

# POLYNOMIAL IDENTITIES AUXILIARY

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ABSTRACT. Polynomial identities auxiliary

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## 1. POLYNOMIAL IDENTITIES AUXILIARY

### 1.1. Central factorial numbers.

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n} \quad (CFNIdentity1)$$

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n} \quad (CFNIdentity2)$$

$$(2k-1)!T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n} \quad (CFNIdentity3)$$

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$$T(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^j \left( \frac{1}{2}k - j \right)^n \quad (\text{CentralFactorialNumber2})$$

### 1.2. Knuth's formula for odd power: approach 1.

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1}$$

Substituting  $(2k-1)! T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n}$  we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \binom{2k}{j} \binom{n+k-1}{2k-1} (k-j)^{2m} \quad (\text{OddPowerIdentity11})$$

By means of binomial identity  $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (\text{OddPowerIdentity12})$$

Collapsing common terms and by means of binomial identity  $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$  we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (\text{OddPowerIdentity13})$$

Because the symmetry of binomial coefficients  $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$  holds, we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (\text{OddPowerIdentity14})$$

By means of binomial identity  $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$  we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (\text{OddPowerIdentity15})$$

### 1.3. Knuth's formula for odd power: approach 2.

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \quad (1.1)$$

Equation (1.1) is validated via Mathematica functions: *OddPowerIdentity1*, *OddPowerIdentity2*, *OddPowerIdentity3*. Substituting  $(2k-1)! T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n}$  we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad (\text{OddPowerIdentity21})$$

By means of binomial identity  $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (\text{OddPowerIdentity22})$$

Collapsing common terms we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (\text{OddPowerIdentity23})$$

By means of binomial identity  $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$  we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \quad (\text{OddPowerIdentity24})$$

Because the symmetry of binomial coefficients  $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$  holds, we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{n+j} \binom{n+j}{k+j} j^{2m} \quad (\text{OddPowerIdentity25})$$

By means of binomial identity  $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$  we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k+j} \binom{n-j}{n-k} j^{2m} \quad (\text{OddPowerIdentity26})$$

**1.4. Knuth's formula for odd power: approach 3.** Let be

$$(2k-1)!T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n}$$

And

$$n^{2m-1} = \sum_{k=1}^m (2k-1)!T(2m, 2k) \binom{n+k-1}{2k-1}$$

Then

$$\begin{aligned} n^{2m-1} &= \sum_{k=1}^m \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2m} \binom{n+k-1}{2k-1} \\ n^{2m-1} &= \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{2k} \binom{n+k-1}{2k-1} \binom{2k}{j} (k-j)^{2m} \quad (\text{OddPowerIdentity31}) \end{aligned}$$

By means of binomial identity  $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{2k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m}$$

Collapsing common terms we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (\text{OddPowerIdentity32})$$

By means of binomial identity  $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$  we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (\text{OddPowerIdentity33})$$

Because the symmetry of binomial coefficients  $\binom{n+k}{j} = \binom{n+k}{n+k-j}$  holds, we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m}$$

By means of binomial identity  $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$  we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (\text{OddPowerIdentity34})$$

### 1.5. Central factorials power identity. Central factorials

$$x^{[n]} = x \left( x + \frac{n}{2} - 1 \right)^{\frac{n-1}{2}} \quad (\text{CentralFactorial1})$$

$$x^{[n]} = x \left( x + \frac{n}{2} - 1 \right) \left( x + \frac{n}{2} - 1 \right) \cdots \left( x + \frac{n}{2} - 1 \right)$$

$$x^{[n]} = x \prod_{k=1}^{n-1} \left( x + \frac{n}{2} - k \right)$$

Then we have power identity given by Knuth

$$x^m = \sum_{k=1}^m T(m, k) x^{[k]} \quad (\text{PowerIdentity1})$$