

POLYNOMIAL IDENTITIES AUXILIARY

PETRO KOLOSOV

ABSTRACT. Polynomial identities auxiliary

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1. POLYNOMIAL IDENTITIES AUXILIARY

1.1. Central factorial numbers.

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n} \quad (CFNIdentity1)$$

$$(2k-1)!T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n} \quad (CFNIdentity2)$$

$$(2k-1)!T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n} \quad (CFNIdentity3)$$

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$$T(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j \right)^n \quad (\text{CentralFactorialNumber2})$$

$$T(2n, 2k) = 2 \sum_{j=1}^k (-1)^{k-j} \frac{j^{2n}}{(k-j)!(k+j)!} \quad (\text{CentralFactorialNumber3})$$

1.2. Central factorial numbers from OEIS. $T(n, k)$ recursively defines central factorial numbers of the second kind. Let be $U(n, k) = T(2n, 2k)$ then

$$U(n, k) = \begin{cases} 1, & \text{if } k = 1; \\ 1, & \text{if } k = n; \\ U(n-1, k-1) + k^2 U(n-1, k), & \text{otherwise} \end{cases} \quad (\text{CentralFactorialNumber1})$$

From OEIS, note that this is not Central factorial number itself

$$T_{\text{OEIS}}(n, k) = \frac{1}{m} \sum_{j=0}^m (-1)^j \binom{2m}{j} (m-j)^{2n}$$

where $m = n - k + 1$. So that

$$T_{\text{OEIS}}(n, k) = \frac{1}{n-k+1} \sum_{j=0}^{n-k+1} (-1)^j \binom{2[n-k+1]}{j} ([n-k+1] - j)^{2n} \quad (1.1)$$

Furthermore, T_{OEIS} may be turned into changing the summation order from $n - k + 1$ to k

$$T_{\text{OEIS}}(n, n-k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n}$$

1.3. Knuth's formula for odd power: approach 1.

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \quad (\text{OddPowerIdentity}(1, 2, 3))$$

$$n^{2m-1} = \sum_{k=1}^m T(2m, 2k) (n+k-1)^{\overline{2k-1}} \quad (\text{OddPowerIdentity4})$$

Substituting $(2k-1)! T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^j \binom{2k}{j} (k-j)^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \binom{2k}{j} \binom{n+k-1}{2k-1} (k-j)^{2m} \quad (\text{OddPowerIdentity11})$$

By means of binomial identity $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (OddPowerIdentity12)$$

Collapsing common terms and by means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (OddPowerIdentity13)$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (OddPowerIdentity14)$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (OddPowerIdentity15)$$

1.4. Knuth's formula for odd power: approach 2.

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \quad (OddPowerIdentity(1, 2, 3))$$

$$n^{2m-1} = \sum_{k=1}^m T(2m, 2k) (n+k-1)^{2k-1} \quad (OddPowerIdentity4)$$

Equation (??) is validated via Mathematica functions: *OddPowerIdentity1*, *OddPowerIdentity2*, *OddPowerIdentity3*. Substituting $(2k-1)! T(2n, 2k) = \frac{1}{k} \sum_{j=0}^k (-1)^{k-j} \binom{2k}{k-j} j^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity21)$$

By means of binomial identity $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity22)$$

Collapsing common terms we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity23)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k-j} \binom{n+j}{k+j} j^{2m} \quad (\text{OddPowerIdentity24})$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{n+j} \binom{n+j}{k+j} j^{2m} \quad (\text{OddPowerIdentity25})$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2 \sum_{k=1}^m \sum_{j=0}^k \frac{(-1)^{k-j}}{n+k} \binom{n+k}{k+j} \binom{n-j}{n-k} j^{2m} \quad (\text{OddPowerIdentity26})$$

1.5. Knuth's formula for odd power: approach 3.

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \quad (\text{OddPowerIdentity}(1, 2, 3))$$

$$n^{2m-1} = \sum_{k=1}^m T(2m, 2k) (n+k-1)^{2k-1} \quad (\text{OddPowerIdentity4})$$

Let be

$$(2k-1)! T(2n, 2k) = \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2n}$$

And

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1}$$

Then

$$n^{2m-1} = \sum_{k=1}^m \frac{1}{2k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (k-j)^{2m} \binom{n+k-1}{2k-1} (k-j)^{2m} \quad (\text{OddPowerIdentity31})$$

By means of binomial identity $\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{2k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m}$$

Collapsing common terms we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (\text{OddPowerIdentity32})$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (\text{OddPowerIdentity33})$$

Because the symmetry of binomial coefficients $\binom{n+k}{j} = \binom{n+k}{n+k-j}$ holds, we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m}$$

By means of binomial identity $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^m \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (\text{OddPowerIdentity34})$$

1.6. Central factorials power identity. Central factorials

$$x^{[n]} = x \left(x + \frac{n}{2} - 1 \right)^{\frac{n-1}{2}} \quad (\text{CentralFactorial1})$$

$$x^{[n]} = x \left(x + \frac{n}{2} - 1 \right) \left(x + \frac{n}{2} - 1 \right) \cdots \left(x + \frac{n}{2} - n + 1 \right)$$

$$x^{[n]} = x \prod_{k=1}^{n-1} \left(x + \frac{n}{2} - k \right) \quad (\text{CentralFactorial2})$$

Then we have power identity given by Knuth

$$x^m = \sum_{k=1}^m T(m, k) x^{[k]} \quad (\text{PowerIdentity1, PowerIdentity2})$$

$$x^m = \sum_{k=1}^m T(m, k) (x-1^2)(x-2^2) \cdots (x-k^2) = \sum_{k=1}^m T(m, k) \prod_{j=0}^{k-1} (x-j^2) \quad (\text{PowerIdentity3})$$

$$\text{PowerProduct}(n, k) = \prod_{j=0}^k (n-j^2)$$