POLYNOMIAL IDENTITIES AUXILIARY

PETRO KOLOSOV

Abstract. Polynomial identities auxiliary

Contents

1. Polynomial identities auxiliary 1
1.1. Central factorial numbers 1
1.2. Knuth's formula - approach 1 (to be verified all) 2
1.3. Knuth's formula - approach 2 (to be verified all) 3

1. Polynomial identities auxiliary

1.1. Central factorial numbers.

$$(2k-1)!T(2n,2k)=\frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n}$$

$$(2k-1)!T(2n,2k)=\frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n}$$

$$(2k-1)!T(2n,2k)=\frac{1}{k}2k!\left(\left\{{2n\atop 2k}\right\}-\sum_{j=k+1}^{2k}(-1)^j\binom{2k}{j}(k-j)^{2n}\right)$$

Date: July 22, 2023.

2010 *Mathematics Subject Classification.* 26E70, 05A30.

Key words and phrases. Polynomials, Polynomial identities, Faulhaber's formula, Cental Factorial Numbers .

1.2. **Knuth's formula - approach 1 (to be verified all).**

$$
\begin{aligned}
n^{2m-1} &= \sum_{k=1}^{m} (2k-1)! T(2m,2k) \binom{n+k-1}{2k-1} \quad \textit{checked} \\
&= \sum_{k=1}^{m} \mathcal{T}(m,k) \binom{n+k-1}{2k-1} \quad \textit{checked knuth1} \\
&= \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad \textit{checked knuth2} \\
&= \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad \textit{checked knuth3}
\end{aligned}
$$

$$
\begin{aligned}
&= 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \\
&= \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \quad \textit{checked knuth4} \\
&= 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \\
&= 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{n+j} \binom{n+j}{k+j} j^{2m} \\
&= 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k+j} \binom{k-j}{n-k} j^{2m}
\end{aligned}
$$

1.3. **Knuth's formula - approach 2 (to be verified all).**

$$
\begin{aligned}
n^{2m-1} &= \sum_{k=1}^{m} (2k-1)! T(2m,2k) \binom{n+k-1}{2k-1} \\
&= \sum_{k=1}^{m} \mathcal{T}(m,k) \binom{n+k-1}{2k-1} \\
&= \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k} \binom{n+k-1}{2k-1} \binom{2k}{j} (k-j)^{2m} \\
&= \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m} \\
&= \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} (k-j)^{2m}
\end{aligned}
$$