

1. FORMULAS

Eulerian numbers identities. All to be verified

$$\begin{aligned}
 \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle &= \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (k+1-j)^n \quad \text{verified} \\
 \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle &= \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{n+1}{k+1-j} j^n \quad \text{verified} \\
 &= \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{n+1}{k-j+1} j^n \\
 &= \sum_{j=0}^{k+1} (-1)^{k+1-j} \sum_{r=0}^n \binom{r}{k-j} j^n \\
 &= \sum_{r=0}^n \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{r}{k-j} j^n
 \end{aligned}$$

Worpitzky identity - approach 1

$$\begin{aligned}
 x^n &= \sum_{k=0}^{n-1} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x+k}{n} = \sum_{k=0}^{n-1} \sum_{r=0}^n \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{x+k}{n} \binom{r}{k-j} j^n \\
 &= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{x+k}{n} \binom{n+1}{k+1-j} j^n \\
 &= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{x+k}{n} \binom{n+1}{k-j+1} j^n \\
 &= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{x+k}{n} \frac{n+1}{k+1-j} \binom{n}{k-j} j^n \\
 &= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} (n+1) \frac{(-1)^{k+1-j}}{k+1-j} \binom{x+k}{n} \binom{n}{k-j} j^n \\
 &= (n+1) \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{k+1-j} \binom{x+k}{n} \binom{n}{k-j} j^n \\
 &= (n+1) \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{k+1-j} \binom{x+k}{k-j} \binom{x}{n-k} j^n
 \end{aligned}$$

However, another anti-derivative identity follows

$$\frac{x^{n+1}}{n+1} = \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{k+1-j} \binom{x+k}{k-j} \binom{x}{n-k} j^n x$$

Stirling numbers of second kind identities

$$\begin{aligned} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} &= \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n \\ \left\{ \begin{matrix} n \\ k \end{matrix} \right\} &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n \end{aligned}$$

Central factorial numbers

$$\begin{aligned} (2k-1)!T(2n, 2k) &= \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n} \\ (2k-1)!T(2n, 2k) &= \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n} \\ (2k-1)!T(2n, 2k) &= \frac{1}{k} 2k! \left(\left\{ \begin{matrix} 2n \\ 2k \end{matrix} \right\} - \sum_{j=k+1}^{2k} (-1)^j \binom{2k}{j} (k-j)^{2n} \right) \end{aligned}$$

Knuth's formula - approach 1 (to be verified all)

$$\begin{aligned} n^{2m-1} &= \sum_{k=1}^m (2k-1)!T(2m, 2k) \binom{n+k-1}{2k-1} \quad \text{checked} \\ &= \sum_{k=1}^m \mathcal{T}(m, k) \binom{n+k-1}{2k-1} \quad \text{checked knuth1} \\ &= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad \text{checked knuth2} \\ &= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad \text{checked knuth3} \end{aligned}$$

$$\begin{aligned}
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \quad \text{checked knuth4} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{n+j} \binom{n+j}{k+j} j^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k+j} \binom{k-j}{n-k} j^{2m}
\end{aligned}$$

Knuth's formula - approach 2 (to be verified all)

$$\begin{aligned}
n^{2m-1} &= \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \\
&= \sum_{k=1}^m \mathcal{T}(m, k) \binom{n+k-1}{2k-1} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \binom{n+k-1}{2k-1} \binom{2k}{j} (k-j)^{2m} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m} \\
&= \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} (k-j)^{2m} \\
&= 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} (k-j)^{2m}
\end{aligned}$$

Binomial theorem

$$\begin{aligned}
(x+y)^n &= \sum_k \binom{n}{k} x^k y^{n-k} \\
(x+y)^n &= \sum_k \binom{n}{k} (x+y-1)^k \\
(x+y)^n &= \sum_k \binom{n}{k} \sum_r (-1)^{k-r} \binom{k}{r} (x+y)^r \\
(x+y)^n &= \sum_k \binom{n}{k} \sum_r \binom{k}{r} x^r (y-1)^{k-r} \\
(x+y)^n &= \sum_k \binom{n}{k} \sum_r \binom{k}{r} y^r (x-1)^{k-r}
\end{aligned}$$

However,

$$\begin{aligned}
x^t &= \sum_r \binom{t}{r} (x-1)^r \\
&= \sum_r \sum_k \binom{t}{r} \binom{r}{k} (-1)^{r-k} x^k \\
&= \sum_r \sum_k \binom{t}{k} \binom{t-k}{r-k} (-1)^{r-k} x^k \\
&= \sum_r \sum_k \binom{t}{k} \binom{t-k}{t-r} (-1)^{r-k} x^k
\end{aligned}$$

Faulhaber's formula Version 1

$$\begin{aligned}
\sum_{k=0}^n k^m &= \frac{1}{m+1} \sum_{k=0}^m (-1)^k \binom{m+1}{k} B_k n^{m-k+1} \\
\sum_{k=0}^n k^m &= \sum_{k=0}^m \frac{(-1)^k}{k} \binom{m}{k-1} B_k n^{m-k+1} \\
\sum_{k=0}^n k^m &= \sum_{k=0}^m \frac{(-1)^k}{k} \binom{m}{k-1} B_k n^{m-(k-1)} \\
\sum_{k=0}^n k^m &= \frac{1}{m+1} \sum_{k=0}^m (-1)^{m-k} \binom{m+1}{m-k} B_k n^{k-1} \\
\sum_{k=0}^n k^m &= \sum_{k=0}^m \frac{(-1)^{m-k}}{m-k} \binom{m}{m-k-1} B_k n^{k-1}
\end{aligned}$$

Two-sided Faulhaber's formula

$$S_p(n) = \sum_{k=1}^n k^p$$

$$n^{2m+1} = \sum_{r=0}^m \sum_{t=0}^r (-1)^t A_{m,r} S_{2t-r}(n) \binom{r}{t} n^r$$