ODD-POWER IDENTITY VIA MULTIPLICATION OF CERTAIN MATRICES

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Abstract. This manuscript establishes an odd-power identity expressed through matrix multiplication. Specifically, we demonstrate that a 1×1 matrix with an entry $a_{1,1} = N^{2M+1}$ can be represented as the product of three matrices: $\mathbf{J}_N$, $\mathbf{K}_{N,M}$, and $\mathbf{T}_M$, as follows

$$\begin{bmatrix} N^{2M+1} \end{bmatrix} = \mathbf{J}_N \times \mathbf{K}_{N,M} \times \mathbf{T}_M$$

Here, $\mathbf{J}_N$ denotes a unit row vector of dimension $1 \times N$, $\mathbf{K}_{N,M}$ is an $N \times M$ matrix, and $\mathbf{T}_M$ represents a column vector of dimension $M \times 1$.

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Sources: https://github.com/kolosovpetro/github-latex-template

Definitions

- $\mathbf{J}_N$ – unit row vector of all 1's having the dimension $1 \times N$. For example,

$$\mathbf{J}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

- $\mathbf{K}_{N,M}$ – matrix of dimension $N \times M$ defined by

$$\mathbf{K}_{N,M} = (k^r(N-k)^r)_{0 \leq k \leq N,\ 0 \leq r \leq M}$$

For example, given $N = 4$

$$\mathbf{K}_{4,M} = \begin{bmatrix} 0^0(4-0)^0 & 0^1(4-0)^1 & 0^2(4-0)^2 & \cdots & 0^M(4-0)^M \\ 1^0(4-1)^0 & 1^1(4-1)^1 & 1^2(4-1)^2 & \cdots & 1^M(4-1)^M \\ 2^0(4-2)^0 & 2^1(4-2)^1 & 2^2(4-2)^2 & \cdots & 2^M(4-2)^M \\ 3^0(4-3)^0 & 3^1(4-3)^1 & 3^2(4-3)^2 & \cdots & 3^M(4-3)^M \\ 4^0(4-4)^0 & 4^1(4-4)^1 & 4^2(4-4)^2 & \cdots & 4^M(4-4)^M \end{bmatrix}$$

- $\mathbf{T}_M$ – column vector of dimension $M \times 1$ defined by

$$\mathbf{T}_M = (\mathbf{A}_{M,r})_{M=\text{const},\ 0 \leq r \leq M}$$

where $\mathbf{A}_{M,r}$ is a real coefficient [1, 2, 3, 4]. For example, given $M = 3$

$$\mathbf{T}_3 = \begin{bmatrix} 1 \\ -14 \\ 0 \\ 140 \end{bmatrix}$$

1. Introduction and main results

Let be a definition of the coefficient $\mathbf{A}_{m,r}$.

Definition 1.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{1.1}$$

Let be a theorem that states the odd-power identity

Theorem 1.2. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively by (1.1).

Proof. The proof is given in [1, 2]. □

Therefore, the claim that a 1×1 matrix with an entry $a_{1,1} = N^{2M+1}$ can be represented as the product of three matrices: $\mathbf{J}_N$, $\mathbf{K}_{N,M}$, and $\mathbf{T}_M$

$$\begin{bmatrix} N^{2M+1} \end{bmatrix} = \mathbf{J}_N \times \mathbf{K}_{N,M} \times \mathbf{T}_M$$

is essentially the odd-power identity (1.2) expressed through matrix multiplication.

2. Examples

$$4^{2\cdot3+1} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 \\ 4^0 & 4^1 & 4^2 & 4^3 \\ 3^0 & 3^1 & 3^2 & 3^3 \\ 0^0 & 0^1 & 0^2 & 0^3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -14 \\ 0 \\ 140 \end{bmatrix}$$

$$4^{2\cdot 4+1} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4^0 & 4^1 & 4^2 & 4^3 \\ 6^0 & 6^1 & 6^2 & 6^3 \\ 6^0 & 6^1 & 6^2 & 6^3 \\ 4^0 & 4^1 & 4^2 & 4^3 \\ 0^0 & 0^1 & 0^2 & 0^3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -120 \\ 0 \\ 0 \\ 630 \end{bmatrix}$$

3. Conclusions

In this manuscript we have successfully established an odd-power identity expressed through matrix multiplication. Specifically, we demonstrate that a 1×1 matrix with an entry $a_{1,1} = N^{2M+1}$ can be represented as the product of three matrices: $\mathbf{J}_N$, $\mathbf{K}_{N,M}$, and $\mathbf{T}_M$, as follows

$$\begin{bmatrix} N^{2M+1} \end{bmatrix} = \mathbf{J}_N \times \mathbf{K}_{N,M} \times \mathbf{T}_M$$

Here, $\mathbf{J}_N$ denotes a unit row vector of dimension $1 \times N$, $\mathbf{K}_{N,M}$ is an $N \times M$ matrix, and $\mathbf{T}_M$ represents a column vector of dimension $M \times 1$.

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