POLYNOMIAL IDENTITIES INVOLVING CENTRAL FACTORIAL NUMBERS

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Abstract. Central factorial numbers often appear in literature, for instance, in Riordan's "Combinatorial identities", D. E. Knuth's work entitled "Johann Faulhaber and Sums of Powers" and many others. In this manuscript, we start our discussion from definition of central factorial numbers (recursive and iterative), continuing with a set of identities used further in the manuscript. Then, based on odd power identities given by D. E. Knuth, we show other variations of these identities rewriting them applying derived previously identities in terms of central factorial numbers. Finally, we provide a comprehensive way to validate the results of the manuscript via supplementary Mathematica programs.

Contents

1. Introduction

Central factorial numbers quietly often appear in literature, like for instance, in Riordan's Combinatorial identities [1]

$$k!T(n,k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j\right)^n \tag{1.1}$$

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where $T(n,k)$ is central factorial number. Also, the book [2] references central factorial numbers as

$$K_{rs} = \frac{1}{(2s)!}\sum_{t=0}^{2s}(-1)^t\binom{2s}{t}(s-t)^{2r+2}$$

D. E. Knuth gives the following recurrence for the central factorial numbers [3]

$$T(2m+2,2k) = k^2T(2m,2k) + T(2m,2k-2)$$

In The On-Line Encyclopedia of Integer Sequences, central factorial numbers [4] appear to be defined via the following recurrence, let be $U(n,k) = T(2n,2k)$ then

$$U(n,k) = \begin{cases} 1, & \text{if } k = 1; \\ 1, & \text{if } k = n; \\ U(n-1,k-1) + k^2U(n-1,k), & \text{otherwise} \end{cases}$$

It is important to note that central factorial numbers are closely related to the central difference operator δ, Newton interpolation formula and central factorials $x^{[n]}$ and could be derived respectively. The derivation of central factorial numbers by means of central difference operator δ, Newton interpolation formula and central factorials $x^{[n]}$ is shown at [5]. So there are few central factorial numbers identities at our disposal. The first is

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n} \tag{1.2}$$

Such that it is derived from the formula given by the OEIS sequence A303675 [6], that is

$$A303675(n,k) = \frac{1}{n-k+1}\sum_{j=0}^{n-k+1}(-1)^j\binom{2[n-k+1]}{j}([n-k+1]-j)^{2n}$$

Note that the term $n-k+1$ is given only to serve proper sequence offset and can be replaced easily by iterator k so that we have arrived to the identity (1.2). The second identity is actively used in current manuscript is that

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n} \tag{1.3}$$

and it is derived from (1.2) by means of symmetry of binomial coefficients $\binom{n}{k} = \binom{n}{n-k}$. The third one and final identity we base our results on involves a partial case of the equation (1.1)

$$(2k-1)!T(2n,2k) = \frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j(k-j)^{2n} \tag{1.4}$$

So that now let's stick to some results of the D. E. Knuth's work [3], in particular to the polynomial identities. For odd powers we have

$$n^{2m-1} = \sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1} \tag{1.5}$$

And for any natural m we have polynomial identity

$$x^m = \sum_{k=1}^{m} T(m,k)x^{[k]} \tag{1.6}$$

where $x^{[k]}$ denotes central factorial defined by

$$x^{[n]} = x\left(x+\frac{n}{2}-1\right)^{\underline{n-1}}$$

where $(n)^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1)$ denotes falling factorial in Knuth's notation. In particular,

$$x^{[n]} = x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-1\right)\cdots\left(x+\frac{n}{2}-n-1\right) = x\prod_{k=1}^{n-1}\left(x+\frac{n}{2}-k\right)$$

So that having the whole context of the topic, we can derive and discuss few other polynomial identities smoothly.

For example, given the Knuth's odd power identity (1.5) and identity in central factorial numbers (1.2) we easily obtain

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k}\binom{2k}{j}\binom{n+k-1}{2k-1}(k-j)^{2m}$$

Furthermore, by means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ we get [1]

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m}$$

[1]The majority of binomial identities used in this paper are given by Gross J. L. in his book [7], some of the chapters available online.

Collapsing common terms k and applying binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ another polynomial identity follows

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{n+k}\binom{n+k}{j}\binom{n+k-j}{2k-j}(k-j)^{2m}$$

It is possible to derive more polynomial identities based on binomial relations of symmetry $\binom{n}{k} = \binom{n}{n-k}$ and an identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

So for now, let's smoothly switch our discussion to another example. Given an odd power identity (1.5) we can observe that odd powered polynomial can be expressed in terms of central factorial numbers and falling factorials as follows

$$n^{2m-1} = \sum_{k=1}^{m} T(2m,2k)\,(n+k-1)^{\underline{2k-1}}$$

Because $\binom{n}{k} = \frac{1}{k!}\,(n)^{\underline{k}}$ where $(n)^{\underline{k}}$ is falling factorial.

Yet another odd power identity follows by means of (1.5) and (1.3), that is

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k}\binom{n+k-1}{2k-1}\binom{2k}{k-j}j^{2m}$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{k-j}j^{2m}$$

Collapsing common terms k and applying the binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ yields

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}j^{2m}$$

As the final part of our discussion, consider the polynomial identities based on odd power identity (1.5) and identity in central factorial numbers (1.4). Replacing the term $(2k-1)!T(2n,2k)$ in (1.5) we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^j}{2k}\binom{n+k-1}{2k-1}\binom{2k}{j}(k-j)^{2m}$$

Continuing similarly with the binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ yields

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m}$$

So that now we are free applying the binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ to obtain

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m}$$

2. Conclusions

Conclusions of your manuscript.

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