PLOTS OF CLOSED FORMS

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CONTENTS

1. INTRODUCTION

$$P(m, X, N) = \sum_{r=0}^{m} \sum_{k=1}^{N} \mathbf{A}_{m,r} k^r (X - k)^r$$

$$Q(m, X, N) = \sum_{r=0}^{m} \sum_{k=0}^{N-1} \mathbf{A}_{m,r} k^r (X - k)^r$$

$$P(m, N, N) = N^{2m+1}$$

$$Q(m, N, N) = N^{2m+1}$$

$$P(m, N+1, N) = (N+1)^{2m+1} - 1 \qquad (verified)$$

$$Q(m, N-1, N) = (N-1)^{2m+1} + 1 \qquad (verified)$$

1.1. Polynomials P(1,n,k).

$$P(1, X, 0) = 0$$

$$P(1, X, 1) = 6X - 5$$

$$P(1, X, 2) = 18X - 28$$

$$P(1, X, 3) = 36X - 81$$

$$P(1, X, 4) = 60X - 176$$

$$P(1, X, 5) = 90X - 325$$

$$P(1, X, 6) = 126X - 540$$

$$P(1, X, 7) = 168X - 833$$

$$P(1, X, 8) = 216X - 1216$$

$$P(1, X, 9) = 270X - 1701$$

$$P(1, X, 10) = 330X - 2300$$

$$P(1, X, 11) = 396X - 3025$$

$$P(1, X, 12) = 468X - 3888$$

$$P(1, X, 13) = 546X - 4901$$

$$P(1, X, 14) = 630X - 6076$$

$$P(1, X, 15) = 720X - 7425$$

$$P(1, X, 16) = 816X - 8960$$

$$P(1, X, 17) = 918X - 10693$$

$$P(1, X, 18) = 1026X - 12636$$

$$P(1, X, 19) = 1140X - 14801$$

$$P(1, X, 20) = 1260X - 17200$$

Figure 1. Polynomials P(1, n, k)

1.2. **Polynomial P(1,n,k) Table n = 6.**

1.3. Polynomials Q(1,n,k).

$$Q(1, X, 0) = 0$$

$$Q(1, X, 1) = 1$$

$$Q(1, X, 2) = 6X - 4$$

$$Q(1, X, 3) = 18X - 27$$

$$Q(1, X, 4) = 36X - 80$$

$$Q(1, X, 5) = 60X - 175$$

$$Q(1, X, 6) = 90X - 324$$

$$Q(1, X, 7) = 126X - 539$$

$$Q(1, X, 8) = 168X - 832$$

$$Q(1, X, 9) = 216X - 1215$$

$$Q(1, X, 10) = 270X - 1700$$

$$Q(1, X, 11) = 330X - 2299$$

$$Q(1, X, 12) = 396X - 3024$$

$$Q(1, X, 13) = 468X - 3887$$

$$Q(1, X, 14) = 546X - 4900$$

$$Q(1, X, 15) = 630X - 6075$$

$$Q(1, X, 16) = 720X - 7424$$

$$Q(1, X, 17) = 816X - 8959$$

$$Q(1, X, 18) = 918X - 10692$$

$$Q(1, X, 19) = 1026X - 12635$$
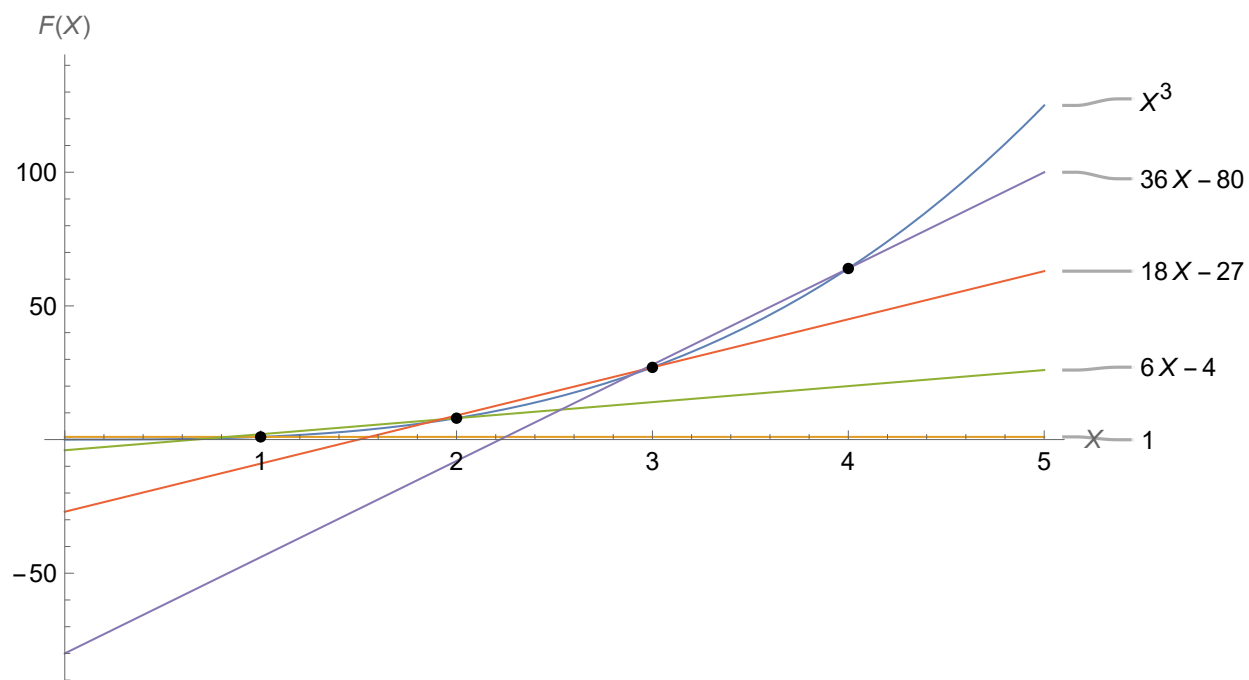
$$Q(1, X, 20) = 1140X - 14800$$

Figure 2. Polynomials Q(1, n, k)

1.4. **Polynomial Q(1,n,k) Table n = 6.**

1.5. **Polynomials P(2,n,k).**

$$P(2, X, 0) = 0$$

$$P(2, X, 1) = 30X^2 - 60X + 31$$

$$P(2, X, 2) = 150X^2 - 540X + 512$$

$$P(2, X, 3) = 420X^2 - 2160X + 2943$$

$$P(2, X, 4) = 900X^2 - 6000X + 10624$$

$$P(2, X, 5) = 1650X^2 - 13500X + 29375$$

$$P(2, X, 6) = 2730X^2 - 26460X + 68256$$

$$P(2, X, 7) = 4200X^2 - 47040X + 140287$$

$$P(2, X, 8) = 6120X^2 - 77760X + 263168$$

$$P(2, X, 9) = 8550X^2 - 121500X + 459999$$

$$P(2, X, 10) = 11550X^2 - 181500X + 760000$$

$$P(2, X, 11) = 15180X^2 - 261360X + 1199231$$

$$P(2, X, 12) = 19500X^2 - 365040X + 1821312$$

$$P(2, X, 13) = 24570X^2 - 496860X + 2678143$$

$$P(2, X, 14) = 30450X^2 - 661500X + 3830624$$

$$P(2, X, 15) = 37200X^2 - 864000X + 5349375$$

$$P(2, X, 16) = 44880X^2 - 1109760X + 7315456$$

$$P(2, X, 17) = 53550X^2 - 1404540X + 9821087$$

$$P(2, X, 18) = 63270X^2 - 1754460X + 12970368$$

$$P(2, X, 19) = 74100X^2 - 2166000X + 16879999$$

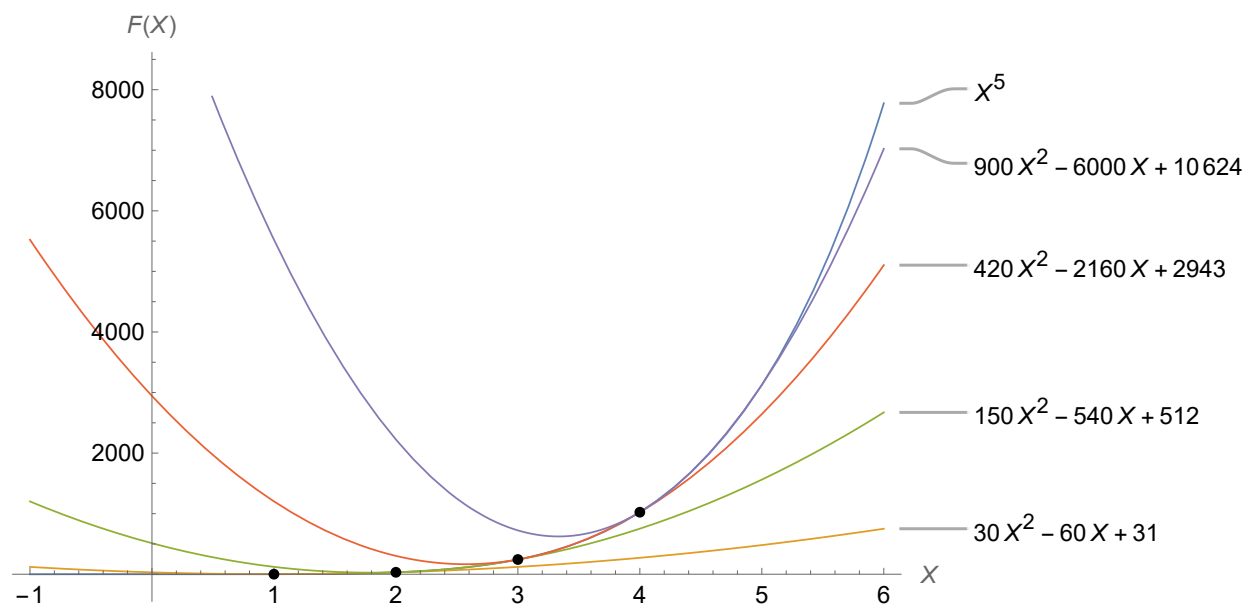$$P(2, X, 20) = 86100X^2 - 2646000X + 21680000$$

Figure 3. Polynomials P(2, n, k)

1.6. **Polynomial P(2,n,k) Table n = 4.**

1.7. Polynomials Q(2,n,k).

$$Q(2, X, 0) = 0$$

$$Q(2, X, 1) = 1$$

$$Q(2, X, 2) = 30X^2 - 60X + 32$$

$$Q(2, X, 3) = 150X^2 - 540X + 513$$

$$Q(2, X, 4) = 420X^2 - 2160X + 2944$$

$$Q(2, X, 5) = 900X^2 - 6000X + 10625$$

$$Q(2, X, 6) = 1650X^2 - 13500X + 29376$$

$$Q(2, X, 7) = 2730X^2 - 26460X + 68257$$

$$Q(2, X, 8) = 4200X^2 - 47040X + 140288$$

$$Q(2, X, 9) = 6120X^2 - 77760X + 263169$$

$$Q(2, X, 10) = 8550X^2 - 121500X + 460000$$

$$Q(2, X, 11) = 11550X^2 - 181500X + 760001$$

$$Q(2, X, 12) = 15180X^2 - 261360X + 1199232$$

$$Q(2, X, 13) = 19500X^2 - 365040X + 1821313$$

$$Q(2, X, 14) = 24570X^2 - 496860X + 2678144$$

$$Q(2, X, 15) = 30450X^2 - 661500X + 3830625$$

$$Q(2, X, 16) = 37200X^2 - 864000X + 5349376$$

$$Q(2, X, 17) = 44880X^2 - 1109760X + 7315457$$

$$Q(2, X, 18) = 53550X^2 - 1404540X + 9821088$$

$$Q(2, X, 19) = 63270X^2 - 1754460X + 12970369$$

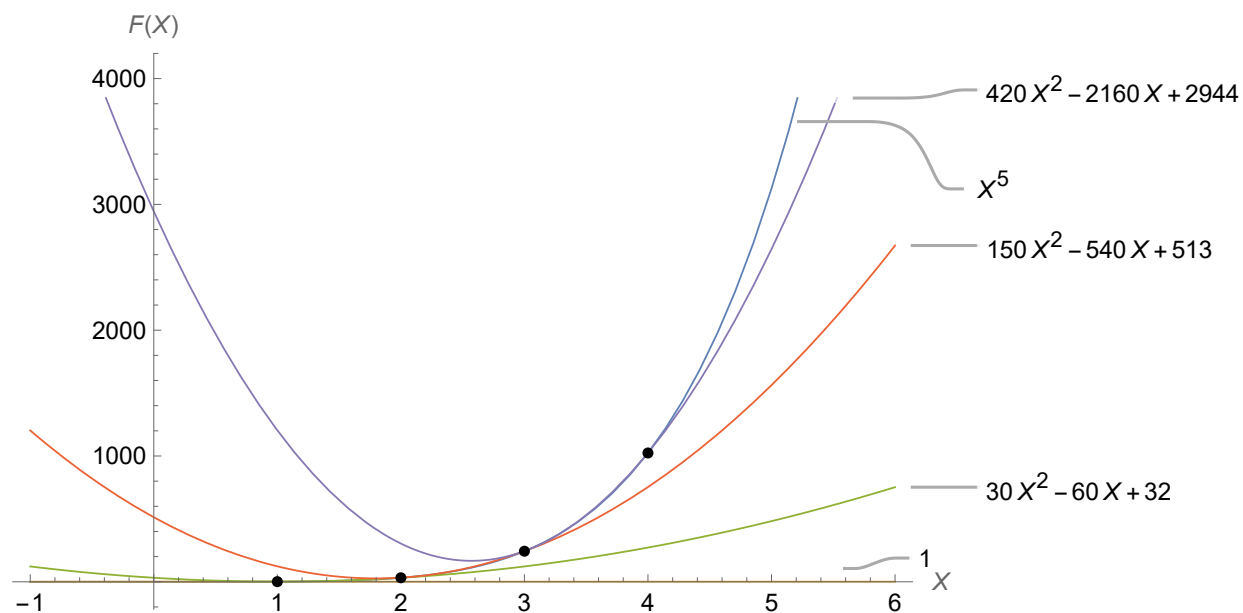$$Q(2, X, 20) = 74100X^2 - 2166000X + 16880000$$

Figure 4. Polynomials $Q(2, n, k)$

1.8. **Polynomial Q(2,n,k) Table n = 4.**

1.9. Polynomials P(3,n,k).

$$P(3, X, 0) = 0$$

$$P(3, X, 1) = 140X^3 - 420X^2 + 406X - 125$$

$$P(3, X, 2) = 1260X^3 - 7140X^2 + 13818X - 9028$$

$$P(3, X, 3) = 5040X^3 - 41160X^2 + 115836X - 110961$$

$$P(3, X, 4) = 14000X^3 - 148680X^2 + 545860X - 684176$$

$$P(3, X, 5) = 31500X^3 - 411180X^2 + 1858290X - 2871325$$

$$P(3, X, 6) = 61740X^3 - 955500X^2 + 5124126X - 9402660$$

$$P(3, X, 7) = 109760X^3 - 1963920X^2 + 12182968X - 25872833$$

$$P(3, X, 8) = 181440X^3 - 3684240X^2 + 25945416X - 62572096$$

$$P(3, X, 9) = 283500X^3 - 6439860X^2 + 50745870X - 136972701$$

$$P(3, X, 10) = 423500X^3 - 10639860X^2 + 92745730X - 276971300$$

$$P(3, X, 11) = 609840X^3 - 16789080X^2 + 160386996X - 524988145$$

$$P(3, X, 12) = 851760X^3 - 25498200X^2 + 264896268X - 943023888$$

$$P(3, X, 13) = 1159340X^3 - 37493820X^2 + 420839146X - 1618774781$$

$$P(3, X, 14) = 1543500X^3 - 53628540X^2 + 646725030X - 2672907076$$

$$P(3, X, 15) = 2016000X^3 - 74891040X^2 + 965662320X - 4267591425$$

$$P(3, X, 16) = 2589440X^3 - 102416160X^2 + 1406064016X - 6616398080$$

$$P(3, X, 17) = 3277260X^3 - 137494980X^2 + 2002403718X - 9995653693$$

$$P(3, X, 18) = 4093740X^3 - 181584900X^2 + 2796022026X - 14757360516$$

$$P(3, X, 19) = 5054000X^3 - 236319720X^2 + 3835983340X - 21343778801$$

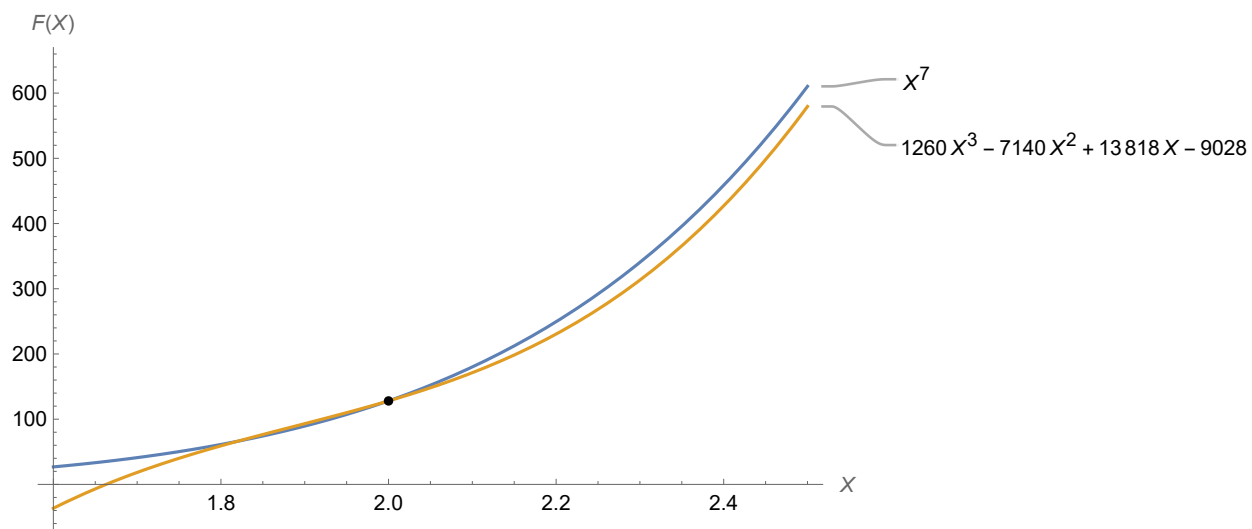$$P(3, X, 20) = 6174000X^3 - 303519720X^2 + 5179983060X - 30303773200$$

Figure 5. Polynomials P(3, n, k)

1.10. Polynomial P(3,n,k) Table n = 3.

1.11. Polynomials Q(3,n,k).

$$Q(3, X, 0) = 0$$

$$Q(3, X, 1) = 1$$

$$Q(3, X, 2) = 140X^3 - 420X^2 + 406X - 124$$

$$Q(3, X, 3) = 1260X^3 - 7140X^2 + 13818X - 9027$$

$$Q(3, X, 4) = 5040X^3 - 41160X^2 + 115836X - 110960$$

$$Q(3, X, 5) = 14000X^3 - 148680X^2 + 545860X - 684175$$

$$Q(3, X, 6) = 31500X^3 - 411180X^2 + 1858290X - 2871324$$

$$Q(3, X, 7) = 61740X^3 - 955500X^2 + 5124126X - 9402659$$

$$Q(3, X, 8) = 109760X^3 - 1963920X^2 + 12182968X - 25872832$$

$$Q(3, X, 9) = 181440X^3 - 3684240X^2 + 25945416X - 62572095$$

$$Q(3, X, 10) = 283500X^3 - 6439860X^2 + 50745870X - 136972700$$

$$Q(3, X, 11) = 423500X^3 - 10639860X^2 + 92745730X - 276971299$$

$$Q(3, X, 12) = 609840X^3 - 16789080X^2 + 160386996X - 524988144$$

$$Q(3, X, 13) = 851760X^3 - 25498200X^2 + 264896268X - 943023887$$

$$Q(3, X, 14) = 1159340X^3 - 37493820X^2 + 420839146X - 1618774780$$

$$Q(3, X, 15) = 1543500X^3 - 53628540X^2 + 646725030X - 2672907075$$

$$Q(3, X, 16) = 2016000X^3 - 74891040X^2 + 965662320X - 4267591424$$

$$Q(3, X, 17) = 2589440X^3 - 102416160X^2 + 1406064016X - 6616398079$$

$$Q(3, X, 18) = 3277260X^3 - 137494980X^2 + 2002403718X - 9995653692$$

$$Q(3, X, 19) = 4093740X^3 - 181584900X^2 + 2796022026X - 14757360515$$

$$Q(3, X, 20) = 5054000X^3 - 236319720X^2 + 3835983340X - 21343778800$$
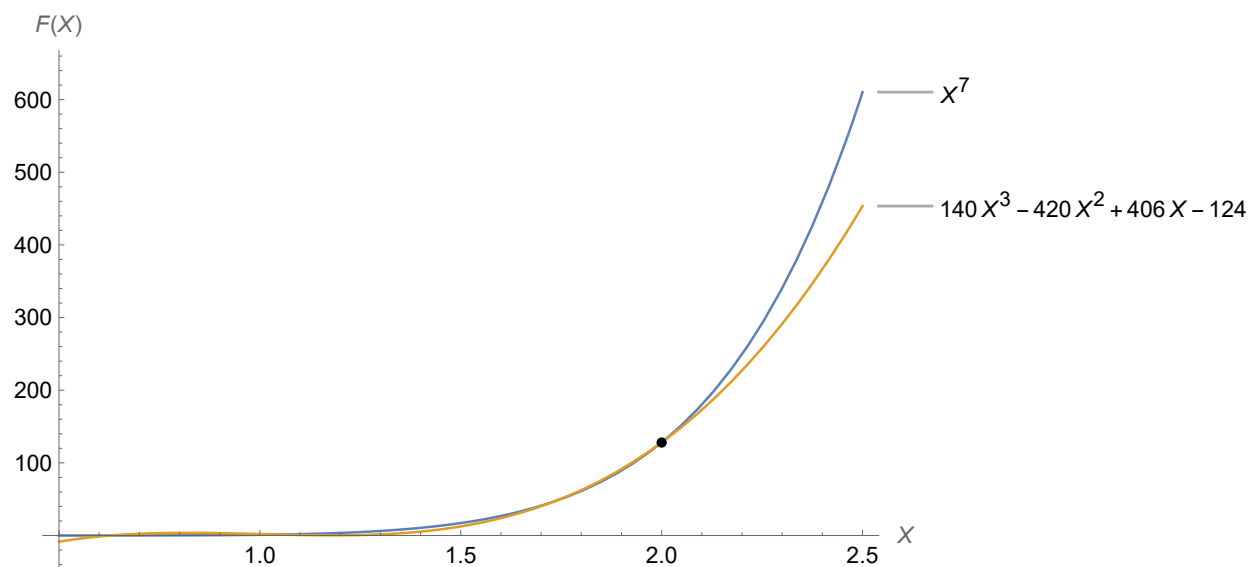
Figure 6. Polynomials Q(3, n, k)

1.12. **Polynomial Q(3,n,k) Table n = 3.**

Table 1. Comparison of X^3, $126X - 540$, Difference, and Absolute Error Percentage

X	X^3	$126X - 540$	Diff	ABS Error %
5.3	148.877	127.800	21.077	14.1573
5.4	157.464	140.400	17.064	10.8368
5.5	166.375	153.000	13.375	8.0391
5.6	175.616	165.600	10.016	5.7034
5.7	185.193	178.200	6.993	3.7761
5.8	195.112	190.800	4.312	2.2100
5.9	205.379	203.400	1.979	0.9636
6.0	216.000	216.000	0.000	0.0000
6.1	226.981	228.600	-1.619	0.7133
6.2	238.328	241.200	-2.872	1.2051
6.3	250.047	253.800	-3.753	1.5009
6.4	262.144	266.400	-4.256	1.6235
6.5	274.625	279.000	-4.375	1.5931
6.6	287.496	291.600	-4.104	1.4275
6.7	300.763	304.200	-3.437	1.1428
6.8	314.432	316.800	-2.368	0.7531
6.9	328.509	329.400	-0.891	0.2712
7.0	343.000	342.000	1.000	0.2915
7.1	357.911	354.600	3.311	0.9251
7.2	373.248	367.200	6.048	1.6204
7.3	389.017	379.800	9.217	2.3693
7.4	405.224	392.400	12.824	3.1647
7.5	421.875	405.000	16.875	4.0000
7.6	438.976	417.600	21.376	4.8695
7.7	456.533	430.200	26.333	5.7680
7.8	474.552	442.800	31.752	6.6909
7.9	493.039	455.400	37.639	7.6341

Table 2. Comparison of X^3, $90X - 324$, Difference, and Absolute Error Percentage

X	X^3	$90X - 324$	Diff	ABS Error %
4.5	91.125	81.000	10.125	11.1111
4.6	97.336	90.000	7.336	7.5368
4.7	103.823	99.000	4.823	4.6454
4.8	110.592	108.000	2.592	2.3438
4.9	117.649	117.000	0.649	0.5516
5.0	125.000	126.000	-1.000	0.8000
5.1	132.651	135.000	-2.349	1.7708
5.2	140.608	144.000	-3.392	2.4124
5.3	148.877	153.000	-4.123	2.7694
5.4	157.464	162.000	-4.536	2.8807
5.5	166.375	171.000	-4.625	2.7799
5.6	175.616	180.000	-4.384	2.4964
5.7	185.193	189.000	-3.807	2.0557
5.8	195.112	198.000	-2.888	1.4802
5.9	205.379	207.000	-1.621	0.7893
6.0	216.000	216.000	0.000	0.0000
6.1	226.981	225.000	1.981	0.8728
6.2	238.328	234.000	4.328	1.8160
6.3	250.047	243.000	7.047	2.8183
6.4	262.144	252.000	10.144	3.8696
6.5	274.625	261.000	13.625	4.9613
6.6	287.496	270.000	17.496	6.0856
6.7	300.763	279.000	21.763	7.2359
6.8	314.432	288.000	26.432	8.4063
6.9	328.509	297.000	31.509	9.5915
7.0	343.000	306.000	37.000	10.7872

Table 3. Comparison of X^5, $900X^2 - 6000X + 10624$, Difference, and Absolute
Error Percentage

X	X^5	$900X^2 - 6000X + 10624$	Diff	ABS Error %
3.6	604.662	688.000	-83.3382	13.7826
3.7	693.440	745.000	-51.5604	7.4355
3.8	792.352	820.000	-27.6483	3.4894
3.9	902.242	913.000	-10.7580	1.1924
4.0	1024.000	1024.000	0.0000	0.0000
4.1	1158.560	1153.000	5.5620	0.4801
4.2	1306.910	1300.000	6.9123	0.5289
4.3	1470.080	1465.000	5.0844	0.3459
4.4	1649.160	1648.000	1.1622	0.0705
4.5	1845.280	1849.000	-3.7188	0.2015
4.6	2059.630	2068.000	-8.3702	0.4064
4.7	2293.450	2305.000	-11.5499	0.5036
4.8	2548.040	2560.000	-11.9603	0.4694
4.9	2824.750	2833.000	-8.2475	0.2920
5.0	3125.000	3124.000	1.0000	0.0320
5.1	3450.250	3433.000	17.2525	0.5000
5.2	3802.040	3760.000	42.0403	1.1057
5.3	4181.950	4105.000	76.9549	1.8402
5.4	4591.650	4468.000	123.6500	2.6929
5.5	5032.840	4849.000	183.8440	3.6529
5.6	5507.320	5248.000	259.3180	4.7086
5.7	6016.920	5665.000	351.9210	5.8489
5.8	6563.570	6100.000	463.5680	7.0627
5.9	7149.240	6553.000	596.2430	8.3399
6.0	7776.000	7024.000	752.0000	9.6708
6.1	8445.960	7513.000	932.9630	11.0463

Table 4. Comparison of X^5, $420X^2 - 2160X + 2944$, Difference, and Absolute Error Percentage

X	X^5	$420X^2 - 2160X + 2944$	Diff	ABS Error %
2.7	143.489	173.800	-30.3109	21.1242
2.8	172.104	188.800	-16.6963	9.7013
2.9	205.111	212.200	-7.0885	3.4559
3.0	243.000	244.000	-1.0000	0.4115
3.1	286.292	284.200	2.0915	0.7306
3.2	335.544	332.800	2.7443	0.8179
3.3	391.354	389.800	1.5539	0.3971
3.4	454.354	455.200	-0.8458	0.1861
3.5	525.219	529.000	-3.7813	0.7199
3.6	604.662	611.200	-6.5382	1.0813
3.7	693.440	701.800	-8.3604	1.2056
3.8	792.352	800.800	-8.4483	1.0662
3.9	902.242	908.200	-5.9580	0.6604
4.0	1024.000	1024.000	0.0000	0.0000
4.1	1158.560	1148.200	10.3620	0.8944
4.2	1306.910	1280.800	26.1123	1.9980
4.3	1470.080	1421.800	48.2844	3.2845
4.4	1649.160	1571.200	77.9622	4.7274
4.5	1845.280	1729.000	116.2810	6.3016
4.6	2059.630	1895.200	164.4300	7.9835
4.7	2293.450	2069.800	223.6500	9.7517
4.8	2548.040	2252.800	295.2400	11.5869

Table 5. Comparison of X^7, $5040X^3 - 41160X^2 + 115836X - 110961$, Difference, and Absolute Error Percentage

X	X^7	$5040X^3 - 41160X^2 + 115836X - 110961$	Diff	ABS Error %
2.7	1046.04	942.12	103.915	9.9342
2.8	1349.29	1323.48	25.8129	1.9131
2.9	1724.99	1728.36	-3.3724	0.1955
3.0	2187.00	2187.00	0.0000	0.0000
3.1	2751.26	2729.64	21.6214	0.7859
3.2	3435.97	3386.52	49.4538	1.4393
3.3	4261.84	4187.88	73.9643	1.7355
3.4	5252.34	5163.96	88.3750	1.6826
3.5	6433.93	6345.00	88.9297	1.3822
3.6	7836.42	7761.24	75.1764	0.9593
3.7	9493.19	9442.92	50.2677	0.5295
3.8	11441.60	11420.30	21.2783	0.1860
3.9	13723.10	13723.60	-0.4593	0.0033
4.0	16384.00	16383.00	1.0000	0.0061
4.1	19475.40	19428.80	46.5874	0.2392
4.2	23053.90	22891.30	162.6130	0.7054
4.3	27181.90	26800.70	381.1810	1.4023
4.4	31927.80	31187.20	740.6210	2.3197
4.5	37366.90	36081.00	1285.9500	3.4414
4.6	43581.80	41512.40	2069.3300	4.7482
4.7	50662.30	47511.70	3150.5900	6.2188
4.8	58706.80	54109.10	4597.7500	7.8317
4.9	67822.30	61334.80	6487.5500	9.5655
5.0	78125.00	69219.00	8906.0000	11.3997
5.1	89741.10	77792.00	11949.0000	13.3150

Table 6. Comparison of X^7, $5040X^3 - 41160X^2 + 115836X - 110961$, Difference, and Absolute Error Percentage

X	X^7	$5040X^3 - 41160X^2 + 115836X - 110961$	Diff	ABS Error %
2.7	1046.04	942.12	103.915	9.9342
2.8	1349.29	1323.48	25.8129	1.9131
2.9	1724.99	1728.36	-3.3724	0.1955
3.0	2187.00	2187.00	0.0000	0.0000
3.1	2751.26	2729.64	21.6214	0.7859
3.2	3435.97	3386.52	49.4538	1.4393
3.3	4261.84	4187.88	73.9643	1.7355
3.4	5252.34	5163.96	88.3750	1.6826
3.5	6433.93	6345.00	88.9297	1.3822
3.6	7836.42	7761.24	75.1764	0.9593
3.7	9493.19	9442.92	50.2677	0.5295
3.8	11441.60	11420.30	21.2783	0.1860
3.9	13723.10	13723.60	-0.4593	0.0033
4.0	16384.00	16383.00	1.0000	0.0061
4.1	19475.40	19428.80	46.5874	0.2392
4.2	23053.90	22891.30	162.6130	0.7054
4.3	27181.90	26800.70	381.1810	1.4023
4.4	31927.80	31187.20	740.6210	2.3197
4.5	37366.90	36081.00	1285.9500	3.4414
4.6	43581.80	41512.40	2069.3300	4.7482
4.7	50662.30	47511.70	3150.5900	6.2188
4.8	58706.80	54109.10	4597.7500	7.8317
4.9	67822.30	61334.80	6487.5500	9.5655
5.0	78125.00	69219.00	8906.0000	11.3997
5.1	89741.10	77792.00	11949.0000	13.3150