ON THE LINK BETWEEN BINOMIAL THEOREM AND DISCRETE CONVOLUTION

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Abstract. Let $\mathbf{P}^m_b(x)$ be a $2m+1$-degree polynomial in x and $b \in \mathbb{R}$

$$\mathbf{P}^m_b(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r$$

where $\mathbf{A}_{m,r}$ are real coefficients. In this manuscript, we introduce the polynomial $\mathbf{P}^m_b(x)$ and study its properties, establishing a polynomial identity for odd-powers in terms of this polynomial. Based on mentioned polynomial identity for odd-powers, we explore the connection between the Binomial theorem and discrete convolution of odd-powers, further extending this relation to the multinomial case. All findings are verified using Mathematica programs.

Contents

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1. Definitions, notations and conventions

We now set the following notation, which remains fixed for the remainder of this manuscript

- $\mathbf{A}_{m,r}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{1.1}$$

  where m is non-negative integer and B_t are Bernoulli numbers [1]. It is assumed that $B_1 = \frac{1}{2}$.

- $\mathbf{P}_b^m(x)$ is a $2m+1$-degree polynomial in $b, x \in \mathbb{R}$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r \tag{1.2}$$

- $\mathbf{H}_{m,t}(b)$ is a polynomial defined as

$$\mathbf{H}_{m,t}(b) = \sum_{j=t}^{m}\binom{j}{t}\mathbf{A}_{m,j}\frac{(-1)^j}{2j-t+1}\binom{2j-t+1}{b}B_{2j-t+1-b} \tag{1.3}$$

  integers m, t, b.

Sources: https://github.com/kolosovpetro/OnTheBinomialTheoremAndDiscreteConvolution

- $\mathbf{X}_{m,t}(j)$ is polynomial of degree $2m+1-t$ in $j \in \mathbb{R}$

$$\mathbf{X}_{m,t}(j) = (-1)^m \sum_{k=1}^{2m+1-t} \mathbf{H}_{m,t}(k) \cdot j^k \tag{1.4}$$

  integers m, t.

- $\mathbf{L}_m(x,k)$ is $2m$ degree polynomial in $x, k \in \mathbb{R}$

$$\mathbf{L}_m(x,k) = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.5}$$

- $(f * f)[n]$ is discrete convolution [2] of function f defined over set of integers $\mathbb{Z}$

$$(f * f)[n] = \sum_k f(k) f(n-k)$$

  and its partial case for polynomials $n^j,\ n \geq a \in \mathbb{R}$

$$(n^j * n^j)[x] = \sum_k k^j (x-k)^j [k \geq a][x-k \geq a] = \sum_{k=a}^{x-a} k^j (x-k)^j$$

2. Introduction and main results

The polynomial $\mathbf{P}_b^m(x)$ is a $2m+1$-degree polynomial in $x, b \in \mathbb{R}$ defined as

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient. By means of Lemma (4.1), the polynomial $\mathbf{P}_b^m(x)$ has the following relation with Binomial theorem [3]

$$\mathbf{P}_{x+y}^m(x+y) = \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

On the other hand, polynomial $\mathbf{P}_b^m(x)$ might be expressed in terms of discrete convolution of polynomial n^j. For every $n \geq 0$

$$\mathbf{P}_{x+1}^m(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x]$$

It is important to notice that n^r of discrete convolution $(n^r * n^r)[x]$ evaluated at x is implicit piecewise-defined polynomial such as

$$n^r = \begin{cases} \underbrace{n \cdot n \cdots n}_{\text{r times}}, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, it is easy to notice the following identities in terms of Binomial theorem and discrete convolution, see the corollaries (6.1) and (6.2). For every $n \geq 0$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x+y] = 1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

For every $n > 0$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x+y] = -1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

Additionally, the following generalizations for the multinomial case are discussed in the corollaries (6.3) and (6.4). For every $n \geq 0$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x_1 + x_2 + \cdots + x_t] = 1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

For every $n > 0$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x_1 + x_2 + \cdots + x_t] = -1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

A few polynomial identities are straightforward by means of the theorems (5.3), (5.5). More precisely, by the theorem (5.3) we have an odd-power identity as follows

$$x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{x-1} k^r (x-k)^r$$

so that

$$1 + x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x] = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{x} k^r (x-k)^r$$

From the other side, the theorem (5.5) provides an odd-power polynomial identity as follows

$$x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{x} k^r (x-k)^r$$

so that

$$-1 + x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x] = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{x-1} k^r (x-k)^r$$

For example,

$$x^3 = \sum_{k=1}^{x} 6k(x-k) + 1$$

$$x^5 = \sum_{k=1}^{x} 30k^2(x-k)^2 + 1$$

$$x^7 = \sum_{k=1}^{x} 140k^3(x-k)^3 - 14k(x-k) + 1$$

$$x^9 = \sum_{k=1}^{x} 630k^4(x-k)^4 - 120k(x-k) + 1$$

$$x^{11} = \sum_{k=1}^{x} 2772k^5(x-k)^5 + 660k^2(x-k)^2 - 1386k(x-k) + 1$$

$$x^{13} = \sum_{k=1}^{x} 51480k^7(x-k)^7 - 60060k^3(x-k)^3 + 491400k^2(x-k)^2 - 450054k(x-k) + 1$$

3. Polynomial $\mathbf{P}_b^m(x)$ and its properties

We continue our mathematical journey from the short overview of polynomial $\mathbf{L}_m(x,k)$ which is an essential part of polynomial $\mathbf{P}_b^m(x)$ since that $\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \mathbf{L}_m(x,k)$. Polynomial $\mathbf{L}_m(x,k)$ is a polynomial of degree $2m$ in $x, k \in \mathbb{R}$, see definition (1.5). In its explicit form the polynomial $\mathbf{L}_m(x,k)$ is as follows

$$\mathbf{L}_m(x,k) = \mathbf{A}_{m,m} k^m (x-k)^m + \mathbf{A}_{m,m-1} k^{m-1} (x-k)^{m-1} + \cdots + \mathbf{A}_{m,0}$$

where $\mathbf{A}_{m,r}$ are real coefficients defined by (1.1). Coefficients $\mathbf{A}_{m,r}$ are nonzero for r only within the range $r \in \{m\} \cup \left[0, \frac{m-1}{2}\right]$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See the OEIS entries A302971, A304042: [4, 5].

Thus, the polynomial $\mathbf{L}_m(x,k)$ may also be written as

$$\mathbf{L}_m(x,k) = \mathbf{A}_{m,m} k^m (x-k)^m + \sum_{r=0}^{\frac{m-1}{2}} \mathbf{A}_{m,r} k^r (x-k)^r$$

For example, the polynomials $\mathbf{L}_m(x,k)$ for $0 \leq m \leq 3$ are

$$\mathbf{L}_0(x,k) = 1,$$

$$\mathbf{L}_1(x,k) = 6k(x-k)+1 = -6k^2+6kx+1,$$

$$\mathbf{L}_2(x,k) = 30k^2(x-k)^2+1 = 30k^4-60k^3x+30k^2x^2+1,$$

$$\begin{aligned}\mathbf{L}_3(x,k) &= 140k^3(x-k)^3-14k(x-k)+1\\ &= -140k^6+420k^5x-420k^4x^2+140k^3x^3+14k^2-14kx+1\end{aligned}$$

It is important to notice that $\mathbf{L}_m(x,k)$ is symmetric over x

Property 3.1. *For every $x, k \in \mathbb{R}$*

$$\mathbf{L}_m(x,k) = \mathbf{L}_m(x,x-k)$$

This might be seen from the following tables

x/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $\mathbf{L}_1(x,k)$. See the OEIS entry A287326, [6].

Another case, given $m=2$ we have the following values of $\mathbf{L}_2(x,k)$

x/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $\mathbf{L}_2(x,k)$. See the OEIS entry A300656, [7].

Note that row sums of the table (2) are cubes of x. Next we discuss the polynomial $\mathbf{P}^m_b(x)$. In its extended form, the polynomial $\mathbf{P}^m_b(x)$ is

$$\mathbf{P}^m_b(x) = \sum_{k=0}^{b-1} \mathbf{L}_m(x,k) = \sum_{k=0}^{b-1}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r$$

By means of binomial theorem $(x-y)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} y^k$,

$$\begin{aligned}\mathbf{P}^m_b(x) &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r \sum_{j=0}^{r} (-1)^j \binom{r}{j} x^{r-j} k^j \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} \sum_{j=0}^{r} (-1)^j \binom{r}{j} x^{r-j} k^{r+j} \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{j=0}^{r} (-1)^j x^{r-j} \binom{r}{j} \sum_{k=0}^{b-1} k^{r+j}\end{aligned}$$

However, by the symmetry (3.1) of $\mathbf{L}_m(x,k)$ the polynomial $\mathbf{P}^m_b(x)$ may also be written in the form

$$\begin{aligned}\mathbf{P}^m_b(x) &= \sum_{k=1}^{b}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r = \sum_{k=1}^{b}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r \sum_{t=0}^{r} (-1)^{r-t} x^t \binom{r}{t} k^{r-t} \\ &= \sum_{t=0}^{m} x^t \underbrace{\sum_{k=1}^{b}\sum_{r=t}^{m} (-1)^{r-t} \binom{r}{t} \mathbf{A}_{m,r} k^{2r-t}}_{(-1)^{m-t}\mathbf{X}_{m,t}(b)}\end{aligned}$$

Note that $\sum_{k=1}^{b}\sum_{r=t}^{m}(-1)^{r-t}\binom{r}{t}\mathbf{A}_{m,r}k^{2r-t}$ is the $(-1)^{m-t}\mathbf{X}_{m,t}(b)$. From this formula it may be not immediately clear why $\mathbf{X}_{m,t}(b)$ represent polynomials in b. However, this can be seen if we change the summation order and use Faulhaber's formula $\sum_{k=1}^{n}k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$ to obtain

$$\mathbf{X}_{m,t}(b) = (-1)^m \sum_{r=t}^{m}\binom{r}{t}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-t+1}\sum_{\ell=0}^{2r-t}\binom{2r-t+1}{\ell}B_\ell b^{2r-t+1-\ell}$$

Introducing $k = 2r-t+1-\ell$ we further get the formula

$$\mathbf{X}_{m,t}(b) = (-1)^m \sum_{k=1}^{2m-t+1} b^k \underbrace{\sum_{r=t}^{m}\binom{r}{t}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-t+1}\binom{2r-t+1}{k}B_{2r-t+1-k}}_{\mathbf{H}_{m,t}(k)}$$

Polynomials $\mathbf{X}_{3,t}(b),\ 0 \le t \le 3$ are

$$\mathbf{X}_{3,0}(j) = 7b^2 - 28b^3 + 70b^5 - 70b^6 + 20b^7,$$

$$\mathbf{X}_{3,1}(j) = 7b - 42b^2 + 175b^4 - 210b^5 + 70b^6,$$

$$\mathbf{X}_{3,2}(j) = -14b + 140b^3 - 210b^4 + 84b^5,$$

$$\mathbf{X}_{3,3}(j) = 35b^2 - 70b^3 + 35b^4$$

Polynomials $\mathbf{H}_{3,t}(k)$ are defined by (1.3) and examples for $m = 3,\ 0 \le t \le 3$ are

$$\mathbf{H}_{3,0}(k) = B_{1-k}\binom{1}{k} + \frac{14}{3}B_{3-k}\binom{3}{k} - 20B_{7-k}\binom{7}{k},$$

$$\mathbf{H}_{3,1}(k) = 7B_{2-k}\binom{2}{k} - 70B_{6-k}\binom{6}{k},$$

$$\mathbf{H}_{3,2}(k) = -84B_{5-k}\binom{5}{k},$$

$$\mathbf{H}_{3,3}(k) = -35B_{4-k}\binom{4}{k}$$

It gives us an opportunity to overview the polynomial $\mathbf{P}_b^m(x)$ from the different prospective, for instance

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m}(-1)^{m-r}\mathbf{X}_{m,r}(b)\cdot x^r = \sum_{r=0}^{m}\sum_{\ell=1}^{2m-r+1}(-1)^{2m-r}\mathbf{H}_{m,r}(\ell)\cdot b^\ell \cdot x^r \tag{3.1}$$

Equation (3.1) clearly states why $\mathbf{P}_b^m(x)$ is polynomial in x, b. For example,

$$\mathbf{P}_b^0(x) = b,$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x,$$

$$\begin{aligned}\mathbf{P}_b^2(x) &= 10b^3 - 15b^4 + 6b^5 \\ &- 15b^2x + 30b^3x - 15b^4x \\ &+ 5bx^2 - 15b^2x^2 + 10b^3x^2\end{aligned}$$

$$\begin{aligned}\mathbf{P}_b^3(x) &= -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 \\ &+ 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x \\ &+ 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 \\ &+ 35b^2x^3 - 70b^3x^3 + 35b^4x^3\end{aligned}$$

The following property is also true in terms of the polynomial $\mathbf{P}_b^m(x)$

Property 3.2. *For every $m \in \mathbb{N},\ x, b \in \mathbb{R}$*

$$\mathbf{P}_{b+1}^m(x) = \mathbf{P}_b^m(x) + \mathbf{L}_m(x, b)$$

4. Relation between the polynomial $\mathbf{P}_b^m(x)$ and Binomial theorem

Lemma 4.1. *For every $m \in \mathbb{N},\ x, y \in \mathbb{R}$*

$$\mathbf{P}_{x+y}^m(x+y) = \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

By means of lemma 4.1 and equation (3.1) the following polynomial identities straightforward

$$x^{2m+1} = \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot x^{\ell+r} = \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x) \cdot x^r$$

For instance,

$$\mathbf{P}_{x+y}^2(x+y) = (x+y)(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4).$$

In addition, the following identities hold

$$
\begin{aligned}
(x+y)^{2m+1} &= \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x+y)^{\ell+r} \\
&= \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x+y) \cdot (x+y)^{r}
\end{aligned}
$$

Obviously, Multinomial expansion of t-fold sum $(x_1+x_2+\cdots+x_t)^{2m+1}$ can be reached by $\mathbf{P}_b^m(x_1+x_2+\cdots+x_t)$ as well

Corollary 4.2. *For all $x_1, x_2, \ldots, x_t \in \mathbb{R},\ m \in \mathbb{N}$*

$$
\mathbf{P}^m_{x_1+x_2+\cdots+x_t}(x_1+x_2+\cdots+x_t) = \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{s=1}^{t} x_t^{k_s}
$$

Moreover, the following multinomial identities hold

$$
\begin{aligned}
(x_1+x_2+\cdots+x_t)^{2m+1} &= \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x_1+x_2+\cdots+x_t)^{\ell+r} \\
&= \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x_1+x_2+\cdots+x_t) \cdot (x_1+x_2+\cdots+x_t)^{r}
\end{aligned}
$$

5. Polynomial $\mathbf{P}_b^m(x)$ in terms of Discrete convolution

In this section we discuss the relation between $\mathbf{P}_b^m(x)$ and discrete convolution of polynomials. To show that $\mathbf{P}_b^m(x)$ involves the discrete convolution of polynomial n^r recall the definition of the polynomial $\mathbf{P}_b^m(x)$

$$
\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r
$$

A discrete convolution of defined over set of integers $\mathbb{Z}$ function f is

$$
(f * f)[n] = \sum_{k} f(k) f(n-k)
$$

General formula of discrete convolution for the polynomial $n^j,\ n \geq a \in \mathbb{R}$ can be derived immediately

$$\begin{aligned}(n^j * n^j)[x] &= \sum_k k^j (x-k)^j [k \geq a][x-k \geq a] \\ &= \sum_k k^j (x-k)^j [k \geq a][k \leq x-a] \\ &= \sum_k k^j (x-k)^j [a \leq k \leq x-a] \\ &= \sum_{k=a}^{x-a} k^j (x-k)^j\end{aligned}$$

where $[a \leq k \leq x-a]$ is Iverson's bracket [8, 9].

Lemma 5.1. *For every* $n \in \mathbb{N},\ x \in \mathbb{R}$ *and* $n \geq 0$

$$(n^r * n^r)[x] = \sum_{k=0}^{x} k^r (x-k)^r$$

It is of first importance to keep in mind that n^r of discrete convolution $(n^r * n^r)[x]$ evaluated at x is an implicit piecewise-defined polynomial such as

$$n^r = \begin{cases} \underbrace{n \cdot n \cdots n}_{\text{r times}}, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the corollary follows

Corollary 5.2. *By Lemma 5.1 the polynomial* $\mathbf{P}^m_b(n)$ *might be expressed in terms of discrete convolution as follows, for every* $n \geq 0$

$$\mathbf{P}^m_{x+1}(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x]$$

Therefore, another polynomial identity follows

Theorem 5.3. *By Lemma 4.1, Corollary 5.2 and property 3.2, for every $m \in \mathbb{N},\ x \in \mathbb{R}$ and $n \geq 0$*

$$1 + x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x] = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{x} k^r (x-k)^r$$

Now we notice the following identity in terms of polynomial $\mathbf{P}_b^m(x)$ and discrete convolution $(n^j * n^j)[x]$

Proposition 5.4. *For every $m \in \mathbb{N},\ x \in \mathbb{R}$ and $n \geq 1$*

$$\begin{aligned}
\mathbf{P}_x^m(x) &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left(0^r x^r + \sum_{k=1}^{x-1} k^r (x-k)^r \right) \\
&= \sum_{r=0}^{m} \mathbf{A}_{m,r} 0^r x^r + \sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x] \\
&= 1 + \sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x]
\end{aligned}$$

Since that for all r in $\mathbf{A}_{m,r} 0^r x^r$ we have

$$\mathbf{A}_{m,r} 0^r x^r = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{if } r > 0 \end{cases}$$

Above is true because $\mathbf{A}_{m,0} = 1$ for every $m \in \mathbb{N}$, and $x^0 = 1$ for every x, see [10]. Hence, the following identity between $\mathbf{P}_b^m(x)$ and discrete convolution $(n^j * n^j)[x]$ holds

Theorem 5.5. *By Lemma 4.1 and Proposition 5.4, for every $m \in \mathbb{N},\ x \in \mathbb{R}$ and $n > 0$*

$$-1 + x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x] = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{x-1} k^r (x-k)^r$$

Corollary 5.6. *By Theorem 5.5, for all $m \in \mathbb{N}$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Corollary 5.6 holds since that convolution $(n^j * n^j)[x] = 1,\ n > 0$ for each r and $x = 2$.

6. Relation between Binomial theorem and Discrete convolution

Corollary 6.1. *(Generalization of Theorem 5.3 for Binomials.) For every $m \in \mathbb{N},\ x, y \in \mathbb{R}$ and $n \geq 0$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x+y] = 1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

For example, given $m = 0, 1, 2$ the Corollary 6.1 yields

$$\sum_{r=0}^{0} \mathbf{A}_{0,r} (n^r * n^r)[x+y] = 1 + x + y$$

$$\begin{aligned}
\sum_{r=0}^{1} \mathbf{A}_{1,r} (n^r * n^r)[x+y] &= 1 + x + y - (x+y)(1+x+y)(1-3x-3y+2(x+y)) \\
&= 1 + x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}$$

$$\begin{aligned}
\sum_{r=0}^{2} \mathbf{A}_{2,r} (n^r * n^r)[x+y] &= 1 + x + y + (x+y)(1+x+y)\big(-1 + x + 5x^2 + y + 10xy + 5y^2 \\
&- 15x(x+y) + 10x^2(x+y) - 15y(x+y) + 20xy(x+y) \\
&+ 10y^2(x+y) + 9(x+y)^2 - 15x(x+y)^2 \\
&-15y(x+y)^2 + 6(x+y)^3\big) \\
&= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 + 1
\end{aligned}$$

Above example could be verified using using the commands defined in Mathematica package at [11]

- `BinomialTheoremAndDiscreteConvolutionTest[0, x + y]`
- `BinomialTheoremAndDiscreteConvolutionTest[1, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[1, x + y]]`
- `BinomialTheoremAndDiscreteConvolutionTest[2, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[2, x + y]]`

Corollary 6.2. *(Generalization of Theorem 5.5 for Binomials.) For every* $m \in \mathbb{N},\ x, y \in \mathbb{R}$ *and* $n > 0$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x+y] = -1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

For example, given $m = 0, 1$ the Corollary 6.2 gives

$$\sum_{r=0}^{0} \mathbf{A}_{0,r}(n^r * n^r)[x+y] = x + y - 1$$

$$\begin{aligned}\sum_{r=0}^{1} \mathbf{A}_{1,r}(n^r * n^r)[x+y] &= -1 + x + y - (-1 + x + y)(x+y)(-1 - 3x - 3y + 2(x+y)) \\ &= x^3 + 3x^2y + 3xy^2 + y^3 - 1\end{aligned}$$

Above example could be verified using using the commands defined in Mathematica package at [11]

- `BinomialTheoremAndDiscreteConvolutionStrictTest[0, x + y]`
- `BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y]]`

From the other prospective, then following binomial holds. For every $n \geq 0$

$$\begin{aligned}(x-2a)^{2m+1} + 1 &= \sum_{r=0}^{m} \mathbf{A}_{m,r}((t-k)^r * (t-k)^r)[x] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=a}^{x-a-1} (k-a)^r (x-k-a)^r\end{aligned} \tag{6.1}$$

Similarly, the following binomial holds. For every $n > 0$

$$\begin{aligned}(x-2a)^{2m+1} - 1 &= \sum_{r=0}^{m} \mathbf{A}_{m,r}((t-k)^r * (t-k)^r)[x] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=a+1}^{x-a-1} (k-a)^r (x-k-a)^r\end{aligned} \tag{6.2}$$

To validate equations (6.1) and (6.2) use the following commands

- `ConvolutionOfBinomial[10, 2, 1]` verifies an equation (6.1).
- `ConvolutionOfBinomial1[10, 2, 1]` verifies an equation (6.2).

6.1. **Generalization for Multinomials.** In this subsection we generalize Theorems (5.3) and (5.5) for multinomial cases.

Corollary 6.3. *(Generalization of Theorem 5.3 for Multinomials.) For every* $x_1, x_2, \ldots, x_t \in \mathbb{R},\ m \in \mathbb{N},\ n \geq 1$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x_1 + x_2 + \cdots + x_t] = 1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

For instance, given $m = 1$ the Corollary 6.3 gives

$$\sum_{r=0}^{1} \mathbf{A}_{1,r}(n^r * n^r)[x + y + z]$$

$$= 1 + x + y + z - (x + y + z)(1 + x + y + z)(1 - 3x - 3y - 3z + 2(x + y + z))$$

$$= 1 + x^3 + 3x^2y + 3xy^2 + y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 + 3yz^2 + z^3.$$

Above example could be verified using using the commands defined in Mathematica package at [11]

- `BinomialTheoremAndDiscreteConvolutionTest[1, x + y + z]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[1, x + y + z]]`

Corollary 6.4. *(Generalization of Theorem 5.5 for Multinomials.) For each* $x_1 + x_2 + \cdots + x_t \geq 1,\ x_1, x_2, \ldots, x_t \in \mathbb{R},\ m \in \mathbb{N},\ n \geq 1$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x_1 + x_2 + \cdots + x_t] = -1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

For example, given $m = 1$ the Corollary 6.4 gives

$$\sum_{r=0}^{1} \mathbf{A}_{1,r}(n^r * n^r)[x + y + z]$$

$$= -1 + x + y + z - (-1 + x + y + z)(x + y + z)(-1 - 3x - 3y - 3z + 2(x + y + z))$$

$$= -1 + x^3 + 3x^2y + 3xy^2 + y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 + 3yz^2 + z^3.$$

Above example could be verified using using the commands defined in Mathematica package at [11]

- `BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y + z]`
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y + z]]`

7. Derivation of the coefficient $\mathbf{A}_{m,r}$

By Lemma 4.1 for every $m \in \mathbb{N},\ n \in \mathbb{R}$

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{n-1} k^r (n-k)^r \tag{7.1}$$

The $\mathbf{A}_{m,r}$ might be evaluated using binomial expansion of $\sum_{k=0}^{n-1} k^r (n-k)^r$

$$\sum_{k=0}^{n-1} k^r (n-k)^r = \sum_{k=0}^{n-1} k^r \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} k^j = \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} \sum_{k=0}^{n-1} k^{r+j}$$

Using Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$ we get

$$\begin{aligned}
\sum_{k=0}^{n-1} k^r (n-k)^r &= \sum_{j=0}^{r} \binom{r}{j} n^{r-j} \frac{(-1)^j}{r+j+1} \left[\sum_{s} \binom{r+j+1}{s} B_s n^{r+j+1-s} - B_{r+j+1} \right] \\
&= \sum_{j,s} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s} B_s n^{2r+1-s} - \sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j} \\
&= \sum_{s} \underbrace{\sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s}}_{S(r)} B_s n^{2r+1-s} \\
&\quad - \sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}
\end{aligned} \tag{7.2}$$

where B_s are Bernoulli numbers and $B_1 = \frac{1}{2}$. Now, we notice that

$$\sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } s = 0; \\ \frac{(-1)^r}{s} \binom{r}{2r-s+1}, & \text{if } s > 0. \end{cases}$$

In particular, the last sum is zero for $0 < s \leq r$. Therefore, expression (7.2) takes the form

$$\sum_{k=0}^{n-1} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\sum_{s\geq 1} \frac{(-1)^r}{s} \binom{r}{2r-s+1} B_s n^{2r+1-s}}_{(\star)}$$
$$- \underbrace{\sum_j \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}}_{(\diamond)}$$

Hence, by introducing $\ell = 2r+1-s$ into $(\star)$ and $\ell = r-j$ into $(\diamond)$, we get

$$\sum_{k=0}^{n-1} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$
$$- \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}$$

$$\sum_{k=0}^{n-1} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + (-1)^r \sum_{\ell} \frac{1}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$
$$- \frac{1}{(-1)^r} \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}$$
$$= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell}^{r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$

Using the definition (7.1) of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^{m} \sum_{\text{odd } \ell}^{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \tag{7.3}$$

Taking the coefficient of n^{2r+1} for $r = m$ in (7.3) we get $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$. Since that odd $\ell \leq r$ in explicit form is $2j+1 \leq r$, it follows that $j \leq \frac{m-1}{2}$, where j is an iterator. Therefore, taking the coefficient of n^{2j+1} for an integer j in the range $\frac{m}{2} \leq j \leq m$, we get $\mathbf{A}_{m,j} = 0$. Taking the coefficient of n^{2d+1} for d in the range $m/4 \leq d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0,$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \le r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

So that

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \le r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

As desired.

8. Conclusion

In this manuscript, we introduced the polynomial $\mathbf{P}_b^m(x)$ and examined its properties. We established a polynomial identity for odd-powers that demonstrates the connection between Binomial theorem and discrete convolution of odd-powered polynomials. This relationship was extended to the multinomial case. All results were verified using Mathematica programs.

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Version: `Local-0.1.0`

10. Addendum 1: Verification of the results

To fulfill our study we provide an opportunity to verify its results by means of Wolfram Mathematica language.

10.1. **Mathematica commands.**

Proceeding to the repository [11] reader is able to find there a folder named `mathematica` that contains the files

- `OnTheBinomialTheoremAndDiscreteConvolution.m` is a package file with definitions
- `OnTheBinomialTheoremAndDiscreteConvolution.nb` is a notebook file with examples.

The following commands may be used to reproduce the results of this manuscript:

- `A[m, r]` returns the real coefficient $\mathbf{A}_{m,r}$ defined by (1.1).
- `PrintTriangleOfA[rows]` prints the table of coefficients $\mathbf{A}_{m,r}$.
  Command `PrintTriangleOfA[7]` reproduces the table (1).
- `PolynomialL[m, n, k]` returns the polynomial $\mathbf{L}_m(n,k)$ defined by (1.5).
- `PolynomialP[m, x, b]` returns the polynomial $\mathbf{P}^m_b(x)$ defined by (1.2).
- `Expand[PolynomialP[m, x + y, x + y]]` verifies the Lemma 4.1.
- `PolynomialH[m, t, j]` returns the polynomial $\mathbf{H}_{m,t}(j)$ defined by (1.3).
- `PolynomialX[m, t, k]` returns the polynomial $\mathbf{X}_{m,t}(k)$ defined by (1.4).
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[m, x + y]]` verifies the Corollary 6.1.
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[m, x + y]]` verifies the Corollary 6.2.
- `DiscreteConvolutionPowerIdentityParametricTest[m, x, a]` verifies an equation (6.1). Usage `Column[Table[DiscreteConvolutionPowerIdentityParametricTest[1, x, 1], x, 3, 20], Left]`.
- `DiscreteConvolutionPowerIdentityStrictParametricTest[m, x, a]` verifies an equation (6.2). Usage `Column[Table[DiscreteConvolutionPowerIdentityStrictParametricTes x, 1], x, 3, 20], Left]`.

- `Expand[PolynomialIdentityOfP[1, n, b]]` validates an identity

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{j=0}^{r} (-1)^j x^{r-j} \binom{r}{j} \sum_{k=0}^{b-1} k^{r+j}$$

- `PolynomialIdentityInvolvingX[m, x, b]` validates an identity (3.1)

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(b) \cdot x^r$$

- `PolynomialIdentityInvolvingH[m, n, b]` validates an identity (3.1).

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot b^\ell \cdot x^r$$

10.2. **Examples.** For example, given $m = 1$ we have the following values of $\mathbf{L}_1(x, k)$

x/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 4. Values of $\mathbf{L}_1(x, k)$. See OEIS entry: `A300656`, [6].

Table 4 can be reproduced using Mathematica command

```
PrintTriangleOfPolynomialL[1, 7]
```

defined in the [11]. From Table 4 it is seen that

$$\mathbf{P}_0^1(0) = 0 = 0^3$$

$$\mathbf{P}_1^1(1) = 1 = 1^3$$

$$\mathbf{P}_2^1(2) = 1 + 7 = 2^3$$

$$\mathbf{P}_3^1(3) = 1 + 13 + 13 = 3^3$$

$$\mathbf{P}_4^1(4) = 1 + 19 + 25 + 19 = 4^3$$

$$\mathbf{P}_5^1(5) = 1 + 25 + 37 + 37 + 25 = 5^3$$

Another case, given $m = 2$ we have the following values of $\mathbf{L}_2(x, k)$

x/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 5. Values of $\mathbf{L}_2(x, k)$. See the OEIS entry A300656, [7].

Table 5 can be reproduced using Mathematica command

```
PrintTriangleOfPolynomialL[2, 7]
```

defined in the [11]. Again, an odd-power identity 4.1 holds

$$\mathbf{P}^2_0(0) = 0 = 0^5$$

$$\mathbf{P}^2_1(1) = 1 = 1^5$$

$$\mathbf{P}^2_2(2) = 1 + 31 = 2^5$$

$$\mathbf{P}^2_3(3) = 1 + 121 + 121 = 3^5$$

$$\mathbf{P}^2_4(4) = 1 + 271 + 481 + 271 = 4^5$$

$$\mathbf{P}^2_5(5) = 1 + 481 + 1081 + 1081 + 481 = 5^5$$

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