PASCALS TRIANGLE AND VOLUME OF HYPERCUBES

PETRO KOLOSOV

ABSTRACT. In this short report famous the binomial identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

is generalized such that relation between Pascal's triangle and volume of m-dimension n-length hypercubes is shown. Where $\binom{n}{k}$ are binomial coefficients and $(m,\ n)$ are positive integers.

CONTENTS

1. INTRODUCTION

We start from the famous relation about row sums of the Pascal triangle, that is

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \tag{1.1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are binomial coefficients [GKPL89]. Identity 1.1 is straightforward because the Pascal's triangle is

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n/k	0	1	2	3	4	5	6	7
1	1							
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Table 1. Pascal's triangle [CG96]. Each $k-$th term of $n-$th row is $\binom{n}{k} \cdot 1^k$.

Consider a generating function such as $f_1(n,k) = \binom{n}{k} \cdot 2^k$. The function $f_1(n,k)$ generates the following Pascal-like triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 2 & & & \\ & & 1 & & 4 & & 4 & & \\ & 1 & & 6 & & 12 & & 8 & \\ 1 & & 8 & & 24 & & 32 & & 16 \end{array}$$

Figure 5. Triangle built by $\binom{n}{k} \cdot 2^k,\ 0 \le k \le n \le 4$.

We can notice that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = 3^n, \quad 0 \le k \le n, \quad (n,\ k) \in \mathbb{N} \tag{1.2}$$

Hereby, let be theorem

Theorem 1.1. *Volume of n-dimension hypercube with length m could be calculated as*

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{1.3}$$

where m and n - positive integers, see [**?**].

Proof. Recall induction over m, in (1.1) is shown a well-known example for $m = 2$.

$$2^n = \sum_{k=0}^{n} \binom{n}{k} (2-1)^k \tag{1.4}$$

Review (1.5) and suppose that

$$(\underbrace{2+1}_{m=3})^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (\underbrace{(2-1)+1}_{m-1})^k \tag{1.5}$$

And, obviously, this statement holds by means of Newton's Binomial Theorem [?], [?] given $m = 3$, more detailed, recall expansion for $(x+1)^n$ to show it.

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1.6}$$

Substituting $x = 2$ to (1.7) we have reached (1.6).
Next, let show example for each $m \in \mathbb{N}$. Recall Binomial theorem to show this

$$m^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k \tag{1.7}$$

Hereby, for $m+1$ we receive Binomial theorem again

$$(m+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot m^k \tag{1.8}$$

Review result from (1.8) and substituting Binomial expansion $\sum_{j=0}^{k} \binom{k}{j} (-1)^{n-k} m^j$ instead $(m-1)^k$ we receive desired result

$$\begin{aligned} m^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot \underbrace{(m-1)^k}_{\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \end{aligned} \tag{1.9}$$

This completes the proof. □

Lemma 1.2. *Number of elements $k-$face elements $\mathscr{E}_k(\mathbf{Y}_n^p)$ of Generalized Hypercube $\mathbf{Y}_n^p$ equals to*

$$\mathscr{E}_k(\mathbf{Y}_n^p) = \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} (p-1)^j \tag{1.10}$$

Expression (1.10) has Multinomial analog

$$\sum_{a_1+a_2+\cdots+a_k=n} \binom{n}{a_1, a_2, \ldots, a_k} = k^n \tag{1.11}$$

It could be found at .txt file in the last line here.

2. Conclusions

Conclusions of your manuscript.

References

[CG96] JH Conway and RK Guy. Pascal's triangle. *The Book of Numbers. New York: Springer-Verlag*, pages 68–70, 1996.

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Email address: `kolosovp94@gmail.com`

URL: `https://yourwebsite.com`