

# FEW POLYNOMIAL IDENTITIES INVOLVING PASCAL'S TRIANGLE ROWS

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ABSTRACT. In this short report famous the binomial identity

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

is generalized such that relation between Pascal's triangle and volume of  $m$ -dimension  $n$ -length hypercubes is shown. Where  $\binom{n}{k}$  are binomial coefficients and  $(m, n)$  are positive integers.

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## 1. INTRODUCTION

We start from the famous relation about row sums of the Pascal triangle, that is

$$\sum_{k=0}^n \binom{n}{k} = 2^n, \tag{1.1}$$

where  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  are binomial coefficients [GKPL89]. Identity (1.1) is straightforward because the Pascal's triangle is

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$n/k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	21	8	1

**Table 1.** Pascal's triangle [CG96]. Each  $k$ -th term of  $n$ -th row is  $\binom{n}{k} \cdot 1^k$ . Sequence A007318 in OEIS [Slo64].

Consider a generating function such as  $f_2(n, k) = \binom{n}{k} \cdot 2^k$ . The function  $f_2(n, k)$  generates the following Pascal-like triangle

$n/k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	2							
2	1	4	4						
3	1	6	12	8					
4	1	8	24	32	16				
5	1	10	40	80	80	32			
6	1	12	60	160	240	192	64		
7	1	14	84	280	560	672	448	128	
8	1	16	112	448	1120	1792	1792	1024	256

**Table 2.** Triangle generated by the function  $\binom{n}{k} \cdot 2^k$ . Can be reproduced using Mathematica function `GeneratePascalLikeTriangle[2, 8]` at [Kol22]. Sequence A013609 in OEIS [Slo64].

Now we can notice that

$$\sum_{k=0}^n \binom{n}{k} \cdot 2^k = 3^n \quad (1.2)$$

Continue similarly we can generalize the equations (1.1), (1.2) as follows

$$\begin{aligned}
 2^n &= \sum_{k=0}^n \binom{n}{k} \cdot 1^k \\
 3^n &= \sum_{k=0}^n \binom{n}{k} \cdot 2^k \\
 4^n &= \sum_{k=0}^n \binom{n}{k} \cdot 3^k \\
 &\dots \\
 m^n &= \sum_{k=0}^n \binom{n}{k} \cdot (m-1)^k
 \end{aligned}$$

Therefore, let be a theorem

**Theorem 1.1.** *The following identity involving polynomial  $m^n$  holds*

$$m^n = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \quad (1.3)$$

where  $(m, n)$  are positive integers.

*Proof.* Recall the induction over  $m$ , let be a base case  $m = 2$ , hereby

$$2^n = \sum_{k=0}^n \binom{n}{k} (2-1)^k \quad (1.4)$$

Reviewing an equation (1.4) we can see that

$$\underbrace{(2+1)}_{m=3}^n = \sum_{k=0}^n \binom{n}{k} \cdot \underbrace{((2-1)+1)}_{m-1}^k \quad (1.5)$$

Continue similarly it is straightforward that  $m^n = \sum_{k=0}^n \binom{n}{k} \cdot (m-1)^k$ . However, we are able to expand the part  $(m-1)^k$  by means of Binomial theorem [AS72], that is

$$(m-1)^k = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} m^j$$

So that now we are able to merge both results  $m^n = \sum_{k=0}^n \binom{n}{k} \cdot (m-1)^k$  and  $(m-1)^k = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} m^j$  to receive

$$\begin{aligned} m^n &= \sum_{k=0}^n \binom{n}{k} \cdot \underbrace{(m-1)^k}_{\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} m^j} = \sum_{k=0}^n \binom{n}{k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} m^j \\ &= \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \end{aligned}$$

Theorem (1.3) may be verified using Mathematica command `PolynomialIdentity[m, n]` at [Kol22]. This completes the proof.  $\square$

Moreover, by means of the binomial identity [[Gro16], Chapter 4]

$$\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}$$

The polynomial  $m^n$  is identical to

$$m^n = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{j} \binom{n-j}{k-j} (-1)^{k-j} m^j$$

## 2. CONCLUSIONS

Conclusions of your manuscript.

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