POLYNOMIAL IDENTITIES AUXILIARY

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Abstract. Polynomial identities auxiliary

Contents

1. Polynomial identities auxiliary

1.1. Central factorial numbers.

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n} \quad (CFNIdentity1)$$

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n} \quad (CFNIdentity2)$$

$$(2k-1)!T(2n,2k) = \frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j(k-j)^{2n} \quad (CFNIdentity3)$$

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$$T(n,k) = \frac{1}{k!}\sum_{j=0}\binom{k}{j}(-1)^j\left(\frac{1}{2}k-j\right)^n \quad (CentralFactorialNumber2)$$

$$T(2n,2k) = 2\sum_{j=1}^{k}(-1)^{k-j}\frac{j^{2n}}{(k-j)!(k+j)!} \quad (CentralFactorialNumber3)$$

1.2. **Central factorial numbers from OEIS.** $T(n,k)$ recursively defines central factorial numbers of the second kind. Let be $U(n,k) = T(2n,2k)$ then

$$U(n,k) = \begin{cases} 1, & \text{if } k = 1; \\ 1, & \text{if } k = n; \\ U(n-1,k-1) + k^2U(n-1,k), & \text{otherwise} \end{cases} \quad (CentralFactorialNumber1)$$

From OEIS, note that this is not Central factorial number itself

$$T_{\text{OEIS}}(n,k) = \frac{1}{m}\sum_{j=0}^{m}(-1)^j\binom{2m}{j}(m-j)^{2n}$$

where $m = n-k+1$. So that

$$T_{\text{OEIS}}(n,k) = \frac{1}{n-k+1}\sum_{j=0}^{n-k+1}(-1)^j\binom{2[n-k+1]}{j}([n-k+1]-j)^{2n} \tag{1.1}$$

Furthermore, T_{OEIS} may be turned into changing the summation order from $n-k+1$ to k

$$T_{\text{OEIS}}(n,n-k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n}$$

1.3. **Knuth's formula for odd power: approach 1.**

$$n^{2m-1} = \sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1} \quad (OddPowerIdentity(1,2,3))$$

$$n^{2m-1} = \sum_{k=1}^{m}T(2m,2k)\,(n+k-1)^{\underline{2k-1}} \quad (OddPowerIdentity4)$$

Substituting $(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k}\binom{2k}{j}\binom{n+k-1}{2k-1}(k-j)^{2m} \quad (OddPowerIdentity11)$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (OddPowerIdentity12)$$

Collapsing common terms and by means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (OddPowerIdentity13)$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j} = \binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (OddPowerIdentity14)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (OddPowerIdentity15)$$

1.4. **Knuth's formula for odd power: approach 2.**

$$n^{2m-1} = \sum_{k=1}^{m} (2k-1)! T(2m, 2k) \binom{n+k-1}{2k-1} \quad (OddPowerIdentity(1, 2, 3))$$

$$n^{2m-1} = \sum_{k=1}^{m} T(2m, 2k)\, (n+k-1)^{\underline{2k-1}} \quad (OddPowerIdentity4)$$

Equation (**??**) is validated via Mathematica functions: *OddPowerIdentity1*, *OddPowerIdentity2*, *OddPowerIdentity3*. Substituting $(2k-1)!T(2n, 2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j} j^{2n}$ we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \binom{n+k-1}{2k-1} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity21)$$

By means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^{k-j}}{k} \frac{2k}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity22)$$

Collapsing common terms we get

$$n^{2m-1} = 2\sum_{k=1}^{m}\sum_{j=0}^{k} \frac{(-1)^{k-j}}{n+k} \binom{n+k}{2k} \binom{2k}{k-j} j^{2m} \quad (OddPowerIdentity23)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k}=\binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1}=2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k-j}\binom{n+j}{k+j}j^{2m} \quad (OddPowerIdentity24)$$

Because the symmetry of binomial coefficients $\binom{n+k}{k-j}=\binom{n+k}{n+k-(k-j)}$ holds, we get

$$n^{2m-1}=2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{n+j}\binom{n+j}{k+j}j^{2m} \quad (OddPowerIdentity25)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k}=\binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1}=2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{n+k}\binom{n+k}{k+j}\binom{n-j}{n-k}j^{2m} \quad (OddPowerIdentity26)$$

1.5. **Knuth's formula for odd power: approach 3.**

$$n^{2m-1}=\sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1} \quad (OddPowerIdentity(1,2,3))$$

$$n^{2m-1}=\sum_{k=1}^{m}T(2m,2k)\,(n+k-1)^{\underline{2k-1}} \quad (OddPowerIdentity4)$$

Let be

$$(2k-1)!T(2n,2k)=\frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j}(k-j)^{2n}$$

And

$$n^{2m-1}=\sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1}$$

Then

$$n^{2m-1}=\sum_{k=1}^{m}\frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j}(k-j)^{2m}\binom{n+k-1}{2k-1}$$

$$n^{2m-1}=\sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^{j}}{2k}\binom{n+k-1}{2k-1}\binom{2k}{j}(k-j)^{2m} \quad (OddPowerIdentity31)$$

By means of binomial identity $\frac{k}{n}\binom{n}{k}=\binom{n-1}{k-1}$

$$n^{2m-1}=\sum_{k=1}^{m}\sum_{j=0}^{2k}\frac{(-1)^{j}}{2k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m}$$

Collapsing common terms we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k} \binom{2k}{j} (k-j)^{2m} \quad (OddPowerIdentity32)$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{j} \binom{n+k-j}{2k-j} (k-j)^{2m} \quad (OddPowerIdentity33)$$

Because the symmetry of binomial coefficients $\binom{n+k}{j} = \binom{n+k}{n+k-j}$ holds, we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{n+k-j} \binom{n+k-j}{2k-j} (k-j)^{2m}$$

By means of binomial identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we get

$$n^{2m-1} = \sum_{k=1}^{m} \sum_{j=0}^{2k} \frac{(-1)^j}{n+k} \binom{n+k}{2k-j} \binom{n-k+j}{n-k} (k-j)^{2m} \quad (OddPowerIdentity34)$$

1.6. **Central factorials power identity.** Central factorials

$$x^{[n]} = x\left(x + \frac{n}{2} - 1\right)^{\underline{n-1}} \quad (CentralFactorial1)$$

$$x^{[n]} = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 1\right)\cdots\left(x + \frac{n}{2} - n - 1\right)$$

$$x^{[n]} = x\prod_{k=1}^{n-1}\left(x + \frac{n}{2} - k\right) (CentralFactorial2)$$

Then we have power identity given by Knuth

$$x^m = \sum_{k=1}^{m} T(m,k) x^{[k]} \quad (PowerIdentity1, PowerIdentity2)$$

$$x^m = \sum_{k=1}^{m} T(m,k)(x-1^2)(x-2^2)\cdots(x-k^2) = \sum_{k=1}^{m} T(m,k)\prod_{j=0}^{k}(n-j^2)$$

$$\texttt{PowerProduct}(n,k) = \prod_{j=0}^{k}(n-j^2)$$