ON THE LINK BETWEEN BINOMIAL THEOREM AND DISCRETE CONVOLUTION

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Abstract. Let $\mathbf{P}_b^m(x)$ be a $2m+1$-degree integer-valued polynomial in $b, x \in \mathbb{R}$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r,$$

where $\mathbf{A}_{m,r}$ is a real coefficient. In this manuscript we establish a relation between Binomial theorem and polynomial $\mathbf{P}_b^m(x)$. Furthermore, a relationship between Binomial theorem and discrete convolution in terms of polynomials is provided.

Contents

1. Definitions, notations and conventions

We now set the following notation, which remains fixed for the remainder of this paper:

- $\mathbf{A}_{m,r},\ m \in \mathbb{N}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \le r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{1.1}$$

  where B_t are Bernoulli numbers [Wei]. It is assumed that $B_1 = \frac{1}{2}$.

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- $\mathbf{P}^m_b(x),\ m\in\mathbb{N}$ is a $2m+1$-degree integer-valued polynomial in $b,x\in\mathbb{R}$

$$\mathbf{P}^m_b(x) := \sum_{k=0}^{b-1}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r \tag{1.2}$$

- $\mathbf{H}_{m,t}(b),\ m,t,b\in\mathbb{N}$ is a polynomial defined as

$$\mathbf{H}_{m,t}(b) := \sum_{j=t}^{m}\binom{j}{t}\mathbf{A}_{m,j}\frac{(-1)^j}{2j-t+1}\binom{2j-t+1}{b}B_{2j-t+1-b} \tag{1.3}$$

- $\mathbf{X}_{m,t}(j),\ m,t\in\mathbb{N}$ is polynomial of degree $2m+1-t$ in $j\in\mathbb{R}$

$$\mathbf{X}_{m,t}(j) := (-1)^m\sum_{k=1}^{2m+1-t}\mathbf{H}_{m,t}(k)\cdot j^k \tag{1.4}$$

- $\mathbf{L}_m(x,k),\ m\in\mathbb{N}$ is $2m$ degree polynomial in $x,k\in\mathbb{R}$

$$\mathbf{L}_m(x,k) := \sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r \tag{1.5}$$

- $(f*f)[n]$ is discrete convolution [BDM11] of function f defined over set of integers $\mathbb{Z}$

$$(f*f)[n] = \sum_k f(k)f(n-k)$$

2. Introduction and main results

The polynomial $\mathbf{P}^m_b(x),\ m\in\mathbb{N}$ is $2m+1$-degree integer-valued polynomial in $x,b\in\mathbb{R}$.

$$\mathbf{P}^m_b(x) = \sum_{k=0}^{b-1}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r,$$

where $\mathbf{A}_{m,r}$ is real coefficient. By means of Lemma 4.1, the polynomial $\mathbf{P}^m_b(x)$ has the following relation with Binomial theorem [AS72]

$$\mathbf{P}^m_{x+y}(x+y) = \sum_{r=0}^{2m+1}\binom{2m+1}{r}x^{2m+1-r}y^r.$$

From the other hand, polynomial $\mathbf{P}^m_b(x)$ might be expressed in terms of discrete convolution of polynomial $n^j,\ j\in\mathbb{N}$

$$\mathbf{P}^m_{x+1}(x) = \sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x], \quad n\geq 0.$$

Therefore, it is easy to notice the following identities in terms of Binomial theorem and discrete convolution, see Corollaries 6.1, 6.2

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x+y] = 1+\sum_{r=0}^{2m+1}\binom{2m+1}{r}x^{2m+1-r}y^r, \quad n\geq 0.$$

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x+y] = -1+\sum_{r=0}^{2m+1}\binom{2m+1}{r}x^{2m+1-r}y^r, \quad n\geq 1.$$

Also, the following generalizations for multinomial case are discussed, see Corollaries 6.3, 6.4

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x_1 + x_2 + \cdots + x_t] = 1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}, \quad n \geq 0.$$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x_1 + x_2 + \cdots + x_t] = -1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}, \quad n \geq 1.$$

A few polynomial identities are straightforward as well, Theorems 5.3, 5.5

$$x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{x-1} k^r (x-k)^r,$$

$$x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{x} k^r (x-k)^r.$$

3. Polynomial $\mathbf{P}_b^m(x)$ and its properties

We continue our mathematical journey from short overview of polynomial $\mathbf{L}_m(x,k)$ that is essential part of polynomial $\mathbf{P}_b^m(x)$ since that $\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \mathbf{L}_m(x,k)$. Polynomial $\mathbf{L}_m(x,k)$, $m \in \mathbb{N}$ is polynomial of degree $2m$ in $x, k \in \mathbb{R}$, see definition (1.5). In explicit form the polynomial $\mathbf{L}_m(x,k)$ is as follows

$$\mathbf{L}_m(x,k) = \mathbf{A}_{m,m} k^m (x-k)^m + \mathbf{A}_{m,m-1} k^{m-1} (x-k)^{m-1} + \cdots + \mathbf{A}_{m,0},$$

where $\mathbf{A}_{m,r}$ are real coefficients by definition (1.1). Coefficients $\mathbf{A}_{m,r}$ are nonzero only for r within the interval $r \in \{m\} \cup \left[0, \frac{m-1}{2}\right]$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$.

Thus, the polynomial $\mathbf{L}_m(x,k)$ may also be written as

$$\mathbf{L}_m(x,k) = \mathbf{A}_{m,m} k^m (x-k)^m + \sum_{r=0}^{\frac{m-1}{2}} \mathbf{A}_{m,r} k^r (x-k)^r$$

For example, the polynomials $\mathbf{L}_m(x,k)$ for $0 \le m \le 3$ are

$$\begin{aligned}
\mathbf{L}_0(x,k) &= 1,\\
\mathbf{L}_1(x,k) &= 6k(x-k)+1 = -6k^2+6kx+1,\\
\mathbf{L}_2(x,k) &= 30k^2(x-k)^2+1 = 30k^4-60k^3x+30k^2x^2+1,\\
\mathbf{L}_3(x,k) &= 140k^3(x-k)^3-14k(x-k)+1\\
&= -140k^6+420k^5x-420k^4x^2+140k^3x^3+14k^2-14kx+1
\end{aligned}$$

It is worth to notice that $\mathbf{L}_m(x,k)$ is symmetrical over x

Property 3.1. *For every $x, k \in \mathbb{R}$*

$$\mathbf{L}_m(x,k) = \mathbf{L}_m(x, x-k)$$

This might be seen in the following table

x/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $\mathbf{L}_1(x,k)$.

Next we discuss the polynomial $\mathbf{P}_b^m(x)$. By definition (1.2), in extended form, the polynomial $\mathbf{P}_b^m(x)$ is

$$\begin{aligned}
\mathbf{P}_b^m(x) &= \sum_{k=0}^{b-1}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r = \sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=0}^{b-1}k^r(x-k)^r\\
&= \sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=0}^{b-1}k^r\sum_{j=0}^{r}(-1)^j\binom{r}{j}x^{r-j}k^j\\
&= \sum_{r=0}^{m}\sum_{j=0}^{r}(-1)^j x^{r-j}\binom{r}{j}\mathbf{A}_{m,r}\sum_{k=0}^{b-1}k^{r+j}\\
&= \sum_{r=0}^{m}\sum_{j=0}^{r}x^{r-j}\binom{r}{j}\mathbf{A}_{m,r}\frac{(-1)^j}{r+j+1}\sum_{s=0}^{r+j}\binom{r+j+1}{s}B_s(b-1)^{r+j-s+1}
\end{aligned}$$

However, by symmetry (3.1) the $\mathbf{P}_b^m(x)$ may also be written in the form

$$\begin{aligned}\mathbf{P}_b^m(x) &= \sum_{k=1}^{b}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r = \sum_{k=1}^{b}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r\sum_{t=0}^{r}(-1)^{r-t}x^t\binom{r}{t}k^{r-t}\\ &= \sum_{t=0}^{m}x^t\underbrace{\sum_{k=1}^{b}\sum_{r=t}^{m}(-1)^{r-t}\binom{r}{t}\mathbf{A}_{m,r}k^{2r-t}}_{(-1)^{m-t}\mathbf{X}_{m,t}(b)}\end{aligned}$$

From this formula it may be not immediately clear why $\mathbf{X}_{m,t}(b)$ represent polynomials in b. However, this can be seen if we change the summation order and use Faulhaber's formula $\sum_{k=1}^{n}k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$ to obtain

$$\mathbf{X}_{m,t}(b) = (-1)^m\sum_{r=t}^{m}\binom{r}{t}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-t+1}\sum_{\ell=0}^{2r-t}\binom{2r-t+1}{\ell}B_\ell b^{2r-t+1-\ell}$$

Introducing $k = 2r-t+1-\ell$ we further get the formula

$$\mathbf{X}_{m,t}(b) = (-1)^m\sum_{k=1}^{2m-t+1}b^k\underbrace{\sum_{r=t}^{m}\binom{r}{t}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-t+1}\binom{2r-t+1}{k}B_{2r-t+1-k}}_{\mathbf{H}_{m,t}(k)}$$

Polynomials $\mathbf{X}_{3,t}(j),\ 0\leq t\leq 3$ are

$$\begin{aligned}\mathbf{X}_{3,0}(j) &= 7j^2-28j^3+70j^5-70j^6+20j^7,\\ \mathbf{X}_{3,1}(j) &= 7j-42j^2+175j^4-210j^5+70j^6,\\ \mathbf{X}_{3,2}(j) &= -14j+140j^3-210j^4+84j^5,\\ \mathbf{X}_{3,3}(j) &= 35j^2-70j^3+35j^4\end{aligned}$$

Polynomials $\mathbf{H}_{3,t}(k)$ are defined by (1.3) and examples for $m=3,\ 0\leq t\leq 3$ are

$$\begin{aligned}\mathbf{H}_{3,0}(k) &= B_{1-k}\binom{1}{k}+\frac{14}{3}B_{3-k}\binom{3}{k}-20B_{7-k}\binom{7}{k},\\ \mathbf{H}_{3,1}(k) &= 7B_{2-k}\binom{2}{k}-70B_{6-k}\binom{6}{k},\\ \mathbf{H}_{3,2}(k) &= -84B_{5-k}\binom{5}{k},\\ \mathbf{H}_{3,3}(k) &= -35B_{4-k}\binom{4}{k}\end{aligned}$$

It gives us an opportunity to review the $\mathbf{P}_b^m(x)$ from different prospective, for instance

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m}(-1)^{m-r}\mathbf{X}_{m,r}(b)\cdot x^r = \sum_{r=0}^{m}\sum_{\ell=1}^{2m-r+1}(-1)^{2m-r}\mathbf{H}_{m,r}(\ell)\cdot b^\ell\cdot x^r \tag{3.1}$$

Equation (3.1) clearly states why $\mathbf{P}_b^m(x)$ is polynomial in x, b. For example,

$$\begin{aligned}
\mathbf{P}_b^0(x) &= b,\\
\mathbf{P}_b^1(x) &= 3b^2 - 2b^3 - 3bx + 3b^2x,\\
\mathbf{P}_b^2(x) &= 10b^3 - 15b^4 + 6b^5\\
&- 15b^2x + 30b^3x - 15b^4x\\
&+ 5bx^2 - 15b^2x^2 + 10b^3x^2,\\
\mathbf{P}_b^3(x) &= -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7\\
&+ 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x\\
&+ 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2\\
&+ 35b^2x^3 - 70b^3x^3 + 35b^4x^3
\end{aligned}$$

The following property also holds for $\mathbf{P}_b^m(x)$

Property 3.2. *For every $m \in \mathbb{N},\ x, b \in \mathbb{R}$*

$$\mathbf{P}_{b+1}^m(x) = \mathbf{P}_b^m(x) + \mathbf{L}_m(x, b)$$

4. Polynomial $\mathbf{P}_b^m(x)$ in terms of Binomial theorem

Lemma 4.1. *For every $m \in \mathbb{N},\ x, y \in \mathbb{R}$*

$$\mathbf{P}_{x+y}^m(x+y) = \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

By Lemma 4.1 and equation (3.1) the following polynomial identities straightforward

$$x^{2m+1} = \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot x^{\ell+r} = \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x) \cdot x^r$$

For instance,

$$\mathbf{P}_{x+y}^2(x+y) = (x+y)(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4).$$

In addition, the following identities hold

$$\begin{aligned}
(x+y)^{2m+1} &= \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x+y)^{\ell+r}\\
&= \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x+y) \cdot (x+y)^r
\end{aligned}$$

Obviously, Multinomial expansion of t-fold sum $(x_1 + x_2 + \cdots + x_t)^{2m+1}$ can be reached by $\mathbf{P}_b^m(x_1 + x_2 + \cdots + x_t)$ as well

Corollary 4.2. *For all $x_1, x_2, \ldots, x_t \in \mathbb{R},\ m \in \mathbb{N}$*

$$\mathbf{P}_{x_1+x_2+\cdots+x_t}^m(x_1 + x_2 + \cdots + x_t) = \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{s=1}^{t} x_t^{k_s}$$

Moreover, the following multinomial identities hold

$$\begin{aligned}(x_1+x_2+\cdots+x_t)^{2m+1} &= \sum_{r=0}^{m}\sum_{\ell=1}^{2m-r+1}(-1)^{2m-r}\mathbf{H}_{m,r}(\ell)\cdot(x_1+x_2+\cdots+x_t)^{\ell+r} \\ &= \sum_{r=0}^{m}(-1)^{m-r}\mathbf{X}_{m,r}(x_1+x_2+\cdots+x_t)\cdot(x_1+x_2+\cdots+x_t)^{r}\end{aligned}$$

5. Polynomial $\mathbf{P}_b^m(x)$ in terms of Discrete convolution

In this section we discuss the relation between $\mathbf{P}_b^m(x)$ and discrete convolution of polynomials. To show that $\mathbf{P}_b^m(x)$ involves the discrete convolution of polynomial n^r let's remind the definition of $\mathbf{P}_b^m(x)$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r = \sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=0}^{b-1}k^r(x-k)^r$$

A discrete convolution of defined over set of integers $\mathbb{Z}$ function f is

$$(f*f)[n] = \sum_{k} f(k)f(n-k)$$

General formula of discrete convolution for polynomials $f(n) = n^j, \ n \geq a \in \mathbb{R}$ may be derived immediately

$$\begin{aligned}(n^j * n^j)[x] &= \sum_k k^j(x-k)^j[k\geq a][x-k\geq a] \\ &= \sum_k k^j(x-k)^j[k\geq a][k\leq x-a] \\ &= \sum_k k^j(x-k)^j[a\leq k\leq x-a] \\ &= \sum_{k=a}^{x-a} k^j(x-k)^j,\end{aligned}$$

where $[a\leq k\leq x-a]$ is Iverson's bracket [Ive62].

Lemma 5.1. *For every $n\in\mathbb{N},\ x\in\mathbb{R}$*

$$(n^r * n^r)[x] = \sum_{k=0}^{x} k^r(x-k)^r, \quad n\geq 0.$$

Thus, the corollary follows

Corollary 5.2. *By Lemma 5.1 the polynomial $\mathbf{P}_b^m(n)$ might be expressed in terms of discrete convolution as follows*

$$\mathbf{P}_{x+1}^m(x) = \sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r * n^r)[x], \quad n\geq 0.$$

Therefore, another polynomial identity follows

Theorem 5.3. *By Lemma 4.1, Corollary 5.2 and property 3.2, for every $m \in \mathbb{N},\ x \in \mathbb{R}$*

$$x^{2m+1} = -1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n \geq 0.$$

Now we notice the following identity in terms of polynomial $\mathbf{P}_b^m(x)$ and discrete convolution $(n^j * n^j)[x]$

Proposition 5.4. *For every $m \in \mathbb{N},\ x \in \mathbb{R}$*

$$\begin{aligned}\mathbf{P}_x^m(x) &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left(0^r x^r + \sum_{k=1}^{x-1} k^r (x-k)^r \right) \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} 0^r x^r + \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x] \\ &= 1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n \geq 1.\end{aligned}$$

Since that for all r in $\mathbf{A}_{m,r} 0^r x^r$ we have

$$\mathbf{A}_{m,r} 0^r x^r = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{if } r > 0 \end{cases}$$

Above is true because $\mathbf{A}_{m,0} = 1$ for every $m \in \mathbb{N}$, and $x^0 = 1$ for every x, [GKP94]. Hence, the following identity between $\mathbf{P}_b^m(x)$ and discrete convolution $(n^j * n^j)[x]$ holds

Theorem 5.5. *By Lemma 4.1 and Proposition 5.4, for every $m \in \mathbb{N},\ x \in \mathbb{R}$*

$$x^{2m+1} = 1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n \geq 1.$$

Corollary 5.6. *By Theorem 5.5, for all $m \in \mathbb{N}$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Corollary 5.6 holds since that convolution $(n^j * n^j)[x] = 1,\ n \geq 1$ for each r when $x = 2$.

6. Relation between Binomial theorem and Discrete convolution

Corollary 6.1. *(Generalization of Theorem 5.3 for Binomials.) For every $m \in \mathbb{N},\ x, y \in \mathbb{R}$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x+y] = 1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r, \quad n \geq 1.$$

For example, given $m = 0, 1, 2$ the Corollary 6.1 gives

$$\sum_{r=0}^{0} \mathbf{A}_{0,r}(n^r * n^r)[x+y] = 1 + x + y$$

$$\begin{aligned}\sum_{r=0}^{1} \mathbf{A}_{1,r}(n^r * n^r)[x+y] &= 1 + x + y - (x+y)(1+x+y)(1-3x-3y+2(x+y)) \\ &= x^3 + 3x^2y + 3xy^2 + y^3 + 1\end{aligned}$$

$$\begin{aligned}\sum_{r=0}^{2} \mathbf{A}_{2,r}(n^r * n^r)[x+y] &= 1 + x + y + (x+y)(1+x+y)\big(-1 + x + 5x^2 + y + 10xy + 5y^2 \\ &- 15x(x+y) + 10x^2(x+y) - 15y(x+y) + 20xy(x+y) \\ &+ 10y^2(x+y) + 9(x+y)^2 - 15x(x+y)^2 \\ &-15y(x+y)^2 + 6(x+y)^3\big) \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 + 1\end{aligned}$$

To get above examples use `ConvPowerIdentity[m, x + y]` command in Mathematica console [Kol20].

Corollary 6.2. *(Generalization of Theorem 5.5 for Binomials.) For every* $m \in \mathbb{N},\ x, y \in \mathbb{R}$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x+y] = -1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r, \quad n \geq 0.$$

For example, given $m = 0, 1$ the Corollary 6.2 gives

$$\sum_{r=0}^{0} \mathbf{A}_{0,r}(n^r * n^r)[x+y] = x + y - 1$$

$$\begin{aligned}\sum_{r=0}^{1} \mathbf{A}_{1,r}(n^r * n^r)[x+y] &= -1 + x + y - (-1+x+y)(x+y)(-1-3x-3y+2(x+y)) \\ &= x^3 + 3x^2y + 3xy^2 + y^3 - 1\end{aligned}$$

To get above examples use `ConvPowerIdentityStrict[m, x + y]` command in Mathematica console [Kol20]. From other prospective, let be a function $f_r(t,k) = (t-k)^r,\ t \geq k$, then following identity holds

$$(x-2a)^{2m+1} + 1 = \sum_{r=0}^{m} \mathbf{A}_{m,r}(f_r(t,k) * f_r(t,k))[x] \tag{6.1}$$

Let be a function $g_r(t,k) = (t-k)^r,\ t > k$, then

$$(x-2a)^{2m+1} - 1 = \sum_{r=0}^{m} \mathbf{A}_{m,r}(g_r(t,k) * g_r(t,k))[x] \tag{6.2}$$

6.1. **Generalization for Multinomials.** In this subsection we generalize Theorems 5.3, 5.5 for multinomial cases.

Corollary 6.3. *(Generalization of Theorem 5.3 for Multinomials.) For every $x_1, x_2, \dots, x_t \in \mathbb{R},\ m \in \mathbb{N},\ n \geq 1 \in \mathbb{N}$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x_1 + x_2 + \cdots + x_t] = 1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \dots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

For instance, given $m = 1$ the Corollary 6.3 gives

$$\begin{aligned}
&\sum_{r=0}^{1} \mathbf{A}_{1,r} (n^r * n^r)[x + y + z] \\
&= 1 + x + y + z - (x + y + z)(1 + x + y + z)(1 - 3x - 3y - 3z + 2(x + y + z)) \\
&= 1 + x^3 + 3x^2 y + 3xy^2 + y^3 + 3x^2 z + 6xyz + 3y^2 z + 3xz^2 + 3yz^2 + z^3,
\end{aligned}$$

it might be verified using `ConvPowerIdentity[m, x + y + z]` command in Mathematica console [Kol20].

Corollary 6.4. *(Generalization of Theorem 5.5 for Multinomials.) For each $x_1 + x_2 + \cdots + x_t \geq 1,\ x_1, x_2, \dots, x_t \in \mathbb{R},\ m \in \mathbb{N},\ n \geq 1 \in \mathbb{N}$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x_1 + x_2 + \cdots + x_t] = -1 + \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \dots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

For example, given $m = 1$ the Corollary 6.4 gives

$$\begin{aligned}
&\sum_{r=0}^{1} \mathbf{A}_{1,r} (n^r * n^r)[x + y + z] \\
&= x + y + z - 1 - (x + y + z - 1)(x + y + z)(2(x + y + z) - 1 - 3x - 3y - 3z) \\
&= x^3 + 3x^2 y + 3xy^2 + y^3 + 3x^2 z + 6xyz + 3y^2 z + 3xz^2 + 3yz^2 + z^3 - 1,
\end{aligned}$$

it might be verified using `ConvPowerIdentityStrict[m, x + y + z]` command in Mathematica console [Kol20].

7. Derivation of coefficient $\mathbf{A}_{m,r}$

By Lemma 4.1 for every $m \in \mathbb{N},\ n \in \mathbb{R}$

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{n-1} k^r (n - k)^r \tag{7.1}$$

The $\mathbf{A}_{m,r}$ might be evaluated using binomial expansion of $\sum_{k=0}^{n-1} k^r (n-k)^r$

$$\sum_{k=0}^{n-1} k^r (n - k)^r = \sum_{k=0}^{n-1} k^r \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} k^j = \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} \sum_{k=0}^{n-1} k^{r+j}$$

Using Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$ we get

$$\begin{aligned}
\sum_{k=0}^{n-1} k^r (n-k)^r &= \sum_{j=0}^{r} \binom{r}{j} n^{r-j} \frac{(-1)^j}{r+j+1} \left[\sum_{s} \binom{r+j+1}{s} B_s n^{r+j+1-s} - B_{r+j+1} \right] \\
&= \sum_{j,s} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s} B_s n^{2r+1-s} - \sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j} \\
&= \sum_{s} \underbrace{\sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s}}_{S(r)} B_s n^{2r+1-s} \\
&\quad - \sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}
\end{aligned} \tag{7.2}$$

where B_s are Bernoulli numbers and $B_1 = \frac{1}{2}$. Now, we notice that

$$\sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } s = 0; \\ \frac{(-1)^r}{s} \binom{r}{2r-s+1}, & \text{if } s > 0. \end{cases}$$

In particular, the last sum is zero for $0 < s \leq r$. Therefore, expression (7.2) takes the form

$$\begin{aligned}
\sum_{k=0}^{n-1} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} &+ \underbrace{\sum_{s \geq 1} \frac{(-1)^r}{s} \binom{r}{2r-s+1} B_s n^{2r+1-s}}_{(\star)} \\
&- \underbrace{\sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}}_{(\diamond)}
\end{aligned}$$

Hence, introducing $\ell = 2r+1-s$ to $(\star)$ and $\ell = r-j$ to $(\diamond)$, we get

$$\begin{aligned}
\sum_{k=0}^{n-1} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} &+ \sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \\
&- \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n-1} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + (-1)^r \sum_{\ell} \frac{1}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \\
&\quad - \frac{1}{(-1)^r} \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \\
&= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell}^{r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}
\end{aligned}$$

Using the definition (7.1) of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{r=0}^{m} \sum_{\text{odd } \ell}^{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \tag{7.3}$$

Taking the coefficient of n^{2r+1} for $r = m$ in (7.3) we get $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$. Since that odd $\ell \le r$ in explicit form is $2j+1 \le r$, it follows that $j \le \frac{m-1}{2}$, where j is iterator. Therefore, taking the coefficient of n^{2j+1} for an integer j in the range $\frac{m}{2} \le j \le m$, we get $\mathbf{A}_{m,j} = 0$. Taking the coefficient of n^{2d+1} for d in the range $m/4 \le d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d} B_{2m-2d} = 0,$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \le r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

8. Verification of the results and examples

To fulfill our study we provide an opportunity to verify its results by means of Wolfram Mathematica language.

8.1. Mathematica commands. Proceeding to the repository [Kol20] the following commands to verify the formulas:

- `A[m,r]` gives $\mathbf{A}_{m,r}$, definition (1.1).
- `L[m,n,k]` gives $\mathbf{L}_m(n,k)$, definition (1.5).
- `P[m,x,b]` gives $\mathbf{P}_x^m(b)$, definition (1.2).
- `P[m,x+y,x+y]` verifies Lemma 4.1.
- `H[m,t,j]` gives $\mathbf{H}_{m,t}(j)$.
- `X[m,t,k]` gives $\mathbf{X}_{m,t}(k)$.
- `ConvPowerIdentityStrict[m, x+y]` verifies the Corollary 6.2.
- `ConvPowerIdentity[m, x+y]` verifies the Corollary 6.1.
- `ConvPowerIdentityParametric[m, x, a]`, m, x, a are constants - verifies equation (6.1).
- `ConvPowerIdentityStrictParametric[m, x, a]`, m, x, a are constants - verifies equation (6.2).

8.2. Examples. For example, given $m = 1$ we have the following values of $\mathbf{L}_1(x, k)$

x/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 3. Values of $\mathbf{L}_1(x,k)$.

From Table 3 it is seen that

$$\begin{aligned}
\mathbf{P}^1_0(0) &= 0 = 0^3\\
\mathbf{P}^1_1(1) &= 1 = 1^3\\
\mathbf{P}^1_2(2) &= 1 + 7 = 2^3\\
\mathbf{P}^1_3(3) &= 1 + 13 + 13 = 3^3\\
\mathbf{P}^1_4(4) &= 1 + 19 + 25 + 19 = 4^3\\
\mathbf{P}^1_5(5) &= 1 + 25 + 37 + 37 + 25 = 5^3
\end{aligned}$$

Another case, given $m = 2$ we have the following values of $\mathbf{L}_2(x,k)$

x/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 4. Values of $\mathbf{L}_2(x,k)$.

Again, an odd-power identity 4.1 holds

$$\begin{aligned}
\mathbf{P}^2_0(0) &= 0 = 0^5\\
\mathbf{P}^2_1(1) &= 1 = 1^5\\
\mathbf{P}^2_2(2) &= 1 + 31 = 2^5\\
\mathbf{P}^2_3(3) &= 1 + 121 + 121 = 3^5\\
\mathbf{P}^2_4(4) &= 1 + 271 + 481 + 271 = 4^5\\
\mathbf{P}^2_5(5) &= 1 + 481 + 1081 + 1081 + 481 = 5^5
\end{aligned}$$

Tables 3, 4 are entries `A287326`, `A300656` in [Slo64].

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10. Conclusion

In this manuscript we have shown that Binomial theorem is partial case of polynomial $\mathbf{P}_b^m(x)$. Furthermore, by means of $\mathbf{P}_b^m(x)$ it is shown a relation between Binomial theorem and discrete convolution of polynomials.

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