

POLYNOMIAL IDENTITIES AUXILIARY

PETRO KOLOSOV

ABSTRACT. Polynomial identities auxiliary

CONTENTS

1. Polynomial identities auxiliary	1
1.1. Central factorial numbers	1
1.2. Knuth's formula - approach 1 (to be verified all)	2

1. POLYNOMIAL IDENTITIES AUXILIARY

1.1. Central factorial numbers.

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n}$$

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n}$$

$$(2k-1)!T(2n, 2k) = \frac{1}{k} 2k! \left(\left\{ \begin{matrix} 2n \\ 2k \end{matrix} \right\} - \sum_{j=k+1}^{2k} (-1)^j \binom{2k}{j} (k-j)^{2n} \right)$$

$$T(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j \right)^n$$

Date: July 22, 2023.

2010 *Mathematics Subject Classification.* 26E70, 05A30.

Key words and phrases. Polynomials, Polynomial identities, Faulhaber's formula, Cental Factorial Numbers .

$$(2k-1)! \cdot T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n}$$

1.2. **Knuth's formula - approach 1 (to be verified all).**

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1}$$

Substituting $(2k-1)! T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity21)$$

By means of binomial identity $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity22)$$

Collapsing common terms we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity23)$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \quad (OddPowerIdentity24)$$

Also holds

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{n+j} \binom{n+j}{k+j} j^{2m} \quad (OddPowerIdentity25)$$