

POLYNOMIAL IDENTITIES INVOLVING CENTRAL FACTORIAL NUMBERS

PETRO KOLOSOV

ABSTRACT. Central factorial numbers often appear in literature, for instance, in Riordan's "Combinatorial identities", D. E. Knuth's work entitled "Johann Faulhaber and Sums of Powers" and many others. In this manuscript, we start our discussion from definition of central factorial numbers (recursive and iterative), continuing with a set of identities used further in the manuscript. Then, based on odd power identities given by D. E. Knuth, we show other variations of these identities rewriting them applying derived previously identities in terms of central factorial numbers. Finally, we provide a comprehensive way to validate the results of the manuscript via supplementary Mathematica programs.

CONTENTS

1. Introduction	1
2. Conclusions	3
References	3

1. INTRODUCTION

Central factorial numbers quietly often appear in literature, like for instance, in Riordan's Combinatorial identities [1]

$$T(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j \right)^n \quad (1.1)$$

Date: July 24, 2023.

2010 Mathematics Subject Classification. 26E70, 05A30.

Key words and phrases. Polynomials, Polynomial identities, Central factorial numbers, Central factorials, Binomial identities, Riordan Combinatorial identities, Falling factorials, Power sums, Faulhaber's formula .

Also, the book [2] references central factorial numbers as

$$K_{rs} = \frac{1}{(2s)!} \sum_{t=0}^{2s} (-1)^t \binom{2s}{t} (s-t)^{2r+2}$$

D. E. Knuth gives the following recurrence for the central factorial numbers [3]

$$T(2m+2, 2k) = k^2 T(2m, 2k) + T(2m, 2k-2)$$

In The On-Line Encyclopedia of Integer Sequences, central factorial numbers appear to be defined via the following recurrence

$$\begin{cases} T(n, 1) &= 1 \\ T(n, n) &= 1 \\ T(n, k) &= T(n-1, k-1) + k^2 T(n-1, k) \end{cases}$$

It is important to note that central factorial numbers are closely related to the central difference operator δ , Newton interpolation formula and central factorials $x^{[n]}$ and could be derived respectively. The derivation of central factorial numbers by means of central difference operator δ , Newton interpolation formula and central factorials $x^{[n]}$ is shown at [4].

So there are few central factorial numbers identities at our disposal. The first is

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n} \quad (1.2)$$

Such that it is derived from the formula given by the OEIS sequence A303675 [5], that is

$$A303675(n, k) = \frac{1}{n-k+1} \sum_{j=0}^{n-k+1} (-1)^j \binom{2[n-k+1]}{j} ([n-k+1]-j)^{2n}$$

Note that the term $n-k+1$ is given only to serve proper sequence offset and can be replaced easily by iterator k so that we have arrived to the identity (1.2). The second identity is actively used in current manuscript is that

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n} \quad (1.3)$$

and it is derived from (1.2) by means of symmetry of binomial coefficients $\binom{n}{k} = \binom{n}{n-k}$. The third one and final identity we base our results on involves a partial case of the equation (1.1)

$$(2k-1)!T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n} \quad (1.4)$$

So that now let's stick to some results of the D. E. Knuth's work [3], in particular to the polynomial identities. For odd powers we have

$$n^{2m-1} = \sum_{k=1}^m (2k-1)!T(2m, 2k) \binom{n+k-1}{2k-1} \quad (1.5)$$

And for any natural m we have polynomial identity

$$x^m = \sum_{k=1}^m T(m, k) x^{[k]} \quad (1.6)$$

where $x^{[k]}$ denotes central factorial defined by

$$x^{[n]} = x \left(x + \frac{n}{2} - 1 \right)^{n-1}$$

where $(n)^k = n(n-1)(n-2) \cdots (n-k+1)$ denotes falling factorial in Knuth's notation. In particular,

$$x^{[n]} = x \left(x + \frac{n}{2} - 1 \right) \left(x + \frac{n}{2} - 1 \right) \cdots \left(x + \frac{n}{2} - n + 1 \right) = x \prod_{k=1}^{n-1} \left(x + \frac{n}{2} - k \right)$$

2. CONCLUSIONS

Conclusions of your manuscript.

REFERENCES

- [1] J. Riordan. *Combinatorial Identities*. Wiley series in probability and mathematical statistics. Wiley, 1968.
- [2] L. Carlitz and John Riordan. The Divided Central Differences of Zero. *Canadian Journal of Mathematics*, 15:94–100, 1963.
- [3] Donald E Knuth. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203):277–294, 1993.
- [4] Scheuer, Markus. MathStackExchange answer 3665722/463487. Published electronically at <https://math.stackexchange.com/a/3665722/463487>, 2020.

- [5] Kolosov, Petro and Luschny, Peter. Coefficients in the sum of odd powers as expressed by Faulhaber's theorem, Entry A303675 in The On-Line Encyclopedia of Integer Sequences. Published electronically at <https://oeis.org/A303675>, 2018.

Version: Local-0.1.0

Email address: kolosovp94@gmail.com

URL: <https://kolosovpetro.github.io>