COUNTING PRIMES AND TWIN-PRIMES USING MINIMAL GOLDBACH PAIRS

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Abstract. Assuming Goldbach's Conjecture holds, every even integer $2N \geq 4$ can be written as $2N = p_i + p_j$ where (p_i, p_j) is called Goldbach pair. The minimal Goldbach pair is a pair (p_i, p_j) having the minimal p_i such that $p_j = 2N - p_i$ is also a prime. We define a function $F_{2N}(P)$ that counts occurrences of $p_j = P$ within the range $6 \leq 2k \leq 2N$. In particular, the function $F_{2N}(P)$ provides the following identities in terms of prime counting $\pi(2N)$ and twin-prime counting $\pi_2(2N)$

$$\pi(2N) = F_{2N}(3) + 1, \quad \pi_2(2N) = F_{2N}(3) - F_{2N}(5)$$

Contents

1. Introduction

This manuscript provides a comprehensive review of the work [1] done by Michel Yamagishi. Goldbach conjecture asserts that every even integer $2N \geq 4$ is a sum of two primes

$$2N = p_i + p_j$$

where (p_i, p_j) is called Goldbach pair.

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Goldbach pair is not unique for even integers greater than 6, meaning that there can be multiple goldbach pairs for even integer $2N \geq 8$.

For example: $10 = 3 + 7$ and $10 = 5 + 5$ and $10 = 7 + 3$ where goldbach pairs are $(3, 7),\ (5, 5),\ (7, 3)$.

Minimal goldbach pair is the pair having minimal p_i across all goldbach pairs for even integer $2N$.

For even integer 10 we have three pairs $(3, 7),\ (5, 5),\ (7, 3)$ while the minimal is $(3, 7)$ because 3 is the minimal value across all the p_i values: $3, 5, 7$.

Consider the following minimal goldbach pairs for even integer $2k$ within the range $6 \leq 2k \leq 50$

$$\begin{aligned}
6 &= 3 + 3, & 8 &= 3 + 5, \\
10 &= 3 + 7, & 12 &= 5 + 7, \\
14 &= 3 + 11, & 16 &= 3 + 13, \\
18 &= 5 + 13, & 20 &= 3 + 17, \\
22 &= 3 + 19, & 24 &= 5 + 19, \\
26 &= 3 + 23, & 28 &= 5 + 23, \\
30 &= 7 + 23, & 32 &= 3 + 29, \\
34 &= 3 + 31, & 36 &= 5 + 31, \\
38 &= 7 + 31, & 40 &= 3 + 37, \\
42 &= 5 + 37, & 44 &= 3 + 41, \\
46 &= 3 + 43, & 48 &= 5 + 43, \\
50 &= 3 + 47
\end{aligned}$$

We can notice that minimal Goldbach pairs having $p_i = 3$ produce a sequence of odd prime numbers $p_j = 3, 5, 7, 11, 13, 17\dots$ which is quite remarkable

$$
\begin{aligned}
6 &= 3 + 3, & 8 &= 3 + 5, \\
10 &= 3 + 7, & 14 &= 3 + 11, \\
16 &= 3 + 13, & 20 &= 3 + 17, \\
22 &= 3 + 19, & 26 &= 3 + 23, \\
32 &= 3 + 29, & 34 &= 3 + 31, \\
40 &= 3 + 37, & 44 &= 3 + 41, \\
46 &= 3 + 43, & 50 &= 3 + 47
\end{aligned}
$$

One more interesting observation is that by selecting the pairs with minimal $p_i = 5$ yields the sequence of primes p_j such that $p_j + 2$ is not a prime.

$$
\begin{aligned}
12 &= 5 + 7, & 18 &= 5 + 13, \\
24 &= 5 + 19, & 28 &= 5 + 23, \\
36 &= 5 + 31, & 42 &= 5 + 37, \\
48 &= 5 + 43
\end{aligned}
$$

For our conversation to be more formal and objective, we define a few functions. Let $G_{\min}(2N)$ be a function that returns a set of minimal Goldbach pairs (p_i, p_j) having $\min p_i$ over the range $6 \le 2k \le 2N$

$$G_{\min}(2N) = \{(p_i, p_j) \mid p_i + p_j = 2k \mid 6 \le 2k \le 2N \mid \min p_i\}.$$

For example,

$$G_{\min}(20) = \{(3,3),\ (3,5),\ (3,7),\ (5,7),\ (3,11),\ (3,13),\ (5,13),\ (3,17)\}$$

Let $W_{2N}(P)$ be a function that returns the set of elements p_j from $G_{\min}(2N)$ having $p_i = P$

$$W_{2N}(P) = \{p_j \mid (p_i, p_j) \in G_{\min}(2N) \text{ and } p_i = P\}$$

Then sequence of odd prime numbers [2] is given by $W_{2N}(3)$

$$\{3,\ 5,\ 7,\ 11,\ \ldots,\ p < 2N - 3\} = W_{2N}(3)$$

Now we can easily count the number of primes within the interval $6 \leq 2k \leq 2N$ because $\pi(2N)$ equals to the total number of elements inside the set $W_{2N}(3)$

$$\pi(2N) = F_{2N}(3) + 1$$

where $F_{2N}(3)$ is the function that counts the number of elements inside the set $W_{2N}(3)$, in general $F_{2N}(P) = |W_{2N}(3)|$.

Taking $P = 5$ in $W_{2N}(P)$ yields a sequence of primes p_j such that $p_j + 2$ is not a prime [3]

$$W_{2N}(5) = \{7,\ 13,\ 19,\ 23,\ 31,\ 37,\ 43,\ 47,\ 53,\ \ldots,\ p < 2N\}$$

Which implies that the number of twin primes in range $6 \leq 2k \leq 2N$ can be expressed in terms of $F_{2N}(P)$

$$\pi_2(2N) = F_{2N}(3) - F_{2N}(5)$$

where $2N = 10^k + 2,\ k = 1, 2, 3, 4, \ldots$. For example,

$$\pi_2(12) = F_{12}(3) - F_{12}(5) = 2$$

$$\pi_2(102) = F_{102}(3) - F_{102}(5) = 8$$

$$\pi_2(1002) = F_{1002}(3) - F_{1002}(5) = 35$$

$$\pi_2(10002) = F_{10002}(3) - F_{10002}(5) = 205$$

$$\pi_2(100002) = F_{100002}(3) - F_{100002}(5) = 1224$$

$$\pi_2(1000002) = F_{1000002}(3) - F_{1000002}(5) = 8169$$

References

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