that for every  $i, k$

$$\sum_{m=0}^k \binom{2i+1-n-k}{m} \binom{k}{m} - \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

For the case  $k < i$  proof is the same as proof of proposition (2.2). For the case  $k = i$  proof is trivial. Thus, the remaining case is  $k > i$  yields

$$\sum_{m=i+1}^k \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

Introducing sum in  $k$  to above equation

$$\sum_{m=i+1}^k \sum_k \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

Implies

$$\sum_{m=i+1}^k \binom{2i+2-n}{2m+1} = 0$$

because  $\sum_{k=0}^l \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1}$ , see equation (5.26) in [7]. Substituting  $m = i + 1 + m$  we get

$$\sum_{m=0}^k \binom{2i+2-n}{2(i+1+m)+1} = \sum_{m=0}^k \binom{2i+2-n}{2i+3+2m} = 0$$

Which is indeed true because  $\binom{2i+2-n}{2i+3+2m} = 0$  for every  $m, n \geq 0$ . □

Moreover, equation (2.4) gives Vandermonde-like identity

**Proposition 2.4.** (*Vandermonde-like identity.*)

$$\binom{2i+1-n}{k} = \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m}$$

$$\begin{aligned}\binom{2i+1}{k} &= \sum_{m=0}^i \binom{2i+1-k}{m} \binom{k}{m} \\ \binom{2i}{k} &= \sum_{m=0}^i \binom{2i-k}{m} \binom{k}{m}\end{aligned}$$

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences  $\binom{n}{k} - \binom{n}{k}_3$



We can spot that having  $i = 3$  the  $k = 4$ -th column gives binomial coefficient  $\binom{n}{4}$ . Indeed, this rule is true for every  $i$ .

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

*Proof.* We have previously stated that iterated rascal numbers are closely related to Vandermonde convolution (2.2). Thus, proposition (2.5) can be rewritten as

$$\sum_{m=0}^i \binom{n+i}{m} \binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m} \binom{i}{i-m} = \binom{n+i}{i} \binom{i}{0}$$

Therefore,  $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$  is indeed true.  $\square$

Proposition (2.5) yields to few more identities. Applying binomial coefficients symmetry

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking  $j = n + i$  gives

$$\binom{j+i}{j} - \binom{j+i}{j}_{i-1} = \binom{j}{j-i}$$

By symmetry

$$\binom{j+i}{i} - \binom{j+i}{i}_{i-1} = \binom{j}{i}$$

Proposition (2.5) can be generalized even further, for every  $i < k$  and  $i > k$ .

**Proposition 2.6.** *(Finite difference of binomial coefficients and iterated rascal numbers for  $i < k$ .) For every  $i < k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^k \binom{n-k}{m} \binom{k}{k-m}$$

*Proof.* It is true by means of Vandermonde convolution.  $\square$

**Proposition 2.7.** *(Finite difference of binomial coefficients and iterated rascal numbers for  $i > k$ .) For every  $i > k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=k+1}^i \binom{n-k}{m} \binom{k}{k-m}$$

*Proof.* It is true by means of Vandermonde convolution.  $\square$

### 3. Q-BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.

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In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame -> All]
```

0																				
0	0																			
0	0	0																		
0	0	0	0																	
0	0	0	0	0																
0	0	0	0	0	0															
0	0	0	0	0	0	0														
0	0	0	0	0	0	0	0													
0	0	0	0	0	0	0	0	0												
0	0	0	0	0	0	0	0	0	0											
0	0	0	0	0	0	0	0	0	0	0										
0	0	0	0	0	0	0	0	0	0	0	0									
0	0	0	0	0	0	0	0	0	0	0	0	0								
0	0	0	0	0	0	0	0	0	0	0	0	0	0							
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0						
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0					
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

**Figure 2.** Difference  $\binom{n}{k} - \binom{n}{k}_3$ . Highlighted column is  $(1, 5)$ -binomial coefficient  $\binom{n}{k}^5$ . Sequence **A096943** in the OEIS [9].

The  $(1, q)$ -binomial coefficients  $\binom{n}{k}^q$  are special kind of binomial coefficients defined by

**Definition 3.1.**  $(1, q)$ -Binomial coefficient

$$\binom{n}{k}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \binom{n-1}{k}^q + \binom{n-1}{k-1}^q & \end{cases} \quad (3.1)$$

Indeed, the relation shown in Figure (2) is true for every  $i$ , so that it establishes a relation between  $(1, q)$ -binomial coefficients and iterated rascal numbers.



**Proposition 3.2.** *(Relation between iterated rascal numbers and  $(1, q)$ -binomial coefficients.)*

For every  $i \geq 0$

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \left[ \begin{matrix} i+2+j \\ i+2 \end{matrix} \right]^{i+2}$$

Taking  $t = i + 2$  in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[ \begin{matrix} t+j \\ t \end{matrix} \right]^t$$

In particular, having  $i = 1$  proposition (3.2) gives the OEIS sequence [A006503](#) [10] such that third column of  $(1, 3)$ -Pascal triangle [A095660](#) [11].

Having  $i = 3$  proposition (3.2) gives the OEIS sequence [A096943](#) [9] such that third column of  $(1, 5)$ -Pascal triangle [A096940](#) [12].

For  $i = 5$ , the proposition (3.2) yields the OEIS sequence [A097297](#) [13] such that seventh column of  $(1, 6)$ -Pascal triangle [A096940](#) [14].

#### 4. ROW SUMS CONJECTURE

In [5] the authors propose the following conjecture for row sums of iterated rascal triangles.

**Conjecture 4.1.** *(Conjecture 7.5 in [5].) For every  $i$*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

*Proof.* Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^i \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^i \sum_k \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2} \quad (4.1)$$

In Concrete mathematics [[7], p. 169, eq (5.26)], Knuth et al. provide the identity for the column sum of binomial coefficients multiplication

$$\sum_{k=0}^l \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1} \quad (4.2)$$

We can observe this pattern in the equation (4.1), thus the sum  $\sum_k \binom{4i+3-k}{m} \binom{k}{m}$  equals to

$$\sum_k \binom{4i+3-k}{m} \binom{k}{m} = \binom{4i+4}{2m+1}$$

Therefore, conjecture (4.1) is equivalent to

$$\sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+3}$$

So that

$$\frac{1}{2} \sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

This completes the proof.  $\square$

**Proposition 4.2.** *For every  $i$*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

In particular, equation (4.2) assumes the following identity in row sums of iterated rascal triangles

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^i \binom{n+1}{2m+1}$$

Decomposing  $\binom{n+1}{2m+1}$  in above equation yields

**Proposition 4.3.** *(Iterated rascal triangles row sums.) For every  $i$*

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^{2i+1} \binom{n}{m}$$

*Proof.*

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^i \binom{n+1}{2m+1} = \sum_{m=0}^i \binom{n}{2m} + \binom{n}{2m+1} = \sum_{m=0}^{2i+1} \binom{n}{m}$$

$\square$

## 5. CONCLUSIONS

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.5), (2.6), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.4). Furthermore, we establish a relation between iterated rascal triangles and  $(1, q)$ -binomial coefficients (3.2). All the results can be validated using supplementary Mathematica scripts at [15].

## 6. ACKNOWLEDGEMENTS

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