1. Formulas

Eulerian numbers identities. All to be verified

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (k+1-j)^n \quad \textit{verified}$$

$$\begin{aligned}
\left\langle {n \atop k} \right\rangle &= \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{n+1}{k+1-j} j^n \quad \textit{verified} \\
&= \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{n+1}{k-j+1} j^n \\
&= \sum_{j=0}^{k+1} (-1)^{k+1-j} \sum_{r=0}^{n} \binom{r}{k-j} j^n \\
&= \sum_{r=0}^{n} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{r}{k-j} j^n
\end{aligned}$$

Worpitzky identity - approach 1

$$\begin{aligned}
x^n &= \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} = \sum_{k=0}^{n-1} \sum_{r=0}^{n} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{x+k}{n} \binom{r}{k-j} j^n \\
&= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{x+k}{n} \binom{n+1}{k+1-j} j^n \\
&= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{x+k}{n} \binom{n+1}{k-j+1} j^n \\
&= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{x+k}{n} \frac{n+1}{k+1-j} \binom{n}{k-j} j^n \\
&= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} (n+1) \frac{(-1)^{k+1-j}}{k+1-j} \binom{x+k}{n} \binom{n}{k-j} j^n \\
&= (n+1) \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{k+1-j} \binom{x+k}{n} \binom{n}{k-j} j^n \\
&= (n+1) \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{k+1-j} \binom{x+k}{k-j} \binom{x}{n-k} j^n
\end{aligned}$$

However, another anti-derivative identity follows

$$\frac{x^{n+1}}{n+1} = \sum_{k=0}^{n-1}\sum_{j=0}^{k+1}\frac{(-1)^{k+1-j}}{k+1-j}\binom{x+k}{k-j}\binom{x}{n-k}j^n x$$

Stirling numbers of second kind identities

$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$$

$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

Central factorial numbers

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n}$$

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n}$$

$$(2k-1)!T(2n,2k) = \frac{1}{k}2k!\left(\left\{ {2n \atop 2k} \right\} - \sum_{j=k+1}^{2k}(-1)^j\binom{2k}{j}(k-j)^{2n}\right)$$

Knuth's formula - approach 1 (to be verified all)

$$\begin{aligned}
n^{2m-1} &= \sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1} \quad \textit{checked} \\
&= \sum_{k=1}^{m}\mathcal{T}(m,k)\binom{n+k-1}{2k-1} \quad \textit{checked knuth1} \\
&= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k}\binom{n+k-1}{2k-1}\binom{2k}{k-j}j^{2m} \quad \textit{checked knuth2} \\
&= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{k-j}j^{2m} \quad \textit{checked knuth3}
\end{aligned}$$

$$= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{2k}\binom{2k}{k-j}j^{2m}$$

$$= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{k}\frac{2k}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}j^{2m} \quad \textit{checked knuth4}$$

$$= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}j^{2m}$$

$$= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{n+j}\binom{n+j}{k+j}j^{2m}$$

$$= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k+j}\binom{k-j}{n-k}j^{2m}$$ Wrong

Knuth's formula - approach 2 (to be verified all)

$$n^{2m-1} = \sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1}$$

$$= \sum_{k=1}^{m}\mathcal{T}(m,k)\binom{n+k-1}{2k-1}$$

$$= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{j}}{k}\binom{n+k-1}{2k-1}\binom{2k}{j}(k-j)^{2m}$$

$$= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{j}}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m}$$

$$= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{j}}{n+k}\binom{n+k}{j}\binom{n+k-j}{2k-j}(k-j)^{2m}$$

$$= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{j}}{n+k}\binom{n+k}{n+k-j}\binom{n+k-j}{2k-j}(k-j)^{2m}$$

$$= \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{j}}{k}\frac{2k}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}(k-j)^{2m}$$

$$= 2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{j}}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}(k-j)^{2m}$$

Binomial theorem

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$
$$(x+y)^n = \sum_k \binom{n}{k} (x+y-1)^k$$
$$(x+y)^n = \sum_k \binom{n}{k} \sum_r (-1)^{k-r} \binom{k}{r} (x+y)^r$$
$$(x+y)^n = \sum_k \binom{n}{k} \sum_r \binom{k}{r} x^r (y-1)^{k-r}$$
$$(x+y)^n = \sum_k \binom{n}{k} \sum_r \binom{k}{r} y^r (x-1)^{k-r}$$

However,

$$\begin{aligned} x^t &= \sum_r \binom{t}{r} (x-1)^r \\ &= \sum_r \sum_k \binom{t}{r} \binom{r}{k} (-1)^{r-k} x^k \\ &= \sum_r \sum_k \binom{t}{k} \binom{t-k}{r-k} (-1)^{r-k} x^k \\ &= \sum_r \sum_k \binom{t}{k} \binom{t-k}{t-r} (-1)^{r-k} x^k \end{aligned}$$

Faulhaber's formula Version 1

$$\sum_{k=0}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} (-1)^k \binom{m+1}{k} B_k n^{m-k+1}$$
$$\sum_{k=0}^{n} k^m = \sum_{k=0}^{m} \frac{(-1)^k}{k} \binom{m}{k-1} B_k n^{m-k+1}$$
$$\sum_{k=0}^{n} k^m = \sum_{k=0}^{m} \frac{(-1)^k}{k} \binom{m}{k-1} B_k n^{m-(k-1)}$$
$$\sum_{k=0}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} (-1)^{m-k} \binom{m+1}{m-k} B_k n^{k-1}$$
$$\sum_{k=0}^{n} k^m = \sum_{k=0}^{m} \frac{(-1)^{m-k}}{m-k} \binom{m}{m-k-1} B_k n^{k-1}$$

Two-sided Faulhaber's formula

$$S_p(n) = \sum_{k=1}^{n} k^p$$

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{t=0}^{r} (-1)^t A_{m,r} S_{2t-r}(n) \binom{r}{t} n^r$$