MINIMAL GOLDBACH PAIRS IN PRIME AND TWIN-PRIME COUNTING

PETRO KOLOSOV

Abstract. Goldbach's Conjecture assumes that every even integer $2N \geq 4$ can be written as the sum of two primes $2N = p_i + p_j$, where (p_i, p_j) is called a Goldbach pair. The minimal Goldbach pair is a pair (p_i, p_j) such that p_i is minimal and $p_j = 2N - p_i$ is also a prime. We define a function $F_{2N}(P)$ that counts the occurrences of $p_i = P$ in a set of minimal Goldbach pairs up to $2N$, where P is a fixed prime number. In particular, the function $F_{2N}(P)$ provides the following identities in terms of prime counting $\pi(2N)$, twin-prime counting $\pi_2(2N)$ and cousin prime counting $\pi_4(2N)$

$$\pi(2N) = F_{2N+3}(3) + 1$$

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$$

$$\pi_4(2N) = F_{2N}(5) - F_{2N}(7)$$

Contents

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Sources: https://github.com/kolosovpetro/MinimalGoldbachPairsInPrimesCounting

1. Introduction

This manuscript provides a comprehensive review of the work [1] done by Michel Yamagishi, extending it with additional results. The Goldbach conjecture asserts that every even integer $2N \geq 4$ is a sum of two primes

$$2N = p_i + p_j$$

where (p_i, p_j) is called a Goldbach pair.

A Goldbach pair is not unique for even integers greater than 6, which means that there can be multiple Goldbach pairs for even integer $2N \geq 8$. For example: $10 = 3 + 7$ and $10 = 5 + 5$ and $10 = 7 + 3$, where the Goldbach pairs are $(3, 7)$, $(5, 5)$, $(7, 3)$.

The minimal Goldbach pair is the pair with the smallest p_i among all goldbach pairs for even integer $2N$. For the even integer 10, we have three pairs: $(3, 7)$, $(5, 5)$, $(7, 3)$ and the minimal one is $(3, 7)$ because 3 is the smallest among all p_i values: $3, 5, 7$. Consider the following minimal Goldbach pairs for even integer $2k$ within the range $6 \leq 2k \leq 50$

$$\begin{array}{lll}
6 = 3 + 3, & 24 = 5 + 19, & 40 = 3 + 37, \\
8 = 3 + 5, & 26 = 3 + 23, & 42 = 5 + 37, \\
10 = 3 + 7, & 28 = 5 + 23, & 44 = 3 + 41, \\
12 = 5 + 7, & 30 = 7 + 23, & 46 = 3 + 43, \\
14 = 3 + 11, & 32 = 3 + 29, & 48 = 5 + 43, \\
16 = 3 + 13, & 34 = 3 + 31, & 50 = 3 + 47, \\
18 = 5 + 13, & 36 = 5 + 31, & \\
20 = 3 + 17, & 38 = 7 + 31, & \\
22 = 3 + 19, & &
\end{array}$$

We can notice that minimal Goldbach pairs with minimal $p_i = 3$ produce a sequence of odd prime numbers $p_j = 3,\ 5,\ 7,\ 11,\ 13,\ 17\ \ldots$ which is quite remarkable:

$$
\begin{aligned}
6 &= 3+3, & 32 &= 3+29,\\
8 &= 3+5, & 34 &= 3+31,\\
10 &= 3+7, & 40 &= 3+37,\\
14 &= 3+11, & 44 &= 3+41,\\
16 &= 3+13, & 46 &= 3+43,\\
20 &= 3+17, & 50 &= 3+47,\\
22 &= 3+19, & &\\
26 &= 3+23, & &
\end{aligned}
$$

Another interesting observation is that by selecting the pairs with minimal $p_i = 5$ yields the sequence of primes p_j such that $p_j + 2$ is not a prime

$$
\begin{aligned}
12 &= 5+7, & 36 &= 5+31,\\
18 &= 5+13, & 42 &= 5+37,\\
24 &= 5+19, & 48 &= 5+43,\\
28 &= 5+23, & &
\end{aligned}
$$

To formalize and clarify our discussion, we define a few functions. Let $G_{\min}(2N)$ be a function that returns a set of minimal Goldbach pairs (p_i, p_j) having $\min p_i$ over the range $6 \le 2k \le 2N$

$$G_{\min}(2N) = \{(p_i, p_j) \mid p_i + p_j = 2k \mid 6 \le 2k \le 2N \mid \min p_i\}$$

For example,

$$G_{\min}(20) = \{(3,3),\ (3,5),\ (3,7),\ (5,7),\ (3,11),\ (3,13),\ (5,13),\ (3,17)\}$$

Let $W_{2N}(P)$ be a function that returns the set of elements p_j from $G_{\min}(2N)$ having $p_i = P$

$$W_{2N}(P) = \{p_j \mid (p_i, p_j) \in G_{\min}(2N) \text{ and } p_i = P\}$$

Then, the sequence of odd prime numbers [2] is given by $W_{2N}(3)$

$$\{3,\ 5,\ 7,\ 11,\ \ldots,\ p \leq 2N-3\} = W_{2N}(3)$$

Now we can easily count the number of primes within the interval $6 \leq 2k \leq 2N$ because $\pi(2N)$ is equal to the total number of elements inside the set $W_{2N}(3)$, which corresponds to the sequence [3]

$$\pi(2N) = F_{2N+3}(3) + 1$$

where $F_{2N}(3)$ is the function that counts the number of elements inside the set $W_{2N}(3)$. In general $F_{2N}(P) = |W_{2N}(P)|$.

Taking $P = 5$ in $W_{2N}(P)$ yields a sequence of primes p_j such that $p_j + 2$ is not a prime [4]

$$W_{2N}(5) = \{7,\ 13,\ 19,\ 23,\ 31,\ 37,\ 43,\ 47,\ 53,\ \ldots,\ p \leq 2N-5\}$$

Hence, by excluding the elements of the set $W_{2N}(5)$ from the set $W_{2N}(3)$ yields the sequence of lesser twin primes [5]

$$W_{2N}(3) \setminus W_{2N}(5) = \{3,\ 5,\ 11,\ 17,\ 29,\ 41,\ 59,\ 71,\ 101,\ 107,\ 137,\ \ldots,\ p \leq 2N-3\}$$

This implies that the number of twin primes in range $6 \leq 2k \leq 2N$ can be expressed in terms of $F_{2N}(P)$

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$$

For example,

$$\pi_2(10) = F_{10+3}(3) - F_{10+5}(5) = 2$$

$$\pi_2(100) = F_{100+3}(3) - F_{100+5}(5) = 8$$

$$\pi_2(1000) = F_{1000+3}(3) - F_{1000+5}(5) = 35$$

$$\pi_2(10000) = F_{10000+3}(3) - F_{10000+5}(5) = 205$$

$$\pi_2(100000) = F_{100000+3}(3) - F_{100000+5}(5) = 1224$$

$$\pi_2(1000000) = F_{1000000+3}(3) - F_{1000000+5}(5) = 8169$$

These results match the sequence [6].

In addition, the functions $W_{2N}(P)$ and $F_{2N}(P)$ provide a way to count cousin primes p such that $p+4$ is a prime. We can observe this by excluding the elements of the set $W_{2N}(7)$ from the set $W_{2N}(5)$ which gives the sequence of cousin primes [7]

$$W_{2N}(5) \setminus W_{2N}(7) = \{7,\ 13,\ 19,\ 37,\ 43,\ 67,\ 79,\ 97,\ 103,\ 109,\ 127,\ 163,\ \ldots,\ p \leq 2N-5\}$$

Therefore, the number of cousin primes up to $2N$ can be calculated as

$$\pi_4(2N) = F_{2N}(5) - F_{2N}(7)$$

For instance,

$$\pi_4(10) = F_{10}(5) - F_{10}(7) = 0$$

$$\pi_4(100) = F_{100}(5) - F_{100}(7) = 7$$

$$\pi_4(1000) = F_{1000}(5) - F_{1000}(7) = 40$$

$$\pi_4(10000) = F_{10000}(5) - F_{10000}(7) = 202$$

$$\pi_4(100000) = F_{100000}(5) - F_{100000}(7) = 1215$$

$$\pi_4(1000000) = F_{1000000}(5) - F_{1000000}(7) = 8143$$

These results match the sequence [8].

Having $P = 7$ function $W_{2N}(P)$ yields the sequence of primes such that $p_j - p_i \geq 6$, where p_j is the next prime after p_i, see [9]

$$W_{2N}(7) = \{23,\ 31,\ 47,\ 53,\ 61,\ 73,\ 83,\ 89,\ 113,\ \ldots,\ p \leq 2N - 7\}$$

This allows us to count the primes for which the next-prime gap at least 6: $\delta_6(2N) = F_{2N+7}(7)$.

2. Discussion on Twin Primes

Twin prime conjecture asserts that there are infinitely many primes p such that $p + 2$ is also a prime. Previously, we have established the following relation in terms of the minimal Goldbach pairs counting function $W_{2N}(P)$. The set of twin primes is the set $W_{2N}(3)$ excluding the set $W_{2N}(5)$

$$\begin{aligned} \texttt{Lesser twin primes} &= \{3,\ 5,\ 11,\ 17,\ 29,\ 41,\ 59,\ 71,\ 101,\ 107,\ 137,\ \ldots,\ p \leq 2N - 3\} \\ &= W_{2N}(3) \setminus W_{2N}(5) \end{aligned}$$

where $W_{2N}(3)$ gives the set of odd primes p, and $W_{2N}(5)$ gives the set of odd primes p such that $p + 2$ is not a prime. Thus, the number of twin primes up to integer $2N$ is

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$$

where $F_{2N}(P)$ counts the elements inside the set $W_{2N}(3)$. Therefore, the twin prime conjecture is identical to the assertion that

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5) = \infty, \qquad 2N \to \infty$$

Thanks to Euclid, we know that there are infinitely many primes, thus there are infinitely many minimal Goldbach pairs $(p_i, p_j) = (3, p_j)$ such that $2N = p_i + p_j$, hence

$$\lim_{2N \to \infty} F_{2N+3}(3) = \infty$$

Consider the set $W_{2N}(5)$ which gives the sequence of odd primes $p = 6n+1$ such that $p+2$ is not a prime. Dirichlet's theorem shows that there are infinitely many primes of the form

$$p = 6n + 1$$

Therefore, there are infinitely many primes p such that $p+2$ is not a prime

$$p + 2 = 6n + 3 = 3(2n + 1)$$

Consequently,

$$\lim_{2N\to\infty} F_{2N+5}(5) = \infty$$

Primes $p = 6n + 1$ are such that either

- $p + 2$ is composite
- $p - 2$ is composite or lesser twin prime

Therefore, the function $W_{2N}(5)$ generates the set of primes of the form $p = 6n + 1$. By excluding the set $W_{2N}(5)$ from the set of odd prime numbers p, we obtain the sequence of lesser twin primes p_ℓ

$$\begin{aligned} \texttt{Lesser twin primes} &= \{3,\ 5,\ 11,\ 17,\ 29,\ 41,\ 59,\ 71,\ 101,\ 107,\ 137,\ \ldots,\ p_\ell \le 2N-3\} \\ &= W_{2N}(3) \setminus W_{2N}(5) \end{aligned}$$

We know that for every $N \ge 3$

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$$

Since $F_{2N+3}(3) \neq 0$ with $N \ge 2$

$$\lim_{2N\to\infty} \frac{\pi_2(2N)}{F_{2N+3}(3)} = 1 - \lim_{2N\to\infty} \frac{F_{2N+5}(5)}{F_{2N+3}(3)}$$

If $\pi_2(2N)$ is finite, then

$$\lim_{2N\to\infty} \frac{\pi_2(2N)}{F_{2N+3}(3)} = 0$$

Which implies

$$\lim_{2N\to\infty} \frac{F_{2N+5}(5)}{F_{2N+3}(3)} = 1$$

By setting $2N = 10^6$, we have

$$\frac{F_{10^6+5}(5)}{F_{10^6+3}(3)} = \frac{78497}{70328} = 0.895932$$

3. Conclusions

Assuming Goldbach's Conjecture holds, we introduced a framework based on minimal Goldbach pairs to derive expressions for key prime-related functions. Specifically, we defined the function $F_{2N}(P)$ that counts occurrences of primes p_j in minimal Goldbach pairs (p_i, p_j) where $p_i = P$. Using this framework, we obtained

- The prime-counting function: $\pi(2N) = F_{2N+3}(3) + 1$
- The twin-prime p_2 counting function: $\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$
- The twin-prime p_4 counting function: $\pi_4(2N) = F_{2N}(5) - F_{2N}(7)$
- The count of primes with next-prime gap at least 6: $\delta_6(2N) = F_{2N+7}(7)$

These identities establish a novel connection between Goldbach partitions and classical prime number theory. Computational examples confirm alignment with known integer sequences, reinforcing the potential of this approach for analytical and numerical exploration of prime distributions. All the results validated up to $N = 10^8$ with source code available on GitHub [10].

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Software Developer, DevOps Engineer

Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io