

POLYNOMIAL IDENTITIES AUXILIARY

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ABSTRACT. Polynomial identities auxiliary

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1. POLYNOMIAL IDENTITIES AUXILIARY

1.1. Central factorial numbers.

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n}$$

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n}$$

$$(2k-1)!T(2n, 2k) = \frac{1}{k} 2k! \left(\left\{ \begin{matrix} 2n \\ 2k \end{matrix} \right\} - \sum_{j=k+1}^{2k} (-1)^j \binom{2k}{j} (k-j)^{2n} \right)$$

$$T(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j \right)^n$$

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$$(2k-1)! \cdot T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n}$$

1.2. **Knuth's formula - approach 1 (to be verified all).**

$$\begin{aligned}
n^{2m-1} &= \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \quad \text{checked} \\
&= \sum_{k=1}^m \mathcal{T}(m, k) \binom{n+k-1}{2k-1} \quad \text{checked knuth1} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad \text{checked knuth2} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad \text{checked knuth3} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \quad \text{checked knuth4} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{n+j} \binom{n+j}{k+j} j^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k+j} \binom{k-j}{n-k} j^{2m}
\end{aligned}$$

1.3. Knuth's formula - approach 2 (to be verified all).

$$\begin{aligned}
n^{2m-1} &= \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \\
&= \sum_{k=1}^m \mathcal{T}(m, k) \binom{n+k-1}{2k-1} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \binom{n+k-1}{2k-1} \binom{2k}{j} (k-j)^{2m} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} (k-j)^{2m}
\end{aligned}$$