POLYNOMIAL IDENTITIES INVOLVING PASCAL'S TRIANGLE ROWS

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Abstract. In this short report we consider the famous binomial identity

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Based on it, the following binomial identities are derived

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j,$$

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j} \binom{n-j}{k-j} (-1)^{k-j} m^j,$$

where $\binom{n}{k}$ are binomial coefficients and $(m, \ n)$ are non-negative integers.

Contents

1. Introduction

We start from the famous relation about row sums of the Pascal triangle, that is

$$2^n = \sum_{k=0}^{n} \binom{n}{k}, \tag{1.1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are binomial coefficients [GKPL89]. Identity (1.1) is straightforward because the Pascal's triangle is

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n/k	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	21	8	1

Table 1. Pascal's triangle [CG96]. Each $k-$th term of $n-$th row is $\binom{n}{k} \cdot 1^k$. Sequence A007318 in OEIS [Slo64].

Consider a generating function such as $f_2(n,k) = \binom{n}{k} \cdot 2^k$. The function $f_2(n,k)$ generates the following Pascal-like triangle

n/k	0	1	2	3	4	5	6	7	8
0	1								
1	1	2							
2	1	4	4						
3	1	6	12	8					
4	1	8	24	32	16				
5	1	10	40	80	80	32			
6	1	12	60	160	240	192	64		
7	1	14	84	280	560	672	448	128	
8	1	16	112	448	1120	1792	1792	1024	256

Table 2. Triangle generated by the function $\binom{n}{k} \cdot 2^k$. Can be reproduced using Mathematica function `GeneratePascalLikeTriangle[2, 8]` at [Kol22]. Sequence A013609 in OEIS [Slo64].

Now we can notice that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \tag{1.2}$$

Continue similarly we can generalize the equations (1.1), (1.2) as follows

$$
\begin{aligned}
2^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \\
3^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \\
4^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot 3^k \\
&\cdots \\
m^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k
\end{aligned}
$$

Obviously, it is simply a form of the Binomial theorem $(m+1)^n = \sum_{k=0}^{n} \binom{n}{k} m^k$. Therefore, we conclude this version of the Binomial theorem as

Theorem 1.1. *(Binomial theorem.) The following identity involving polynomial m^n holds*

$$
m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{1.3}
$$

where (m,n) are non-negative integers.

Proof. Recall the induction over m, let be a base case $m=2$, hereby

$$
2^n = \sum_{k=0}^{n} \binom{n}{k} (2-1)^k \tag{1.4}
$$

Reviewing an equation (1.4) we can see that

$$
(\underbrace{2+1}_{m=3})^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (\underbrace{(2-1)+1}_{m-1})^k \tag{1.5}
$$

Continue similarly it is straightforward that $m^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k$. However, we are able to expand the part $(m-1)^k$ by means of Binomial theorem [AS72], that is

$$
(m-1)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j = \sum_{j=0}^{k} \binom{k}{j} (-1)^k m^{k-j}
$$

So that now we are able to merge both results $m^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k$ and $(m-1)^k = \sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j}m^j = \sum_{j=0}^{k} \binom{k}{j}(-1)^{k}m^{k-j}$ to receive

$$\begin{aligned} m^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j}m^j \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k}\binom{k}{j}(-1)^{k-j}m^j \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k}\binom{k}{j}(-1)^{k}m^{k-j} \end{aligned}$$

Theorem (1.1) may be verified using Mathematica command `PolynomialIdentity[m, n]` at [Kol22]. This completes the proof. □

Moreover, by means of the binomial identity [[Gro16], Chapter 4]

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$$

The polynomial m^n is identical to

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j}\binom{n-j}{k-j}(-1)^{k-j}m^j = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j}\binom{n-j}{k-j}(-1)^{k}m^{k-j}$$

2. Conclusions

The following binomial identities are derived

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k}\binom{k}{j}(-1)^{k-j}m^j = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k}\binom{k}{j}(-1)^{k}m^{k-j}$$

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j}\binom{n-j}{k-j}(-1)^{k-j}m^j = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j}\binom{n-j}{k-j}(-1)^{k}m^{k-j}$$

Moreover, above results are verified by means of specified Mathematica scripts available at `github.com/kolosovpetro/PolynomialIdentitiesInvolvingPascalsTriangleRows`.

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