POLYNOMIAL IDENTITIES INVOLVING CENTRAL FACTORIAL NUMBERS

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Abstract. Central factorial numbers appear in mathematical literature quite frequently. Few of prominent works are Riordan's "Combinatorial identities" and Knuth's "Johann Faulhaber and sums of powers" along as many others. In this manuscript, we start our discussion from the definition of central factorial numbers (both, recursive and iterative) continuing with a set of identities shown further in this manuscript. Afterward, based on odd-power identities given by Knuth, we show new variations of odd-power identities applying certain relations in terms of central factorial numbers. Finally, we provide a comprehensive way to validate the results of the manuscript via supplementary Mathematica programs.

Contents

1. Introduction

Central factorial numbers appear in mathematical literature quite frequently. In Riordan's Combinatorial identities [1, p. 217] they defined as

$$k!T(n,k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j\right)^n \tag{1.1}$$

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where $T(n,k)$ is central factorial number. Also, in the book *The Divided Central Differences of Zero* [2, eq. (10a)] central factorial numbers are referenced as

$$K_{rs} = \frac{1}{(2s)!}\sum_{t=0}^{2s}(-1)^t\binom{2s}{t}(s-t)^{2r+2}$$

J.F. Steffenson mentions central factorial numbers in his book *Interpolation* [3] in context of central difference of polynomial x^r at zero

$$k!T(n,k) = \delta^m 0^r = \sum_{v=0}^{m}(-1)^v\binom{m}{v}\left(\frac{m}{2}-v\right)^r$$

D. E. Knuth gives the following recurrence for the central factorial numbers

$$T(2m+2,2k) = k^2T(2m,2k) + T(2m,2k-2)$$

See *Johann Faulhaber and sums of powers* [4, p. 284].

Central factorial numbers appear in The On-Line Encyclopedia of Integer Sequences [5] as the following recurrence relation. Let be $U(n,k) = T(2n,2k)$ then

$$U(n,k) = \begin{cases} 1, & \text{if } k=1; \\ 1, & \text{if } k=n; \\ U(n-1,k-1)+k^2U(n-1,k), & \text{otherwise} \end{cases}$$

Moreover, the OEIS sequence [6] gives polynomial representation of central factorial numbers

$$T(2n,2k) = 2\sum_{j=1}^{k}(-1)^{k-j}\frac{j^{2n}}{(k-j)!(k+j)!}$$

It is important to note that central factorial numbers are closely related to the central difference operator δ, Newton interpolation formula and central factorials $x^{[n]}$. A process of derivation of central factorial numbers by applying central difference operator δ, Newton interpolation formula and central factorials $x^{[n]}$ is provided by the MathStackExchange discussion [7]. So there are few central factorial numbers identities at our disposal. The first is

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n} \tag{1.2}$$

Such that it is derived from the formula given by the OEIS sequence A303675 [8], that is

$$A303675(n,k) = \frac{1}{n-k+1}\sum_{j=0}^{n-k+1}(-1)^j\binom{2[n-k+1]}{j}([n-k+1]-j)^{2n}$$

Note that the term $n-k+1$ is given only to serve proper sequence offset and can be replaced easily by iterator k so that we have arrived to the identity (1.2). The second identity is actively used in current manuscript is that

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n} \tag{1.3}$$

and it is derived from (1.2) by means of symmetry of binomial coefficients $\binom{n}{k} = \binom{n}{n-k}$. The third one and final identity we base our results on involves a partial case of the equation (1.1)

$$(2k-1)!T(2n,2k) = \frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j(k-j)^{2n} \tag{1.4}$$

So that now let's stick to the results of D. E. Knuth's work [4], in particular to the polynomial identities. For odd powers we have

$$n^{2m-1} = \sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1} \tag{1.5}$$

And for any natural m we have polynomial identity

$$x^m = \sum_{k=1}^{m}T(m,k)x^{[k]} \tag{1.6}$$

where $x^{[k]}$ denotes central factorial defined by

$$x^{[n]} = x\left(x+\frac{n}{2}-1\right)^{\underline{n-1}}$$

where $(n)^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1)$ denotes falling factorial in Knuth's notation. In particular,

$$x^{[n]} = x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-1\right)\cdots\left(x+\frac{n}{2}-n-1\right) = x\prod_{k=1}^{n-1}\left(x+\frac{n}{2}-k\right)$$

So that having the whole context of the topic, we can derive and discuss few other polynomial identities smoothly.

For example, given the Knuth's odd power identity (1.5) and identity in central factorial numbers (1.2) we easily obtain

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k} \binom{2k}{j} \binom{n+k-1}{2k-1} (k-j)^{2m}$$

Furthermore, by means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ we get [1]

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m}$$

Collapsing common terms k and applying binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ another polynomial identity follows

$$n^{2m-1} = 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m}$$

It is possible to derive more polynomial identities based on binomial relations of symmetry $\binom{n}{k} = \binom{n}{n-k}$ and an identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

So for now, let's smoothly switch our discussion to another example. Given an odd power identity (1.5) we can observe that odd powered polynomial can be expressed in terms of central factorial numbers and falling factorials as follows

$$n^{2m-1} = \sum_{k=1}^{m} T(2m, 2k)\, (n+k-1)^{\underline{2k-1}}$$

Because $\binom{n}{k} = \frac{1}{k!}\, (n)^{\underline{k}}$ where $(n)^{\underline{k}}$ is falling factorial.

Yet another odd power identity follows by means of (1.5) and (1.3), that is

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m}$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m}$$

[1] The majority of binomial identities used in this paper are given by Gross J. L. in his book [9], some of the chapters available online.

Collapsing common terms k and applying the binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ yields

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}j^{2m}$$

As the final part of our discussion on odd power identities, consider the polynomial identities based on odd power identity (1.5) and identity in central factorial numbers (1.4). Replacing the term $(2k-1)!T(2n,2k)$ in (1.5) we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^{j}}{2k}\binom{n+k-1}{2k-1}\binom{2k}{j}(k-j)^{2m}$$

Continuing similarly with the binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ yields

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^{j}}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m}$$

So that now we are free applying the binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ to obtain

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^{j}}{n+k}\binom{n+k}{j}\binom{n+k-j}{2k-j}(k-j)^{2m}$$

The remaining opportunity to show a few more power identities lays in replacing the corresponding coefficient $T(n,k)$ in (1.6) by identities in central factorial numbers (1.2), (1.3), (1.4)

2. Conclusions

In this manuscript, we have provided and discussed polynomial identities in terms of central factorial number. We started from literature overview including various forms and definitions of central factorial numbers continuing with polynomial identities based on those definitions. Finally, all the results of this manuscript can be validated using supplementary Mathematica programs at [10].

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