NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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Abstract.

1. Introduction

Proposition 1.1. *(Newton's series around arbitrary point* [1, Lemma V]*.)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

Example 1.2 (Newton series for cubes monomial)**.**

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$
$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$
$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

In general,

$$n^3 = \Delta^0 t^3 \binom{n-t}{0} + \Delta^1 t^3 \binom{n-t}{1} + \Delta^2 t^3 \binom{n-t}{2} + \Delta^3 t^3 \binom{n-t}{3}$$

Corollary 1.3 (Newton series for binomial reversed)**.**

$$(n+t)^m = \sum_{k=0}^{m} \binom{n}{k} \Delta^k t^m$$

Proposition 1.4 (Newton series for monomial reversed)**.**

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m$$

Proof. By setting $n \to n-t$ into (1.3). □

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Definition 1.5 (Multifold sum of powers recurrence)**.**

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

Thus, for arbitrary integer t

$$\Sigma^1 n^m = \sum_{k=1}^{n} \sum_{j=0}^{m} \binom{-t+k}{j} \Delta^j t^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$$

Proposition 1.6 (Segmented Hockey stick identity)**.** *For integers n, t and j*

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Therefore,

Proposition 1.7 (Ordinary sums of powers via Newton's series)**.** *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right]$$

Proof. Ordinary sum of powers is given by $\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$, where $\sum_{k=1}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$ by means of segmented hockey stick identity (1.6). □

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently. For $t = 0$ and $m = 3$ we have the famous identity

$$\Sigma^1 n^3 = 0\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$$

which was discussed in [2, p. 190] and in [3]. The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were discussed in [4]. For instance,

$$\Sigma^1 n^3 = 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}$$

$$\Sigma^1 n^4 = 1\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}$$

The coefficients $1, 7, 12, 6, 1, 15, \dots$ are given by the sequence [ID] in the OEIS [5]. Interestingly enough that the paper [4] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ {k \atop j} \right\}_r$$

where $\left\{ {k \atop j} \right\}_r$ are generalized Stirling numbers of the second kind. The formula above is identical to the proposition (1.7), which yields that finite differences can be expressed in terms of generalized Stirling numbers of the second kind, that is $\Delta^j t^m = j! \left\{ {m \atop j} \right\}_t$.

By considering the special cases of the proposition (1.7) for $t = 4$, we observe rather unexpected formulas for sums of powers, that are

$$\Sigma^1 n^0 = 1 \left(\binom{n-3}{1} + \binom{3}{1} \right)$$

$$\Sigma^1 n^1 = 4 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 1 \left(\binom{n-3}{2} - \binom{4}{2} \right)$$

$$\Sigma^1 n^2 = 16 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 9 \left(\binom{n-3}{2} - \binom{4}{2} \right) + 2 \left(\binom{n-2}{3} + \binom{5}{3} \right)$$

$$\Sigma^1 n^3 = 64 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 61 \left(\binom{n-3}{2} - \binom{4}{2} \right) + 30 \left(\binom{n-3}{3} + \binom{5}{3} \right) + 6 \left(\binom{n-3}{4} - \binom{6}{4} \right)$$

The coefficients $1, 4, 1, 16, 9, \dots$ are given by the sequence [ID] in the OEIS [5]. To obtain the formula for double sum of powers, we simply apply summation operator over the ordinary sum again, thus

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \sum_{k=1}^{n} \binom{j+t-1}{j+1} + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$$

which yields

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$$

Thus,

Proposition 1.8 (Double sums of powers via Newton's series)**.**

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}\right]$$

Proof. We have $\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1}\right]$, where $\sum_{k=1}^{n} \binom{k-t+1}{j+1} = (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}$ by means of segmented hockey stick identity (1.6). $\square$

For example, given $t = 5$, the double sums of powers are

$$\Sigma^2 n^0 = 1 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2}\right)$$

$$\Sigma^2 n^1 = 5 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2}\right) + 1 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3}\right)$$

$$\Sigma^2 n^2 = 25 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2}\right) + 11 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3}\right) + 2 \left(\binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4}\right)$$

$$\Sigma^2 n^3 = 125 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2}\right) + 91 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3}\right) + 36 \left(\binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4}\right) + 6 \left(\binom{n-3}{5} - \binom{7}{4} n + \binom{7}{5}\right)$$

Similarly, we obtain the formula for the triple sums of powers

Proposition 1.9 (Triple sums of powers via Newton's series)**.**

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 + {} + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3}\right]$$

Proof. By summing up the double powers sums, we get

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[(-1)^j \binom{j+t-1}{j+1} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} k^0 + \binom{k-t+2}{j+2}\right]$$

$$= \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \sum_{k=1}^{n} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} \sum_{k=1}^{n} k^0 + \sum_{k=1}^{n} \binom{k-t+2}{j+2}\right]$$

Note that $\sum_{k=1}^{n} k^1 = \Sigma^2\, n^0$ and $\sum_{k=1}^{n} k^0 = \Sigma^1\, n^0$. By segmented hockey stick identity (1.6)

$$\sum_{k=1}^{n} \binom{k-t+2}{j+2} = (-1)^{j+2}\binom{j+t-1}{j+3}\Sigma^0\, n^0 + \binom{n-t+3}{j+3}$$

This completes the proof. □

2. Proof of Segmented hockey stick identity

First we split the sum $\sum_{k=0}^{n}\binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^{n}\binom{-t+k}{j}=\sum_{k=0}^{t-1}\binom{-t+k}{j}+\sum_{k=t}^{n}\binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0,1,2,\cdots,t,\cdots,n\}$ such that $t<n$. Considering the sum $\sum_{k=0}^{t-1}\binom{-t+k}{j}$ we notice that

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=\binom{-t}{j}+\binom{-t+1}{j}+\binom{-t+2}{j}+\cdots+$$
$$+\binom{-t+t-2}{j}+\binom{-t+t-1}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=\sum_{k=1}^{t}\binom{-k}{j}=\sum_{k=0}^{t-1}\binom{-k-1}{j}$$

By means of $\binom{-k}{j}=(-1)^j\binom{j+k-1}{j}$

$$\binom{-k-1}{j}=\binom{-(k+1)}{j}=(-1)^j\binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=(-1)^j\sum_{k=0}^{t-1}\binom{j+k}{j}=(-1)^j\binom{j+t}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t}\binom{j+k}{j}=\binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^{n}\binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^{n}\binom{-t+k}{j}=\sum_{k=0}^{n-t}\binom{k}{j}$$

Thus

$$\sum_{k=t}^{n}\binom{-t+k}{j}=\sum_{k=0}^{n-t}\binom{k}{j}=\binom{n-t+1}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. Thus

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

This completes the proof.

References

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Sources: github.com/kolosovpetro/github-latex-template

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