PLOTS OF CLOSED FORMS

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ABSTRACT. Let $P(m, X, N)$ be an m-degree polynomials in $X \in \mathbb{R}$ having fixed non-negative integers m and N. In this manuscript an efficient method of spline approximation for power function is shown and discussed. Approximation technique is based on the fact that polynomial $P(m, X, N)$ approximates odd-power function X^{2m+1} for a in some neighborhood of fixed N.

CONTENTS

1. Introduction

1.1. Definitions.

Definition 1.1. *(Polynomial P(m,X,N)).*

$$P(m,X,N)=\sum_{r=0}^{m}\sum_{k=1}^{N}\mathbf{A}_{m,r}k^r(X-k)^r$$

fixed non-negative integers m and N. For example

$$P(2,X,0)=0$$

$$P(2,X,1)=30X^2-60X+31$$

$$P(2,X,2)=150X^2-540X+512$$

$$P(2,X,3)=420X^2-2160X+2943$$

Definition 1.2. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r}=\begin{cases}(2r+1)\binom{2r}{r} & \text{if } r=m\\ (2r+1)\binom{2r}{r}\sum_{d\geq 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0\leq r<m\\ 0 & \text{if } r<0 \text{ or } r>m\end{cases} \tag{1}$$

where B_t are Bernoulli numbers [1]. It is assumed that $B_1=\frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [2, 3].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

More detailed discussions about these coefficients at [4, 5, 6, 7].

Polynomial identities found, these polynomial identities allow us to assume that $N-1, N, N+1$ interval has quite precise convergence.

$$P(m,N,N) = N^{2m+1}$$

$$Q(m,N,N) = N^{2m+1}$$

$$P(m,N+1,N) = (N+1)^{2m+1} - 1 \qquad (verified)$$

$$Q(m,N-1,N) = (N-1)^{2m+1} + 1 \qquad (verified)$$

The function $U(m,N,r)$ rises as $O(N^{2m+1-r})$ having fixed values for m and r

$$U(m,N,r) = O(N^{2m+1-r})$$

$$U(3,N,1) = 70N^6 + 210N^5 + 175N^4 - 42N^2 - 7N$$

The function $V(m,N,r)$ rises as $O(N^{2m+1-r})$ having fixed values for m and r

$$V(m,N,r) = O(N^{2m+1-r})$$

$$V(3,N,2) = -14N + 140N^3 - 210N^4 + 84N^5$$

Error of approximation, fixed m and N

$$E = (X+1)^{2m+1} - P(m,X,N)$$

$$E = \sum_{k=0}^{N} \binom{N}{k} X^k - P(m,X,N)$$

About interval of convergence, we say that having fixed points m and N, the polynomial $P(m,X,N)$ approximates odd power function X^{2m+1} in some interval of convergence $a_1 \leq N \leq b_1$. For example,

$$P(1,X,6) = 126X - 540$$

so that it approximates odd power function X^3 in some neighborhood of point $X = 6$, more precisely $5.5 \leq X \leq 7.9$ with the maximal percentage error 8%.

Having $N = 10$ the convergence interval with cubes in neighborhood of $X = 10$ is: $8.9 \leq X \leq 13$ with maximal percentage error $E \leq 10\%$.

Having $N = 70$ the convergence interval with cubes in neighborhood of $X = 70$ is: $60.1 \leq X \leq 87.6$ with maximal percentage error $E \leq 10\%$.

Having $N = 150$ the convergence interval with cubes in neighborhood of $X = 150$ is: $128.4 \leq X \leq 187.1$ with maximal percentage error $E \leq 10\%$. Within interval $142.5 \leq X \leq 159.9$ the maximal percentage error $E < 1\%$.

Which implies that convergence interval rises as N rise.

Which makes the method quite fit for spline approximation.

1.2. Polynomials P(1,X,N).

$$P(1,X,0) = 0$$

$$P(1,X,1) = 6X - 5$$

$$P(1,X,2) = 18X - 28$$

$$P(1,X,3) = 36X - 81$$

$$P(1,X,4) = 60X - 176$$

$$P(1,X,5) = 90X - 325$$

$$P(1,X,6) = 126X - 540$$

$$P(1,X,7) = 168X - 833$$

$$P(1,X,8) = 216X - 1216$$

$$P(1,X,9) = 270X - 1701$$

$$P(1,X,10) = 330X - 2300$$

$$P(1,X,11) = 396X - 3025$$

$$P(1,X,12) = 468X - 3888$$

$$P(1,X,13) = 546X - 4901$$

$$P(1,X,14) = 630X - 6076$$

$$P(1,X,15) = 720X - 7425$$

$$P(1,X,16) = 816X - 8960$$

$$P(1,X,17) = 918X - 10693$$

$$P(1,X,18) = 1026X - 12636$$

$$P(1,X,19) = 1140X - 14801$$

$$P(1,X,20) = 1260X - 17200$$

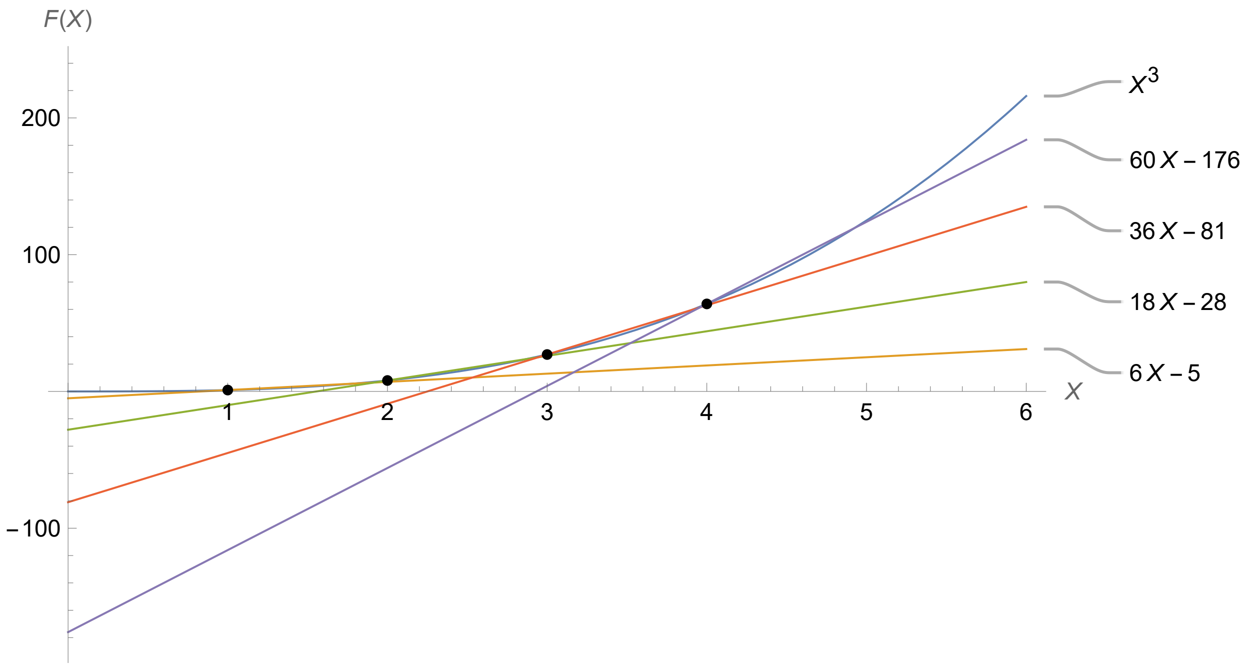

Figure 1. Polynomials $P(1, X, N)$ for N=1..4

Intervals of convergence:

- $6X - 5$: $1 \leq X \leq 1$ with $E \leq 0\%$
- $18X - 28$: $2 \leq X \leq 3$ with $E \leq 10\%$
- $36X - 81$: $2.9 \leq X \leq 4.1$ with $E \leq 5\%$
- $60X - 176$: $3.9 \leq X \leq 5.3$ with $E \leq 5\%$

1.3. **Polynomial P(1,X,6) Table of values.**

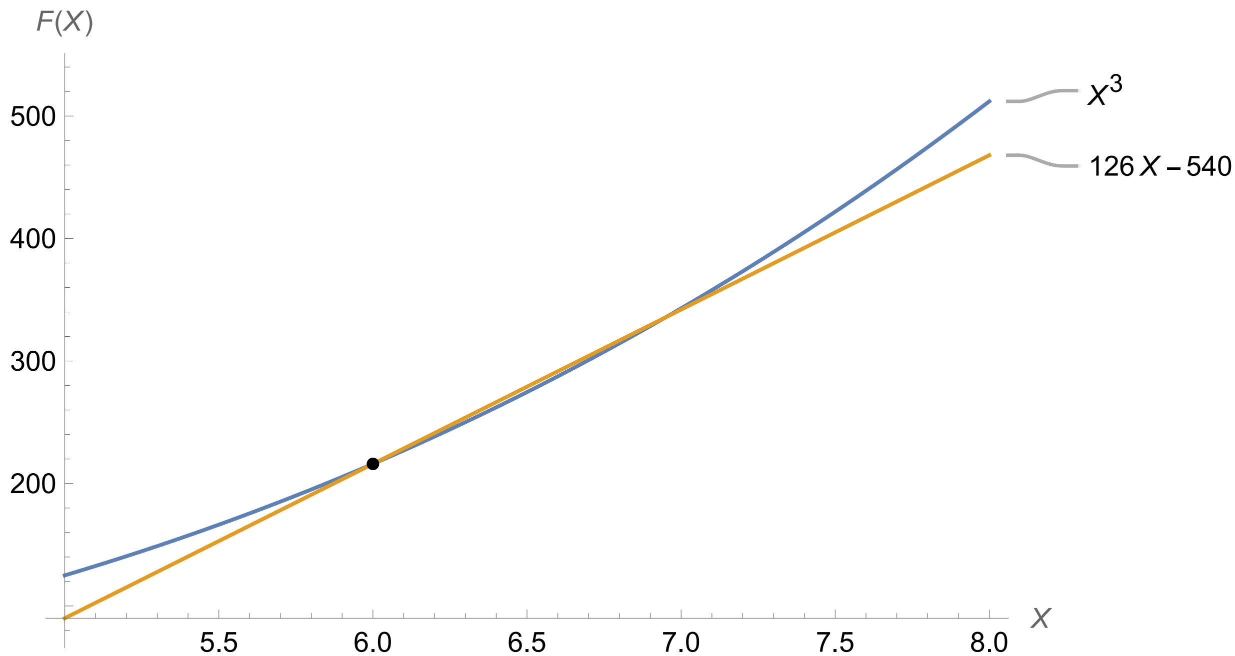

Figure 2. Polynomial plot $P(1, X, 6)$ with cubes X^3. Points of intersection $X = 6$, $X = 6.94987$. Interval of convergence: $5.9 \leq X \leq 7.2$ with $E \leq 2\%$.

1.4. **Polynomial P(1,X,6) plot with cubes.**

1.5. Polynomials Q(1,X,N).

$$Q(1,X,0) = 0$$

$$Q(1,X,1) = 1$$

$$Q(1,X,2) = 6X - 4$$

$$Q(1,X,3) = 18X - 27$$

$$Q(1,X,4) = 36X - 80$$

$$Q(1,X,5) = 60X - 175$$

$$Q(1,X,6) = 90X - 324$$

$$Q(1,X,7) = 126X - 539$$

$$Q(1,X,8) = 168X - 832$$

$$Q(1,X,9) = 216X - 1215$$

$$Q(1,X,10) = 270X - 1700$$

$$Q(1,X,11) = 330X - 2299$$

$$Q(1,X,12) = 396X - 3024$$

$$Q(1,X,13) = 468X - 3887$$

$$Q(1,X,14) = 546X - 4900$$

$$Q(1,X,15) = 630X - 6075$$

$$Q(1,X,16) = 720X - 7424$$

$$Q(1,X,17) = 816X - 8959$$

$$Q(1,X,18) = 918X - 10692$$

$$Q(1,X,19) = 1026X - 12635$$

$$Q(1,X,20) = 1140X - 14800$$

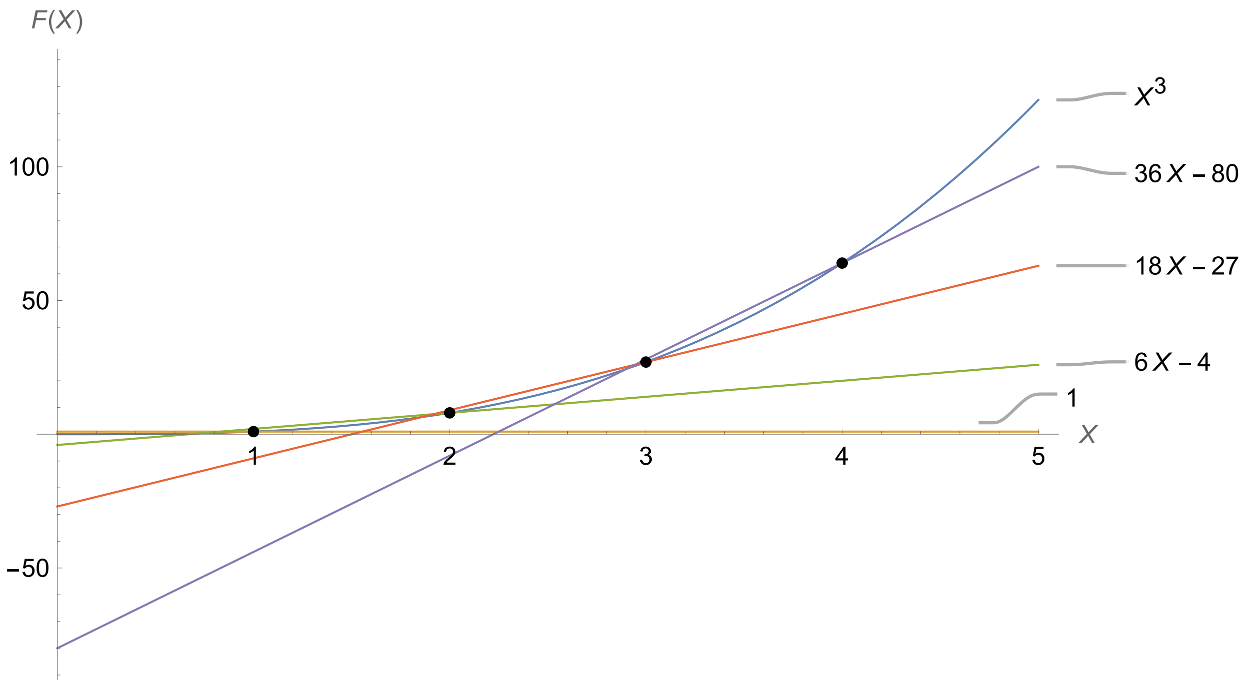

Figure 3. Polynomials Q(1, n, k)

1.6. **Polynomial Q(1,X,6) Table of values.**

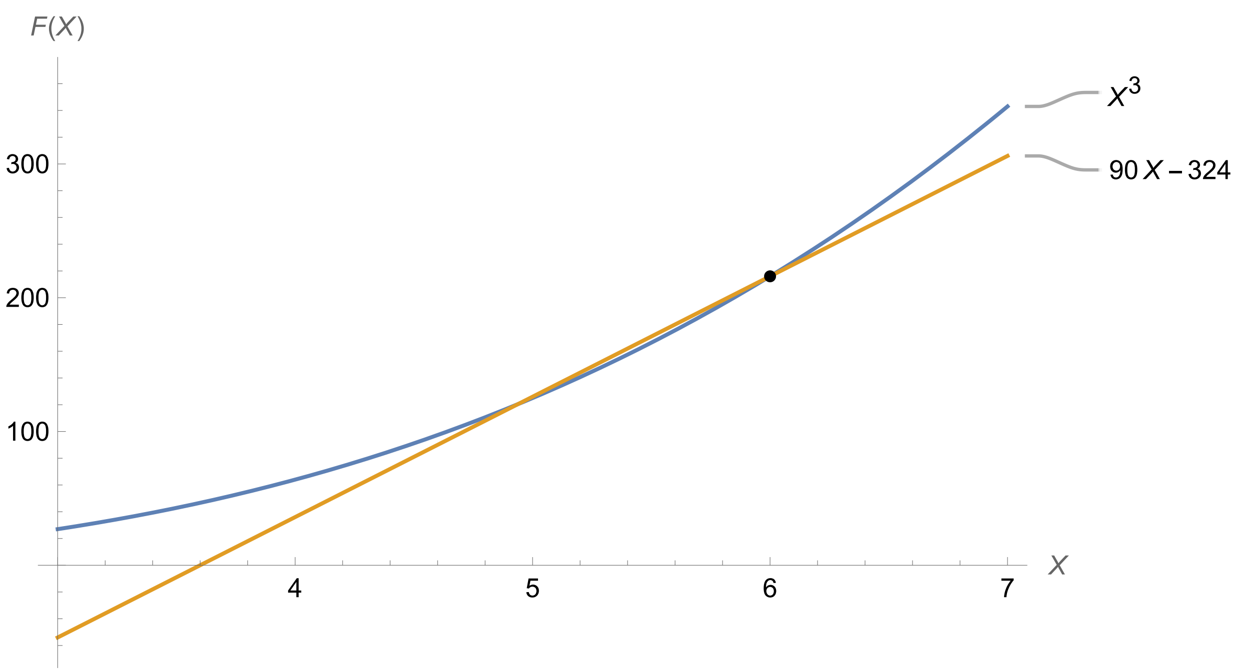

Figure 4. Polynomial plot $Q(1, X, 6)$ with cubes X^3. Points of intersection: $X = 6$, $X = 4.93725$. Interval of convergence: $4.9 \leq X \leq 6.3$ with $E \leq 3\%$.

1.7. Polynomial Q(1,X,6) plot with cubes.

1.8. Polynomials P(2,X,N).

$$P(2,X,0) = 0$$

$$P(2,X,1) = 30X^2 - 60X + 31$$

$$P(2,X,2) = 150X^2 - 540X + 512$$

$$P(2,X,3) = 420X^2 - 2160X + 2943$$

$$P(2,X,4) = 900X^2 - 6000X + 10624$$

$$P(2,X,5) = 1650X^2 - 13500X + 29375$$

$$P(2,X,6) = 2730X^2 - 26460X + 68256$$

$$P(2,X,7) = 4200X^2 - 47040X + 140287$$

$$P(2,X,8) = 6120X^2 - 77760X + 263168$$

$$P(2,X,9) = 8550X^2 - 121500X + 459999$$

$$P(2,X,10) = 11550X^2 - 181500X + 760000$$

$$P(2,X,11) = 15180X^2 - 261360X + 1199231$$

$$P(2,X,12) = 19500X^2 - 365040X + 1821312$$

$$P(2,X,13) = 24570X^2 - 496860X + 2678143$$

$$P(2,X,14) = 30450X^2 - 661500X + 3830624$$

$$P(2,X,15) = 37200X^2 - 864000X + 5349375$$

$$P(2,X,16) = 44880X^2 - 1109760X + 7315456$$

$$P(2,X,17) = 53550X^2 - 1404540X + 9821087$$

$$P(2,X,18) = 63270X^2 - 1754460X + 12970368$$

$$P(2,X,19) = 74100X^2 - 2166000X + 16879999$$

$$P(2,X,20) = 86100X^2 - 2646000X + 21680000$$

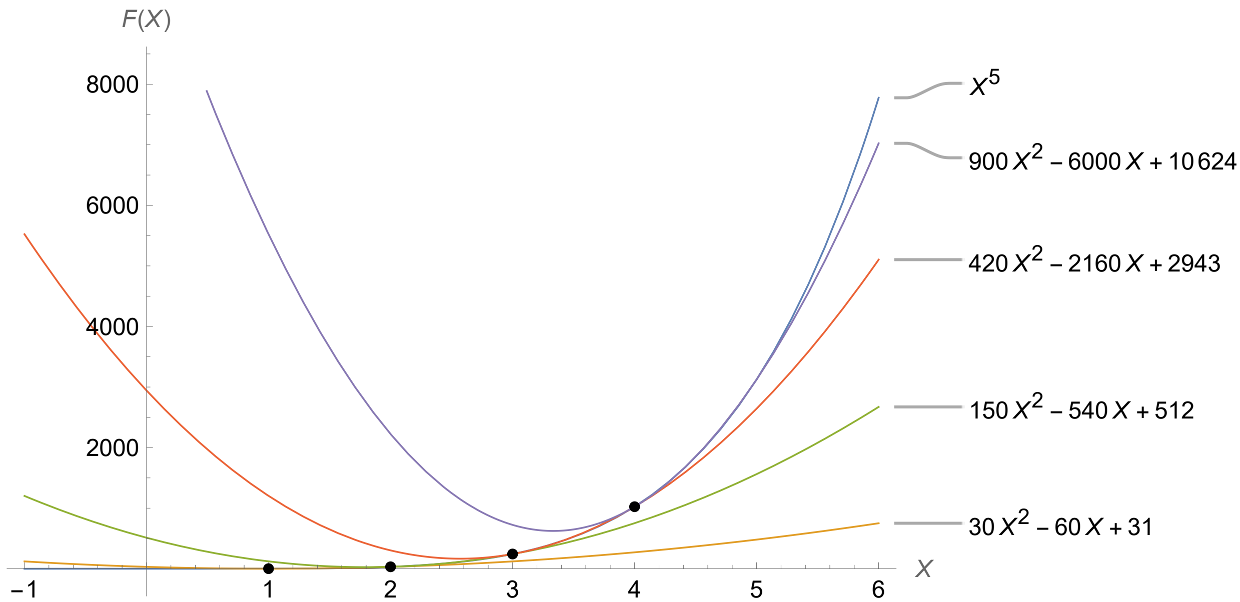

Figure 5. Polynomials P(2, n, k)

1.9. Polynomial P(2,X,4) Table of values.

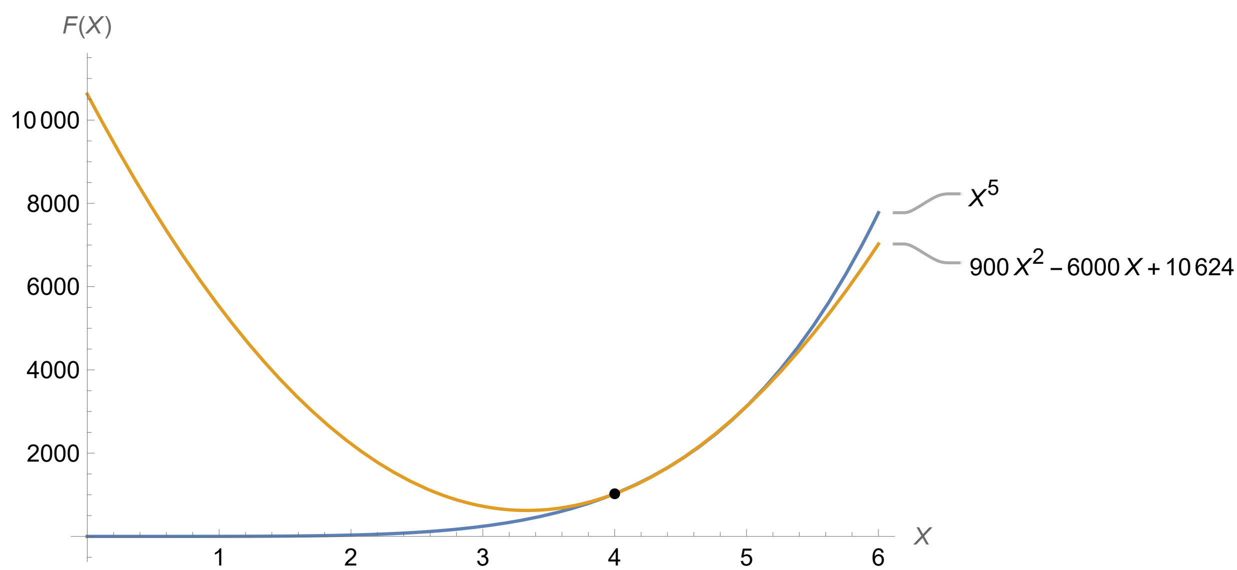

Figure 6. Polynomial plot $P(2, X, 4)$ with fifth power X^5. Points of intersection $X = 4$, $X = 4.42472$, $X = 4.99181$. Interval of convergence: $3.9 \leq X \leq 5.3$ with $E \leq 2\%$.

1.10. Polynomial P(2,X,4) plot with fifth.

1.11. **Polynomials Q(2,X,N).**

$$Q(2,X,0) = 0$$

$$Q(2,X,1) = 1$$

$$Q(2,X,2) = 30X^2 - 60X + 32$$

$$Q(2,X,3) = 150X^2 - 540X + 513$$

$$Q(2,X,4) = 420X^2 - 2160X + 2944$$

$$Q(2,X,5) = 900X^2 - 6000X + 10625$$

$$Q(2,X,6) = 1650X^2 - 13500X + 29376$$

$$Q(2,X,7) = 2730X^2 - 26460X + 68257$$

$$Q(2,X,8) = 4200X^2 - 47040X + 140288$$

$$Q(2,X,9) = 6120X^2 - 77760X + 263169$$

$$Q(2,X,10) = 8550X^2 - 121500X + 460000$$

$$Q(2,X,11) = 11550X^2 - 181500X + 760001$$

$$Q(2,X,12) = 15180X^2 - 261360X + 1199232$$

$$Q(2,X,13) = 19500X^2 - 365040X + 1821313$$

$$Q(2,X,14) = 24570X^2 - 496860X + 2678144$$

$$Q(2,X,15) = 30450X^2 - 661500X + 3830625$$

$$Q(2,X,16) = 37200X^2 - 864000X + 5349376$$

$$Q(2,X,17) = 44880X^2 - 1109760X + 7315457$$

$$Q(2,X,18) = 53550X^2 - 1404540X + 9821088$$

$$Q(2,X,19) = 63270X^2 - 1754460X + 12970369$$

$$Q(2,X,20) = 74100X^2 - 2166000X + 16880000$$

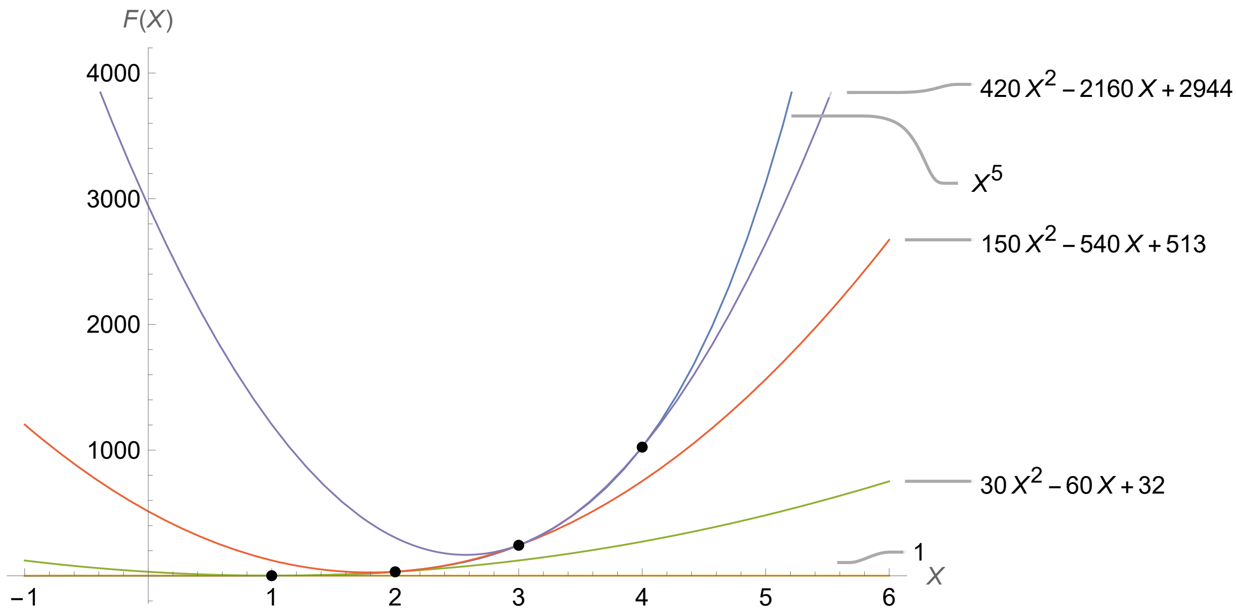

Figure 7. Polynomials Q(2, n, k)

1.12. Polynomial Q(2,X,4) Table of values.

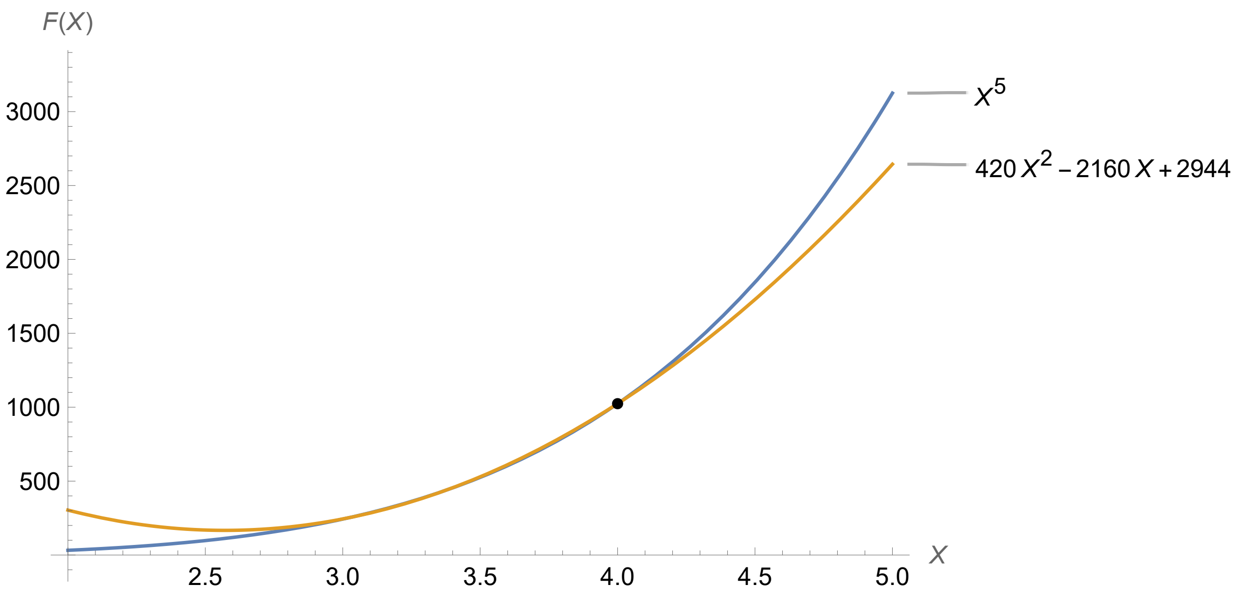

Figure 8. Polynomial plot $Q(2, X, 4)$ with fifth power X^5. Points of intersection $X = 3.02414$, $X = 3.36852$, $X = 4$. Interval of convergence: $3.0 \le X \le 4.2$ with $E \le 2\%$.

1.13. Polynomial Q(2,X,4) plot with fifth.

1.14. Polynomials P(3,X,N).

$$P(3,X,0) = 0$$

$$P(3,X,1) = 140X^3 - 420X^2 + 406X - 125$$

$$P(3,X,2) = 1260X^3 - 7140X^2 + 13818X - 9028$$

$$P(3,X,3) = 5040X^3 - 41160X^2 + 115836X - 110961$$

$$P(3,X,4) = 14000X^3 - 148680X^2 + 545860X - 684176$$

$$P(3,X,5) = 31500X^3 - 411180X^2 + 1858290X - 2871325$$

$$P(3,X,6) = 61740X^3 - 955500X^2 + 5124126X - 9402660$$

$$P(3,X,7) = 109760X^3 - 1963920X^2 + 12182968X - 25872833$$

$$P(3,X,8) = 181440X^3 - 3684240X^2 + 25945416X - 62572096$$

$$P(3,X,9) = 283500X^3 - 6439860X^2 + 50745870X - 136972701$$

$$P(3,X,10) = 423500X^3 - 10639860X^2 + 92745730X - 276971300$$

$$P(3,X,11) = 609840X^3 - 16789080X^2 + 160386996X - 524988145$$

$$P(3,X,12) = 851760X^3 - 25498200X^2 + 264896268X - 943023888$$

$$P(3,X,13) = 1159340X^3 - 37493820X^2 + 420839146X - 1618774781$$

$$P(3,X,14) = 1543500X^3 - 53628540X^2 + 646725030X - 2672907076$$

$$P(3,X,15) = 2016000X^3 - 74891040X^2 + 965662320X - 4267591425$$

$$P(3,X,16) = 2589440X^3 - 102416160X^2 + 1406064016X - 6616398080$$

$$P(3,X,17) = 3277260X^3 - 137494980X^2 + 2002403718X - 9995653693$$

$$P(3,X,18) = 4093740X^3 - 181584900X^2 + 2796022026X - 14757360516$$

$$P(3,X,19) = 5054000X^3 - 236319720X^2 + 3835983340X - 21343778801$$

$$P(3,X,20) = 6174000X^3 - 303519720X^2 + 5179983060X - 30303773200$$

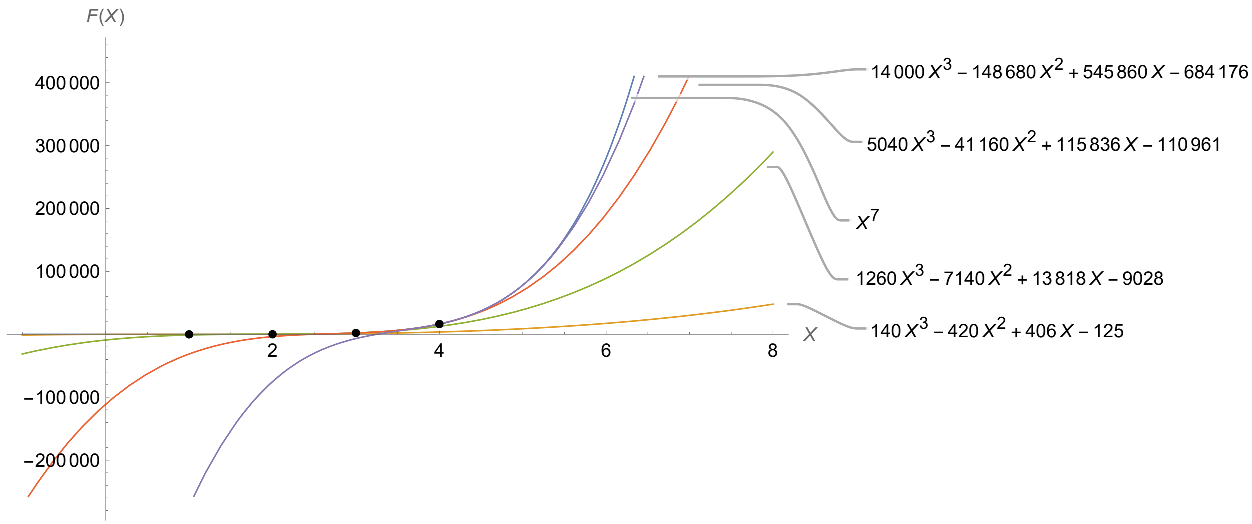

Figure 9. Polynomials P(3, n, k)

1.15. Polynomial P(3,X,3) Table of values.

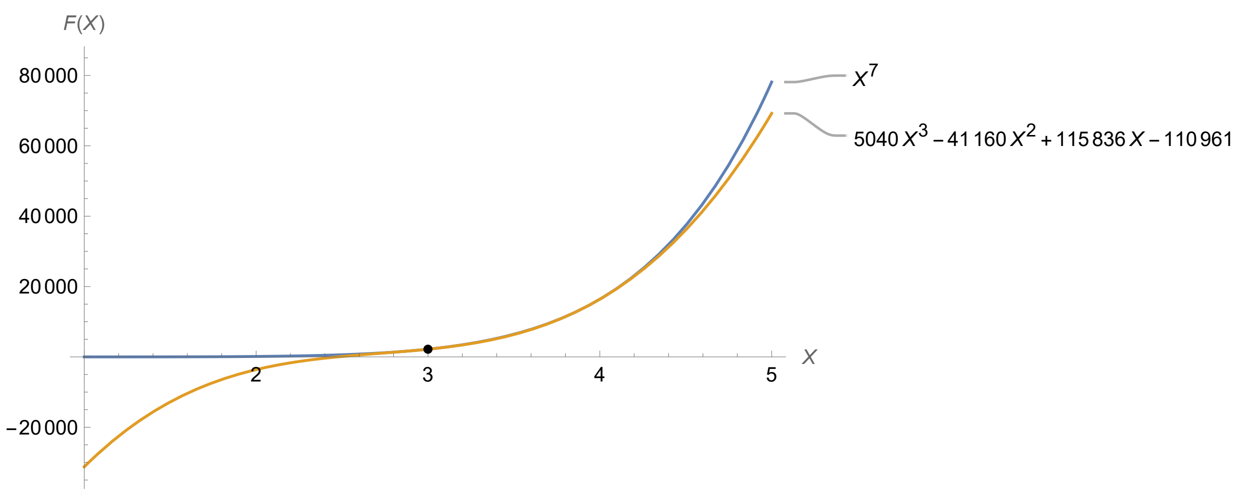

Figure 10. Polynomial plot $P(3, X, 3)$ with seventh power X^7. Points of intersection $X = 2.87643$, $X = 3$, $X = 3.89662$, $X = 3.99457$. Interval of convergence: $2.8 \leq X \leq 4.3$ with $E \leq 2\%$.

1.16. Polynomial P(3,X,3) plot with seventh.

1.17. Polynomials Q(3,X,N).

$$Q(3,X,0) = 0$$

$$Q(3,X,1) = 1$$

$$Q(3,X,2) = 140X^3 - 420X^2 + 406X - 124$$

$$Q(3,X,3) = 1260X^3 - 7140X^2 + 13818X - 9027$$

$$Q(3,X,4) = 5040X^3 - 41160X^2 + 115836X - 110960$$

$$Q(3,X,5) = 14000X^3 - 148680X^2 + 545860X - 684175$$

$$Q(3,X,6) = 31500X^3 - 411180X^2 + 1858290X - 2871324$$

$$Q(3,X,7) = 61740X^3 - 955500X^2 + 5124126X - 9402659$$

$$Q(3,X,8) = 109760X^3 - 1963920X^2 + 12182968X - 25872832$$

$$Q(3,X,9) = 181440X^3 - 3684240X^2 + 25945416X - 62572095$$

$$Q(3,X,10) = 283500X^3 - 6439860X^2 + 50745870X - 136972700$$

$$Q(3,X,11) = 423500X^3 - 10639860X^2 + 92745730X - 276971299$$

$$Q(3,X,12) = 609840X^3 - 16789080X^2 + 160386996X - 524988144$$

$$Q(3,X,13) = 851760X^3 - 25498200X^2 + 264896268X - 943023887$$

$$Q(3,X,14) = 1159340X^3 - 37493820X^2 + 420839146X - 1618774780$$

$$Q(3,X,15) = 1543500X^3 - 53628540X^2 + 646725030X - 2672907075$$

$$Q(3,X,16) = 2016000X^3 - 74891040X^2 + 965662320X - 4267591424$$

$$Q(3,X,17) = 2589440X^3 - 102416160X^2 + 1406064016X - 6616398079$$

$$Q(3,X,18) = 3277260X^3 - 137494980X^2 + 2002403718X - 9995653692$$

$$Q(3,X,19) = 4093740X^3 - 181584900X^2 + 2796022026X - 14757360515$$

$$Q(3,X,20) = 5054000X^3 - 236319720X^2 + 3835983340X - 21343778800$$

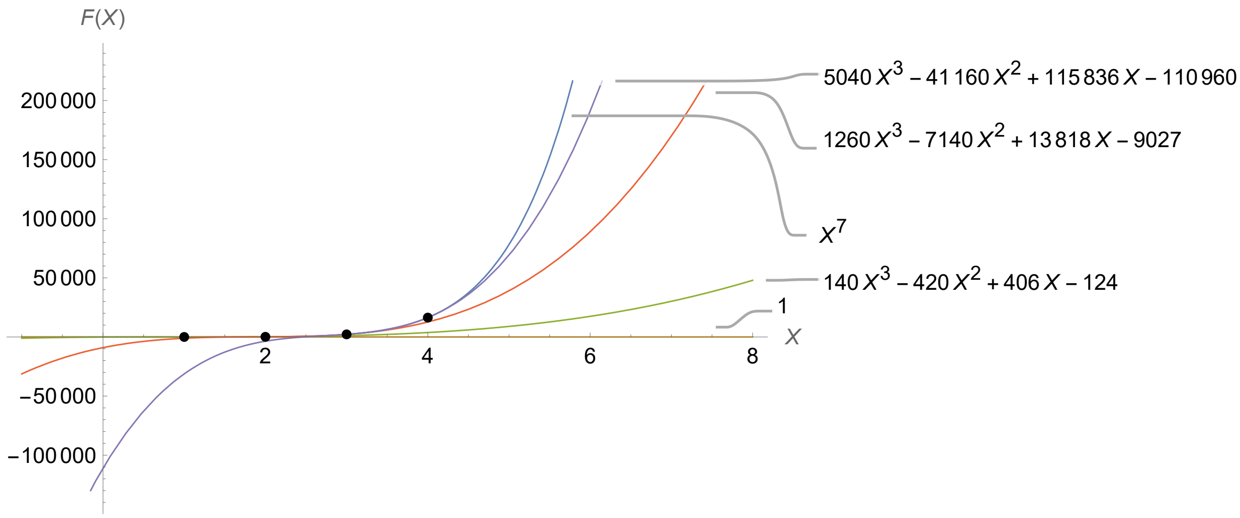

Figure 11. Polynomials Q(3, n, k)

1.18. **Polynomial Q(3,X,3) Table of values.**

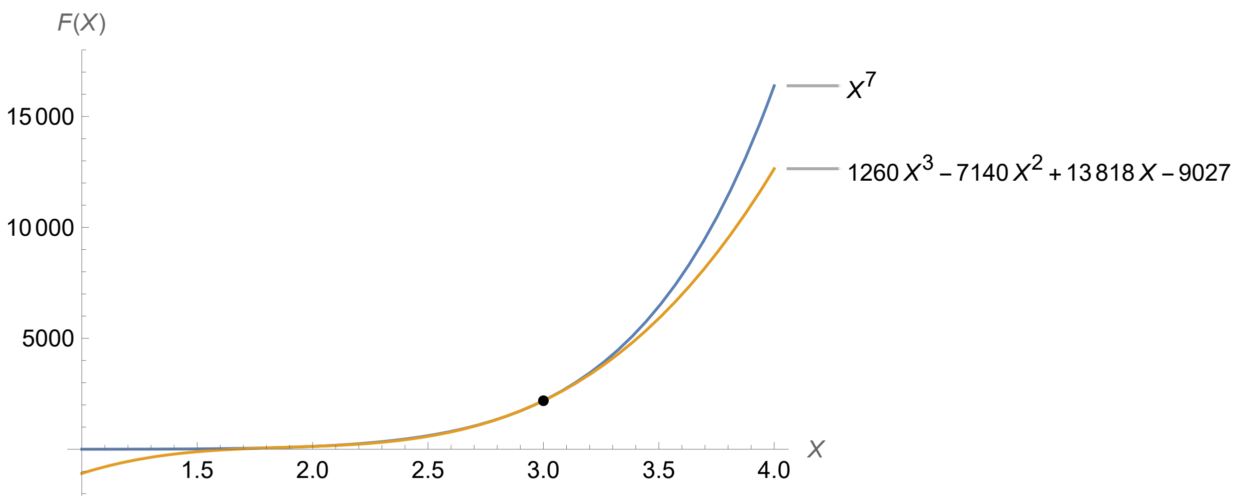

Figure 12. Polynomial plot $Q(3, X, 3)$ with seventh power X^7. Points of intersection $X = 1.80948$, $X = 2.01364$, $X = 2.84612$, $X = 3$. Interval of convergence: $2.7 \leq X \leq 3.1$ with $E \leq 2\%$.

1.19. **Polynomial Q(3,X,3) plot with seventh.**

References

[1] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book company, 1953.

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Table 2. Comparison of X^3, $P(1, X, 6) = 126X - 540$, Absolute, Relative, and Percentage Error

X	X^3	$126X - 540$	**ABS**	**Relative**	**% Error**
5.3	148.877	127.8	21.077	0.141573	14.1573
5.4	157.464	140.4	17.064	0.108368	10.8368
5.5	166.375	153.0	13.375	0.0803907	8.03907
5.6	175.616	165.6	10.016	0.0570335	5.70335
5.7	185.193	178.2	6.993	0.0377606	3.77606
5.8	195.112	190.8	4.312	0.0221001	2.21001
5.9	205.379	203.4	1.979	0.00963584	0.963584
6.0	216.0	216.0	0.0	0.0	0.0
6.1	226.981	228.6	1.619	0.00713276	0.713276
6.2	238.328	241.2	2.872	0.0120506	1.20506
6.3	250.047	253.8	3.753	0.0150092	1.50092
6.4	262.144	266.4	4.256	0.0162354	1.62354
6.5	274.625	279.0	4.375	0.0159308	1.59308
6.6	287.496	291.6	4.104	0.014275	1.4275
6.7	300.763	304.2	3.437	0.0114276	1.14276
6.8	314.432	316.8	2.368	0.00753104	0.753104
6.9	328.509	329.4	0.891	0.00271225	0.271225
7.0	343.0	342.0	1.0	0.00291545	0.291545
7.1	357.911	354.6	3.311	0.0092509	0.92509
7.2	373.248	367.2	6.048	0.0162037	1.62037
7.3	389.017	379.8	9.217	0.0236931	2.36931
7.4	405.224	392.4	12.824	0.0316467	3.16467
7.5	421.875	405.0	16.875	0.04	4.0
7.6	438.976	417.6	21.376	0.0486951	4.86951
7.7	456.533	430.2	26.333	0.0576804	5.76804
7.8	474.552	442.8	31.752	0.0669094	6.69094
7.9	493.039	455.4	37.639	0.0763408	7.63408

Table 3. Comparison of X^3, $Q(1, X, 6) = 90X - 324$, Absolute, Relative, and Percentage Error

X	X^3	$90X - 324$	**ABS**	**Relative**	**% Error**
4.5	91.125	81.0	10.125	0.111111	11.1111
4.6	97.336	90.0	7.336	0.0753678	7.53678
4.7	103.823	99.0	4.823	0.0464541	4.64541
4.8	110.592	108.0	2.592	0.0234375	2.34375
4.9	117.649	117.0	0.649	0.00551641	0.551641
5.0	125.0	126.0	1.0	0.008	0.8
5.1	132.651	135.0	2.349	0.0177081	1.77081
5.2	140.608	144.0	3.392	0.0241238	2.41238
5.3	148.877	153.0	4.123	0.027694	2.7694
5.4	157.464	162.0	4.536	0.0288066	2.88066
5.5	166.375	171.0	4.625	0.0277986	2.77986
5.6	175.616	180.0	4.384	0.0249636	2.49636
5.7	185.193	189.0	3.807	0.0205569	2.05569
5.8	195.112	198.0	2.888	0.0148018	1.48018
5.9	205.379	207.0	1.621	0.00789273	0.789273
6.0	216.0	216.0	0.0	0.0	0.0
6.1	226.981	225.0	1.981	0.0087276	0.87276
6.2	238.328	234.0	4.328	0.0181598	1.81598
6.3	250.047	243.0	7.047	0.0281827	2.81827
6.4	262.144	252.0	10.144	0.0386963	3.86963
6.5	274.625	261.0	13.625	0.0496131	4.96131
6.6	287.496	270.0	17.496	0.0608565	6.08565
6.7	300.763	279.0	21.763	0.0723593	7.23593
6.8	314.432	288.0	26.432	0.0840627	8.40627
6.9	328.509	297.0	31.509	0.0959152	9.59152
7.0	343.0	306.0	37.0	0.107872	10.7872

Table 4. Comparison of X^5, $P(2, X, 4) = 900X^2 - 6000X + 10624$, Absolute, Relative, and Percentage Error

X	X^5	$900X^2 - 6000X + 10624$	**ABS**	**Relative**	**% Error**
3.6	604.662	688.0	83.3382	0.137826	13.7826
3.7	693.44	745.0	51.5604	0.0743546	7.43546
3.8	792.352	820.0	27.6483	0.034894	3.4894
3.9	902.242	913.0	10.758	0.0119236	1.19236
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1153.0	5.56201	0.00480079	0.480079
4.2	1306.91	1300.0	6.91232	0.00528905	0.528905
4.3	1470.08	1465.0	5.08443	0.0034586	0.34586
4.4	1649.16	1648.0	1.16224	0.000704746	0.0704746
4.5	1845.28	1849.0	3.71875	0.00201528	0.201528
4.6	2059.63	2068.0	8.37024	0.00406395	0.406395
4.7	2293.45	2305.0	11.5499	0.00503605	0.503605
4.8	2548.04	2560.0	11.9603	0.00469393	0.469393
4.9	2824.75	2833.0	8.24751	0.00291973	0.291973
5.0	3125.0	3124.0	1.0	0.00032	0.032
5.1	3450.25	3433.0	17.2525	0.00500036	0.500036
5.2	3802.04	3760.0	42.0403	0.0110573	1.10573
5.3	4181.95	4105.0	76.9549	0.0184017	1.84017
5.4	4591.65	4468.0	123.65	0.0269294	2.69294
5.5	5032.84	4849.0	183.844	0.0365288	3.65288
5.6	5507.32	5248.0	259.318	0.047086	4.7086
5.7	6016.92	5665.0	351.921	0.0584885	5.84885
5.8	6563.57	6100.0	463.568	0.0706274	7.06274
5.9	7149.24	6553.0	596.243	0.0833995	8.33995
6.0	7776.0	7024.0	752.0	0.0967078	9.67078
6.1	8445.96	7513.0	932.963	0.110463	11.0463

Table 5. Comparison of X^5, $Q(2, X, 4) = 420X^2 - 2160X + 2944$, Absolute, Relative, and Percentage Error

X	X^5	$420X^2 - 2160X + 2944$	**ABS**	**Relative**	**% Error**
2.7	143.489	173.8	30.3109	0.211242	21.1242
2.8	172.104	188.8	16.6963	0.0970131	9.70131
2.9	205.111	212.2	7.08851	0.0345593	3.45593
3.0	243.0	244.0	1.0	0.00411523	0.411523
3.1	286.292	284.2	2.09151	0.00730553	0.730553
3.2	335.544	332.8	2.74432	0.00817871	0.817871
3.3	391.354	389.8	1.55393	0.00397065	0.397065
3.4	454.354	455.2	0.84576	0.00186146	0.186146
3.5	525.219	529.0	3.78125	0.00719938	0.719938
3.6	604.662	611.2	6.53824	0.0108131	1.08131
3.7	693.44	701.8	8.36043	0.0120565	1.20565
3.8	792.352	800.8	8.44832	0.0106623	1.06623
3.9	902.242	908.2	5.95801	0.00660356	0.660356
4.0	1024.0	1024.0	0.0	0.0	0.0
4.1	1158.56	1148.2	10.362	0.00894385	0.894385
4.2	1306.91	1280.8	26.1123	0.0199802	1.99802
4.3	1470.08	1421.8	48.2844	0.0328447	3.28447
4.4	1649.16	1571.2	77.9622	0.0472738	4.72738
4.5	1845.28	1729.0	116.281	0.0630155	6.30155
4.6	2059.63	1895.2	164.43	0.0798346	7.98346
4.7	2293.45	2069.8	223.65	0.0975169	9.75169
4.8	2548.04	2252.8	295.24	0.115869	11.5869

Table 6. Comparison of X^7, $P(3,X,3) = 5040X^3 - 41160X^2 + 115836X - 110961$, Absolute, Relative, and Percentage Error

X	X^7	$5040X^3 - 41160X^2 + 115836X - 110961$	ABS	Relative	% Error
2.7	1046.04	942.12	103.915	0.0993421	9.93421
2.8	1349.29	1323.48	25.8129	0.0191307	1.91307
2.9	1724.99	1728.36	3.37237	0.00195501	0.195501
3.0	2187.00	2187.00	0.0	0.0	0.0
3.1	2751.26	2729.64	21.6214	0.00785873	0.785873
3.2	3435.97	3386.52	49.4538	0.014393	1.4393
3.3	4261.84	4187.88	73.9643	0.017355	1.7355
3.4	5252.34	5163.96	88.375	0.0168259	1.68259
3.5	6433.93	6345.00	88.9297	0.013822	1.3822
3.6	7836.42	7761.24	75.1764	0.00959321	0.959321
3.7	9493.19	9442.92	50.2677	0.00529514	0.529514
3.8	11441.6	11420.3	21.2783	0.00185973	0.185973
3.9	13723.1	13723.6	0.459332	0.0000334715	0.00334715
4.0	16384.0	16383.0	1.0	0.0000610352	0.00610352
4.1	19475.4	19428.8	46.5874	0.00239211	0.239211
4.2	23053.9	22891.3	162.613	0.0070536	0.70536
4.3	27181.9	26800.7	381.181	0.0140234	1.40234
4.4	31927.8	31187.2	740.621	0.0231968	2.31968
4.5	37366.9	36081.0	1285.95	0.034414	3.4414
4.6	43581.8	41512.4	2069.33	0.0474815	4.74815
4.7	50662.3	47511.7	3150.59	0.0621881	6.21881
4.8	58706.8	54109.1	4597.75	0.0783172	7.83172
4.9	67822.3	61334.8	6487.55	0.0956551	9.56551
5.0	78125.0	69219.0	8906.0	0.113997	11.3997
5.1	89741.1	77792.0	11949.0	0.13315	13.315

Table 7. Comparison of X^7, $Q(3,X,3) = 1260X^3 - 7140X^2 + 13818X - 9027$, Absolute, Relative, and Percentage Error

X	X^7	$1260X^3 - 7140X^2 + 13818X - 9027$	**ABS**	**Relative**	**% Error**
1.7	41.0339	19.38	21.6539	0.527707	52.7707
1.8	61.222	60.12	1.102	0.0180001	1.80001
1.9	89.3872	94.14	4.75283	0.0531712	5.31712
2.0	128.0	129.0	1.0	0.0078125	0.78125
2.1	180.109	172.26	7.84885	0.0435784	4.35784
2.2	249.436	231.48	17.9558	0.0719856	7.19856
2.3	340.483	314.22	26.2625	0.0771333	7.71333
2.4	458.647	428.04	30.6071	0.0667335	6.67335
2.5	610.352	580.5	29.8516	0.0489088	4.89088
2.6	803.181	779.16	24.021	0.0299074	2.99074
2.7	1046.04	1031.58	14.4553	0.0138192	1.38192
2.8	1349.29	1345.32	3.97285	0.0029444	0.29444
2.9	1724.99	1727.94	2.95237	0.00171153	0.171153
3.0	2187.0	2187.0	0.0	0.0	0.0
3.1	2751.26	2730.06	21.2014	0.00770607	0.770607
3.2	3435.97	3364.68	71.2938	0.0207492	2.07492
3.3	4261.84	4098.42	163.424	0.0383459	3.83459
3.4	5252.34	4938.84	313.495	0.0596868	5.96868
3.5	6433.93	5893.5	540.43	0.0839968	8.39968
3.6	7836.42	6969.96	866.456	0.110568	11.0568
3.7	9493.19	8175.78	1317.41	0.138774	13.8774