# ON THE LINK BETWEEN BINOMIAL THEOREM AND DISCRETE CONVOLUTION

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Abstract. Let $\mathbf{P}_b^m(x)$ be a $2m+1$-degree integer-valued polynomial in $b, x \in \mathbb{R}$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r,$$

where $\mathbf{A}_{m,r}$ is a real coefficient. In this manuscript we establish a relation between Binomial theorem and polynomial $\mathbf{P}_b^m(x)$. Furthermore, a relationship between Binomial theorem and discrete convolution in terms of polynomials is provided.

## Contents



## 1. Definitions, notations and conventions

We now set the following notation, which remains fixed for the remainder of this paper:

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- $\mathbf{A}_{m,r},\ m \in \mathbb{N}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{1.1}$$

where $B_t$ are Bernoulli numbers [Wei]. It is assumed that $B_1 = \frac{1}{2}$.

- $\mathbf{P}_b^m(x),\ m \in \mathbb{N}$ is a $2m+1$-degree integer-valued polynomial in $b, x \in \mathbb{R}$

$$\mathbf{P}_b^m(x) := \sum_{k=0}^{b-1}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.2}$$

- $\mathbf{H}_{m,t}(b),\ m, t, b \in \mathbb{N}$ is a polynomial defined as

$$\mathbf{H}_{m,t}(b) := \sum_{j=t}^{m} \binom{j}{t} \mathbf{A}_{m,j} \frac{(-1)^j}{2j-t+1} \binom{2j-t+1}{b} B_{2j-t+1-b} \tag{1.3}$$

- $\mathbf{X}_{m,t}(j),\ m, t \in \mathbb{N}$ is polynomial of degree $2m+1-t$ in $j \in \mathbb{R}$

$$\mathbf{X}_{m,t}(j) := (-1)^m \sum_{k=1}^{2m+1-t} \mathbf{H}_{m,t}(k) \cdot j^k \tag{1.4}$$

- $\mathbf{L}_m(x,k),\ m \in \mathbb{N}$ is $2m$ degree polynomial in $x, k \in \mathbb{R}$

$$\mathbf{L}_m(x,k) := \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.5}$$

- $(f*f)[n]$ is discrete convolution [BDM11] of function $f$ defined over set of integers $\mathbb{Z}$

$$(f*f)[n] = \sum_{k} f(k) f(n-k)$$

## 2. Introduction and main results

The polynomial $\mathbf{P}_b^m(x),\ m \in \mathbb{N}$ is $2m+1$-degree integer-valued polynomial in $x, b \in \mathbb{R}$.

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r,$$

where $\mathbf{A}_{m,r}$ is real coefficient. By means of Lemma 4.1, the polynomial $\mathbf{P}_b^m(x)$ has the following relation with Binomial theorem [AS72]

$$\mathbf{P}_{x+y}^m(x+y) = \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r.$$

From the other hand, polynomial $\mathbf{P}^m_b(x)$ might be expressed in terms of discrete convolution of polynomial $n^j,\ j\in\mathbb{N}$

$$\mathbf{P}^m_{x+1}(x)=\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x],\quad n\geq 0.$$

Therefore, it is easy to notice the following identities in terms of Binomial theorem and discrete convolution, see Corollaries 6.1, 6.2

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x+y]=1+\sum_{r=0}^{2m+1}\binom{2m+1}{r}x^{2m+1-r}y^r,\quad n\geq 0,$$

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x+y]=-1+\sum_{r=0}^{2m+1}\binom{2m+1}{r}x^{2m+1-r}y^r,\quad n> 0.$$

Also, the following generalizations for multinomial case are discussed, see Corollaries 6.3, 6.4

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x_1+x_2+\cdots+x_t]=1+\sum_{k_1+k_2+\cdots+k_t=2m+1}\binom{2m+1}{k_1,k_2,\ldots,k_t}\prod_{\ell=1}^{t}x_\ell^{k_\ell},\quad n\geq 0,$$

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x_1+x_2+\cdots+x_t]=-1+\sum_{k_1+k_2+\cdots+k_t=2m+1}\binom{2m+1}{k_1,k_2,\ldots,k_t}\prod_{\ell=1}^{t}x_\ell^{k_\ell},\quad n> 0.$$

A few polynomial identities are straightforward as well, Theorems 5.3, 5.5

$$x^{2m+1}=\sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=0}^{x-1}k^r(x-k)^r,$$

$$x^{2m+1}=\sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=1}^{x}k^r(x-k)^r.$$

## 3. Polynomial $\mathbf{P}^m_b(x)$ and its properties

We continue our mathematical journey from short overview of polynomial $\mathbf{L}_m(x,k)$ that is essential part of polynomial $\mathbf{P}^m_b(x)$ since that $\mathbf{P}^m_b(x)=\sum_{k=0}^{b-1}\mathbf{L}_m(x,k)$. Polynomial $\mathbf{L}_m(x,k),\ m\in\mathbb{N}$ is polynomial of degree $2m$ in $x,k\in\mathbb{R}$, see definition (1.5). In explicit form the polynomial $\mathbf{L}_m(x,k)$ is as follows

$$\mathbf{L}_m(x,k)=\mathbf{A}_{m,m}k^m(x-k)^m+\mathbf{A}_{m,m-1}k^{m-1}(x-k)^{m-1}+\cdots+\mathbf{A}_{m,0},$$

where $\mathbf{A}_{m,r}$ are real coefficients defined by (1.1). Coefficients $\mathbf{A}_{m,r}$ are nonzero only for $r$ within the interval $r\in\{m\}\cup\left[0,\frac{m-1}{2}\right]$. For example,

| $m/r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 6 | | | | | | |
| 2 | 1 | 0 | 30 | | | | | |
| 3 | 1 | -14 | 0 | 140 | | | | |
| 4 | 1 | -120 | 0 | 0 | 630 | | | |
| 5 | 1 | -1386 | 660 | 0 | 0 | 2772 | | |
| 6 | 1 | -21840 | 18018 | 0 | 0 | 0 | 12012 | |
| 7 | 1 | -450054 | 491400 | -60060 | 0 | 0 | 0 | 51480 |

**Table 1.** Coefficients $\mathbf{A}_{m,r}$. See the OEIS entries: A302971, A304042.

Thus, the polynomial $\mathbf{L}_m(x,k)$ may also be written as

$$\mathbf{L}_m(x,k) = \mathbf{A}_{m,m} k^m (x-k)^m + \sum_{r=0}^{\frac{m-1}{2}} \mathbf{A}_{m,r} k^r (x-k)^r$$

For example, the polynomials $\mathbf{L}_m(x,k)$ for $0 \leq m \leq 3$ are

$$\begin{aligned}
\mathbf{L}_0(x,k) &= 1, \\
\mathbf{L}_1(x,k) &= 6k(x-k) + 1 = -6k^2 + 6kx + 1, \\
\mathbf{L}_2(x,k) &= 30k^2(x-k)^2 + 1 = 30k^4 - 60k^3x + 30k^2x^2 + 1, \\
\mathbf{L}_3(x,k) &= 140k^3(x-k)^3 - 14k(x-k) + 1 \\
&= -140k^6 + 420k^5x - 420k^4x^2 + 140k^3x^3 + 14k^2 - 14kx + 1
\end{aligned}$$

It is worth to notice that $\mathbf{L}_m(x,k)$ is symmetrical over $x$

**Property 3.1.** *For every $x, k \in \mathbb{R}$*

$$\mathbf{L}_m(x,k) = \mathbf{L}_m(x, x-k)$$

This might be seen in the following table

| $x/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 1 | 7 | 1 | | | | | |
| 3 | 1 | 13 | 13 | 1 | | | | |
| 4 | 1 | 19 | 25 | 19 | 1 | | | |
| 5 | 1 | 25 | 37 | 37 | 25 | 1 | | |
| 6 | 1 | 31 | 49 | 55 | 49 | 31 | 1 | |
| 7 | 1 | 37 | 61 | 73 | 73 | 61 | 37 | 1 |

**Table 2.** Values of $\mathbf{L}_1(x,k)$. See the OEIS entry: A287326.

Next we discuss the polynomial $\mathbf{P}_b^m(x)$. In its extended form, the polynomial $\mathbf{P}_b^m(x)$ is

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r = \sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=0}^{b-1}k^r(x-k)^r$$

By the binomial theorem $(x-y)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^k y^{n-k}$,

$$\begin{aligned}\mathbf{P}_b^m(x) &= \sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=0}^{b-1}k^r\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}x^r k^{r-j}\\ &= \sum_{r=0}^{m}\sum_{j=0}^{r}(-1)^{r-j}x^r\binom{r}{j}\mathbf{A}_{m,r}\sum_{k=0}^{b-1}k^{2r-j}\\ &= \sum_{r=0}^{m}x^r\mathbf{A}_{m,r}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\sum_{k=0}^{b-1}k^{2r-j}\\ &= \sum_{r=0}^{m}(-1)^r x^r\mathbf{A}_{m,r}\left[\sum_{j=0}^{r}\frac{1}{(-1)^j}\binom{r}{j}\sum_{k=0}^{b-1}k^{2r-j}\right]\end{aligned}$$

Given the power sum $S_r(b) = \sum_{k=0}^{b}k^r$ we get

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m}(-1)^r x^r\mathbf{A}_{m,r}\sum_{j=0}^{r}\frac{1}{(-1)^j}\binom{r}{j}S_{2r-j}(b-1)$$

However, by the symmetry (3.1) of $\mathbf{L}_m(x,k)$ the polynomial $\mathbf{P}_b^m(x)$ may also be written in the form

$$\begin{aligned}\mathbf{P}_b^m(x) &= \sum_{k=1}^{b}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r = \sum_{k=1}^{b}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r\sum_{t=0}^{r}(-1)^{r-t}x^t\binom{r}{t}k^{r-t}\\ &= \sum_{t=0}^{m}x^t\underbrace{\sum_{k=1}^{b}\sum_{r=t}^{m}(-1)^{r-t}\binom{r}{t}\mathbf{A}_{m,r}k^{2r-t}}_{(-1)^{m-t}\mathbf{X}_{m,t}(b)}\end{aligned}$$

Note that $\sum_{k=1}^{b}\sum_{r=t}^{m}(-1)^{r-t}\binom{r}{t}\mathbf{A}_{m,r}k^{2r-t}$ is the $(-1)^{m-t}\mathbf{X}_{m,t}(b)$. From this formula it may be not immediately clear why $\mathbf{X}_{m,t}(b)$ represent polynomials in $b$. However, this can be seen if we change the summation order and use Faulhaber's formula $\sum_{k=1}^{n}k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$ to obtain

$$\mathbf{X}_{m,t}(b) = (-1)^m\sum_{r=t}^{m}\binom{r}{t}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-t+1}\sum_{\ell=0}^{2r-t}\binom{2r-t+1}{\ell}B_\ell b^{2r-t+1-\ell}$$

Introducing $k = 2r - t + 1 - \ell$ we further get the formula

$$\mathbf{X}_{m,t}(b) = (-1)^m \sum_{k=1}^{2m-t+1} b^k \underbrace{\sum_{r=t}^{m} \binom{r}{t} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-t+1} \binom{2r-t+1}{k} B_{2r-t+1-k}}_{\mathbf{H}_{m,t}(k)}$$

Polynomials $\mathbf{X}_{3,t}(b),\ 0 \leq t \leq 3$ are

$$\begin{aligned}
\mathbf{X}_{3,0}(j) &= 7b^2 - 28b^3 + 70b^5 - 70b^6 + 20b^7,\\
\mathbf{X}_{3,1}(j) &= 7b - 42b^2 + 175b^4 - 210b^5 + 70b^6,\\
\mathbf{X}_{3,2}(j) &= -14b + 140b^3 - 210b^4 + 84b^5,\\
\mathbf{X}_{3,3}(j) &= 35b^2 - 70b^3 + 35b^4
\end{aligned}$$

Polynomials $\mathbf{H}_{3,t}(k)$ are defined by (1.3) and examples for $m = 3,\ 0 \leq t \leq 3$ are

$$\begin{aligned}
\mathbf{H}_{3,0}(k) &= B_{1-k}\binom{1}{k} + \frac{14}{3} B_{3-k}\binom{3}{k} - 20B_{7-k}\binom{7}{k},\\
\mathbf{H}_{3,1}(k) &= 7B_{2-k}\binom{2}{k} - 70B_{6-k}\binom{6}{k},\\
\mathbf{H}_{3,2}(k) &= -84B_{5-k}\binom{5}{k},\\
\mathbf{H}_{3,3}(k) &= -35B_{4-k}\binom{4}{k}
\end{aligned}$$

It gives us an opportunity to overview the polynomial $\mathbf{P}_b^m(x)$ from the different prospective, for instance

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(b) \cdot x^r = \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot b^\ell \cdot x^r \tag{3.1}$$

Equation (3.1) clearly states why $\mathbf{P}_b^m(x)$ is polynomial in $x, b$. For example,

$$\begin{aligned}
\mathbf{P}_b^0(x) &= b,\\
\mathbf{P}_b^1(x) &= 3b^2 - 2b^3 - 3bx + 3b^2x,\\
\mathbf{P}_b^2(x) &= 10b^3 - 15b^4 + 6b^5\\
&\quad - 15b^2x + 30b^3x - 15b^4x\\
&\quad + 5bx^2 - 15b^2x^2 + 10b^3x^2,\\
\mathbf{P}_b^3(x) &= -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7\\
&\quad + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x\\
&\quad + 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2\\
&\quad + 35b^2x^3 - 70b^3x^3 + 35b^4x^3
\end{aligned}$$

The following property also holds for $\mathbf{P}_b^m(x)$

**Property 3.2.** *For every $m \in \mathbb{N},\ x, b \in \mathbb{R}$*

$$\mathbf{P}_{b+1}^m(x) = \mathbf{P}_b^m(x) + \mathbf{L}_m(x, b)$$

## 4. Polynomial $\mathbf{P}_b^m(x)$ in terms of Binomial theorem

**Lemma 4.1.** *For every $m \in \mathbb{N},\ x, y \in \mathbb{R}$*

$$\mathbf{P}_{x+y}^m(x+y) = \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r$$

By Lemma 4.1 and equation (3.1) the following polynomial identities straightforward

$$x^{2m+1} = \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot x^{\ell+r} = \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x) \cdot x^r$$

For instance,

$$\mathbf{P}_{x+y}^2(x+y) = (x+y)(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4).$$

In addition, the following identities hold

$$\begin{aligned}(x+y)^{2m+1} &= \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x+y)^{\ell+r} \\ &= \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x+y) \cdot (x+y)^r\end{aligned}$$

Obviously, Multinomial expansion of $t$-fold sum $(x_1 + x_2 + \cdots + x_t)^{2m+1}$ can be reached by $\mathbf{P}_b^m(x_1 + x_2 + \cdots + x_t)$ as well

**Corollary 4.2.** *For all $x_1, x_2, \ldots, x_t \in \mathbb{R},\ m \in \mathbb{N}$*

$$\mathbf{P}_{x_1+x_2+\cdots+x_t}^m(x_1 + x_2 + \cdots + x_t) = \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{s=1}^{t} x_t^{k_s}$$

Moreover, the following multinomial identities hold

$$\begin{aligned}(x_1 + x_2 + \cdots + x_t)^{2m+1} &= \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x_1 + x_2 + \cdots + x_t)^{\ell+r} \\ &= \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x_1 + x_2 + \cdots + x_t) \cdot (x_1 + x_2 + \cdots + x_t)^r\end{aligned}$$

## 5. Polynomial $\mathbf{P}_b^m(x)$ in terms of Discrete convolution

In this section we discuss the relation between $\mathbf{P}_b^m(x)$ and discrete convolution of polynomials. To show that $\mathbf{P}_b^m(x)$ involves the discrete convolution of polynomial $n^r$ let's remind the definition of $\mathbf{P}_b^m(x)$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r$$

A discrete convolution of defined over set of integers $\mathbb{Z}$ function $f$ is

$$(f * f)[n] = \sum_k f(k) f(n-k)$$

General formula of discrete convolution for polynomials $f(n) = n^j, \ n \geq a \in \mathbb{R}$ may be derived immediately

$$\begin{aligned}(n^j * n^j)[x] &= \sum_k k^j (x-k)^j [k \geq a][x-k \geq a] \\ &= \sum_k k^j (x-k)^j [k \geq a][k \leq x-a] \\ &= \sum_k k^j (x-k)^j [a \leq k \leq x-a] \\ &= \sum_{k=a}^{x-a} k^j (x-k)^j,\end{aligned}$$

where $[a \leq k \leq x-a]$ is Iverson's bracket [Ive62].

**Lemma 5.1.** *For every* $n \in \mathbb{N}, \ x \in \mathbb{R}$

$$(n^r * n^r)[x] = \sum_{k=0}^{x} k^r (x-k)^r, \quad n \geq 0.$$

It is of first necessity to notice that $n^r$ of discrete convolution $(n^r * n^r)[x]$ evaluated at $x$ is implicit piecewise-defined polynomial such as

$$n^r = \begin{cases} \underbrace{n \cdot n \cdots n}_{\text{r times}}, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the corollary follows

**Corollary 5.2.** *By Lemma 5.1 the polynomial* $\mathbf{P}_b^m(n)$ *might be expressed in terms of discrete convolution as follows*

$$\mathbf{P}_{x+1}^m(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} (n^r * n^r)[x], \quad n \geq 0.$$

Therefore, another polynomial identity follows

**Theorem 5.3.** *By Lemma 4.1, Corollary 5.2 and property 3.2, for every $m \in \mathbb{N},\ x \in \mathbb{R}$*

$$1 + x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x], \quad n \geq 0.$$

Now we notice the following identity in terms of polynomial $\mathbf{P}_b^m(x)$ and discrete convolution $(n^j * n^j)[x]$

**Proposition 5.4.** *For every $m \in \mathbb{N},\ x \in \mathbb{R}$*

$$\begin{aligned}
\mathbf{P}_x^m(x) &= \sum_{r=0}^{m} \mathbf{A}_{m,r}\left(0^r x^r + \sum_{k=1}^{x-1} k^r (x-k)^r\right) \\
&= \sum_{r=0}^{m} \mathbf{A}_{m,r} 0^r x^r + \sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x] \\
&= 1 + \sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x], \quad n \geq 1.
\end{aligned}$$

Since that for all $r$ in $\mathbf{A}_{m,r}0^r x^r$ we have

$$\mathbf{A}_{m,r}0^r x^r = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{if } r > 0 \end{cases}$$

Above is true because $\mathbf{A}_{m,0} = 1$ for every $m \in \mathbb{N}$, and $x^0 = 1$ for every $x$, [GKP94]. Hence, the following identity between $\mathbf{P}_b^m(x)$ and discrete convolution $(n^j * n^j)[x]$ holds

**Theorem 5.5.** *By Lemma 4.1 and Proposition 5.4, for every $m \in \mathbb{N},\ x \in \mathbb{R}$*

$$-1 + x^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x], \quad n > 0.$$

**Corollary 5.6.** *By Theorem 5.5, for all $m \in \mathbb{N}$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Corollary 5.6 holds since that convolution $(n^j * n^j)[x] = 1,\ n > 0$ for each $r$ and $x = 2$.

## 6. Relation between Binomial theorem and Discrete convolution

**Corollary 6.1.** *(Generalization of Theorem 5.3 for Binomials.) For every $m \in \mathbb{N},\ x, y \in \mathbb{R}$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x+y] = 1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r, \quad n \geq 0.$$

For example, given $m = 0, 1, 2$ the Corollary 6.1 gives

$$\sum_{r=0}^{0} \mathbf{A}_{0,r}(n^r * n^r)[x+y] = 1 + x + y$$

$$\begin{aligned}\sum_{r=0}^{1} \mathbf{A}_{1,r}(n^r * n^r)[x+y] &= 1 + x + y - (x+y)(1+x+y)(1-3x-3y+2(x+y)) \\ &= 1 + x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

$$\begin{aligned}\sum_{r=0}^{2} \mathbf{A}_{2,r}(n^r * n^r)[x+y] &= 1 + x + y + (x+y)(1+x+y)\left(-1 + x + 5x^2 + y + 10xy + 5y^2\right. \\ &- 15x(x+y) + 10x^2(x+y) - 15y(x+y) + 20xy(x+y) \\ &+ 10y^2(x+y) + 9(x+y)^2 - 15x(x+y)^2 \\ &\left.-15y(x+y)^2 + 6(x+y)^3\right) \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 + 1\end{aligned}$$

Above example could be verified using using the commands

- `BinomialTheoremAndDiscreteConvolutionTest[0, x + y]`
- `BinomialTheoremAndDiscreteConvolutionTest[1, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[1, x + y]]`
- `BinomialTheoremAndDiscreteConvolutionTest[2, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[2, x + y]]`

defined in Mathematica package at [Kol20].

**Corollary 6.2.** *(Generalization of Theorem 5.5 for Binomials.) For every $m \in \mathbb{N}$, $x, y \in \mathbb{R}$*

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}(n^r * n^r)[x+y] = -1 + \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r, \quad n > 0.$$

For example, given $m = 0, 1$ the Corollary 6.2 gives

$$\sum_{r=0}^{0} \mathbf{A}_{0,r}(n^r * n^r)[x+y] = x + y - 1$$

$$\begin{aligned}\sum_{r=0}^{1} \mathbf{A}_{1,r}(n^r * n^r)[x+y] &= -1 + x + y - (-1+x+y)(x+y)(-1-3x-3y+2(x+y)) \\ &= x^3 + 3x^2y + 3xy^2 + y^3 - 1\end{aligned}$$

Above example could be verified using using the commands

- `BinomialTheoremAndDiscreteConvolutionStrictTest[0, x + y]`
- `BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y]`
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y]]`

defined in Mathematica package at [Kol20]. From the other prospective, let be a function $f_r(t,k)=(t-k)^r,\ t\geq k$, then following identity holds

$$(x-2a)^{2m+1}+1=\sum_{r=0}^{m}\mathbf{A}_{m,r}(f_r(t,k)*f_r(t,k))[x] \tag{6.1}$$

Let be a function $g_r(t,k)=(t-k)^r,\ t>k$, then

$$(x-2a)^{2m+1}-1=\sum_{r=0}^{m}\mathbf{A}_{m,r}(g_r(t,k)*g_r(t,k))[x] \tag{6.2}$$

6.1. **Generalization for Multinomials.** In this subsection we generalize Theorems 5.3, 5.5 for multinomial cases.

**Corollary 6.3.** *(Generalization of Theorem 5.3 for Multinomials.) For every* $x_1,x_2,\dots,x_t\in\mathbb{R},\ m\in\mathbb{N},\ n\geq 1\in\mathbb{N}$

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x_1+x_2+\cdots+x_t]=1+\sum_{k_1+k_2+\cdots+k_t=2m+1}\binom{2m+1}{k_1,k_2,\dots,k_t}\prod_{\ell=1}^{t}x_\ell^{k_\ell}$$

For instance, given $m=1$ the Corollary 6.3 gives

$$\begin{aligned}&\sum_{r=0}^{1}\mathbf{A}_{1,r}(n^r*n^r)[x+y+z]\\&=1+x+y+z-(x+y+z)(1+x+y+z)(1-3x-3y-3z+2(x+y+z))\\&=1+x^3+3x^2y+3xy^2+y^3+3x^2z+6xyz+3y^2z+3xz^2+3yz^2+z^3.\end{aligned}$$

Above example could be verified using using the commands

- `BinomialTheoremAndDiscreteConvolutionTest[1, x + y + z]`
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[1, x + y + z]]`

defined in Mathematica package at [Kol20].

**Corollary 6.4.** *(Generalization of Theorem 5.5 for Multinomials.) For each* $x_1+x_2+\cdots+x_t\geq 1,\ x_1,x_2,\dots,x_t\in\mathbb{R},\ m\in\mathbb{N},\ n\geq 1\in\mathbb{N}$

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}(n^r*n^r)[x_1+x_2+\cdots+x_t]=-1+\sum_{k_1+k_2+\cdots+k_t=2m+1}\binom{2m+1}{k_1,k_2,\dots,k_t}\prod_{\ell=1}^{t}x_\ell^{k_\ell}$$

For example, given $m=1$ the Corollary 6.4 gives

$$\begin{aligned}&\sum_{r=0}^{1}\mathbf{A}_{1,r}(n^r*n^r)[x+y+z]\\&=-1+x+y+z-(-1+x+y+z)(x+y+z)(-1-3x-3y-3z+2(x+y+z))\\&=-1+x^3+3x^2y+3xy^2+y^3+3x^2z+6xyz+3y^2z+3xz^2+3yz^2+z^3.\end{aligned}$$

Above example could be verified using using the commands

- `BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y + z]`
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[1, x + y + z]]`

defined in Mathematica package at [Kol20].

## 7. Derivation of coefficient $\mathbf{A}_{m,r}$

By Lemma 4.1 for every $m \in \mathbb{N},\ n \in \mathbb{R}$

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{n-1} k^r (n-k)^r \tag{7.1}$$

The $\mathbf{A}_{m,r}$ might be evaluated using binomial expansion of $\sum_{k=0}^{n-1} k^r (n-k)^r$

$$\sum_{k=0}^{n-1} k^r (n-k)^r = \sum_{k=0}^{n-1} k^r \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} k^j = \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} \sum_{k=0}^{n-1} k^{r+j}$$

Using Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$ we get

$$\begin{aligned}
\sum_{k=0}^{n-1} k^r (n-k)^r &= \sum_{j=0}^{r} \binom{r}{j} n^{r-j} \frac{(-1)^j}{r+j+1} \left[ \sum_{s} \binom{r+j+1}{s} B_s n^{r+j+1-s} - B_{r+j+1} \right] \\
&= \sum_{j,s} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s} B_s n^{2r+1-s} - \sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j} \\
&= \sum_{s} \underbrace{\sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s}}_{S(r)} B_s n^{2r+1-s} \\
&\quad - \sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}
\end{aligned} \tag{7.2}$$

where $B_s$ are Bernoulli numbers and $B_1 = \frac{1}{2}$. Now, we notice that

$$\sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} \binom{r+j+1}{s} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } s = 0; \\ \frac{(-1)^r}{s} \binom{r}{2r-s+1}, & \text{if } s > 0. \end{cases}$$

In particular, the last sum is zero for $0 < s \leq r$. Therefore, expression (7.2) takes the form

$$\sum_{k=0}^{n-1} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\sum_{s\geq 1} \frac{(-1)^r}{s} \binom{r}{2r-s+1} B_s n^{2r+1-s}}_{(\star)}$$
$$- \underbrace{\sum_{j} \binom{r}{j} \frac{(-1)^j}{r+j+1} B_{r+j+1} n^{r-j}}_{(\diamond)}$$

Hence, introducing $\ell = 2r+1-s$ to $(\star)$ and $\ell = r-j$ to $(\diamond)$, we get

$$\sum_{k=0}^{n-1} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$
$$- \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}$$

$$\sum_{k=0}^{n-1} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + (-1)^r \sum_{\ell} \frac{1}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$
$$- \frac{1}{(-1)^r} \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{j-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}$$
$$= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell}^{r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$

Using the definition (7.1) of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in $n$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^{m} \sum_{\text{odd } \ell}^{r} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \tag{7.3}$$

Taking the coefficient of $n^{2r+1}$ for $r = m$ in (7.3) we get $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$. Since that odd $\ell \leq r$ in explicit form is $2j+1 \leq r$, it follows that $j \leq \frac{m-1}{2}$, where $j$ is iterator. Therefore, taking the coefficient of $n^{2j+1}$ for an integer $j$ in the range $\frac{m}{2} \leq j \leq m$, we get $\mathbf{A}_{m,j} = 0$. Taking the coefficient of $n^{2d+1}$ for $d$ in the range $m/4 \leq d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1) \binom{2m}{m} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0,$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer $r$ in range $m/2^{s+1} \leq r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

## 8. Verification of the results and examples

To fulfill our study we provide an opportunity to verify its results by means of Wolfram Mathematica language.

### 8.1. Mathematica commands.

Proceeding to the repository [Kol20] reader is able to find there a folder named `mathematica` that contains the files

- `OnTheBinomialTheoremAndDiscreteConvolution.m` is a package file with definitions
- `OnTheBinomialTheoremAndDiscreteConvolution.nb` is a notebook file with examples.

The following commands may be used to reproduce the results of this manuscript:

- `A[m, r]` returns the real coefficient $\mathbf{A}_{m,r}$ defined by (1.1).
- `PolynomialL[m, n, k]` returns the polynomial $\mathbf{L}_m(n,k)$ defined by (1.5).
- `PolynomialP[m, x, b]` returns the polynomial $\mathbf{P}^m_b(x)$ defined by (1.2).
- `Expand[PolynomialP[m, x + y, x + y]]` verifies the Lemma 4.1.
- `PolynomialH[m, t, j]` returns the polynomial $\mathbf{H}_{m,t}(j)$ defined by (1.3).
- `PolynomialX[m, t, k]` returns the polynomial $\mathbf{X}_{m,t}(k)$ defined by (1.4).
- `Expand[BinomialTheoremAndDiscreteConvolutionTest[m, x + y]]` verifies the Corollary 6.1.
- `Expand[BinomialTheoremAndDiscreteConvolutionStrictTest[m, x + y]]` verifies the Corollary 6.2.
- `DiscreteConvolutionPowerIdentityParametricTest[m, x, a]` verifies an equation (6.1). Usage `Column[Table[DiscreteConvolutionPowerIdentityParametricTest[1, x, 1], x, 3, 20], Left]`.
- `DiscreteConvolutionPowerIdentityStrictParametricTest[m, x, a]` verifies an equation (6.2). Usage `Column[Table[DiscreteConvolutionPowerIdentityStrictParametricTes x, 1], x, 3, 20], Left]`.
- `PolynomialIdentityInvolvingX[m, x, b]` validates an identity at (3.1)

$$\mathbf{P}^m_b(x) = \sum_{r=0}^{m}(-1)^{m-r}\mathbf{X}_{m,r}(b)\cdot x^r$$

- `PolynomialIdentityInvolvingH[m, n, b]` validates an identity at (3.1).

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot b^\ell \cdot x^r$$

8.2. **Examples.** For example, given $m = 1$ we have the following values of $\mathbf{L}_1(x, k)$

| $x/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 1 | 7 | 1 | | | | | |
| 3 | 1 | 13 | 13 | 1 | | | | |
| 4 | 1 | 19 | 25 | 19 | 1 | | | |
| 5 | 1 | 25 | 37 | 37 | 25 | 1 | | |
| 6 | 1 | 31 | 49 | 55 | 49 | 31 | 1 | |
| 7 | 1 | 37 | 61 | 73 | 73 | 61 | 37 | 1 |

**Table 3.** Values of $\mathbf{L}_1(x, k)$. See OEIS entry: `A300656`.

Table 3 can be reproduced using Mathematica command

`PrintTriangleOfPolynomialL[1, 7]`

defined in the [Kol20]. From Table 3 it is seen that

$$\mathbf{P}_0^1(0) = 0 = 0^3$$
$$\mathbf{P}_1^1(1) = 1 = 1^3$$
$$\mathbf{P}_2^1(2) = 1 + 7 = 2^3$$
$$\mathbf{P}_3^1(3) = 1 + 13 + 13 = 3^3$$
$$\mathbf{P}_4^1(4) = 1 + 19 + 25 + 19 = 4^3$$
$$\mathbf{P}_5^1(5) = 1 + 25 + 37 + 37 + 25 = 5^3$$

Another case, given $m = 2$ we have the following values of $\mathbf{L}_2(x, k)$

| $x/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 1 | 31 | 1 | | | | | |
| 3 | 1 | 121 | 121 | 1 | | | | |
| 4 | 1 | 271 | 481 | 271 | 1 | | | |
| 5 | 1 | 481 | 1081 | 1081 | 481 | 1 | | |
| 6 | 1 | 751 | 1921 | 2431 | 1921 | 751 | 1 | |
| 7 | 1 | 1081 | 3001 | 4321 | 4321 | 3001 | 1081 | 1 |

**Table 4.** Values of $\mathbf{L}_2(x, k)$. See OEIS entry: `A300656`.

Table 4 can be reproduced using Mathematica command

```
PrintTriangleOfPolynomialL[2, 7]
```

defined in the [Kol20]. Again, an odd-power identity 4.1 holds

$$
\begin{aligned}
&\mathbf{P}^2_0(0) = 0 = 0^5\\
&\mathbf{P}^2_1(1) = 1 = 1^5\\
&\mathbf{P}^2_2(2) = 1 + 31 = 2^5\\
&\mathbf{P}^2_3(3) = 1 + 121 + 121 = 3^5\\
&\mathbf{P}^2_4(4) = 1 + 271 + 481 + 271 = 4^5\\
&\mathbf{P}^2_5(5) = 1 + 481 + 1081 + 1081 + 481 = 5^5
\end{aligned}
$$

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## 10. Conclusion

In this manuscript we have shown that Binomial theorem is partial case of polynomial $\mathbf{P}^m_b(x)$. Furthermore, by means of $\mathbf{P}^m_b(x)$ it is shown a relation between Binomial theorem and discrete convolution of polynomials.

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