

COEFFICIENTS IN POLYNOMIAL IDENTITY FOR ODD POWERS

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ABSTRACT. <https://mathoverflow.net/q/485764/113033>

1. INTRODUCTION

It is widely known fact that finite difference of cubes can be expressed in terms of triangular numbers

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6 \binom{n+1}{2}$$

where $\binom{n+1}{2}$ are triangular numbers. Apart that, triangular numbers themselves are equivalent to the sum of first n non-negative integers

$$\binom{n+1}{2} = \sum_{k=0}^n k$$

Which leads to identity in terms of finite differences of cubes

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6 \sum_{k=0}^n k$$

It is obvious that n^3 evaluates to the sum of its $n-1$ finite differences, so that

$$\Delta n^3 = \sum_{k=0}^{n-1} \Delta k^3 = \sum_{k=0}^{n-1} \left(1 + 6 \sum_{t=0}^k t \right) \quad (1)$$

In its explicit form

$$\begin{aligned} n^3 &= [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] \\ &\quad + \cdots + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned} \quad (2)$$

We could use Faulhaber's formula on $\sum_{t=0}^k t$ in (1), which leads to well known and expected identity in cubes $n^3 = \sum_{k=0}^{n-1} \sum_{t=0}^2 \binom{3}{t} k^t$.

Instead, let's rearrange the terms in (2) to get

$$\begin{aligned} n^3 = n &+ [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] \\ &+ \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)] \end{aligned}$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term n in the sum above can be moved under sigma notation, because there is exactly n iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1 \quad (3)$$

By inspecting the expression $6k(n-k) + 1$ we iterate under summation, we can notice that it is symmetric over k , let be $T(n, k) = 6k(n-k) + 1$, then

$$T(n, k) = T(n, n-k)$$

This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^n 6k(n-k) + 1 \quad (4)$$

Assume that polynomial identities $n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$ and $n^3 = \sum_{k=1}^n 6k(n-k) + 1$ have explicit form as follows

$$n^3 = \sum_k A_{1,1} k^1 (n-k)^1 + A_{1,0} k^0 (n-k)^0$$

where $A_{1,1} = 6$ and $A_{1,0} = 1$, respectively.

It could be generalized even further, for every odd power $2m+1$, giving a set of real coefficients $A_{m,0}, A_{m,1}, A_{m,2}, A_{m,3}, \dots, A_{m,m}$ such that

$$n^{2m+1} = \sum_{k=1}^n A_{m,0} k^0 (n-k)^0 + A_{m,1} k^1 (n-k)^1 + \cdots + A_{m,m} k^m (n-k)^m \quad (5)$$

Leading to numerous polynomial identities, including its compact form

$$n^{2m+1} = \sum_{r=0}^m \sum_{k=1}^n A_{m,r} k^r (n-k)^r; \quad n^{2m+1} = \sum_{r=0}^m \sum_{k=0}^{n-1} A_{m,r} k^r (n-k)^r$$

For example,

$$\begin{aligned} n^3 &= \sum_{k=1}^n 6k(n-k) + 1 \\ n^5 &= \sum_{k=1}^n 30k^2(n-k)^2 + 1 \\ n^7 &= \sum_{k=1}^n 140k^3(n-k)^3 - 14k(n-k) + 1 \\ n^9 &= \sum_{k=1}^n 630k^4(n-k)^4 - 120k(n-k) + 1 \\ n^{11} &= \sum_{k=1}^n 2772k^5(n-k)^5 + 660k^2(n-k)^2 - 1386k(n-k) + 1 \end{aligned}$$

These coefficients $A_{m,r}$ are registered in OEIS: <https://oeis.org/A302971>, <https://oeis.org/A304042>.

Recurrence relation for $A_{m,r}$ is given by: <https://mathoverflow.net/q/297900/113033>

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****Question 1**:** The algorithm we used to obtain identities for cubes (3), (4) is quite simple, if not naive. I believe it should be discussed in mathematical literature, as well as identity that gives a set of real coefficients $A_{m,r}$ such that

$$n^{2m+1} = \sum_{k=1}^n A_{m,0} k^0 (n-k)^0 + A_{m,1} (n-k)^1 + \cdots + A_{m,m} k^m (n-k)^m$$

However, I was not able to find any references that in particular mention coefficients $A_{m,r}$, which is one of open questions.

****Question 2**:** Can we consider the process of obtaining the identities (3), (4) as an interpolation technique?

****Question 3**:** If the question 2 is true, can we consider equation (5) as an interpolation technique too?