POLYNOMIAL IDENTITIES INVOLVING RASCAL TRIANGLE

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Abstract. Abstract

Contents

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1. Definitions

Definition of generalized Rascal triangle

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m} \tag{1.1}$$

$$= \binom{n-k}{0}\binom{k}{0} + \binom{n-k}{1}\binom{k}{1} + \binom{n-k}{2}\binom{k}{2} + \ldots + \binom{n-k}{i}\binom{k}{i} \tag{1.2}$$

Definition of $(1,q)$-Pascal triangle

$$\begin{bmatrix} n \\ k \end{bmatrix}^q = \begin{cases} q & \text{if } k=0, n=0 \\ 1 & \text{if } k=0 \\ 0 & \text{if } k>n \\ \begin{bmatrix} n-1 \\ k \end{bmatrix}^q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^q & \end{cases}$$

Pascals triangle as polynomial

$$\binom{n}{k} = \frac{(n)^{\underline{k}}}{k!} = \frac{1}{k!} n(n-1)(n-2)\cdots(n-(k-1)) = \prod_{i=1}^{k} \frac{n-i+1}{i} \tag{1.3}$$

2. Formulae

2.1. **Claim 0. Vandermonde convolution.** Generalized rascal number is partial case of Vandermonde convolution Consider Vandermonde convolution

$$\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m}\binom{b}{r-m}$$

Then generalized rascal number is partial case of Vandermonde convolution with upper summation bound equals i

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m}$$

2.2. **Claim 1. I-th column identity.** Generalized rascal triangle equals to Pascal's triangle up to i-th column

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \le k \le i \tag{2.1}$$

$$\binom{n}{i-j}_i = \binom{n}{i-j}, \quad ColumnIdentity1 \tag{2.2}$$

$$\binom{n}{n-i+j}_i = \binom{n}{n-i+j}, \quad ColumnIdentity2 \tag{2.3}$$

2.3. **Claim 2. 2i+1 row identity.** Generalized rascal triangle equals to Pascal's triangle up to $2i+1$-th row

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \le n \le 2i+1 \tag{2.4}$$

For every fixed $i \ge 0$

$$\binom{2i+1-j}{k}_i = \binom{2i+1-j}{k} \quad RowIdentity1 \tag{2.5}$$

$$\binom{(2i+1)-j}{k}_{(2i+1)-i-1} = \binom{(2i+1)-j}{k} \tag{2.6}$$

For every fixed $i \ge 0$ and $t \ge 2i+1$

$$\binom{t-j}{k}_{t-i-1} = \binom{t-j}{k} \quad RowIdentity2 \tag{2.7}$$

For $k = j$

$$\binom{2i+1-j}{j}_i = \binom{2i+1-j}{j}, \quad 0 \le j \le i \quad RowIdentity3$$

$$\binom{2i+1-j}{2i+1-2j}_i = \binom{2i+1-j}{2i+1-2j}$$

$$\binom{(2i+1)-j}{(2i+1)-2j}_{(2i+1)-i-1} = \binom{(2i+1)-j}{(2i+1)-2j}$$

$$\binom{t-j}{t-2j}_{t-i-1} = \binom{t-j}{t-2j}, \quad t \ge 2i+1, \quad 0 \le j \le t-i-1, \quad RowIdentity4$$

2.4. **Proof of 2i+1 row identity.** Let be definition of rascal number

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m}$$

Then for every $0 \leq n \leq 2i+1$ holds

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq n \leq 2i+1 \tag{2.8}$$

Let be Vandermonde formula

$$\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m}\binom{b}{r-m}$$

By Vandermonde formula

$$\binom{n}{k} = \sum_{m=0}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

Then for every $0 \leq n \leq 2i+1$ and $0 \leq k \leq 2i+1-n$

$$\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k}$$

Thus we have to prove that

$$\binom{2i+1-n}{k}_i = \sum_{m=0}^{i} \binom{2i+1-n-k}{m}\binom{k}{m} = \binom{2i+1-n}{k}$$

$$\binom{2i+1-n}{k} = \sum_{m=0}^{i} \binom{2i+1-n-k}{m}\binom{k}{m} = \sum_{m=0}^{k} \binom{2i+1-n-k}{m}\binom{k}{k-m}$$

Rewrite explicitly gives for i

$$\begin{aligned}\sum_{m=0}^{i} \binom{2i+1-n-k}{m}\binom{k}{m} &= \binom{2i+1-n-k}{0}\binom{k}{0} + \binom{2i+1-n-k}{1}\binom{k}{1} \\ &+ \binom{2i+1-n-k}{2}\binom{k}{2} + \binom{2i+1-n-k}{3}\binom{k}{3} \\ &+ \cdots + \binom{2i+1-n-k}{i}\binom{k}{i}\end{aligned}$$

Rewrite explicitly gives for k

$$\begin{aligned}\sum_{m=0}^{k}\binom{2i+1-n-k}{m}\binom{k}{m} &= \binom{2i+1-n-k}{0}\binom{k}{0}+\binom{2i+1-n-k}{1}\binom{k}{1}\\ &+\binom{2i+1-n-k}{2}\binom{k}{2}+\binom{2i+1-n-k}{3}\binom{k}{3}\\ &+\cdots+\binom{2i+1-n-k}{k}\binom{k}{k}\end{aligned}$$

We have three cases in general $k<i$, $k=i$, $k>i$, so we have to prove that

$$\sum_{m=0}^{k}\binom{2i+1-n-k}{m}\binom{k}{m}-\sum_{m=0}^{i}\binom{2i+1-n-k}{m}\binom{k}{m}=0$$

For the case $k<i$ proof in Jenna et all paper. For the case $k=i$ proof is trivial. Thus,

$$\sum_{m=i+1}^{k}\binom{2i+1-n-k}{m}\binom{k}{m}=0$$

Considering the constraints,

$$\begin{cases} n\geq 0\\ k\geq i+1\\ 2i+1-n-k\leq i-n\\ m\geq i+1\end{cases}$$

Thus,

$$\sum_{m=i+1}^{k}\binom{2i+1-n-k}{m}\binom{k}{m}=0$$

because binomial coefficients $\binom{i-n-s}{i+1+s}$ are zero for each $i,n,s\geq 0$.

2.5. **Claim 3. Row-column difference binomial identity.** Row-column difference identity. Proof via Vandermonde's identity. For every fixed $i \geq 1$

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i} \quad RowColumnDifferenceIdentity1$$

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

$$\binom{(n+i)+i}{(n+i)} - \binom{(n+i)+i}{(n+i)}_{i-1} = \binom{(n+i)}{(n+i)-i}$$

$$\binom{j+i}{j} - \binom{j+i}{j}_{i-1} = \binom{j}{j-i}, \quad RowColumnDifferenceIdentity2$$

2.6. **Proof of Row-column difference binomial identity.** Let be Vandermonde formula

$$\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m}\binom{b}{r-m}$$

Let be definition of rascal number

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m}$$

Then

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

$$\sum_{m=0}^{i} \binom{n+i}{m}\binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m}\binom{i}{m} = \binom{n+i}{i}$$

2.7. **Difference between binomial coefficients and iterated rascal numbers.** For every $i < k$

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^{k} \binom{n-k}{m}\binom{k}{m}$$

2.8. **Claim 4.** Relation between $(1, q)$-Pascal's triangle

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \begin{bmatrix} i+2+j \\ i+2 \end{bmatrix}^{i+2}, \quad OneQPascalIdentity1$$

$$\binom{2(i+2)-1+j}{i+2} - \binom{2(i+2)-1+j}{i+2}_i = \begin{bmatrix} i+2+j \\ i+2 \end{bmatrix}^{i+2}$$

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \begin{bmatrix} t+j \\ t \end{bmatrix}^{t}, \quad OneQPascalIdentity2$$

3. Row sums power of 2 identity

By definition

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} \tag{3.1}$$

Consider Vandermonde convolution

$$\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m}\binom{b}{r-m}$$

Then generalized rascal number is partial case of Vandermonde convolution with upper summation bound equals i

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m}$$

Difference between binomial coefficients and iterated rascal numbers

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

Proposition 3.1. *Row* $4i+3$ *sum gives* 2^{4i+2}

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = \sum_{k=0}^{4i+3} \sum_{m=0}^{i} \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

We know that sum of binomial coefficients equals 2^n

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k} = 2^{4i+3}$$

If conjecture holds, then it must be true that

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k} - \binom{4i+3}{k}_i = 2^{4i+2}$$

because $2^n - 2^n = 2^{n-1}$. Thus,

$$\sum_{k=0}^{4i+3} \sum_{m=i+1}^{k} \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

In case $k < i$ the sum $\sum_{m=i+1}^{k} \binom{4i+3-k}{m}$ is always zero.

$$\sum_{k=i+1}^{4i+3} \sum_{m=i+1}^{k} \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

$$\sum_{k=0}^{4i+3} \sum_{m=i+1}^{k} \binom{4i+3-(i+1+k)}{m} \binom{k}{m} = 2^{4i+2}$$

$$\sum_{k=0}^{4i+3} \sum_{m=i+1}^{k} \binom{3i+2-k}{m} \binom{k}{m} = 2^{4i+2}$$

$$\sum_{k=0}^{4i+3} \sum_{m=0}^{k} \binom{3i+2-k}{i+1+m} \binom{k}{i+1+m} = 2^{4i+2}$$

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