NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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ABSTRACT.

1. INTRODUCTION

Proposition 1.1. *(Newton's series around arbitrary point* [1, *Lemma V*].*)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

Example 1.2 (Newton series for cubes monomial).

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$

$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$

$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

In general,

$$n^3 = \Delta^0 t^3 \binom{n-t}{0} + \Delta^1 t^3 \binom{n-t}{1} + \Delta^2 t^3 \binom{n-t}{2} + \Delta^3 t^3 \binom{n-t}{3}$$

Corollary 1.3 (Newton series for binomial reversed).

$$(n+t)^m = \sum_{k=0}^{m} \binom{n}{k} \Delta^k t^m$$

Proposition 1.4 (Newton series for monomial reversed).

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m$$

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Proof. By setting $n \to n - t$ into (1.3). □

Definition 1.5 (Multifold sum of powers recurrence).

$$\Sigma^0 \, n^m = n^m$$

$$\Sigma^1 \, n^m = \Sigma^0 \, 1^m + \Sigma^0 \, 2^m + \cdots + \Sigma^0 \, n^m$$

$$\Sigma^{r+1} \, n^m = \Sigma^r \, 1^m + \Sigma^r \, 2^m + \cdots + \Sigma^r \, n^m$$

Thus, for arbitrary integer t

$$\Sigma^1 \, n^m = \sum_{k=1}^{n} \sum_{j=0}^{m} \binom{-t+k}{j} \Delta^j t^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$$

Proposition 1.6 (Segmented Hockey stick identity). *For integers n, t and j*

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Theorem 1.7 (Ordinary sums of powers via Newton's series). *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 \, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right]$$

Proof. Ordinary sum of powers is given by $\Sigma^1 \, n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$, where $\sum_{k=1}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$ by means of segmented hockey stick identity (1.6). □

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently. For $t = 0$ and $m = 3$ we have the famous identity

$$\Sigma^1 \, n^3 = 0 \binom{n+1}{1} + 1 \binom{n+1}{2} + 6 \binom{n+1}{3} + 6 \binom{n+1}{4}$$

which was discussed in [2, p. 190] and in [3]. The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were discussed in [4]. For instance,

$$\Sigma^1 n^3 = 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}$$

$$\Sigma^1 n^4 = 1\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}$$

The coefficients $1, 7, 12, \ldots$ are given by the sequence [ID] in the OEIS [5]. Interestingly enough that the paper [4] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ {k \atop j} \right\}_r$$

where $\left\{ {k \atop j} \right\}_r$ are generalized Stirling numbers of the second kind. The result above is essentially identical to (1.7). Let be a conjecture

Conjecture 1.8.

$$\Delta^j t^m = j! \left\{ {m \atop j} \right\}_t$$

By considering the special cases of the theorem (1.7) for $t = 4$, we observe rather unexpected results, that are

$$\Sigma^1 n^0 = 1\left(\binom{n-3}{1} + \binom{3}{1} \right)$$

$$\Sigma^1 n^1 = 4\left(\binom{n-3}{1} + \binom{3}{1} \right) + 1\left(\binom{n-3}{2} - \binom{4}{2} \right)$$

$$\Sigma^1 n^2 = 16\left(\binom{n-3}{1} + \binom{3}{1} \right) + 9\left(\binom{n-3}{2} - \binom{4}{2} \right) + 2\left(\binom{n-2}{3} + \binom{5}{3} \right)$$

$$\Sigma^1 n^3 = 64\left(\binom{n-3}{1} + \binom{3}{1} \right) + 61\left(\binom{n-3}{2} - \binom{4}{2} \right) + 30\left(\binom{n-3}{3} + \binom{5}{3} \right)$$
$$+ 6\left(\binom{n-3}{4} - \binom{6}{4} \right)$$

2. Proof of Segmented hockey stick identity

First we split the sum $\sum_{k=0}^{n} \binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^{n} \binom{-t+k}{j} = \sum_{k=0}^{t-1} \binom{-t+k}{j} + \sum_{k=t}^{n} \binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0, 1, 2, \cdots, t, \cdots, n\}$ such that $t < n$. Considering the sum $\sum_{k=0}^{t-1} \binom{-t+k}{j}$ we notice that

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = \binom{-t}{j} + \binom{-t+1}{j} + \binom{-t+2}{j} + \cdots +$$
$$+ \binom{-t+t-2}{j} + \binom{-t+t-1}{j}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = \sum_{k=1}^{t} \binom{-k}{j} = \sum_{k=0}^{t-1} \binom{-k-1}{j}$$

By means of $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$

$$\binom{-k-1}{j} = \binom{-(k+1)}{j} = (-1)^j \binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = (-1)^j \sum_{k=0}^{t-1} \binom{j+k}{j} = (-1)^j \binom{j+t}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^{n} \binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^{n} \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j}$$

Thus

$$\sum_{k=t}^{n} \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j} = \binom{n-t+1}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. Thus

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

This completes the proof.

References

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Sources: github.com/kolosovpetro/github-latex-template

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