POLYNOMIAL IDENTITIES INVOLVING CENTRAL FACTORIAL NUMBERS

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Abstract. Central factorial numbers often appear in literature, for instance, in Riordan's "Combinatorial identities", D. E. Knuth's work entitled "Johann Faulhaber and Sums of Powers" and many others. In this manuscript, we start our discussion from definition of central factorial numbers (recursive and iterative), continuing with a set of identities used further in the manuscript. Then, based on odd power identities given by D. E. Knuth, we show other variations of these identities rewriting them applying derived previously identities in terms of central factorial numbers. Finally, we provide a comprehensive way to validate the results of the manuscript via supplementary Mathematica programs.

Contents

1. Introduction

Central factorial numbers quietly often appear in literature, like for instance, in Riordan's Combinatorial identities [1]

$$k!T(n,k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j\right)^n \tag{1.1}$$

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Also, the book [2] references central factorial numbers as

$$K_{rs} = \frac{1}{(2s)!}\sum_{t=0}^{2s}(-1)^t\binom{2s}{t}(s-t)^{2r+2}$$

D. E. Knuth gives the following recurrence for the central factorial numbers [4]

$$T(2m+2,2k) = k^2T(2m,2k) + T(2m,2k-2)$$

In The On-Line Encyclopedia of Integer Sequences, central factorial numbers [5] appear to be defined via the following recurrence

$$\begin{cases} T(n,1) & = 1 \\ T(n,n) & = 1 \\ T(n,k) & = T(n-1,k-1) + k^2T(n-1,k) \end{cases}$$

It is important to note that central factorial numbers are closely related to the central difference operator δ, Newton interpolation formula and central factorials $x^{[n]}$ and could be derived respectively. The derivation of central factorial numbers by means of central difference operator δ, Newton interpolation formula and central factorials $x^{[n]}$ is shown at [6]. So there are few central factorial numbers identities at our disposal. The first is

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}(k-j)^{2n} \tag{1.2}$$

Such that it is derived from the formula given by the OEIS sequence A303675 [7], that is

$$A303675(n,k) = \frac{1}{n-k+1}\sum_{j=0}^{n-k+1}(-1)^j\binom{2[n-k+1]}{j}([n-k+1]-j)^{2n}$$

Note that the term $n-k+1$ is given only to serve proper sequence offset and can be replaced easily by iterator k so that we have arrived to the identity (1.2). The second identity is actively used in current manuscript is that

$$(2k-1)!T(2n,2k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2n} \tag{1.3}$$

and it is derived from (1.2) by means of symmetry of binomial coefficients $\binom{n}{k} = \binom{n}{n-k}$. The third one and final identity we base our results on involves a partial case of the equation (1.1)

$$(2k-1)!T(2n,2k) = \frac{1}{2k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j(k-j)^{2n} \tag{1.4}$$

So that now let's stick to some results of the D. E. Knuth's work [4], in particular to the polynomial identities. For odd powers we have

$$n^{2m-1} = \sum_{k=1}^{m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1} \tag{1.5}$$

And for any natural m we have polynomial identity

$$x^m = \sum_{k=1}^{m} T(m,k)x^{[k]} \tag{1.6}$$

where $x^{[k]}$ denotes central factorial defined by

$$x^{[n]} = x\left(x+\frac{n}{2}-1\right)^{\underline{n-1}}$$

where $(n)^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1)$ denotes falling factorial in Knuth's notation. In particular,

$$x^{[n]} = x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-1\right)\cdots\left(x+\frac{n}{2}-n-1\right) = x\prod_{k=1}^{n-1}\left(x+\frac{n}{2}-k\right)$$

So that having the whole context of the topic, we can derive and discuss few other polynomial identities smoothly. For example, given the Knuth's odd power identity (1.5) and identity in central factorial numbers (1.2) we easily obtain

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k}\binom{2k}{j}\binom{n+k-1}{2k-1}(k-j)^{2m}$$

Furthermore, by means of binomial identity $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ we get

$$n^{2m-1} = \sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k}\frac{2k}{n+k}\binom{n+k}{2k}\binom{2k}{j}(k-j)^{2m}$$

Collapsing common terms and applying binomial identity $\binom{n}{m}\binom{m}{k}=\binom{n}{k}\binom{n-k}{m-k}$ another polynomial identity follows

$$n^{2m-1}=2\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{n+k}\binom{n+k}{j}\binom{n+k-j}{2k-j}(k-j)^{2m}$$

2. Conclusions

Conclusions of your manuscript.

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