COEFFICIENTS IN POLYNOMIAL IDENTITY FOR ODD POWERS

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1. Introduction

It is widely known fact that finite difference of cubes can be expressed in terms of triangular numbers

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6\binom{n+1}{2}$$

where $\binom{n+1}{2}$ are triangular numbers. Apart that, triangular numbers themselves are equivalent to the sum of first n natural numbers

$$\binom{n+1}{2} = \sum_{k=1}^{n} k$$

Which leads to identity in terms of finite differences of cubes

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6\sum_{k=1}^{n} k$$

It is obvious that n^3 evaluates to the sum of its $n-1$ finite differences, so that

$$\Delta n^3 = \sum_{k=0}^{n-1} \Delta k^3 = \sum_{k=0}^{n-1} \left(1 + 6\sum_{k=1}^{n} k\right) \qquad (1)$$

In its explicit form

$$n^3 = [1 + 6\cdot 0] + [1 + 6\cdot 0 + 6\cdot 1] + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2]$$
$$+ \cdots + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + \cdots + 6\cdot (n-1)] \qquad (2)$$

Now the tricky part starts. We could use Faulhaber's formula on $\sum_{k=1}^{n} k$ in (1), which leads to well known and expected identity in cubes $n^3 = \sum_{k=0}^{n-1}\sum_{r=0}^{2}\binom{3}{r}k^r$.

Instead, let's rearrange the terms in (2) to get

$$n^3 = n + [(n-0)\cdot 6\cdot 0] + [(n-1)\cdot 6\cdot 1] + [(n-2)\cdot 6\cdot 2]$$
$$+ \cdots + [(n-k)\cdot 6\cdot k] + \cdots + [1\cdot 6\cdot (n-1)]$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term n in the sum above can be moved under sigma notation, because there is exactly n iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1 \qquad (3)$$

By inspecting the expression $6k(n-k)+1$ we iterate under summation, we can notice that it is symmetric over k, let be $T(n,k) = 6k(n-k)+1$, then

$$T(n,k) = T(n, n-k)$$

This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \qquad (4)$$

Assume that polynomial identities $n^3 = \sum_{k=0}^{n-1} 6k(n-k)+1$ and $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$ have explicit form as follows

$$n^3 = \sum_{k} A_{1,1}k^1(n-k)^1 + A_{1,0}k^0(n-k)^0$$

where $A_{1,1} = 6$ and $A_{1,0} = 1$, respectively.

It could be generalized even further, for every odd power $2m+1$, giving a set of real coefficients $A_{m,0}, A_{m,1}, A_{m,2}, A_{m,3}, \ldots, A_{m,m}$ such that

$$n^{2m+1} = \sum_{k=1}^{n} A_{m,0}k^0(n-k)^0 + A_{m,1}(n-k)^1 + \cdots + A_{m,m}k^m(n-k)^m \qquad (5)$$

Leading to numerous polynomial identities, including its compact form

$$n^{2m+1}=\sum_{r=0}^{m}\sum_{k=1}^{n}A_{m,r}k^r(n-k)^r; \quad n^{2m+1}=\sum_{r=0}^{m}\sum_{k=0}^{n-1}A_{m,r}k^r(n-k)^r$$

For example,

$$n^3=\sum_{k=1}^{n}6k(n-k)+1$$

$$n^5=\sum_{k=1}^{n}30k^2(n-k)^2+1$$

$$n^7=\sum_{k=1}^{n}140k^3(n-k)^3-14k(n-k)+1$$

$$n^9=\sum_{k=1}^{n}630k^4(n-k)^4-120k(n-k)+1$$

$$n^{11}=\sum_{k=1}^{n}2772k^5(n-k)^5+660k^2(n-k)^2-1386k(n-k)+1$$

These coefficients $A_{m,r}$ are registered in OEIS: https://oeis.org/A302971, https://oeis.org/A304042.

Recurrence relation for $A_{m,r}$ is given by: https://mathoverflow.net/q/297900/113033

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Question 1: The algorithm we used to obtain identities for cubes (3), (4) is quite simple, if not naive. I believe it should be discussed in mathematical literature, as well as identity that gives a set of real coefficients $A_{m,r}$ such that

$$n^{2m+1}=\sum_{k=1}^{n}A_{m,0}k^0(n-k)^0+A_{m,1}(n-k)^1+\cdots+A_{m,m}k^m(n-k)^m$$

However, I was not able to find any references that in particular mention coefficients $A_{m,r}$, which is one of open questions.

Question 2: Can we consider the process of obtaining the identities (3), (4) as an interpolation technique?

Question 3: If the question 2 is true, can we consider equation (5) as an interpolation technique too?