POLYNOMIAL IDENTITIES AUXILIARY

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Abstract. Polynomial identities auxiliary

Contents

1. Polynomial identities auxiliary

1.1. Central factorial numbers.

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n}$$

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n}$$

$$(2k-1)!T(2n,2k) = \frac{1}{k}2k!\left(\left\{{2n \atop 2k}\right\} - \sum_{j=k+1}^{2k}(-1)^j\binom{2k}{j}(k-j)^{2n}\right)$$

$$T(n,k) = \frac{1}{k!}\sum_{j=0}\binom{k}{j}(-1)^j\left(\frac{1}{2}k-j\right)^n$$

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$$(2k-1)!\cdot T(2n,2k)=\frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j(k-j)^{2n}$$

1.2. **Knuth's formula - approach 1 (to be verified all).**

$$n^{2m-1}=\sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1}$$

Substituting $(2k-1)!T(2n,2k)=\frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n}$ we get

$$n^{2m-1}=\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k}\binom{n+k-1}{2k-1}\binom{2k}{k-j}j^{2m}$$

By means of binomial identity $\frac{k}{n}\binom{n}{k}=\binom{n-1}{k-1}$

$$n^{2m-1}=\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{k-j}j^{2m}$$

Collapsing common terms we get

$$n^{2m-1}=2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{2k}\binom{2k}{k-j}j^{2m}$$

$$n^{2m-1}=2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{n+j}\binom{n+j}{k+j}j^{2m}$$

$$n^{2m-1}=2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k+j}\binom{k-j}{n-k}j^{2m}$$

1.3. **Knuth's formula - approach 2 (to be verified all).**

$$
\begin{aligned}
n^{2m-1} &= \sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1} \\
&= \sum_{k=1}^{m}\mathcal{T}(m,k)\binom{n+k-1}{2k-1} \\
&= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k}\binom{n+k-1}{2k-1}\binom{2k}{j}(k-j)^{2m} \\
&= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m} \\
&= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{n+k}\binom{n+k}{j}\binom{n+k-j}{2k-j}(k-j)^{2m} \\
&= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{n+k}\binom{n+k}{n+k-j}\binom{n+k-j}{2k-j}(k-j)^{2m} \\
&= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k}\frac{2k}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}(k-j)^{2m} \\
&= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}(k-j)^{2m}
\end{aligned}
$$