POLYNOMIAL IDENTITIES AUXILIARY

PETRO KOLOSOV

ABSTRACT. Polynomial identities auxiliary

CONTENTS

1. POLYNOMIAL IDENTITIES AUXILIARY

1.1. Central factorial numbers.

$$(2k-1)!\,T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{j}\binom{2k}{j}(k-j)^{2n} \quad (CFNIdentity1)$$

$$(2k-1)!\,T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n} \quad (CFNIdentity2)$$

$$(2k-1)!\,T(2n,2k) = \frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j}(k-j)^{2n} \quad (CFNIdentity3)$$

$$T(n, k) = \frac{1}{k!} \sum_{j=0} \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j\right)^n \quad (Central Factorial Number 2)$$

1.2. Knuth's formula for odd power: approach 1.

$$n^{2m-1} = \sum_{k=1}^{m} (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \quad (OddPowerIdentity(1,2,3))$$

$$n^{2m-1} = \sum_{k=1}^{m} T(2m, 2k) (n+k-1)^{\underline{2k-1}} \quad (OddPowerIdentity4)$$

Substituting $(2k-1)! T(2n, 2k) = \frac{1}{k} \sum_{j=0}^{k} (-1)^j \binom{2k}{j} (k-j)^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k} \binom{2k}{j} \binom{n+k-1}{2k-1} (k-j)^{2m} \quad (OddPowerIdentity11)$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (OddPowerIdentity12)$$

Collapsing common terms and by means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2\sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (OddPowerIdentity13)$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2\sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (OddPowerIdentity14)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2\sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (OddPowerIdentity15)$$

1.3. Knuth's formula for odd power: approach 2.

$$n^{2m-1} = \sum_{k=1}^{m} (2k-1)! \, T(2m, 2k) \binom{n+k-1}{2k-1} \quad (OddPowerIdentity(1,2,3))$$

$$n^{2m-1} = \sum_{k=1}^{m} T(2m, 2k) \, (n+k-1)^{\underline{2k-1}} \quad (OddPowerIdentity4)$$

Equation (**??**) is validated via Mathematica functions: *OddPowerIdentity1, OddPowerIdentity2, OddPowerIdentity3*. Substituting $(2k-1)! \, T(2n, 2k) = \frac{1}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{k-j} j^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity21)$$

By means of binomial identity $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity22)$$

Collapsing common terms we get

$$n^{2m-1} = 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity23)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \quad (OddPowerIdentity24)$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{n+j} \binom{n+j}{k+j} j^{2m} \quad (OddPowerIdentity25)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k+j} \binom{n-j}{n-k} j^{2m} \quad (OddPowerIdentity26)$$

1.4. **Knuth's formula for odd power: approach 3.**

$$n^{2m-1} = \sum_{k=1}^{m} (2k-1)!\,T(2m,2k)\binom{n+k-1}{2k-1} \quad (OddPowerIdentity(1,2,3))$$

$$n^{2m-1} = \sum_{k=1}^{m} T(2m,2k)\,(n+k-1)^{\underline{2k-1}} \quad (OddPowerIdentity4)$$

Let be

$$(2k-1)!\,T(2n,2k) = \frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j(k-j)^{2n}$$

And

$$n^{2m-1} = \sum_{k=1}^{m} (2k-1)!\,T(2m,2k)\binom{n+k-1}{2k-1}$$

Then

$$n^{2m-1} = \sum_{k=1}^{m}\frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j(k-j)^{2m}\binom{n+k-1}{2k-1}$$

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^j}{2k}\binom{n+k-1}{2k-1}\binom{2k}{j}(k-j)^{2m} \quad (OddPowerIdentity31)$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^j}{2k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m}$$

Collapsing common terms we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^j}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m} \quad (OddPowerIdentity32)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^j}{n+k}\binom{n+k}{j}\binom{n+k-j}{2k-j}(k-j)^{2m} \quad (OddPowerIdentity33)$$

Because the symmetry of binomial coefficients $\binom{n+k}{j} = \binom{n+k}{n+k-j}$ holds, we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^j}{n+k}\binom{n+k}{n+k-j}\binom{n+k-j}{2k-j}(k-j)^{2m}$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k} \frac{(-1)^j}{n+k}\binom{n+k}{2k-j}\binom{n-k+j}{n-k}(k-j)^{2m} \quad (OddPowerIdentity34)$$

1.5. **Central factorials power identity.** Central factorials

$$x^{[n]} = x\left(x + \frac{n}{2} - 1\right)^{\overline{n-1}} \quad (CentralFactorial1)$$

$$x^{[n]} = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 1\right)\cdots\left(x + \frac{n}{2} - 1\right)$$

$$x^{[n]} = x\prod_{k=1}^{n-1}\left(x + \frac{n}{2} - k\right) (CentralFactorial2)$$

Then we have power identity given by Knuth

$$x^m = \sum_{k=1}^{m} T(m,k)x^{[k]} \quad (PowerIdentity1, PowerIdentity2)$$