PLOTS OF CLOSED FORMS

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Contents

1. Introduction

$$P(m,X,N)=\sum_{r=0}^{m}\sum_{k=1}^{N}\mathbf{A}_{m,r}k^r(X-k)^r$$

$$Q(m,X,N)=\sum_{r=0}^{m}\sum_{k=0}^{N-1}\mathbf{A}_{m,r}k^r(X-k)^r$$

$$P(m,N,N)=N^{2m+1}$$

$$Q(m,N,N)=N^{2m+1}$$

$$P(m,N+1,N)=(N+1)^{2m+1}-1 \qquad (verified)$$

$$Q(m,N-1,N)=(N-1)^{2m+1}+1 \qquad (verified)$$

1.1. **Polynomials P(1,X,N).**

$$P(1,X,0) = 0$$

$$P(1,X,1) = 6X - 5$$

$$P(1,X,2) = 18X - 28$$

$$P(1,X,3) = 36X - 81$$

$$P(1,X,4) = 60X - 176$$

$$P(1,X,5) = 90X - 325$$

$$P(1,X,6) = 126X - 540$$

$$P(1,X,7) = 168X - 833$$

$$P(1,X,8) = 216X - 1216$$

$$P(1,X,9) = 270X - 1701$$

$$P(1,X,10) = 330X - 2300$$

$$P(1,X,11) = 396X - 3025$$

$$P(1,X,12) = 468X - 3888$$

$$P(1,X,13) = 546X - 4901$$

$$P(1,X,14) = 630X - 6076$$

$$P(1,X,15) = 720X - 7425$$

$$P(1,X,16) = 816X - 8960$$

$$P(1,X,17) = 918X - 10693$$

$$P(1,X,18) = 1026X - 12636$$

$$P(1,X,19) = 1140X - 14801$$

$$P(1,X,20) = 1260X - 17200$$

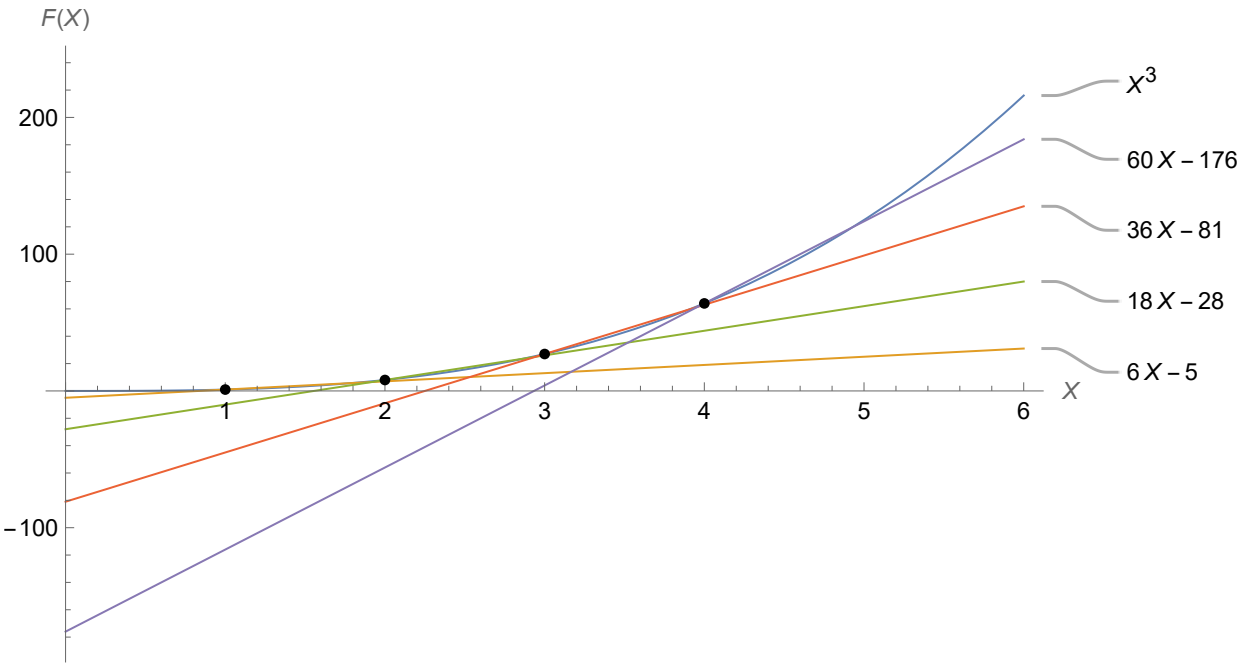

Figure 1. Polynomials P(1, X, N) for N=1..4

1.2. Polynomial P(1,X,N) Table of values for N = 6.

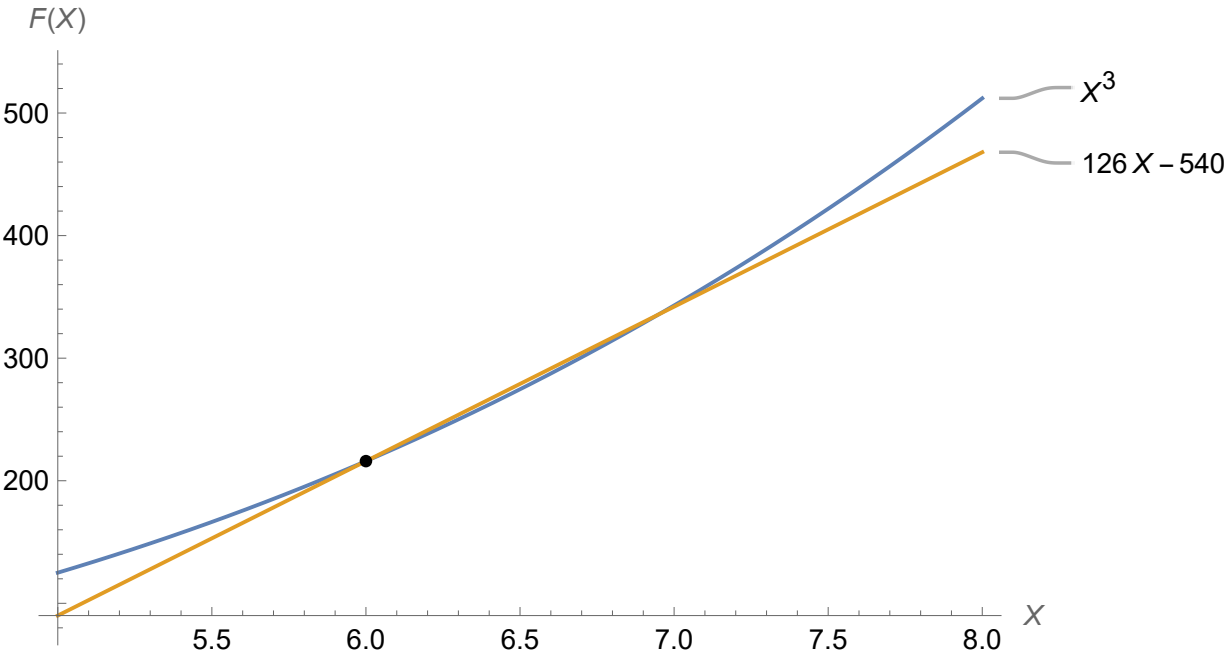

Figure 2. Polynomial plot P(1, X, 6) with cubes

1.3. Polynomial P(1,X,6) plot with cubes.

1.4. **Polynomials Q(1,X,N).**

$$Q(1,X,0) = 0$$

$$Q(1,X,1) = 1$$

$$Q(1,X,2) = 6X - 4$$

$$Q(1,X,3) = 18X - 27$$

$$Q(1,X,4) = 36X - 80$$

$$Q(1,X,5) = 60X - 175$$

$$Q(1,X,6) = 90X - 324$$

$$Q(1,X,7) = 126X - 539$$

$$Q(1,X,8) = 168X - 832$$

$$Q(1,X,9) = 216X - 1215$$

$$Q(1,X,10) = 270X - 1700$$

$$Q(1,X,11) = 330X - 2299$$

$$Q(1,X,12) = 396X - 3024$$

$$Q(1,X,13) = 468X - 3887$$

$$Q(1,X,14) = 546X - 4900$$

$$Q(1,X,15) = 630X - 6075$$

$$Q(1,X,16) = 720X - 7424$$

$$Q(1,X,17) = 816X - 8959$$

$$Q(1,X,18) = 918X - 10692$$

$$Q(1,X,19) = 1026X - 12635$$

$$Q(1,X,20) = 1140X - 14800$$

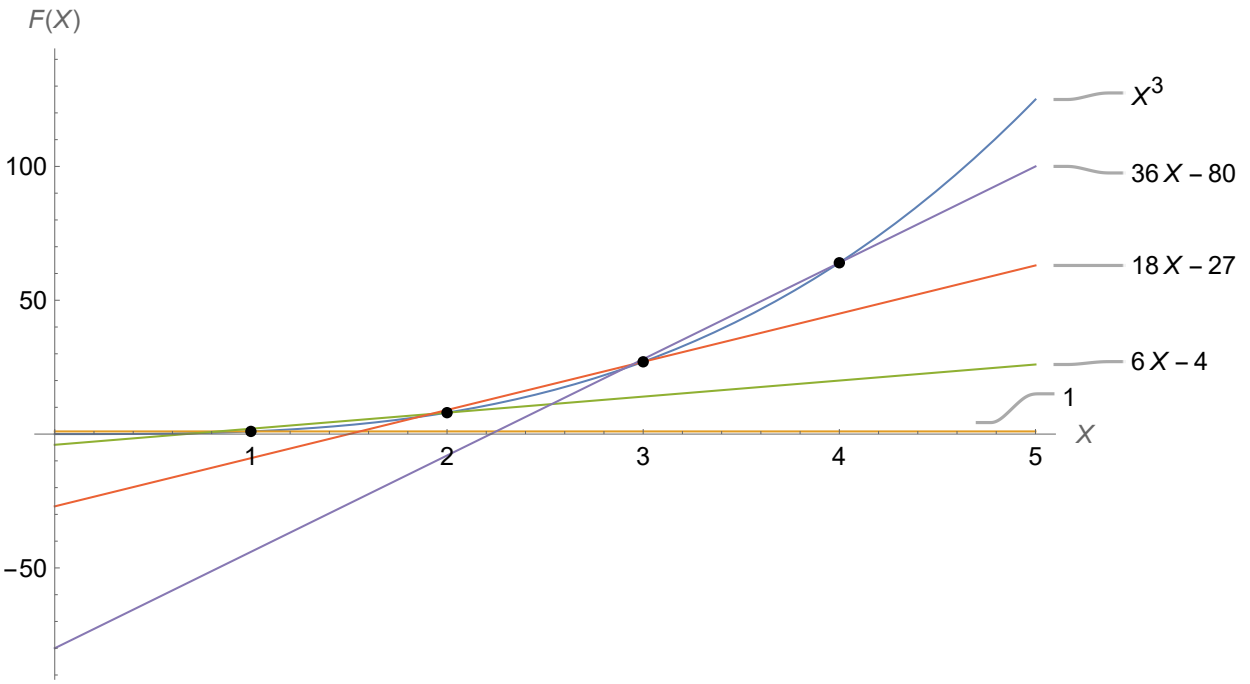

Figure 3. Polynomials Q(1, n, k)

1.5. **Polynomial Q(1,X,N) Table of values for N = 6.**

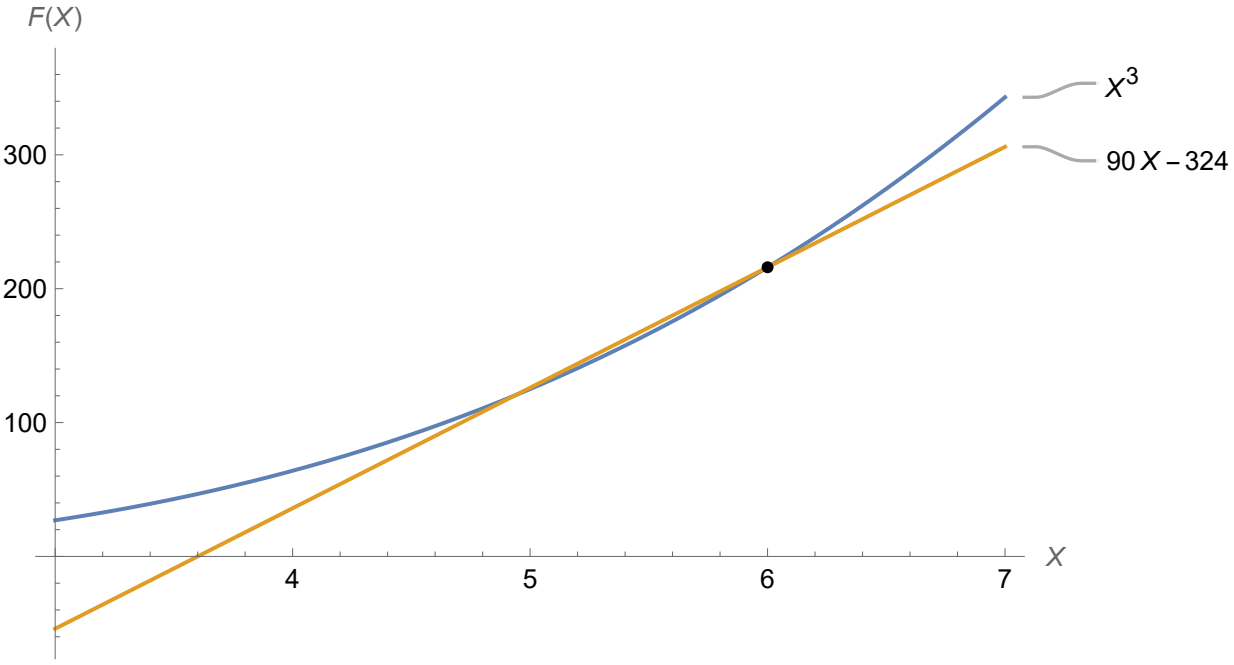

Figure 4. Polynomial plot Q(1, X, 6) with cubes

1.6. **Polynomial Q(1,X,6) plot with cubes.**

1.7. **Polynomials P(2,X,N).**

$$P(2,X,0) = 0$$

$$P(2,X,1) = 30X^2 - 60X + 31$$

$$P(2,X,2) = 150X^2 - 540X + 512$$

$$P(2,X,3) = 420X^2 - 2160X + 2943$$

$$P(2,X,4) = 900X^2 - 6000X + 10624$$

$$P(2,X,5) = 1650X^2 - 13500X + 29375$$

$$P(2,X,6) = 2730X^2 - 26460X + 68256$$

$$P(2,X,7) = 4200X^2 - 47040X + 140287$$

$$P(2,X,8) = 6120X^2 - 77760X + 263168$$

$$P(2,X,9) = 8550X^2 - 121500X + 459999$$

$$P(2,X,10) = 11550X^2 - 181500X + 760000$$

$$P(2,X,11) = 15180X^2 - 261360X + 1199231$$

$$P(2,X,12) = 19500X^2 - 365040X + 1821312$$

$$P(2,X,13) = 24570X^2 - 496860X + 2678143$$

$$P(2,X,14) = 30450X^2 - 661500X + 3830624$$

$$P(2,X,15) = 37200X^2 - 864000X + 5349375$$

$$P(2,X,16) = 44880X^2 - 1109760X + 7315456$$

$$P(2,X,17) = 53550X^2 - 1404540X + 9821087$$

$$P(2,X,18) = 63270X^2 - 1754460X + 12970368$$

$$P(2,X,19) = 74100X^2 - 2166000X + 16879999$$

$$P(2,X,20) = 86100X^2 - 2646000X + 21680000$$

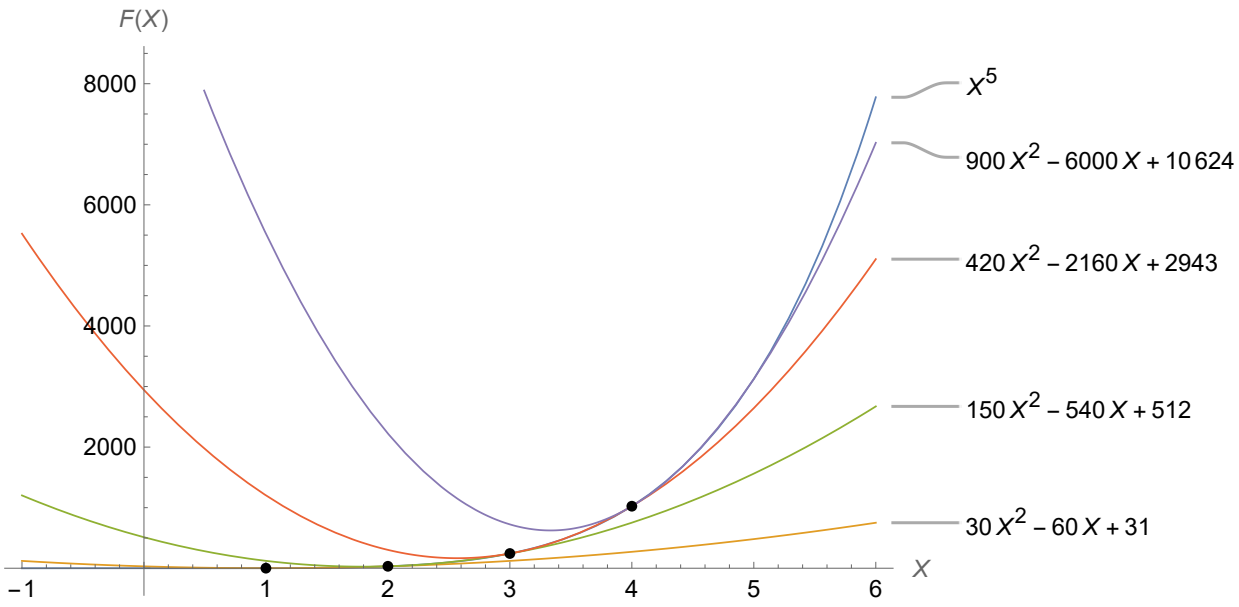

Figure 5. Polynomials P(2, n, k)

1.8. **Polynomial P(2,X,N) Table of values for N = 4.**

1.9. **Polynomials Q(2,X,N).**

$$Q(2,X,0) = 0$$

$$Q(2,X,1) = 1$$

$$Q(2,X,2) = 30X^2 - 60X + 32$$

$$Q(2,X,3) = 150X^2 - 540X + 513$$

$$Q(2,X,4) = 420X^2 - 2160X + 2944$$

$$Q(2,X,5) = 900X^2 - 6000X + 10625$$

$$Q(2,X,6) = 1650X^2 - 13500X + 29376$$

$$Q(2,X,7) = 2730X^2 - 26460X + 68257$$

$$Q(2,X,8) = 4200X^2 - 47040X + 140288$$

$$Q(2,X,9) = 6120X^2 - 77760X + 263169$$

$$Q(2,X,10) = 8550X^2 - 121500X + 460000$$

$$Q(2,X,11) = 11550X^2 - 181500X + 760001$$

$$Q(2,X,12) = 15180X^2 - 261360X + 1199232$$

$$Q(2,X,13) = 19500X^2 - 365040X + 1821313$$

$$Q(2,X,14) = 24570X^2 - 496860X + 2678144$$

$$Q(2,X,15) = 30450X^2 - 661500X + 3830625$$

$$Q(2,X,16) = 37200X^2 - 864000X + 5349376$$

$$Q(2,X,17) = 44880X^2 - 1109760X + 7315457$$

$$Q(2,X,18) = 53550X^2 - 1404540X + 9821088$$

$$Q(2,X,19) = 63270X^2 - 1754460X + 12970369$$

$$Q(2,X,20) = 74100X^2 - 2166000X + 16880000$$

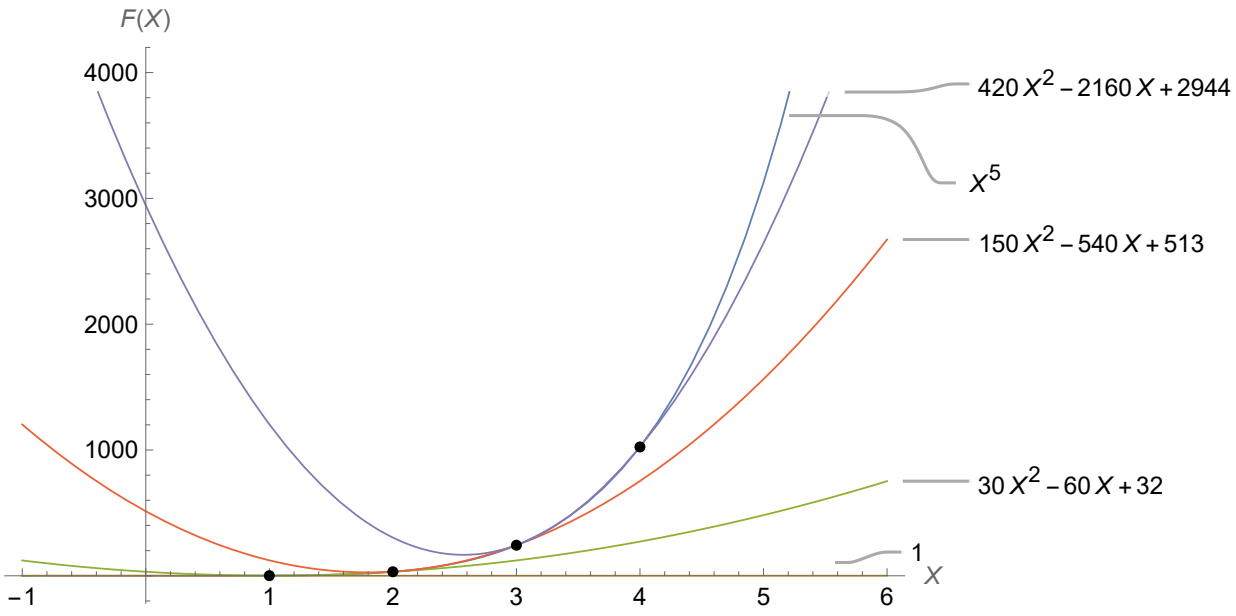

Figure 6. Polynomials Q(2, n, k)

1.10. **Polynomial Q(2,X,N) Table of values for N = 4.**

1.11. **Polynomials P(3,X,N).**

$$P(3,X,0) = 0$$

$$P(3,X,1) = 140X^3 - 420X^2 + 406X - 125$$

$$P(3,X,2) = 1260X^3 - 7140X^2 + 13818X - 9028$$

$$P(3,X,3) = 5040X^3 - 41160X^2 + 115836X - 110961$$

$$P(3,X,4) = 14000X^3 - 148680X^2 + 545860X - 684176$$

$$P(3,X,5) = 31500X^3 - 411180X^2 + 1858290X - 2871325$$

$$P(3,X,6) = 61740X^3 - 955500X^2 + 5124126X - 9402660$$

$$P(3,X,7) = 109760X^3 - 1963920X^2 + 12182968X - 25872833$$

$$P(3,X,8) = 181440X^3 - 3684240X^2 + 25945416X - 62572096$$

$$P(3,X,9) = 283500X^3 - 6439860X^2 + 50745870X - 136972701$$

$$P(3,X,10) = 423500X^3 - 10639860X^2 + 92745730X - 276971300$$

$$P(3,X,11) = 609840X^3 - 16789080X^2 + 160386996X - 524988145$$

$$P(3,X,12) = 851760X^3 - 25498200X^2 + 264896268X - 943023888$$

$$P(3,X,13) = 1159340X^3 - 37493820X^2 + 420839146X - 1618774781$$

$$P(3,X,14) = 1543500X^3 - 53628540X^2 + 646725030X - 2672907076$$

$$P(3,X,15) = 2016000X^3 - 74891040X^2 + 965662320X - 4267591425$$

$$P(3,X,16) = 2589440X^3 - 102416160X^2 + 1406064016X - 6616398080$$

$$P(3,X,17) = 3277260X^3 - 137494980X^2 + 2002403718X - 9995653693$$

$$P(3,X,18) = 4093740X^3 - 181584900X^2 + 2796022026X - 14757360516$$

$$P(3,X,19) = 5054000X^3 - 236319720X^2 + 3835983340X - 21343778801$$

$$P(3,X,20) = 6174000X^3 - 303519720X^2 + 5179983060X - 30303773200$$

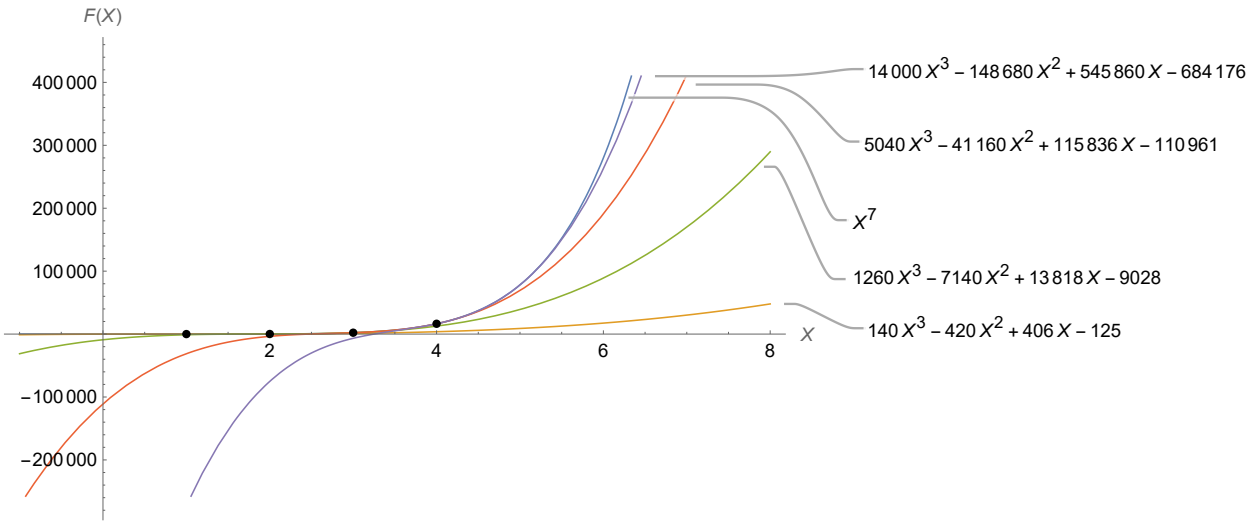

Figure 7. Polynomials P(3, n, k)

1.12. **Polynomial P(3,X,N) Table of values for N = 3.**

1.13. **Polynomials Q(3,X,N).**

$$Q(3,X,0) = 0$$

$$Q(3,X,1) = 1$$

$$Q(3,X,2) = 140X^3 - 420X^2 + 406X - 124$$

$$Q(3,X,3) = 1260X^3 - 7140X^2 + 13818X - 9027$$

$$Q(3,X,4) = 5040X^3 - 41160X^2 + 115836X - 110960$$

$$Q(3,X,5) = 14000X^3 - 148680X^2 + 545860X - 684175$$

$$Q(3,X,6) = 31500X^3 - 411180X^2 + 1858290X - 2871324$$

$$Q(3,X,7) = 61740X^3 - 955500X^2 + 5124126X - 9402659$$

$$Q(3,X,8) = 109760X^3 - 1963920X^2 + 12182968X - 25872832$$

$$Q(3,X,9) = 181440X^3 - 3684240X^2 + 25945416X - 62572095$$

$$Q(3,X,10) = 283500X^3 - 6439860X^2 + 50745870X - 136972700$$

$$Q(3,X,11) = 423500X^3 - 10639860X^2 + 92745730X - 276971299$$

$$Q(3,X,12) = 609840X^3 - 16789080X^2 + 160386996X - 524988144$$

$$Q(3,X,13) = 851760X^3 - 25498200X^2 + 264896268X - 943023887$$

$$Q(3,X,14) = 1159340X^3 - 37493820X^2 + 420839146X - 1618774780$$

$$Q(3,X,15) = 1543500X^3 - 53628540X^2 + 646725030X - 2672907075$$

$$Q(3,X,16) = 2016000X^3 - 74891040X^2 + 965662320X - 4267591424$$

$$Q(3,X,17) = 2589440X^3 - 102416160X^2 + 1406064016X - 6616398079$$

$$Q(3,X,18) = 3277260X^3 - 137494980X^2 + 2002403718X - 9995653692$$

$$Q(3,X,19) = 4093740X^3 - 181584900X^2 + 2796022026X - 14757360515$$

$$Q(3,X,20) = 5054000X^3 - 236319720X^2 + 3835983340X - 21343778800$$

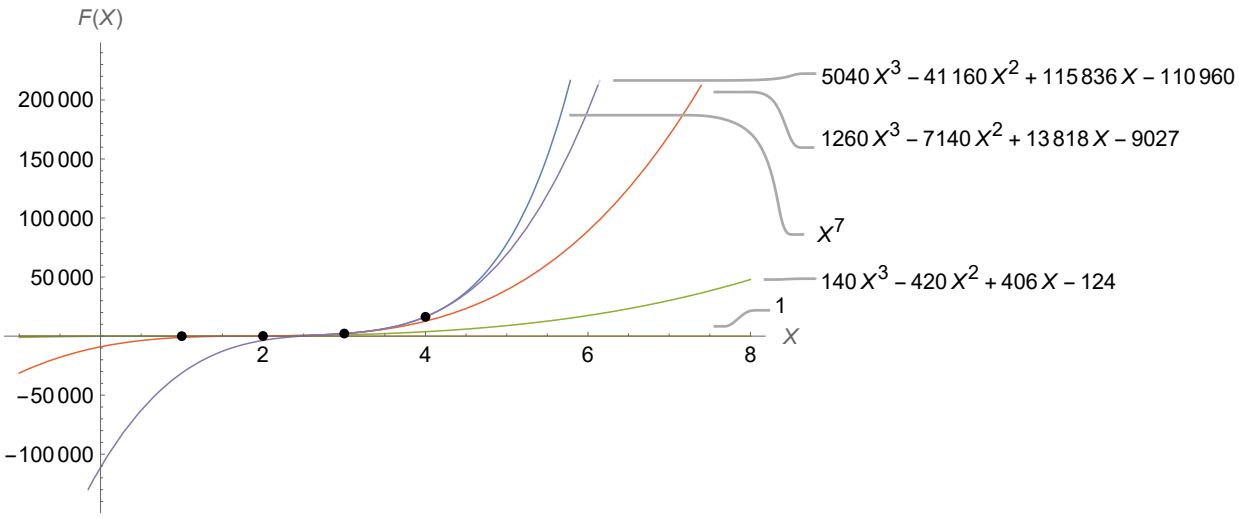

Figure 8. Polynomials Q(3, n, k)

1.14. **Polynomial Q(3,X,N) Table of values for N = 3.**

Table 1. Comparison of X^3, $P(1, X, 6) = 126X - 540$, Absolute, Relative, and Percentage Error

X	X^3	$126X - 540$	**ABS**	**Relative**	**% Error**
5.3	148.877	127.8	21.077	0.141573	14.1573
5.4	157.464	140.4	17.064	0.108368	10.8368
5.5	166.375	153.0	13.375	0.0803907	8.03907
5.6	175.616	165.6	10.016	0.0570335	5.70335
5.7	185.193	178.2	6.993	0.0377606	3.77606
5.8	195.112	190.8	4.312	0.0221001	2.21001
5.9	205.379	203.4	1.979	0.00963584	0.963584
6.0	216.0	216.0	0.0	0.0	0.0
6.1	226.981	228.6	1.619	0.00713276	0.713276
6.2	238.328	241.2	2.872	0.0120506	1.20506
6.3	250.047	253.8	3.753	0.0150092	1.50092
6.4	262.144	266.4	4.256	0.0162354	1.62354
6.5	274.625	279.0	4.375	0.0159308	1.59308
6.6	287.496	291.6	4.104	0.014275	1.4275
6.7	300.763	304.2	3.437	0.0114276	1.14276
6.8	314.432	316.8	2.368	0.00753104	0.753104
6.9	328.509	329.4	0.891	0.00271225	0.271225
7.0	343.0	342.0	1.0	0.00291545	0.291545
7.1	357.911	354.6	3.311	0.0092509	0.92509
7.2	373.248	367.2	6.048	0.0162037	1.62037
7.3	389.017	379.8	9.217	0.0236931	2.36931
7.4	405.224	392.4	12.824	0.0316467	3.16467
7.5	421.875	405.0	16.875	0.04	4.0
7.6	438.976	417.6	21.376	0.0486951	4.86951
7.7	456.533	430.2	26.333	0.0576804	5.76804
7.8	474.552	442.8	31.752	0.0669094	6.69094
7.9	493.039	455.4	37.639	0.0763408	7.63408

Table 2. Comparison of X^3, $Q(1, X, 6) = 90X - 324$, Absolute, Relative, and Percentage Error

X	X^3	$90X - 324$	**ABS**	**Relative**	**% Error**
4.5	91.125	81.0	10.125	0.111111	11.1111
4.6	97.336	90.0	7.336	0.0753678	7.53678
4.7	103.823	99.0	4.823	0.0464541	4.64541
4.8	110.592	108.0	2.592	0.0234375	2.34375
4.9	117.649	117.0	0.649	0.00551641	0.551641
5.0	125.0	126.0	1.0	0.008	0.8
5.1	132.651	135.0	2.349	0.0177081	1.77081
5.2	140.608	144.0	3.392	0.0241238	2.41238
5.3	148.877	153.0	4.123	0.027694	2.7694
5.4	157.464	162.0	4.536	0.0288066	2.88066
5.5	166.375	171.0	4.625	0.0277986	2.77986
5.6	175.616	180.0	4.384	0.0249636	2.49636
5.7	185.193	189.0	3.807	0.0205569	2.05569
5.8	195.112	198.0	2.888	0.0148018	1.48018
5.9	205.379	207.0	1.621	0.00789273	0.789273
6.0	216.0	216.0	0.0	0.0	0.0
6.1	226.981	225.0	1.981	0.0087276	0.87276
6.2	238.328	234.0	4.328	0.0181598	1.81598
6.3	250.047	243.0	7.047	0.0281827	2.81827
6.4	262.144	252.0	10.144	0.0386963	3.86963
6.5	274.625	261.0	13.625	0.0496131	4.96131
6.6	287.496	270.0	17.496	0.0608565	6.08565
6.7	300.763	279.0	21.763	0.0723593	7.23593
6.8	314.432	288.0	26.432	0.0840627	8.40627
6.9	328.509	297.0	31.509	0.0959152	9.59152
7.0	343.0	306.0	37.0	0.107872	10.7872

Table 3. Comparison of X^5, $P(2, X, 4) = 900X^2 - 6000X + 10624$, Absolute, Relative, and Percentage Error

X	X^5	$900X^2 - 6000X + 10624$	**ABS**	**Relative**	**% Error**
3.6	604.662	688.0	83.3382	0.137826	13.7826
3.7	693.44	745.0	51.5604	0.0743546	7.43546
3.8	792.352	820.0	27.6483	0.034894	3.4894
3.9	902.242	913.0	10.758	0.0119236	1.19236
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1153.0	5.56201	0.00480079	0.480079
4.2	1306.91	1300.0	6.91232	0.00528905	0.528905
4.3	1470.08	1465.0	5.08443	0.0034586	0.34586
4.4	1649.16	1648.0	1.16224	0.000704746	0.0704746
4.5	1845.28	1849.0	3.71875	0.00201528	0.201528
4.6	2059.63	2068.0	8.37024	0.00406395	0.406395
4.7	2293.45	2305.0	11.5499	0.00503605	0.503605
4.8	2548.04	2560.0	11.9603	0.00469393	0.469393
4.9	2824.75	2833.0	8.24751	0.00291973	0.291973
5.0	3125.0	3124.0	1.0	0.00032	0.032
5.1	3450.25	3433.0	17.2525	0.00500036	0.500036
5.2	3802.04	3760.0	42.0403	0.0110573	1.10573
5.3	4181.95	4105.0	76.9549	0.0184017	1.84017
5.4	4591.65	4468.0	123.65	0.0269294	2.69294
5.5	5032.84	4849.0	183.844	0.0365288	3.65288
5.6	5507.32	5248.0	259.318	0.047086	4.7086
5.7	6016.92	5665.0	351.921	0.0584885	5.84885
5.8	6563.57	6100.0	463.568	0.0706274	7.06274
5.9	7149.24	6553.0	596.243	0.0833995	8.33995
6.0	7776.0	7024.0	752.0	0.0967078	9.67078
6.1	8445.96	7513.0	932.963	0.110463	11.0463

Table 4. Comparison of X^5, $Q(2, X, 4) = 420X^2 - 2160X + 2944$, Absolute, Relative, and Percentage Error

X	X^5	$420X^2 - 2160X + 2944$	**ABS**	**Relative**	**% Error**
2.7	143.489	173.8	30.3109	0.211242	21.1242
2.8	172.104	188.8	16.6963	0.0970131	9.70131
2.9	205.111	212.2	7.08851	0.0345593	3.45593
3.0	243.0	244.0	1.0	0.00411523	0.411523
3.1	286.292	284.2	2.09151	0.00730553	0.730553
3.2	335.544	332.8	2.74432	0.00817871	0.817871
3.3	391.354	389.8	1.55393	0.00397065	0.397065
3.4	454.354	455.2	0.84576	0.00186146	0.186146
3.5	525.219	529.0	3.78125	0.00719938	0.719938
3.6	604.662	611.2	6.53824	0.0108131	1.08131
3.7	693.44	701.8	8.36043	0.0120565	1.20565
3.8	792.352	800.8	8.44832	0.0106623	1.06623
3.9	902.242	908.2	5.95801	0.00660356	0.660356
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1148.2	10.362	0.00894385	0.894385
4.2	1306.91	1280.8	26.1123	0.0199802	1.99802
4.3	1470.08	1421.8	48.2844	0.0328447	3.28447
4.4	1649.16	1571.2	77.9622	0.0472738	4.72738
4.5	1845.28	1729.0	116.281	0.0630155	6.30155
4.6	2059.63	1895.2	164.43	0.0798346	7.98346
4.7	2293.45	2069.8	223.65	0.0975169	9.75169
4.8	2548.04	2252.8	295.24	0.115869	11.5869

Table 5. Comparison of X^7, $P(3, X, 3) = 5040X^3 - 41160X^2 + 115836X - 110961$, Absolute, Relative, and Percentage Error

X	X^7	$5040X^3 - 41160X^2 + 115836X - 110961$	**ABS**	**Relative**	**% Error**
2.7	1046.04	942.12	103.915	0.0993421	9.93421
2.8	1349.29	1323.48	25.8129	0.0191307	1.91307
2.9	1724.99	1728.36	3.37237	0.00195501	0.195501
3.0	2187.00	2187.00	0.0	0.0	0.0
3.1	2751.26	2729.64	21.6214	0.00785873	0.785873
3.2	3435.97	3386.52	49.4538	0.014393	1.4393
3.3	4261.84	4187.88	73.9643	0.017355	1.7355
3.4	5252.34	5163.96	88.375	0.0168259	1.68259
3.5	6433.93	6345.00	88.9297	0.013822	1.3822
3.6	7836.42	7761.24	75.1764	0.00959321	0.959321
3.7	9493.19	9442.92	50.2677	0.00529514	0.529514
3.8	11441.6	11420.3	21.2783	0.00185973	0.185973
3.9	13723.1	13723.6	0.459332	0.0000334715	0.00334715
4.0	16384.0	16383.0	1.0	0.0000610352	0.00610352
4.1	19475.4	19428.8	46.5874	0.00239211	0.239211
4.2	23053.9	22891.3	162.613	0.0070536	0.70536
4.3	27181.9	26800.7	381.181	0.0140234	1.40234
4.4	31927.8	31187.2	740.621	0.0231968	2.31968
4.5	37366.9	36081.0	1285.95	0.034414	3.4414
4.6	43581.8	41512.4	2069.33	0.0474815	4.74815
4.7	50662.3	47511.7	3150.59	0.0621881	6.21881
4.8	58706.8	54109.1	4597.75	0.0783172	7.83172
4.9	67822.3	61334.8	6487.55	0.0956551	9.56551
5.0	78125.0	69219.0	8906.0	0.113997	11.3997
5.1	89741.1	77792.0	11949.0	0.13315	13.315

Table 6. Comparison of X^7, $Q(3, X, 3) = 1260X^3 - 7140X^2 + 13818X - 9027$, Absolute, Relative, and Percentage Error

X	X^7	$1260X^3 - 7140X^2 + 13818X - 9027$	**ABS**	**Relative**	**% Error**
1.7	41.0339	19.38	21.6539	0.527707	52.7707
1.8	61.222	60.12	1.102	0.0180001	1.80001
1.9	89.3872	94.14	4.75283	0.0531712	5.31712
2.0	128.0	129.0	1.0	0.0078125	0.78125
2.1	180.109	172.26	7.84885	0.0435784	4.35784
2.2	249.436	231.48	17.9558	0.0719856	7.19856
2.3	340.483	314.22	26.2625	0.0771333	7.71333
2.4	458.647	428.04	30.6071	0.0667335	6.67335
2.5	610.352	580.5	29.8516	0.0489088	4.89088
2.6	803.181	779.16	24.021	0.0299074	2.99074
2.7	1046.04	1031.58	14.4553	0.0138192	1.38192
2.8	1349.29	1345.32	3.97285	0.0029444	0.29444
2.9	1724.99	1727.94	2.95237	0.00171153	0.171153
3.0	2187.0	2187.0	0.0	0.0	0.0
3.1	2751.26	2730.06	21.2014	0.00770607	0.770607
3.2	3435.97	3364.68	71.2938	0.0207492	2.07492
3.3	4261.84	4098.42	163.424	0.0383459	3.83459
3.4	5252.34	4938.84	313.495	0.0596868	5.96868
3.5	6433.93	5893.5	540.43	0.0839968	8.39968
3.6	7836.42	6969.96	866.456	0.110568	11.0568
3.7	9493.19	8175.78	1317.41	0.138774	13.8774