NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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Abstract.

1. Introduction

Proposition 1.1. *(Newton's series around arbitrary point* [1, Lemma V]*.)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

Example 1.2 (Newton series for cubes monomial)**.**

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$
$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$
$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

In general,

$$n^3 = \Delta^0 t^3 \binom{n-t}{0} + \Delta^1 t^3 \binom{n-t}{1} + \Delta^2 t^3 \binom{n-t}{2} + \Delta^3 t^3 \binom{n-t}{3}$$

Corollary 1.3 (Newton series for binomial reversed)**.**

$$(n+t)^m = \sum_{k=0}^{m} \binom{n}{k} \Delta^k t^m$$

Proposition 1.4 (Newton series for monomial reversed)**.**

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m$$

Proof. By setting $n \to n-t$ into (1.3). $\square$

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Definition 1.5 (Multifold sum of powers recurrence)**.**

$$\Sigma^0\, n^m = n^m$$

$$\Sigma^1\, n^m = \Sigma^0\, 1^m + \Sigma^0\, 2^m + \cdots + \Sigma^0\, n^m$$

$$\Sigma^{r+1}\, n^m = \Sigma^r\, 1^m + \Sigma^r\, 2^m + \cdots + \Sigma^r\, n^m$$

Thus, for arbitrary integer t

$$\Sigma^1\, n^m = \sum_{k=1}^{n}\sum_{j=0}^{m}\binom{-t+k}{j}\Delta^j t^m = \sum_{j=0}^{m}\Delta^j t^m \sum_{k=1}^{n}\binom{-t+k}{j}$$

Proposition 1.6 (Segmented Hockey stick identity)**.** *For integers n, t and j*

$$\sum_{k=0}^{n}\binom{-t+k}{j} = (-1)^j\binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Therefore,

Proposition 1.7 (Ordinary sums of powers via Newton's series)**.** *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1\, n^m = \sum_{j=0}^{m}\Delta^j t^m\left[(-1)^j\binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}\right]$$

Proof. Ordinary sum of powers is given by $\Sigma^1\, n^m = \sum_{j=0}^{m}\Delta^j t^m \sum_{k=1}^{n}\binom{-t+k}{j}$, where $\sum_{k=1}^{n}\binom{-t+k}{j} = (-1)^j\binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$ by means of segmented hockey stick identity (1.6). $\square$

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently. For $t = 0$ and $m = 3$ we have the famous identity

$$\Sigma^1\, n^3 = 0\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$$

which was discussed in [2, p. 190] and in [3]. The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were discussed in [4]. For instance,

$$\Sigma^1\, n^3 = 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}$$

$$\Sigma^1\, n^4 = 1\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}$$

The coefficients $1, 7, 12, \ldots$ are given by the sequence [ID] in the OEIS [5]. Interestingly enough that the paper [4] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] {k \brace j}_r$$

where ${k \brace j}_r$ are generalized Stirling numbers of the second kind. The formula above is essentially identical to (1.7), which yields the identity for finite differences $\Delta^j t^m = j! {m \brace j}_t$.

By considering the special cases of the theorem (1.7) for $t = 4$, we observe rather unexpected formulas for sums of powers, that are

$$\begin{aligned}
\Sigma^1 n^0 &= 1 \left(\binom{n-3}{1} + \binom{3}{1} \right) \\
\Sigma^1 n^1 &= 4 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 1 \left(\binom{n-3}{2} - \binom{4}{2} \right) \\
\Sigma^1 n^2 &= 16 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 9 \left(\binom{n-3}{2} - \binom{4}{2} \right) + 2 \left(\binom{n-2}{3} + \binom{5}{3} \right) \\
\Sigma^1 n^3 &= 64 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 61 \left(\binom{n-3}{2} - \binom{4}{2} \right) + 30 \left(\binom{n-3}{3} + \binom{5}{3} \right) \\
&\quad + 6 \left(\binom{n-3}{4} - \binom{6}{4} \right)
\end{aligned}$$

The coefficients $1, 4, 1, 16, 9, \ldots$ are given by the sequence [ID] in the OEIS [5]. To obtain the formula for double sum of powers, we simply apply summation operator over the ordinary sum again, thus

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \sum_{k=1}^{n} \binom{j+t-1}{j+1} + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$$

which yields

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$$

Thus,

Theorem 1.8 (Double sums of powers via finite difference 2)**.**

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2} \right]$$

Proof. We have $\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1}\right]$, where $\sum_{k=1}^{n} \binom{k-t+1}{j+1} = (-1)^{j+1}\binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}$ by means of segmented hockey stick identity (1.6). □

For example, given $t = 5$, the double sums of powers are

$$\Sigma^2 n^0 = 1\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right)$$

$$\Sigma^2 n^1 = 5\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 1\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)$$

$$\begin{aligned}\Sigma^2 n^2 &= 25\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 11\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)\\ &\quad + 2\left(\binom{n-3}{4} + \binom{6}{3}n - \binom{6}{4}\right)\end{aligned}$$

$$\begin{aligned}\Sigma^2 n^3 &= 125\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 91\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)\\ &\quad + 36\left(\binom{n-3}{4} + \binom{6}{3}n - \binom{6}{4}\right) + 6\left(\binom{n-3}{5} - \binom{7}{4}n + \binom{7}{5}\right)\end{aligned}$$

2. Proof of Segmented hockey stick identity

First we split the sum $\sum_{k=0}^{n}\binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^{n}\binom{-t+k}{j}=\sum_{k=0}^{t-1}\binom{-t+k}{j}+\sum_{k=t}^{n}\binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0,1,2,\cdots,t,\cdots,n\}$ such that $t<n$. Considering the sum $\sum_{k=0}^{t-1}\binom{-t+k}{j}$ we notice that

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=\binom{-t}{j}+\binom{-t+1}{j}+\binom{-t+2}{j}+\cdots+$$
$$+\binom{-t+t-2}{j}+\binom{-t+t-1}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=\sum_{k=1}^{t}\binom{-k}{j}=\sum_{k=0}^{t-1}\binom{-k-1}{j}$$

By means of $\binom{-k}{j}=(-1)^j\binom{j+k-1}{j}$

$$\binom{-k-1}{j}=\binom{-(k+1)}{j}=(-1)^j\binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j}=(-1)^j\sum_{k=0}^{t-1}\binom{j+k}{j}=(-1)^j\binom{j+t}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t}\binom{j+k}{j}=\binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^{n}\binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^{n}\binom{-t+k}{j}=\sum_{k=0}^{n-t}\binom{k}{j}$$

Thus

$$\sum_{k=t}^{n}\binom{-t+k}{j}=\sum_{k=0}^{n-t}\binom{k}{j}=\binom{n-t+1}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t}\binom{j+k}{j}=\binom{j+t+1}{j+1}$. Thus

$$\sum_{k=0}^{n}\binom{-t+k}{j}=(-1)^{j}\binom{j+t}{j+1}+\binom{n-t+1}{j+1}$$

This completes the proof.

References

[1] Newton, Isaac and Chittenden, N.W. *Newton's Principia: the mathematical principles of natural philosophy*. New-York, D. Adee, 1850. https://archive.org/details/bub_gb_KaAIAAAAIAAJ/page/466/mode/2up.

[2] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. https://archive.org/details/concrete-mathematics.

[3] Thomas J Pfaff. Deriving a formula for sums of powers of integers. *Pi Mu Epsilon Journal*, 12(7):425–430, 2007. https://www.jstor.org/stable/24340705.

[4] Cereceda, José L. Sums of powers of integers and generalized Stirling numbers of the second kind. *arXiv preprint arXiv:2211.11648*, 2022. https://arxiv.org/abs/2211.11648.

[5] Sloane, Neil JA and others. The on-line encyclopedia of integer sequences, 2003. https://oeis.org/.

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Sources: github.com/kolosovpetro/github-latex-template

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