NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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ABSTRACT. In this manuscript we derive the formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, this manuscript provides the formulas for multifold sums of powers in terms of Stirling numbers of the second kind, and Eulerian numbers.

1. Introduction

In this manuscript we derive the formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, this manuscript provides the formulas for multifold sums of powers in terms of Stirling numbers of the second kind, and Eulerian numbers.

Allow us to start from the definition of multifold sums of powers. We utilize the recurrence proposed by Donald Knuth in his article *Johann Faulhaber and sums of powers*, see [1]

$$\Sigma^0 \, n^m = n^m$$

$$\Sigma^1 \, n^m = \Sigma^0 \, 1^m + \Sigma^0 \, 2^m + \cdots + \Sigma^0 \, n^m$$

$$\Sigma^{r+1} \, n^m = \Sigma^r \, 1^m + \Sigma^r \, 2^m + \cdots + \Sigma^r \, n^m$$

Throughout the paper, we utilize the Newton's interpolation formula as stated below

Proposition 1.1. *(Newton's series around arbitrary point* [2, Lemma V]*.)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

where $\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j)$ *is* k*-degree forward finite difference of* f*.*

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Which indeed holds, because

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$
$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$
$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

Proposition 1.2 (Newton's series for power)**.** *For non-negative integers m and n*

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m$$

Thus, for arbitrary integer t, the ordinary sum of powers is

$$\Sigma^1 n^m = \sum_{k=1}^{n} \sum_{j=0}^{m} \binom{-t+k}{j} \Delta^j t^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$$

Proposition 1.3 (Segmented Hockey stick identity)**.** *For integers n, t and j*

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Therefore,

Proposition 1.4 (Ordinary sums of powers via Newton's series)**.** *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right]$$

Proof. Ordinary sum of powers is given by $\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$, where $\sum_{k=1}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$ by means of segmented hockey stick identity (1.3). □

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently. For $t = 0$ and $m = 3$ we have the famous identity

$$\Sigma^1 n^3 = 0\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$$

which was discussed in [3, p. 190] and in [4]. The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were discussed in [5]. For instance,

$$\Sigma^1 n^3 = 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}$$
$$\Sigma^1 n^4 = 1\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}$$

The coefficients $1, 7, 12, 6, 1, 15, \ldots$ are given by the sequence A028246 in the OEIS [6]. Interestingly enough that the paper [5] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1}\right] \left\{ k \atop j \right\}_r$$

where $\left\{ k \atop j \right\}_r$ are generalized Stirling numbers of the second kind. The formula above is identical to the proposition (1.4), which yields that finite differences can be expressed in terms of generalized Stirling numbers of the second kind, that is $\Delta^j t^m = j! \left\{ m \atop j \right\}_t$.

By considering the special cases of the proposition (1.4) for $t = 4$, we observe rather unexpected formulas for sums of powers, that are

$$\Sigma^1 n^0 = 1\left(\binom{n-3}{1} + \binom{3}{1}\right)$$
$$\Sigma^1 n^1 = 4\left(\binom{n-3}{1} + \binom{3}{1}\right) + 1\left(\binom{n-3}{2} - \binom{4}{2}\right)$$
$$\Sigma^1 n^2 = 16\left(\binom{n-3}{1} + \binom{3}{1}\right) + 9\left(\binom{n-3}{2} - \binom{4}{2}\right) + 2\left(\binom{n-2}{3} + \binom{5}{3}\right)$$
$$\Sigma^1 n^3 = 64\left(\binom{n-3}{1} + \binom{3}{1}\right) + 61\left(\binom{n-3}{2} - \binom{4}{2}\right) + 30\left(\binom{n-3}{3} + \binom{5}{3}\right)$$
$$+ 6\left(\binom{n-3}{4} - \binom{6}{4}\right)$$

The coefficients $1, 4, 1, 16, 9, \ldots$ are given by the sequence A391633 in the OEIS [6]. To obtain the formula for double sum of powers, we simply apply summation operator over the ordinary sum again, thus

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \sum_{k=1}^{n} \binom{j+t-1}{j+1} + \sum_{k=1}^{n} \binom{k-t+1}{j+1}\right]$$

which yields

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1}\right]$$

Thus,

Proposition 1.5 (Double sums of powers via Newton's series)**.**

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}\right]$$

Proof. We have $\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1}\right]$, where $\sum_{k=1}^{n} \binom{k-t+1}{j+1} = (-1)^{j+1}\binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}$ by means of segmented hockey stick identity (1.3). □

For example, given $t = 5$, the double sums of powers are

$$\begin{aligned}
\Sigma^2 n^0 &= 1\left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2}\right) \\
\Sigma^2 n^1 &= 5\left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2}\right) + 1\left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3}\right) \\
\Sigma^2 n^2 &= 25\left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2}\right) + 11\left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3}\right) \\
&\quad + 2\left(\binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4}\right) \\
\Sigma^2 n^3 &= 125\left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2}\right) + 91\left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3}\right) \\
&\quad + 36\left(\binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4}\right) + 6\left(\binom{n-3}{5} - \binom{7}{4} n + \binom{7}{5}\right)
\end{aligned}$$

The coefficients $1, 5, 1, 25, 11, 2, \ldots$ are given by the sequence A391635 in the OEIS [6].

Similarly, we obtain the formula for the triple sums of powers

Proposition 1.6 (Triple sums of powers via Newton's series)**.**

$$\begin{aligned}
\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \Bigg[&(-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 + \\
&+ (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3}\Bigg]
\end{aligned}$$

Proof. By summing up the double powers sums, we get

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[(-1)^j \binom{j+t-1}{j+1} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} k^0 + \binom{k-t+2}{j+2} \right]$$

$$= \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \sum_{k=1}^{n} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} \sum_{k=1}^{n} k^0 + \sum_{k=1}^{n} \binom{k-t+2}{j+2} \right]$$

Note that $\sum_{k=1}^{n} k^1 = \Sigma^2 n^0$ and $\sum_{k=1}^{n} k^0 = \Sigma^1 n^0$. Thus,

$$\sum_{k=1}^{n} \binom{k-t+2}{j+2} = (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3}$$

by segmented hockey stick identity (1.3). This completes the proof. ◻

Theorem 1.7 (Multifold sums of powers via Newton's series)**.**

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right]$$

Proof. By Newton's series for power (1.2) and repeated segmented hockey stick identity (1.3).

◻

We may observe that

Proposition 1.8. *For integers r and n*

$$\Sigma^r n^0 = \binom{r+n-1}{r}$$

Proof. By hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. ◻

Which yields the following binomial variations of the Multifold sums of powers (1.7)

Corollary 1.9 (Multifold sums of powers binomial form)**.** *For non-negative integers r, n and m*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right) + \binom{n-t+r}{j+r} \right]$$

Corollary 1.10 (Multifold sums of powers binomial form reindexed)**.** *For non-negative integers r, n and m*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right) + \binom{n-t+r}{j+r} \right]$$

2. Proof of Segmented hockey stick identity

First we split the sum $\sum_{k=0}^{n}\binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^{n}\binom{-t+k}{j}=\sum_{k=0}^{t-1}\binom{-t+k}{j}+\sum_{k=t}^{n}\binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0,1,2,\cdots,t,\cdots,n\}$ such that $t<n$. Considering the sum $\sum_{k=0}^{t-1}\binom{-t+k}{j}$ we notice that

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=\binom{-t}{j}+\binom{-t+1}{j}+\binom{-t+2}{j}+\cdots+$$
$$+\binom{-t+t-2}{j}+\binom{-t+t-1}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=\sum_{k=1}^{t}\binom{-k}{j}=\sum_{k=0}^{t-1}\binom{-k-1}{j}$$

By means of $\binom{-k}{j}=(-1)^j\binom{j+k-1}{j}$

$$\binom{-k-1}{j}=\binom{-(k+1)}{j}=(-1)^j\binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=(-1)^j\sum_{k=0}^{t-1}\binom{j+k}{j}=(-1)^j\binom{j+t}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t}\binom{j+k}{j}=\binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^{n}\binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^{n}\binom{-t+k}{j}=\sum_{k=0}^{n-t}\binom{k}{j}$$

Thus

$$\sum_{k=t}^{n}\binom{-t+k}{j}=\sum_{k=0}^{n-t}\binom{k}{j}=\binom{n-t+1}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. Thus

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

This completes the proof.

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Sources: github.com/kolosovpetro/github-latex-template

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