

POLYNOMIAL IDENTITIES AUXILIARY

PETRO KOLOSOV

ABSTRACT. Polynomial identities auxiliary

CONTENTS

1. Polynomial identities auxiliary	1
1.1. Central factorial numbers	1
1.2. Knuth's formula for odd power: approach 1	2
1.3. Knuth's formula for odd power: approach 2	2
1.4. Knuth's formula for odd power: approach 3	3
1.5. Central factorials power identity	4

1. POLYNOMIAL IDENTITIES AUXILIARY

1.1. Central factorial numbers.

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n} \quad (CFNIdentity1)$$

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n} \quad (CFNIdentity2)$$

$$(2k-1)!T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n} \quad (CFNIdentity3)$$

Date: July 23, 2023.

2010 *Mathematics Subject Classification.* 26E70, 05A30.

Key words and phrases. Polynomials, Polynomial identities, Faulhaber's formula, Cental Factorial Numbers .

$$T(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j \right)^n \quad (\text{CentralFactorialNumber2})$$

1.2. Knuth's formula for odd power: approach 1.

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1}$$

Substituting $(2k-1)! T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \binom{2k}{j} \binom{n+k-1}{2k-1} (k-j)^{2m} \quad (\text{OddPowerIdentity11})$$

By means of binomial identity $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (\text{OddPowerIdentity12})$$

Collapsing common terms and by means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (\text{OddPowerIdentity13})$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (\text{OddPowerIdentity14})$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (\text{OddPowerIdentity15})$$

1.3. Knuth's formula for odd power: approach 2.

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \quad (1.1)$$

Equation (1.1) is validated via Mathematica functions: *OddPowerIdentity1*, *OddPowerIdentity2*, *OddPowerIdentity3*. Substituting $(2k-1)! T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad (\text{OddPowerIdentity21})$$

By means of binomial identity $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (\text{OddPowerIdentity22})$$

Collapsing common terms we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (\text{OddPowerIdentity23})$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \quad (\text{OddPowerIdentity24})$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{n+j} \binom{n+j}{k+j} j^{2m} \quad (\text{OddPowerIdentity25})$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k+j} \binom{n-j}{n-k} j^{2m} \quad (\text{OddPowerIdentity26})$$

1.4. Knuth's formula for odd power: approach 3. Let be

$$(2k-1)!T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n}$$

And

$$n^{2m-1} = \sum_{k=1}^m (2k-1)!T(2m, 2k) \binom{n+k-1}{2k-1}$$

Then

$$\begin{aligned} n^{2m-1} &= \sum_{k=1}^m \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2m} \binom{n+k-1}{2k-1} \\ n^{2m-1} &= \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{2k} \binom{n+k-1}{2k-1} \binom{2k}{j} (k-j)^{2m} \quad (\text{OddPowerIdentity31}) \end{aligned}$$

By means of binomial identity $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{2k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m}$$

Collapsing common terms we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (\text{OddPowerIdentity32})$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (\text{OddPowerIdentity33})$$

Because the symmetry of binomial coefficients $\binom{n+k}{j} = \binom{n+k}{n+k-j}$ holds, we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m}$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (\text{OddPowerIdentity34})$$

1.5. Central factorials power identity. Central factorials

$$x^{[n]} = x \left(x + \frac{n}{2} - 1 \right)^{\frac{n-1}{2}} \quad (\text{CentralFactorial1})$$

$$x^{[n]} = x \left(x + \frac{n}{2} - 1 \right) \left(x + \frac{n}{2} - 1 \right) \cdots \left(x + \frac{n}{2} - 1 \right)$$

$$x^{[n]} = x \prod_{k=1}^{n-1} \left(x + \frac{n}{2} - k \right) \quad (\text{CentralFactorial2})$$

Then we have power identity given by Knuth

$$x^m = \sum_{k=1}^m T(m, k) x^{[k]} \quad (\text{PowerIdentity1, PowerIdentity2})$$