NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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ABSTRACT. In this manuscript we derive the formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, this manuscript provides the formulas for multifold sums of powers in terms of Stirling numbers of the second kind, and Eulerian numbers.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this manuscript we derive the formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, this manuscript provides the formulas for multifold sums of powers in terms of Stirling numbers of the second kind, and Eulerian numbers.

Allow us to start from the definition of multifold sums of powers. We utilize the recurrence proposed by Donald Knuth in his article *Johann Faulhaber and sums of powers*, see [1]

$$\Sigma^0 \, n^m = n^m$$

$$\Sigma^1 \, n^m = \Sigma^0 \, 1^m + \Sigma^0 \, 2^m + \cdots + \Sigma^0 \, n^m$$

$$\Sigma^{r+1} \, n^m = \Sigma^r \, 1^m + \Sigma^r \, 2^m + \cdots + \Sigma^r \, n^m$$

Throughout the paper, we utilize the Newton's interpolation formula as stated below

Proposition 1.1. *(Newton's series around arbitrary point* [2, Lemma V]*.)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

where $\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j)$ is k-degree forward finite difference of f.

Which indeed holds, because

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$
$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$
$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

Proposition 1.2 (Newton's series for power). *For non-negative integers m, n and arbitrary integer t*

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m$$

Thus, for arbitrary integer t, the ordinary sum of powers is

$$\Sigma^1 \, n^m = \sum_{k=1}^{n} \sum_{j=0}^{m} \binom{-t+k}{j} \Delta^j t^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$$

Proposition 1.3 (Segmented Hockey stick identity). *For integers n, t and j*

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Therefore,

Proposition 1.4 (Ordinary sums of powers via Newton's series). *For non-negative integers* n, m *and arbitrary integer* t

$$\Sigma^1 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right]$$

Proof. Ordinary sum of powers is given by $\Sigma^1 n^m = \sum_{j=0}^m \Delta^j t^m \sum_{k=1}^n \binom{-t+k}{j}$, where $\sum_{k=1}^n \binom{-t+k}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$ by means of segmented hockey stick identity (1.3). $\square$

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently. For $t = 0$ and $m = 3$ we have the famous identity

$$\Sigma^1 n^3 = 0\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$$

which was discussed in [3, p. 190] and in [4]. The coefficients $0, 1, 6, 6, 0, 1, 14, 36, 24, \ldots$ are given by the sequence A131689 in the OEIS [5].

The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were discussed in [6]. For instance,

$$\Sigma^1 n^3 = 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}$$

$$\Sigma^1 n^4 = 1\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}$$

The coefficients $1, 7, 12, 6, 1, 15, \ldots$ are given by the sequence A028246 in the OEIS [5]. Interestingly enough that the paper [6] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^k j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ {k \atop j} \right\}_r$$

where $\left\{ {k \atop j} \right\}_r$ are generalized Stirling numbers of the second kind. The formula above is identical to the proposition (1.4), which yields that finite differences can be expressed in terms of generalized Stirling numbers of the second kind, that is $\Delta^j t^m = j! \left\{ {m \atop j} \right\}_t$.

By considering the special cases of the proposition (1.4) for $t = 4$, we observe rather unexpected formulas for sums of powers, that are

$$\Sigma^1 n^0 = 1\left(\binom{n-3}{1} + \binom{3}{1}\right)$$

$$\Sigma^1 n^1 = 4\left(\binom{n-3}{1} + \binom{3}{1}\right) + 1\left(\binom{n-3}{2} - \binom{4}{2}\right)$$

$$\Sigma^1 n^2 = 16\left(\binom{n-3}{1} + \binom{3}{1}\right) + 9\left(\binom{n-3}{2} - \binom{4}{2}\right) + 2\left(\binom{n-2}{3} + \binom{5}{3}\right)$$

$$\Sigma^1 n^3 = 64\left(\binom{n-3}{1} + \binom{3}{1}\right) + 61\left(\binom{n-3}{2} - \binom{4}{2}\right) + 30\left(\binom{n-3}{3} + \binom{5}{3}\right)$$
$$+ 6\left(\binom{n-3}{4} - \binom{6}{4}\right)$$

The coefficients $1, 4, 1, 16, 9, \ldots$ are given by the sequence A391633 in the OEIS [5]. To obtain the formula for double sum of powers, we simply apply summation operator over the ordinary sum of powers again, thus

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \sum_{k=1}^n \binom{j+t-1}{j+1} + \sum_{k=1}^n \binom{k-t+1}{j+1}\right]$$

which yields

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1}n + \sum_{k=1}^n \binom{k-t+1}{j+1}\right]$$

Thus,

Proposition 1.5 (Double sums of powers via Newton's series). *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1}n + (-1)^{j+1}\binom{j+t-1}{j+2}n^0 + \binom{n-t+2}{j+2}\right]$$

Proof. We have $\Sigma^2 n^m = \sum_{j=0}^m \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1}n + \sum_{k=1}^n \binom{k-t+1}{j+1}\right]$, where $\sum_{k=1}^n \binom{k-t+1}{j+1} = (-1)^{j+1}\binom{j+t-1}{j+2}n^0 + \binom{n-t+2}{j+2}$ by means of segmented hockey stick identity (1.3). $\square$

For example, given $t = 5$, the double sums of powers are

$$\Sigma^2 n^0 = 1\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right)$$

$$\Sigma^2 n^1 = 5\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 1\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)$$

$$\Sigma^2 n^2 = 25\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 11\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)$$
$$+ 2\left(\binom{n-3}{4} + \binom{6}{3}n - \binom{6}{4}\right)$$

$$\Sigma^2 n^3 = 125\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 91\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)$$
$$+ 36\left(\binom{n-3}{4} + \binom{6}{3}n - \binom{6}{4}\right) + 6\left(\binom{n-3}{5} - \binom{7}{4}n + \binom{7}{5}\right)$$

The coefficients $1, 5, 1, 25, 11, 2, \ldots$ are given by the sequence A391635 in the OEIS [5].

Similarly, we obtain the formula for the triple sums of powers

Proposition 1.6 (Triple sums of powers via Newton's series). *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 + \right.$$
$$\left. + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3} \right]$$

Proof. By summing up the double powers sums, we get

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[(-1)^j \binom{j+t-1}{j+1} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} k^0 + \binom{k-t+2}{j+2} \right]$$

$$= \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \sum_{k=1}^{n} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} \sum_{k=1}^{n} k^0 + \sum_{k=1}^{n} \binom{k-t+2}{j+2} \right]$$

Note that $\sum_{k=1}^{n} k^1 = \Sigma^2 n^0$ and $\sum_{k=1}^{n} k^0 = \Sigma^1 n^0$. Thus,

$$\sum_{k=1}^{n} \binom{k-t+2}{j+2} = (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3}$$

by segmented hockey stick identity (1.3). This completes the proof. $\square$

Theorem 1.7 (Multifold sums of powers via Newton's series). *For non-negative integers* r, n, m *and arbitrary integer* t

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right]$$

Proof. By Newton's series for power (1.2) and repeated segmented hockey stick identity (1.3). □

We may observe that

Proposition 1.8 (Multifold sum of zero powers). *For integers* r *and* n

$$\Sigma^r n^0 = \binom{r+n-1}{r}$$

Proof. By hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. □

Which yields the following binomial variations of the Multifold sums of powers (1.7)

Proposition 1.9 (Multifold sums of powers binomial form). *For non-negative integers* r, n, m *and arbitrary integer* t

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right) + \binom{n-t+r}{j+r} \right]$$

Proposition 1.10 (Multifold sums of powers binomial form reindexed). *For non-negative integers* r, n, m *and arbitrary integer* t

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right) + \binom{n-t+r}{j+r} \right]$$

Finite difference of power is closely related to Stirling numbers of the second kind

Lemma 1.11 (Finite difference via Stirling numbers). *For non-negative integers* j, m *and arbitrary integer* t

$$\Delta^j t^m = \sum_{k=0}^{m} \binom{t}{k} \left\{ \begin{matrix} m \\ j+k \end{matrix} \right\} (j+k)!$$

which implies the variations of the formulas for sums of powers

Proposition 1.12 (Ordinary sums of powers via Stirling numbers). *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[(-1)^j\binom{j+t-1}{j+1}+\binom{n-t+1}{j+1}\right]\binom{t}{k}\begin{Bmatrix}m\\j+k\end{Bmatrix}(j+k)!$$

Proof. By ordinary sums of powers via Newton's series (1.4) and finite difference via Stirling numbers of the second kind (1.11). $\square$

In general,

Proposition 1.13 (Multifold sums of powers via Stirling numbers). *For non-negative integers r, n, m and arbitrary integer t*

$$\Sigma^r n^m$$

$$= \sum_{j=0}^{m}\left[\left(\sum_{s=1}^{r}(-1)^{j+s-1}\binom{j+t-1}{j+s}\Sigma^{r-s} n^0\right)+\binom{n-t+r}{j+r}\right]\sum_{k=0}^{m}\binom{t}{k}\begin{Bmatrix}m\\j+k\end{Bmatrix}(j+k)!$$

Proof. By multifold sums of powers via Newton's series (1.7) and finite difference via Stirling numbers of the second kind (1.11). $\square$

The proposition above can be presented in a pure binomial form as well, by means of the identity (1.8): $\Sigma^r n^0 = \binom{r+n-1}{r}$.

2. Backward difference form

The formula for multifold sums of powers via Newton's series (1.7) can be altered to be in terms of backward differences easily, because

$$\nabla^j(t+1)^m = \Delta^j t^m$$

Thus,

Proposition 2.1 (Multifold sums of powers via backward difference). *For non-negative integers r, n, m and arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m}\nabla^j(t+1)^m\left[\left(\sum_{s=1}^{r}(-1)^{j+s-1}\binom{j+t-1}{j+s}\Sigma^{r-s} n^0\right)+\binom{n-t+r}{j+r}\right]$$

Proof. By multifold sums of powers via Newton's series (1.7) and $\nabla^j (t+1)^m = \Delta^j t^m$. $\qquad \square$

3. Future research

In this manuscript we focus on the idea to combine the Newton's interpolation formula and Hockey-stick identity for binomial coefficients to express the sums of powers seamlessly.

This particular idea is great, however it can be generalized even further, so that the main aim is to utilize an interpolation formula for power n^m in terms of *abstract difference operator* $D(n^m)$ and binomial coefficients $\binom{f(n)}{k}$ such that n indicates the variable of power function. The difference operator can be arbitrary, for example: forward, backward, central differences etc. For example, the abstract interpolation formula is

$$n^m = \sum_k \binom{f(n)}{k} D(n^m, k)$$

Thus, the formula of sums of powers involves the abstract difference operator D in some point k and hockey stick identity over the binomial coefficients $\binom{n}{k}$

$$\Sigma^1 \, n^m = \sum_k D(n^m, k) \sum_{j \le n} \binom{f(j)}{k}$$

Similarly, for multifold sums of powers

$$\Sigma^r \, n^m = \sum_k D(n^m, k) \binom{f(n+r)}{k}$$

Many of interpolation approaches involve rising factorials $x_{(n)}$, falling factorials $(x)_n$ or usual factorials $n!$, and thus can be expressed in terms of binomial coefficients, because

$$\frac{(x)_n}{n!} = \binom{x}{n}; \quad \frac{x_{(n)}}{n!} = \binom{x+n-1}{n}.$$

In particular, Donald Knuth provides the formula multifold sums of odd powers [1] such that based on the operator of central finite differences of power evaluated in zero, that is

Proposition 3.1 (Multifold sums of odd powers).

$$\Sigma^r n^{2m-1} = \sum_{k=1}^{m} (2k-1)! T(2m,2k) \binom{n+k-1+r}{2k-1+r}$$

$$= \sum_{k=1}^{m} \binom{n+k-1+r}{2k-1+r} \frac{1}{2k} \delta^{2k} 0^{2m}$$

where $T(n,k)$ are central factorial numbers of the second kind, see [7, section 58] and [8, formula (10a)], such that

$$T(n,k) = \frac{1}{k!} \delta^k 0^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^n$$

In general, the central factorial numbers of the second kind $T(n,k)$ were defined by Riordan in his fundamental work *Combinatorial identities* [9, ch. 6.5, formula (24)], via the polynomial identity

Lemma 3.2 (Riordan power identity).

$$n^m = \sum_{k=1}^{m} T(m,k) n^{[k]}$$

where $n^{[k]}$ are central factorials $n^{[k]} = n \prod_{j=0}^{k-1} \left(n + \frac{k}{2} - j\right)$. The sequence A008957 in the OEIS [5] provides non-zero central factorial numbers of the second kind $T(2n,2k)$.

The Knuth's formula (3.1) utilizes the operator of central finite differences of power evaluated in zero, it is worth to research the existence of the sums of odd powers involving the central differences evaluated in arbitrary integer point t, similar to multifold sums of powers via Newton's series (1.7).

4. Proof of Segmented hockey stick identity

First we split the sum $\sum_{k=0}^{n} \binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^{n} \binom{-t+k}{j} = \sum_{k=0}^{t-1} \binom{-t+k}{j} + \sum_{k=t}^{n} \binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0, 1, 2, \cdots, t, \cdots, n\}$ such that $t < n$. Considering the sum $\sum_{k=0}^{t-1} \binom{-t+k}{j}$ we notice that

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = \binom{-t}{j} + \binom{-t+1}{j} + \binom{-t+2}{j} + \cdots +$$
$$+ \binom{-t+t-2}{j} + \binom{-t+t-1}{j}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = \sum_{k=1}^{t} \binom{-k}{j} = \sum_{k=0}^{t-1} \binom{-k-1}{j}$$

By means of $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$

$$\binom{-k-1}{j} = \binom{-(k+1)}{j} = (-1)^j \binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = (-1)^j \sum_{k=0}^{t-1} \binom{j+k}{j} = (-1)^j \binom{j+t}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^{n} \binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^{n} \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j}$$

Thus

$$\sum_{k=t}^{n} \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j} = \binom{n-t+1}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. Thus

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

This completes the proof.

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Sources: github.com/kolosovpetro/NewtonsInterpolationFormulaAndSumsOfPowers

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