

# ON THE LINK BETWEEN BINOMIAL THEOREM AND DISCRETE CONVOLUTION

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ABSTRACT. Let  $\mathbf{P}_b^m(x)$  be a  $2m + 1$ -degree integer-valued polynomial in  $b, x \in \mathbb{R}$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x - k)^r,$$

where  $\mathbf{A}_{m,r}$  is a real coefficient. In this manuscript we establish a relation between Binomial theorem and polynomial  $\mathbf{P}_b^m(x)$ . Furthermore, a relationship between Binomial theorem and discrete convolution in terms of polynomials is provided.

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## 1. DEFINITIONS, NOTATIONS AND CONVENTIONS

We now set the following notation, which remains fixed for the remainder of this paper:

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- $\mathbf{A}_{m,r}$ ,  $m \in \mathbb{N}$  is a real coefficient defined recursively

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1) \binom{2r}{r}, & \text{if } r = m; \\ (2r+1) \binom{2r}{r} \sum_{d=2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \quad (1.1)$$

where  $B_t$  are Bernoulli numbers [Wei]. It is assumed that  $B_1 = \frac{1}{2}$ .

- $\mathbf{P}_b^m(x)$ ,  $m \in \mathbb{N}$  is a  $2m+1$ -degree integer-valued polynomial in  $b, x \in \mathbb{R}$

$$\mathbf{P}_b^m(x) := \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \quad (1.2)$$

- $\mathbf{H}_{m,t}(b)$ ,  $m, t, b \in \mathbb{N}$  is a polynomial defined as

$$\mathbf{H}_{m,t}(b) := \sum_{j=t}^m \binom{j}{t} \mathbf{A}_{m,j} \frac{(-1)^j}{2j-t+1} \binom{2j-t+1}{b} B_{2j-t+1-b} \quad (1.3)$$

- $\mathbf{X}_{m,t}(j)$ ,  $m, t \in \mathbb{N}$  is polynomial of degree  $2m+1-t$  in  $j \in \mathbb{R}$

$$\mathbf{X}_{m,t}(j) := (-1)^m \sum_{k=1}^{2m+1-t} \mathbf{H}_{m,t}(k) \cdot j^k \quad (1.4)$$

- $\mathbf{L}_m(x, k)$ ,  $m \in \mathbb{N}$  is  $2m$  degree polynomial in  $x, k \in \mathbb{R}$

$$\mathbf{L}_m(x, k) := \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \quad (1.5)$$

- $\mathbf{T}^r(b)$ ,  $r \in \mathbb{N}$ ,  $b \in \mathbb{R}$  is polynomial defined as follows

$$\mathbf{T}^r(b) := \sum_{j=0}^r \frac{1}{(-1)^j} \binom{r}{j} \sum_{k=0}^b k^{2r-j} \quad (1.6)$$

- $(f * f)[n]$  is discrete convolution [BDM11] of function  $f$  defined over set of integers  $\mathbb{Z}$

$$(f * f)[n] = \sum_k f(k) f(n-k)$$

## 2. INTRODUCTION AND MAIN RESULTS

The polynomial  $\mathbf{P}_b^m(x)$ ,  $m \in \mathbb{N}$  is  $2m+1$ -degree integer-valued polynomial in  $x, b \in \mathbb{R}$ .

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r,$$

where  $\mathbf{A}_{m,r}$  is real coefficient. By means of Lemma 4.1, the polynomial  $\mathbf{P}_b^m(x)$  has the following relation with Binomial theorem [AS72]

$$\mathbf{P}_{x+y}^m(x+y) = \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r.$$

From the other hand, polynomial  $\mathbf{P}_b^m(x)$  might be expressed in terms of discrete convolution of polynomial  $n^j$ ,  $j \in \mathbb{N}$

$$\mathbf{P}_{x+1}^m(x) = \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n \geq 0.$$

It is of first necessity to notice that  $n^r$  of discrete convolution  $(n^r * n^r)[x]$  evaluated at  $x$  is implicit piecewise-defined polynomial such as

$$n^r = \begin{cases} \underbrace{n \cdot n \cdots n}_{r \text{ times}}, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, it is easy to notice the following identities in terms of Binomial theorem and discrete convolution, see Corollaries 6.1, 6.2

$$\begin{aligned} \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x+y] &= 1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r, \quad n \geq 0, \\ \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x+y] &= -1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r, \quad n > 0. \end{aligned}$$

Also, the following generalizations for multinomial case are discussed, see Corollaries 6.3, 6.4

$$\begin{aligned} \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x_1 + x_2 + \cdots + x_t] &= 1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \dots, k_t} \prod_{\ell=1}^t x_{\ell}^{k_{\ell}}, \quad n \geq 0, \\ \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x_1 + x_2 + \cdots + x_t] &= -1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \dots, k_t} \prod_{\ell=1}^t x_{\ell}^{k_{\ell}}, \quad n > 0. \end{aligned}$$

A few polynomial identities are straightforward as well by means of Theorems 5.3, 5.5. Precisely, by the theorem 5.3 we have an odd-power identity as follows

$$x^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=0}^{x-1} k^r (x-k)^r$$

From the other prospective, the theorem 5.5 concludes as follows

$$x^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=1}^x k^r (x-k)^r$$

In its explicit form an identity  $x^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=0}^{x-1} k^r (x-k)^r$  looks like as follows

$$x^3 = \sum_{k=1}^x 6k(x-k) + 1$$

$$x^5 = \sum_{k=1}^x 30k^2(x-k)^2 + 1$$

$$x^7 = \sum_{k=1}^x 140k^3(x-k)^3 - 14k(x-k) + 1$$

$$x^9 = \sum_{k=1}^x 630k^4(x-k)^4 - 120k(x-k) + 1$$

$$x^{11} = \sum_{k=1}^x 2772k^5(x-k)^5 + 660k^2(x-k)^2 - 1386k(x-k) + 1$$

$$x^{13} = \sum_{k=1}^x 51480k^7(x-k)^7 - 60060k^3(x-k)^3 + 491400k^2(x-k)^2 - 450054k(x-k) + 1$$

### 3. POLYNOMIAL $\mathbf{P}_b^m(x)$ AND ITS PROPERTIES

We continue our mathematical journey from short overview of polynomial  $\mathbf{L}_m(x, k)$  that is essential part of polynomial  $\mathbf{P}_b^m(x)$  since that  $\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \mathbf{L}_m(x, k)$ . Polynomial  $\mathbf{L}_m(x, k)$ ,  $m \in \mathbb{N}$  is polynomial of degree  $2m$  in  $x, k \in \mathbb{R}$ , see definition (1.5). In explicit form the polynomial  $\mathbf{L}_m(x, k)$  is as follows

$$\mathbf{L}_m(x, k) = \mathbf{A}_{m,m}k^m(x-k)^m + \mathbf{A}_{m,m-1}k^{m-1}(x-k)^{m-1} + \cdots + \mathbf{A}_{m,0},$$

where  $\mathbf{A}_{m,r}$  are real coefficients defined by (1.1). Coefficients  $\mathbf{A}_{m,r}$  are nonzero only for  $r$  within the interval  $r \in \{m\} \cup [0, \frac{m-1}{2}]$ . For example,

$m/r$	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

**Table 1.** Coefficients  $\mathbf{A}_{m,r}$ . See the OEIS entries: [A302971](#), [A304042](#).

Thus, the polynomial  $\mathbf{L}_m(x, k)$  may also be written as

$$\mathbf{L}_m(x, k) = \mathbf{A}_{m,m}k^m(x-k)^m + \sum_{r=0}^{\frac{m-1}{2}} \mathbf{A}_{m,r}k^r(x-k)^r$$

For example, the polynomials  $\mathbf{L}_m(x, k)$  for  $0 \leq m \leq 3$  are

$$\mathbf{L}_0(x, k) = 1,$$

$$\mathbf{L}_1(x, k) = 6k(x-k) + 1 = -6k^2 + 6kx + 1,$$

$$\mathbf{L}_2(x, k) = 30k^2(x-k)^2 + 1 = 30k^4 - 60k^3x + 30k^2x^2 + 1,$$

$$\begin{aligned} \mathbf{L}_3(x, k) &= 140k^3(x-k)^3 - 14k(x-k) + 1 \\ &= -140k^6 + 420k^5x - 420k^4x^2 + 140k^3x^3 + 14k^2 - 14kx + 1 \end{aligned}$$

It is worth to notice that  $\mathbf{L}_m(x, k)$  is symmetrical over  $x$

**Property 3.1.** *For every  $x, k \in \mathbb{R}$*

$$\mathbf{L}_m(x, k) = \mathbf{L}_m(x, x-k)$$

This might be seen in the following table

$x/k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

**Table 2.** Values of  $\mathbf{L}_1(x, k)$ . See the OEIS entry: [A287326](#).

Next we discuss the polynomial  $\mathbf{P}_b^m(x)$ . In its extended form, the polynomial  $\mathbf{P}_b^m(x)$  is

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r}k^r(x-k)^r = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r(x-k)^r$$

By the binomial theorem  $(x - y)^n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} x^k y^{n-k}$ ,

$$\begin{aligned}
\mathbf{P}_b^m(x) &= \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} x^r k^{r-j} \\
&= \sum_{r=0}^m \sum_{j=0}^r (-1)^{r-j} x^r \binom{r}{j} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^{2r-j} \\
&= \sum_{r=0}^m x^r \mathbf{A}_{m,r} \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \sum_{k=0}^{b-1} k^{2r-j} \\
&= \sum_{r=0}^m (-1)^r x^r \mathbf{A}_{m,r} \left[ \sum_{j=0}^r \frac{1}{(-1)^j} \binom{r}{j} \sum_{k=0}^{b-1} k^{2r-j} \right] \\
&= \sum_{r=0}^m (-1)^r x^r \mathbf{A}_{m,r} \mathbf{T}^r(b-1),
\end{aligned}$$

where  $\mathbf{T}^r(b)$  is defined by (1.6). However, by the symmetry (3.1) of  $\mathbf{L}_m(x, k)$  the polynomial  $\mathbf{P}_b^m(x)$  may also be written in the form

$$\begin{aligned}
\mathbf{P}_b^m(x) &= \sum_{k=1}^b \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x - k)^r = \sum_{k=1}^b \sum_{r=0}^m \mathbf{A}_{m,r} k^r \sum_{t=0}^r (-1)^{r-t} x^t \binom{r}{t} k^{r-t} \\
&= \sum_{t=0}^m x^t \underbrace{\sum_{k=1}^b \sum_{r=t}^m (-1)^{r-t} \binom{r}{t} \mathbf{A}_{m,r} k^{2r-t}}_{(-1)^{m-t} \mathbf{X}_{m,t}(b)}
\end{aligned}$$

Note that  $\sum_{k=1}^b \sum_{r=t}^m (-1)^{r-t} \binom{r}{t} \mathbf{A}_{m,r} k^{2r-t}$  is the  $(-1)^{m-t} \mathbf{X}_{m,t}(b)$ . From this formula it may be not immediately clear why  $\mathbf{X}_{m,t}(b)$  represent polynomials in  $b$ . However, this can be seen if we change the summation order and use Faulhaber's formula  $\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j}$  to obtain

$$\mathbf{X}_{m,t}(b) = (-1)^m \sum_{r=t}^m \binom{r}{t} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-t+1} \sum_{\ell=0}^{2r-t} \binom{2r-t+1}{\ell} B_\ell b^{2r-t+1-\ell}$$

Introducing  $k = 2r - t + 1 - \ell$  we further get the formula

$$\mathbf{X}_{m,t}(b) = (-1)^m \sum_{k=1}^{2m-t+1} b^k \underbrace{\sum_{r=t}^m \binom{r}{t} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-t+1} \binom{2r-t+1}{k} B_{2r-t+1-k}}_{\mathbf{H}_{m,t}(k)}$$

Polynomials  $\mathbf{X}_{3,t}(b)$ ,  $0 \leq t \leq 3$  are

$$\begin{aligned}\mathbf{X}_{3,0}(j) &= 7b^2 - 28b^3 + 70b^5 - 70b^6 + 20b^7, \\ \mathbf{X}_{3,1}(j) &= 7b - 42b^2 + 175b^4 - 210b^5 + 70b^6, \\ \mathbf{X}_{3,2}(j) &= -14b + 140b^3 - 210b^4 + 84b^5, \\ \mathbf{X}_{3,3}(j) &= 35b^2 - 70b^3 + 35b^4\end{aligned}$$

Polynomials  $\mathbf{H}_{3,t}(k)$  are defined by (1.3) and examples for  $m = 3$ ,  $0 \leq t \leq 3$  are

$$\begin{aligned}\mathbf{H}_{3,0}(k) &= B_{1-k} \binom{1}{k} + \frac{14}{3} B_{3-k} \binom{3}{k} - 20 B_{7-k} \binom{7}{k}, \\ \mathbf{H}_{3,1}(k) &= 7 B_{2-k} \binom{2}{k} - 70 B_{6-k} \binom{6}{k}, \\ \mathbf{H}_{3,2}(k) &= -84 B_{5-k} \binom{5}{k}, \\ \mathbf{H}_{3,3}(k) &= -35 B_{4-k} \binom{4}{k}\end{aligned}$$

It gives us an opportunity to overview the polynomial  $\mathbf{P}_b^m(x)$  from the different prospective, for instance

$$\mathbf{P}_b^m(x) = \sum_{r=0}^m (-1)^{m-r} \mathbf{X}_{m,r}(b) \cdot x^r = \sum_{r=0}^m \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot b^\ell \cdot x^r \quad (3.1)$$

Equation (3.1) clearly states why  $\mathbf{P}_b^m(x)$  is polynomial in  $x, b$ . For example,

$$\begin{aligned}\mathbf{P}_b^0(x) &= b, \\ \mathbf{P}_b^1(x) &= 3b^2 - 2b^3 - 3bx + 3b^2x, \\ \mathbf{P}_b^2(x) &= 10b^3 - 15b^4 + 6b^5 \\ &\quad - 15b^2x + 30b^3x - 15b^4x \\ &\quad + 5bx^2 - 15b^2x^2 + 10b^3x^2, \\ \mathbf{P}_b^3(x) &= -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 \\ &\quad + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x \\ &\quad + 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 \\ &\quad + 35b^2x^3 - 70b^3x^3 + 35b^4x^3\end{aligned}$$

The following property also holds for  $\mathbf{P}_b^m(x)$

**Property 3.2.** For every  $m \in \mathbb{N}$ ,  $x, b \in \mathbb{R}$

$$\mathbf{P}_{b+1}^m(x) = \mathbf{P}_b^m(x) + \mathbf{L}_m(x, b)$$

4. POLYNOMIAL  $\mathbf{P}_b^m(x)$  IN TERMS OF BINOMIAL THEOREM

**Lemma 4.1.** *For every  $m \in \mathbb{N}$ ,  $x, y \in \mathbb{R}$*

$$\mathbf{P}_{x+y}^m(x+y) = \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

By Lemma 4.1 and equation (3.1) the following polynomial identities straightforward

$$x^{2m+1} = \sum_{r=0}^m \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot x^{\ell+r} = \sum_{r=0}^m (-1)^{m-r} \mathbf{X}_{m,r}(x) \cdot x^r$$

For instance,

$$\mathbf{P}_{x+y}^2(x+y) = (x+y)(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4).$$

In addition, the following identities hold

$$\begin{aligned} (x+y)^{2m+1} &= \sum_{r=0}^m \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x+y)^{\ell+r} \\ &= \sum_{r=0}^m (-1)^{m-r} \mathbf{X}_{m,r}(x+y) \cdot (x+y)^r \end{aligned}$$

Obviously, Multinomial expansion of  $t$ -fold sum  $(x_1 + x_2 + \cdots + x_t)^{2m+1}$  can be reached by  $\mathbf{P}_b^m(x_1 + x_2 + \cdots + x_t)$  as well

**Corollary 4.2.** *For all  $x_1, x_2, \dots, x_t \in \mathbb{R}$ ,  $m \in \mathbb{N}$*

$$\mathbf{P}_{x_1+x_2+\dots+x_t}^m(x_1+x_2+\dots+x_t) = \sum_{k_1+k_2+\dots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \dots, k_t} \prod_{s=1}^t x_s^{k_s}$$

Moreover, the following multinomial identities hold

$$\begin{aligned} (x_1 + x_2 + \cdots + x_t)^{2m+1} &= \sum_{r=0}^m \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x_1 + x_2 + \cdots + x_t)^{\ell+r} \\ &= \sum_{r=0}^m (-1)^{m-r} \mathbf{X}_{m,r}(x_1 + x_2 + \cdots + x_t) \cdot (x_1 + x_2 + \cdots + x_t)^r \end{aligned}$$

5. POLYNOMIAL  $\mathbf{P}_b^m(x)$  IN TERMS OF DISCRETE CONVOLUTION

In this section we discuss the relation between  $\mathbf{P}_b^m(x)$  and discrete convolution of polynomials. To show that  $\mathbf{P}_b^m(x)$  involves the discrete convolution of polynomial  $n^r$  let's remind the definition of  $\mathbf{P}_b^m(x)$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r$$



A discrete convolution of defined over set of integers  $\mathbb{Z}$  function  $f$  is

$$(f * f)[n] = \sum_k f(k)f(n - k)$$

General formula of discrete convolution for polynomials  $f(n) = n^j$ ,  $n \geq a \in \mathbb{R}$  may be derived immediately

$$\begin{aligned} (n^j * n^j)[x] &= \sum_k k^j (x - k)^j [k \geq a][x - k \geq a] \\ &= \sum_k k^j (x - k)^j [k \geq a][k \leq x - a] \\ &= \sum_k k^j (x - k)^j [a \leq k \leq x - a] \\ &= \sum_{k=a}^{x-a} k^j (x - k)^j, \end{aligned}$$

where  $[a \leq k \leq x - a]$  is Iverson's bracket [Ive62].

**Lemma 5.1.** *For every  $n \in \mathbb{N}$ ,  $x \in \mathbb{R}$*

$$(n^r * n^r)[x] = \sum_{k=0}^x k^r (x - k)^r, \quad n \geq 0.$$

It is of first importance to keep in mind that  $n^r$  of discrete convolution  $(n^r * n^r)[x]$  evaluated at  $x$  is implicit piecewise-defined polynomial such as

$$n^r = \begin{cases} \underbrace{n \cdot n \cdots n}_{r \text{ times}}, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the corollary follows

**Corollary 5.2.** *By Lemma 5.1 the polynomial  $\mathbf{P}_b^m(n)$  might be expressed in terms of discrete convolution as follows*

$$\mathbf{P}_{x+1}^m(x) = \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n \geq 0.$$

Therefore, another polynomial identity follows

**Theorem 5.3.** *By Lemma 4.1, Corollary 5.2 and property 3.2, for every  $m \in \mathbb{N}$ ,  $x \in \mathbb{R}$*

$$1 + x^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n \geq 0.$$

Now we notice the following identity in terms of polynomial  $\mathbf{P}_b^m(x)$  and discrete convolution  $(n^j * n^j)[x]$

**Proposition 5.4.** *For every  $m \in \mathbb{N}$ ,  $x \in \mathbb{R}$*

$$\begin{aligned} \mathbf{P}_x^m(x) &= \sum_{r=0}^m \mathbf{A}_{m,r} \left( 0^r x^r + \sum_{k=1}^{x-1} k^r (x-k)^r \right) \\ &= \sum_{r=0}^m \mathbf{A}_{m,r} 0^r x^r + \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x] \\ &= 1 + \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n \geq 1. \end{aligned}$$

Since that for all  $r$  in  $\mathbf{A}_{m,r} 0^r x^r$  we have

$$\mathbf{A}_{m,r} 0^r x^r = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{if } r > 0 \end{cases}$$

Above is true because  $\mathbf{A}_{m,0} = 1$  for every  $m \in \mathbb{N}$ , and  $x^0 = 1$  for every  $x$ , [GKP94]. Hence, the following identity between  $\mathbf{P}_b^m(x)$  and discrete convolution  $(n^j * n^j)[x]$  holds

**Theorem 5.5.** *By Lemma 4.1 and Proposition 5.4, for every  $m \in \mathbb{N}$ ,  $x \in \mathbb{R}$*

$$-1 + x^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n > 0.$$

**Corollary 5.6.** *By Theorem 5.5, for all  $m \in \mathbb{N}$*

$$\sum_{r=0}^m \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Corollary 5.6 holds since that convolution  $(n^j * n^j)[x] = 1$ ,  $n > 0$  for each  $r$  and  $x = 2$ .

## 6. RELATION BETWEEN BINOMIAL THEOREM AND DISCRETE CONVOLUTION

**Corollary 6.1.** *(Generalization of Theorem 5.3 for Binomials.) For every  $m \in \mathbb{N}$ ,  $x, y \in \mathbb{R}$*

$$\sum_{r=0}^m \mathbf{A}_{m,r} (n^r * n^r)[x+y] = 1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r, \quad n \geq 0.$$

For example, given  $m = 0, 1, 2$  the Corollary 6.1 gives

$$\begin{aligned}
\sum_{r=0}^0 \mathbf{A}_{0,r}(n^r * n^r)[x+y] &= 1 + x + y \\
\sum_{r=0}^1 \mathbf{A}_{1,r}(n^r * n^r)[x+y] &= 1 + x + y - (x+y)(1+x+y)(1-3x-3y+2(x+y)) \\
&= 1 + x^3 + 3x^2y + 3xy^2 + y^3 \\
\sum_{r=0}^2 \mathbf{A}_{2,r}(n^r * n^r)[x+y] &= 1 + x + y + (x+y)(1+x+y)(-1+x+5x^2+y+10xy+5y^2 \\
&\quad - 15x(x+y) + 10x^2(x+y) - 15y(x+y) + 20xy(x+y) \\
&\quad + 10y^2(x+y) + 9(x+y)^2 - 15x(x+y)^2 \\
&\quad - 15y(x+y)^2 + 6(x+y)^3) \\
&= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 + 1
\end{aligned}$$

Above example could be verified using using the commands

- `BinomialTheoremAndDiscreteConvolutionTest[0, x + y]`
- `BinomialTheoremAndDiscreteConvolutionTest[1, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[1, x + y]]`
- `BinomialTheoremAndDiscreteConvolutionTest[2, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[2, x + y]]`

defined in Mathematica package at [Kol20].

**Corollary 6.2.** (*Generalization of Theorem 5.5 for Binomials.*) For every  $m \in \mathbb{N}$ ,  $x, y \in \mathbb{R}$

$$\sum_{r=0}^m \mathbf{A}_{m,r}(n^r * n^r)[x+y] = -1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r, \quad n > 0.$$

For example, given  $m = 0, 1$  the Corollary 6.2 gives

$$\begin{aligned}
\sum_{r=0}^0 \mathbf{A}_{0,r}(n^r * n^r)[x+y] &= x + y - 1 \\
\sum_{r=0}^1 \mathbf{A}_{1,r}(n^r * n^r)[x+y] &= -1 + x + y - (-1+x+y)(x+y)(-1-3x-3y+2(x+y)) \\
&= x^3 + 3x^2y + 3xy^2 + y^3 - 1
\end{aligned}$$

Above example could be verified using using the commands

- `BinomialTheoremAndDiscreteConvolutionStrictTest[0, x + y]`
- `BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y]]`

defined in Mathematica package at [Kol20]. From the other prospective, let be a function  $f_r(t, k) = (t - k)^r$ ,  $t \geq k$ , then following identity holds

$$(x - 2a)^{2m+1} + 1 = \sum_{r=0}^m \mathbf{A}_{m,r}(f_r(t, k) * f_r(t, k))[x] \quad (6.1)$$

Let be a function  $g_r(t, k) = (t - k)^r$ ,  $t > k$ , then

$$(x - 2a)^{2m+1} - 1 = \sum_{r=0}^m \mathbf{A}_{m,r}(g_r(t, k) * g_r(t, k))[x] \quad (6.2)$$

**6.1. Generalization for Multinomials.** In this subsection we generalize Theorems 5.3, 5.5 for multinomial cases.

**Corollary 6.3.** (Generalization of Theorem 5.3 for Multinomials.) For every  $x_1, x_2, \dots, x_t \in \mathbb{R}$ ,  $m \in \mathbb{N}$ ,  $n \geq 1 \in \mathbb{N}$

$$\sum_{r=0}^m \mathbf{A}_{m,r}(n^r * n^r)[x_1 + x_2 + \dots + x_t] = 1 + \sum_{k_1+k_2+\dots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \dots, k_t} \prod_{\ell=1}^t x_\ell^{k_\ell}$$

For instance, given  $m = 1$  the Corollary 6.3 gives

$$\begin{aligned} & \sum_{r=0}^1 \mathbf{A}_{1,r}(n^r * n^r)[x + y + z] \\ &= 1 + x + y + z - (x + y + z)(1 + x + y + z)(1 - 3x - 3y - 3z + 2(x + y + z)) \\ &= 1 + x^3 + 3x^2y + 3xy^2 + y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 + 3yz^2 + z^3. \end{aligned}$$

Above example could be verified using using the commands

- `BinomialTheoremAndDiscreteConvolutionTest[1, x + y + z]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[1, x + y + z]]`

defined in Mathematica package at [Kol20].

**Corollary 6.4.** (Generalization of Theorem 5.5 for Multinomials.) For each  $x_1 + x_2 + \dots + x_t \geq 1$ ,  $x_1, x_2, \dots, x_t \in \mathbb{R}$ ,  $m \in \mathbb{N}$ ,  $n \geq 1 \in \mathbb{N}$

$$\sum_{r=0}^m \mathbf{A}_{m,r}(n^r * n^r)[x_1 + x_2 + \dots + x_t] = -1 + \sum_{k_1+k_2+\dots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \dots, k_t} \prod_{\ell=1}^t x_\ell^{k_\ell}$$

For example, given  $m = 1$  the Corollary 6.4 gives

$$\begin{aligned} & \sum_{r=0}^1 \mathbf{A}_{1,r}(n^r * n^r)[x + y + z] \\ &= -1 + x + y + z - (-1 + x + y + z)(x + y + z)(-1 - 3x - 3y - 3z + 2(x + y + z)) \\ &= -1 + x^3 + 3x^2y + 3xy^2 + y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 + 3yz^2 + z^3. \end{aligned}$$

Above example could be verified using the commands

- `BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y + z]`
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y + z]]`

defined in Mathematica package at [\[Kol20\]](#).

## 7. DERIVATION OF COEFFICIENT $\mathbf{A}_{m,r}$

By Lemma 4.1 for every  $m \in \mathbb{N}$ ,  $n \in \mathbb{R}$

$$n^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=0}^{n-1} k^r (n-k)^r \quad (7.1)$$

The  $\mathbf{A}_{m,r}$  might be evaluated using binomial expansion of  $\sum_{k=0}^{n-1} k^r (n-k)^r$

$$\sum_{k=0}^{n-1} k^r (n-k)^r = \sum_{k=0}^{n-1} k^r \sum_{j=0}^r (-1)^j \binom{r}{j} n^{r-j} k^j = \sum_{j=0}^r (-1)^j \binom{r}{j} n^{r-j} \sum_{k=0}^{n-1} k^{r+j}$$

Using Faulhaber's formula  $\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j}$  we get

$$\begin{aligned} \sum_{k=0}^{n-1} k^r (n-k)^r &= \sum_{j=0}^r \binom{r}{j} n^{r-j} \frac{(-1)^j}{r+j+1} \left[ \sum_s \binom{r+j+1}{s} B_s n^{r+j+1-s} - B_{r+j+1} \right] \\ &= \sum_{j,s} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s} B_s n^{2r+1-s} - \sum_j \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j} \\ &= \sum_s \underbrace{\sum_j \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s}}_{S(r)} B_s n^{2r+1-s} \\ &\quad - \sum_j \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j} \end{aligned} \quad (7.2)$$

where  $B_s$  are Bernoulli numbers and  $B_1 = \frac{1}{2}$ . Now, we notice that

$$\sum_j \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s} = \begin{cases} \frac{1}{(2r+1) \binom{2r}{r}}, & \text{if } s = 0; \\ \frac{(-1)^r}{s} \binom{r}{2r-s+1}, & \text{if } s > 0. \end{cases}$$

In particular, the last sum is zero for  $0 < s \leq r$ . Therefore, expression (7.2) takes the form

$$\begin{aligned} \sum_{k=0}^{n-1} k^r (n-k)^r &= \frac{1}{(2r+1) \binom{2r}{r}} n^{2r+1} + \underbrace{\sum_{s \geq 1} \frac{(-1)^r}{s} \binom{r}{2r-s+1} B_s n^{2r+1-s}}_{(\star)} \\ &\quad - \underbrace{\sum_j \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}}_{(\diamond)} \end{aligned}$$

Hence, introducing  $\ell = 2r+1-s$  to  $(\star)$  and  $\ell = r-j$  to  $(\diamond)$ , we get

$$\begin{aligned} \sum_{k=0}^{n-1} k^r (n-k)^r &= \frac{1}{(2r+1) \binom{2r}{r}} n^{2r+1} + \sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \\ &\quad - \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \\ \sum_{k=0}^{n-1} k^r (n-k)^r &= \frac{1}{(2r+1) \binom{2r}{r}} n^{2r+1} + (-1)^r \sum_{\ell} \frac{1}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \\ &\quad - \frac{1}{(-1)^r} \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \\ &= \frac{1}{(2r+1) \binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell}^r \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned}$$

Using the definition (7.1) of  $\mathbf{A}_{m,r}$ , we obtain the following identity for polynomials in  $n$

$$\sum_{r=0}^m \mathbf{A}_{m,r} \frac{1}{(2r+1) \binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^m \sum_{\text{odd } \ell}^r \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \quad (7.3)$$

Taking the coefficient of  $n^{2r+1}$  for  $r = m$  in (7.3) we get  $\mathbf{A}_{m,m} = (2m+1) \binom{2m}{m}$ . Since that odd  $\ell \leq r$  in explicit form is  $2j+1 \leq r$ , it follows that  $j \leq \frac{m-1}{2}$ , where  $j$  is iterator. Therefore, taking the coefficient of  $n^{2j+1}$  for an integer  $j$  in the range  $\frac{m}{2} \leq j \leq m$ , we get  $\mathbf{A}_{m,j} = 0$ . Taking the coefficient of  $n^{2d+1}$  for  $d$  in the range  $m/4 \leq d < m/2$  we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1) \binom{2d}{d}} + 2(2m+1) \binom{2m}{m} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0,$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can express  $\mathbf{A}_{m,r}$  for each integer  $r$  in range  $m/2^{s+1} \leq r < m/2^s$  (iterating consecutively  $s = 1, 2, \dots$ ) via previously determined values of  $\mathbf{A}_{m,d}$  as follows

$$\mathbf{A}_{m,r} = (2r+1) \binom{2r}{r} \sum_{d=2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

## 8. VERIFICATION OF THE RESULTS AND EXAMPLES

To fulfill our study we provide an opportunity to verify its results by means of Wolfram Mathematica language.

**8.1. Mathematica commands.** Proceeding to the repository [Kol20] reader is able to find there a folder named `mathematica` that contains the files

- `OnTheBinomialTheoremAndDiscreteConvolution.m` is a package file with definitions
- `OnTheBinomialTheoremAndDiscreteConvolution.nb` is a notebook file with examples.

The following commands may be used to reproduce the results of this manuscript:

- `A[m, r]` returns the real coefficient  $\mathbf{A}_{m,r}$  defined by (1.1).
- `PrintTriangleOfA[rows]` prints the table of coefficients  $\mathbf{A}_{m,r}$ .  
Command `PrintTriangleOfA[7]` reproduces the table (1).
- `PolynomialL[m, n, k]` returns the polynomial  $\mathbf{L}_m(n, k)$  defined by (1.5).
- `PolynomialP[m, x, b]` returns the polynomial  $\mathbf{P}_b^m(x)$  defined by (1.2).
- `Expand[PolynomialP[m, x + y, x + y]]` verifies the Lemma 4.1.
- `PolynomialH[m, t, j]` returns the polynomial  $\mathbf{H}_{m,t}(j)$  defined by (1.3).
- `PolynomialX[m, t, k]` returns the polynomial  $\mathbf{X}_{m,t}(k)$  defined by (1.4).
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[m, x + y]]` verifies the Corollary 6.1.
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[m, x + y]]` verifies the Corollary 6.2.
- `DiscreteConvolutionPowerIdentityParametricTest[m, x, a]` verifies an equation (6.1). Usage `Column[Table[DiscreteConvolutionPowerIdentityParametricTest[1, x, 1], x, 3, 20], Left]`.
- `DiscreteConvolutionPowerIdentityStrictParametricTest[m, x, a]` verifies an equation (6.2). Usage `Column[Table[DiscreteConvolutionPowerIdentityStrictParametricTest[x, 1], x, 3, 20], Left]`.
- `PolynomialIdentityInvolvingX[m, x, b]` validates an identity at (3.1)

$$\mathbf{P}_b^m(x) = \sum_{r=0}^m (-1)^{m-r} \mathbf{X}_{m,r}(b) \cdot x^r$$

- `PolynomialIdentityInvolvingH[m, n, b]` validates an identity at (3.1).

$$\mathbf{P}_b^m(x) = \sum_{r=0}^m \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot b^\ell \cdot x^r$$

8.2. **Examples.** For example, given  $m = 1$  we have the following values of  $\mathbf{L}_1(x, k)$

$x/k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

**Table 3.** Values of  $\mathbf{L}_1(x, k)$ . See OEIS entry: [A300656](#).

Table 3 can be reproduced using Mathematica command

`PrintTriangleOfPolynomialL[1, 7]`

defined in the [\[Kol20\]](#). From Table 3 it is seen that

$$\mathbf{P}_0^1(0) = 0 = 0^3$$

$$\mathbf{P}_1^1(1) = 1 = 1^3$$

$$\mathbf{P}_2^1(2) = 1 + 7 = 2^3$$

$$\mathbf{P}_3^1(3) = 1 + 13 + 13 = 3^3$$

$$\mathbf{P}_4^1(4) = 1 + 19 + 25 + 19 = 4^3$$

$$\mathbf{P}_5^1(5) = 1 + 25 + 37 + 37 + 25 = 5^3$$

Another case, given  $m = 2$  we have the following values of  $\mathbf{L}_2(x, k)$

$x/k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

**Table 4.** Values of  $\mathbf{L}_2(x, k)$ . See OEIS entry: [A300656](#).



Table 4 can be reproduced using Mathematica command

`PrintTriangleOfPolynomialL[2, 7]`

defined in the [Kol20]. Again, an odd-power identity 4.1 holds

$$\mathbf{P}_0^2(0) = 0 = 0^5$$

$$\mathbf{P}_1^2(1) = 1 = 1^5$$

$$\mathbf{P}_2^2(2) = 1 + 31 = 2^5$$

$$\mathbf{P}_3^2(3) = 1 + 121 + 121 = 3^5$$

$$\mathbf{P}_4^2(4) = 1 + 271 + 481 + 271 = 4^5$$

$$\mathbf{P}_5^2(5) = 1 + 481 + 1081 + 1081 + 481 = 5^5$$

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## 10. CONCLUSION

In this manuscript we have shown that Binomial theorem is partial case of polynomial  $\mathbf{P}_b^m(x)$ . Furthermore, by means of  $\mathbf{P}_b^m(x)$  it is shown a relation between Binomial theorem and discrete convolution of polynomials.

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