POLYNOMIAL IDENTITIES AUXILIARY

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ABSTRACT. Polynomial identities auxiliary

CONTENTS

1. POLYNOMIAL IDENTITIES AUXILIARY

1.1. Central factorial numbers.

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n} \quad (CFNIdentity1)$$

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n} \quad (CFNIdentity2)$$

$$(2k-1)!T(2n,2k) = \frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j(k-j)^{2n} \quad (CFNIdentity3)$$

$$T(n,k) = \frac{1}{k!}\sum_{j=0}\binom{k}{j}(-1)^j\left(\frac{1}{2}k-j\right)^n \quad (CentralFactorialNumber2)$$

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1.2. **Knuth's formula - approach 1 (to be verified all).**

$$n^{2m-1} = \sum_{k=1}^{m} (2k-1)! T(2m,2k) \binom{n+k-1}{2k-1}$$

Substituting $(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j \binom{2k}{j}(k-j)^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{k}\binom{2k}{j}\binom{n+k-1}{2k-1}(k-j)^{2m} \quad (OddPowerIdentity11)$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m} \quad (OddPowerIdentity12)$$

Collapsing common terms and by means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{n+k}\binom{n+k}{j}\binom{n+k-j}{2k-j}(k-j)^{2m} \quad (OddPowerIdentity13)$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{n+k}\binom{n+k}{n+k-j}\binom{n+k-j}{2k-j}(k-j)^{2m} \quad (OddPowerIdentity14)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{n+k}\binom{n+k}{2k-j}\binom{n-k+j}{n-k}(k-j)^{2m} \quad (OddPowerIdentity15)$$

1.3. **Knuth's formula - approach 2 (to be verified all).**

$$n^{2m-1} = \sum_{k=1}^{m} (2k-1)! T(2m,2k) \binom{n+k-1}{2k-1} \tag{1.1}$$

Equation (1.1) is validated via Mathematica functions: *OddPowerIdentity1*, *OddPowerIdentity2*, *OddPowerIdentity3*. Substituting $(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^{k-j}}{k}\binom{n+k-1}{2k-1}\binom{2k}{k-j}j^{2m} \quad (OddPowerIdentity21)$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{k-j}j^{2m} \quad (OddPowerIdentity22)$$

Collapsing common terms we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{2k}\binom{2k}{k-j}j^{2m} \quad (OddPowerIdentity23)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}j^{2m} \quad (OddPowerIdentity24)$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{n+j}\binom{n+j}{k+j}j^{2m} \quad (OddPowerIdentity25)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k+j}\binom{n-j}{n-k}j^{2m} \quad (OddPowerIdentity26)$$

1.4. **Knuth's formula - approach 3 (to be verified all).** Let be

$$(2k-1)!T(2n,2k) = \frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j}(k-j)^{2n}$$

And

$$n^{2m-1} = \sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1}$$

Then

$$n^{2m-1} = \sum_{k=1}^{m}\frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j}(k-j)^{2m}\binom{n+k-1}{2k-1}$$

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^{j}}{2k}\binom{n+k-1}{2k-1}\binom{2k}{j}(k-j)^{2m} \quad (OddPowerIdentity31)$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^{j}}{2k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m}$$

Collapsing common terms we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (OddPowerIdentity32)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (OddPowerIdentity33)$$

Because the symmetry of binomial coefficients $\binom{n+k}{j} = \binom{n+k}{n+k-j}$ holds, we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m}$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (OddPowerIdentity34)$$