

NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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ABSTRACT.

1. INTRODUCTION

Proposition 1.1. (*Newton's series around arbitrary point [1, Lemma V].*)

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

Example 1.2 (Newton series for cubes monomial).

$$\begin{aligned} n^3 &= 0 \binom{n}{0} + 1 \binom{n}{1} + 6 \binom{n}{2} + 6 \binom{n}{3} \\ n^3 &= 1 \binom{n-1}{0} + 7 \binom{n-1}{1} + 12 \binom{n-1}{2} + 6 \binom{n-1}{3} \\ n^3 &= 8 \binom{n-2}{0} + 19 \binom{n-2}{1} + 18 \binom{n-2}{2} + 6 \binom{n-2}{3} \end{aligned}$$

In general,

$$n^3 = \Delta^0 t^3 \binom{n-t}{0} + \Delta^1 t^3 \binom{n-t}{1} + \Delta^2 t^3 \binom{n-t}{2} + \Delta^3 t^3 \binom{n-t}{3}$$

Corollary 1.3 (Newton series for binomial reversed).

$$(n+t)^m = \sum_{k=0}^m \binom{n}{k} \Delta^k t^m$$

Proposition 1.4 (Newton series for monomial reversed).

$$n^m = \sum_{k=0}^m \binom{n-t}{k} \Delta^k t^m$$

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Proof. By setting $n \rightarrow n - t$ into (1.3). □

Definition 1.5 (Multifold sum of powers recurrence).

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

Thus, for arbitrary integer t

$$\Sigma^1 n^m = \sum_{k=1}^n \sum_{j=0}^m \binom{-t+k}{j} \Delta^j t^m = \sum_{j=0}^m \Delta^j t^m \sum_{k=1}^n \binom{-t+k}{j}$$

Proposition 1.6 (Segmented Hockey stick identity). *For integers n, t and j*

$$\sum_{k=0}^n \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Theorem 1.7 (Ordinary sums of powers via Newton's series). *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right]$$

Proof. Ordinary sum of powers is given by $\Sigma^1 n^m = \sum_{j=0}^m \Delta^j t^m \sum_{k=1}^n \binom{-t+k}{j}$, where $\sum_{k=1}^n \binom{-t+k}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$ by means of segmented hockey stick identity (1.6). □

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently. For $t = 0$ and $m = 3$ we have the famous identity

$$\Sigma^1 n^3 = 0 \binom{n+1}{1} + 1 \binom{n+1}{2} + 6 \binom{n+1}{3} + 6 \binom{n+1}{4}$$

which was discussed in [2, p. 190] and in [3]. The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were discussed in [4]. For instance,

$$\begin{aligned}\Sigma^1 n^3 &= 1 \binom{n}{1} + 7 \binom{n}{2} + 12 \binom{n}{3} + 6 \binom{n}{4} \\ \Sigma^1 n^4 &= 1 \binom{n}{1} + 15 \binom{n}{2} + 50 \binom{n}{3} + 60 \binom{n}{4} + 24 \binom{n}{5}\end{aligned}$$

The coefficients $1, 7, 12, \dots$ are given by the sequence [ID] in the OEIS [5]. Interestingly enough that the paper [4] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^k j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_r$$

where $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}_r$ are generalized Stirling numbers of the second kind. The formula above is essentially identical to (1.7). Let be a conjecture

Conjecture 1.8.

$$\Delta^j t^m = j! \left\{ \begin{matrix} m \\ j \end{matrix} \right\}_t$$

By considering the special cases of the theorem (1.7) for $t = 4$, we observe rather unexpected formulas for sums of powers, that are

$$\begin{aligned}\Sigma^1 n^0 &= 1 \left(\binom{n-3}{1} + \binom{3}{1} \right) \\ \Sigma^1 n^1 &= 4 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 1 \left(\binom{n-3}{2} - \binom{4}{2} \right) \\ \Sigma^1 n^2 &= 16 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 9 \left(\binom{n-3}{2} - \binom{4}{2} \right) + 2 \left(\binom{n-2}{3} + \binom{5}{3} \right) \\ \Sigma^1 n^3 &= 64 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 61 \left(\binom{n-3}{2} - \binom{4}{2} \right) + 30 \left(\binom{n-3}{3} + \binom{5}{3} \right) \\ &\quad + 6 \left(\binom{n-3}{4} - \binom{6}{4} \right)\end{aligned}$$

To obtain the formula for double sum of powers, we simply apply summation operator over the ordinary sum again, thus

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \sum_{k=1}^n \binom{j+t-1}{j+1} + \sum_{k=1}^n \binom{k-t+1}{j+1} \right]$$

which yields

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^n \binom{k-t+1}{j+1} \right]$$

Thus,

Theorem 1.9 (Double sums of powers via finite difference 2).

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2} \right]$$

Proof. We have $\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^n \binom{k-t+1}{j+1} \right]$, where $\sum_{k=1}^n \binom{k-t+1}{j+1} = (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}$ by means of segmented hockey stick identity (1.6). \square

For example, given $t = 5$, the double sums of powers are

$$\begin{aligned} \Sigma^2 n^0 &= 1 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) \\ \Sigma^2 n^1 &= 5 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 1 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\ \Sigma^2 n^2 &= 25 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 11 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\ &\quad + 2 \left(\binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4} \right) \\ \Sigma^2 n^3 &= 125 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 91 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\ &\quad + 36 \left(\binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4} \right) + 6 \left(\binom{n-3}{5} - \binom{7}{4} n + \binom{7}{5} \right) \end{aligned}$$

2. PROOF OF SEGMENTED HOCKEY STICK IDENTITY

First we split the sum $\sum_{k=0}^n \binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^n \binom{-t+k}{j} = \sum_{k=0}^{t-1} \binom{-t+k}{j} + \sum_{k=t}^n \binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0, 1, 2, \dots, t, \dots, n\}$ such that $t < n$. Considering the sum $\sum_{k=0}^{t-1} \binom{-t+k}{j}$ we notice that

$$\begin{aligned} \sum_{k=0}^{t-1} \binom{-t+k}{j} &= \binom{-t}{j} + \binom{-t+1}{j} + \binom{-t+2}{j} + \dots + \\ &\quad + \binom{-t+t-2}{j} + \binom{-t+t-1}{j} \end{aligned}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = \sum_{k=1}^t \binom{-k}{j} = \sum_{k=0}^{t-1} \binom{-k-1}{j}$$

By means of $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$

$$\binom{-k-1}{j} = \binom{-(k+1)}{j} = (-1)^j \binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = (-1)^j \sum_{k=0}^{t-1} \binom{j+k}{j} = (-1)^j \binom{j+t}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^t \binom{j+k}{j} = \binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^n \binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^n \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j}$$

Thus

$$\sum_{k=t}^n \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j} = \binom{n-t+1}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^t \binom{j+k}{j} = \binom{j+t+1}{j+1}$. Thus

$$\sum_{k=0}^n \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

This completes the proof.

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Sources: github.com/kolosovpetro/github-latex-template

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