

# NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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ABSTRACT.

## 1. INTRODUCTION

**Proposition 1.1.** (*Newton's series around arbitrary point [1, Lemma V].*)

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

**Example 1.2** (Newton series for cubes monomial).

$$\begin{aligned} n^3 &= 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} \\ n^3 &= 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3} \\ n^3 &= 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3} \end{aligned}$$

In general,

$$n^3 = \Delta^0 t^3 \binom{n-t}{0} + \Delta^1 t^3 \binom{n-t}{1} + \Delta^2 t^3 \binom{n-t}{2} + \Delta^3 t^3 \binom{n-t}{3}$$

**Corollary 1.3** (Newton series for binomial reversed).

$$(n+t)^m = \sum_{k=0}^m \binom{n}{k} \Delta^k t^m$$

**Proposition 1.4** (Newton series for monomial reversed).

$$n^m = \sum_{k=0}^m \binom{n-t}{k} \Delta^k t^m$$

*Proof.* By setting  $n \rightarrow n - t$  into (1.3).  $\square$

**Definition 1.5** (Multifold sum of powers recurrence).

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

Thus, for arbitrary integer  $t$

$$\Sigma^1 n^m = \sum_{k=1}^n \sum_{j=0}^m \binom{-t+k}{j} \Delta^j t^m = \sum_{j=0}^m \Delta^j t^m \sum_{k=1}^n \binom{-t+k}{j}$$

**Proposition 1.6** (Segmented Hockey stick identity). *For integers  $n, t$  and  $j$*

$$\sum_{k=0}^n \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

**Theorem 1.7** (Ordinary sums of powers via Newton's series). *For non-negative integers  $n, m$  and arbitrary integer  $t$*

$$\Sigma^1 n^m = \sum_{j=0}^m \Delta^j t^m \left[ (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right]$$

*Proof.* Ordinary sum of powers is given by  $\Sigma^1 n^m = \sum_{j=0}^m \Delta^j t^m \sum_{k=1}^n \binom{-t+k}{j}$ , where  $\sum_{k=1}^n \binom{-t+k}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$  by means of segmented hockey stick identity (1.6).  $\square$

The special cases for  $t = 0$  and  $t = 1$  are widely known and appear in literature quite frequently. For  $t = 0$  and  $m = 3$  we have the famous identity

$$\Sigma^1 n^3 = 0 \binom{n+1}{1} + 1 \binom{n+1}{2} + 6 \binom{n+1}{3} + 6 \binom{n+1}{4}$$

which was discussed in [2, p. 190] and in [3]. The special cases for  $t = 1$  and  $m = 2, 3, 4, 5$  were discussed in [4]. For instance,

$$\begin{aligned}\Sigma^1 n^3 &= 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4} \\ \Sigma^1 n^4 &= 1\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}\end{aligned}$$

The coefficients  $1, 7, 12, \dots$  are given by the sequence [ID] in the OEIS [5]. Interestingly enough that the paper [4] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^k j! \left[ \binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_r$$

where  $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}_r$  are generalized Stirling numbers of the second kind. The formula above is essentially identical to (1.7). Let be a conjecture

### Conjecture 1.8.

$$\Delta^j t^m = j! \left\{ \begin{matrix} m \\ j \end{matrix} \right\}_t$$

By considering the special cases of the theorem (1.7) for  $t = 4$ , we observe rather unexpected formulas for sums of powers, that are

$$\begin{aligned}\Sigma^1 n^0 &= 1 \left( \binom{n-3}{1} + \binom{3}{1} \right) \\ \Sigma^1 n^1 &= 4 \left( \binom{n-3}{1} + \binom{3}{1} \right) + 1 \left( \binom{n-3}{2} - \binom{4}{2} \right) \\ \Sigma^1 n^2 &= 16 \left( \binom{n-3}{1} + \binom{3}{1} \right) + 9 \left( \binom{n-3}{2} - \binom{4}{2} \right) + 2 \left( \binom{n-2}{3} + \binom{5}{3} \right) \\ \Sigma^1 n^3 &= 64 \left( \binom{n-3}{1} + \binom{3}{1} \right) + 61 \left( \binom{n-3}{2} - \binom{4}{2} \right) + 30 \left( \binom{n-3}{3} + \binom{5}{3} \right) \\ &\quad + 6 \left( \binom{n-3}{4} - \binom{6}{4} \right)\end{aligned}$$

To obtain the formula for double sum of powers, we simply apply summation operator over the ordinary sum again, thus

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[ (-1)^j \sum_{k=1}^n \binom{j+t-1}{j+1} + \sum_{k=1}^n \binom{k-t+1}{j+1} \right]$$

which yields

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[ (-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^n \binom{k-t+1}{j+1} \right]$$

Thus,

**Theorem 1.9** (Double sums of powers via finite difference 2).

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[ (-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2} \right]$$

*Proof.* We have  $\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[ (-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^n \binom{k-t+1}{j+1} \right]$ , where  $\sum_{k=1}^n \binom{k-t+1}{j+1} = (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}$  by means of segmented hockey stick identity (1.6).  $\square$

For example, given  $t = 5$ , the double sums of powers are

$$\begin{aligned} \Sigma^2 n^0 &= 1 \left( \binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) \\ \Sigma^2 n^1 &= 5 \left( \binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 1 \left( \binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\ \Sigma^2 n^2 &= 25 \left( \binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 11 \left( \binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\ &\quad + 2 \left( \binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4} \right) \\ \Sigma^2 n^3 &= 125 \left( \binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 91 \left( \binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\ &\quad + 36 \left( \binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4} \right) + 6 \left( \binom{n-3}{5} - \binom{7}{4} n + \binom{7}{5} \right) \end{aligned}$$

## 2. PROOF OF SEGMENTED HOCKEY STICK IDENTITY

First we split the sum  $\sum_{k=0}^n \binom{-t+k}{j}$  into two sub-sums so that we discuss them separately

$$\sum_{k=0}^n \binom{-t+k}{j} = \sum_{k=0}^{t-1} \binom{-t+k}{j} + \sum_{k=t}^n \binom{-t+k}{j}$$

We assume that the two sums above run over the partition  $\{0, 1, 2, \dots, t, \dots, n\}$  such that  $t < n$ . Considering the sum  $\sum_{k=0}^{t-1} \binom{-t+k}{j}$  we notice that

$$\begin{aligned} \sum_{k=0}^{t-1} \binom{-t+k}{j} &= \binom{-t}{j} + \binom{-t+1}{j} + \binom{-t+2}{j} + \dots + \\ &\quad + \binom{-t+t-2}{j} + \binom{-t+t-1}{j} \end{aligned}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = \sum_{k=1}^t \binom{-k}{j} = \sum_{k=0}^{t-1} \binom{-k-1}{j}$$

By means of  $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$

$$\binom{-k-1}{j} = \binom{-(k+1)}{j} = (-1)^j \binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = (-1)^j \sum_{k=0}^{t-1} \binom{j+k}{j} = (-1)^j \binom{j+t}{j+1}$$

By means of Hockey stick identity  $\sum_{k=0}^t \binom{j+k}{j} = \binom{j+t+1}{j+1}$ .

Considering the sum  $\sum_{k=t}^n \binom{-t+k}{j}$  we notice that

$$\sum_{k=t}^n \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j}$$

Thus

$$\sum_{k=t}^n \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j} = \binom{n-t+1}{j+1}$$

By means of Hockey stick identity  $\sum_{k=0}^t \binom{j+k}{j} = \binom{j+t+1}{j+1}$ . Thus

$$\sum_{k=0}^n \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

This completes the proof.

## REFERENCES

- [1] Newton, Isaac and Chittenden, N.W. *Newton's Principia: the mathematical principles of natural philosophy*. New-York, D. Adee, 1850. [https://archive.org/details/bub\\_gb\\_KaAIAAAIAAJ/page/466/mode/2up](https://archive.org/details/bub_gb_KaAIAAAIAAJ/page/466/mode/2up).
- [2] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. <https://archive.org/details/concrete-mathematics>.
- [3] Thomas J Pfaff. Deriving a formula for sums of powers of integers. *Pi Mu Epsilon Journal*, 12(7):425–430, 2007. <https://www.jstor.org/stable/24340705>.
- [4] Cereceda, José L. Sums of powers of integers and generalized Stirling numbers of the second kind. *arXiv preprint arXiv:2211.11648*, 2022. <https://arxiv.org/abs/2211.11648>.
- [5] Sloane, Neil JA and others. The on-line encyclopedia of integer sequences, 2003. <https://oeis.org/>.

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