MATH STACKOVERFLOW QUESTION

PETRO KOLOSOV

1. Introduction

It is widely known fact that finite difference of cubes can be expressed in terms of triangular numbers

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6\binom{n+1}{2}$$

where $\binom{n+1}{2}$ are triangular numbers. Apart that, triangular numbers themselves are equivalent to the sum of first n natural numbers

$$\binom{n+1}{2} = \sum_{k=1}^{n} k$$

Which leads to identity in terms of finite differences of cubes

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6\sum_{k=1}^{n} k$$

It is obvious that n^3 evaluates to the sum of its $n-1$ finite differences, so that

$$\Delta n^3 = \sum_{k=0}^{n-1} \Delta k^3 = \sum_{k=0}^{n-1} \left(1 + 6\sum_{k=1}^{n} k\right) \qquad (1)$$

In its explicit form

$$n^3 = [1 + 6\cdot 0] + [1 + 6\cdot 0 + 6\cdot 1] + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2]$$
$$+ \cdots + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + \cdots + 6\cdot(n-1)] \qquad (2)$$

Now the tricky part starts. We could use Faulhaber's formula on $\sum_{k=1}^{n} k$ in (1), which leads to well known and expected identity in cubes $n^3 = \sum_{k=0}^{n-1}\sum_{r=0}^{2}\binom{3}{r}k^r$.

Instead, let's rearrange the terms in (2) to get

$$n^3 = n + [(n-0)\cdot 6 \cdot 0] + [(n-1)\cdot 6\cdot 1] + [(n-2)\cdot 6 \cdot 2]$$
$$+ \cdots + [(n-k)\cdot 6\cdot k] + \cdots + [1\cdot 6\cdot (n-1)]$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term n in the sum above can be moved under sigma notation, because there is exactly n iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k)+1$$

By inspecting the expression $6k(n-k)+1$ we iterate under summation, we can notice that it is symmetric over k, let be $T(n,k) = 6k(n-k)+1$, then

$$T(n,k) = T(n,n-k)$$

This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^{n} 6k(n-k)+1$$

Assume that polynomial identities $n^3 = \sum_{k=0}^{n-1} 6k(n-k)+1$ and $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$ have explicit form as follows

$$n^3 = \sum_{k} A_{1,1}k^1(n-k)^1 + A_{1,0}k^0(n-k)^0$$

where $A_{1,1} = 6$ and $A_{1,0} = 1$, respectively.

It could be generalized even further, for every odd power $2m+1$, giving a set of real coefficients $A_{m,0}, A_{m,1}, A_{m,2}, A_{m,3}, \ldots, A_{m,m}$ such that

$$n^{2m+1} = \sum_{k=1}^{n} A_{m,0}k^0(n-k)^0 + A_{m,1}(n-k)^1 + \cdots + A_{m,m}k^m(n-k)^m$$

Leading to numerous polynomial identities, including its compact form

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} A_{m,r} k^r (n-k)^r; \quad n^{2m+1} = \sum_{r=0}^{m} \sum_{k=0}^{n-1} A_{m,r} k^r (n-k)^r$$

For example,

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k) + 1$$

$$n^9 = \sum_{k=1}^{n} 630k^4(n-k)^4 - 120k(n-k) + 1$$

$$n^{11} = \sum_{k=1}^{n} 2772k^5(n-k)^5 + 660k^2(n-k)^2 - 1386k(n-k) + 1$$