PASCAL'S TRIANGLE AND VOLUME OF HYPERCUBES

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Abstract. In this short report famous the binomial identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

is generalized such that relation between Pascal's triangle and volume of m-dimension n-length hypercubes is shown. Where $\binom{n}{k}$ are binomial coefficients and $(m,\ n)$ are positive integers.

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1. Introduction

We start from the famous relation about row sums of the Pascal triangle, that is

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \tag{1.1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are binomial coefficients [GKPL89]. Identity (1.1) is straightforward because the Pascal's triangle is

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n/k	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	21	8	1

Table 1. Pascal's triangle [CG96]. Each $k-$th term of $n-$th row is $\binom{n}{k} \cdot 1^k$.

Consider a generating function such as $f_2(n,k) = \binom{n}{k} \cdot 2^k$. The function $f_2(n,k)$ generates the following Pascal-like triangle

n/k	0	1	2	3	4	5	6	7	8
0	1								
1	1	2							
2	1	4	4						
3	1	6	12	8					
4	1	8	24	32	16				
5	1	10	40	80	80	32			
6	1	12	60	160	240	192	64		
7	1	14	84	280	560	672	448	128	
8	1	16	112	448	1120	1792	1792	1024	256

Table 2. Triangle generated by the function $\binom{n}{k} \cdot 2^k$. Can be reproduced using Mathematica function `GeneratePascalLikeTriangle[2, 8]` at [Kol22].

Now we can notice that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = 3^n \tag{1.2}$$

Continue similarly we can generalize the equations (1.1), (1.2) as follows

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k$$

$$3^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k$$

$$4^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 3^k$$

$$\cdots$$

$$m^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k$$

Theorem 1.1. *Volume of n-dimension hypercube with length m could be calculated as*

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{1.3}$$

where m and n - positive integers, see [**?**].

Proof. Recall induction over m, in (1.1) is shown a well-known example for $m = 2$.

$$2^n = \sum_{k=0}^{n} \binom{n}{k} (2-1)^k \tag{1.4}$$

Review (1.5) and suppose that

$$(\underbrace{2+1}_{m=3})^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (\underbrace{(2-1)+1}_{m-1})^k \tag{1.5}$$

And, obviously, this statement holds by means of Newton's Binomial Theorem [**?**], [**?**] given $m = 3$, more detailed, recall expansion for $(x+1)^n$ to show it.

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1.6}$$

Substituting $x = 2$ to (1.7) we have reached (1.6).
Next, let show example for each $m \in \mathbb{N}$. Recall Binomial theorem to show this

$$m^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k \tag{1.7}$$

Hereby, for $m+1$ we receive Binomial theorem again

$$(m+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot m^k \tag{1.8}$$

Review result from (1.8) and substituting Binomial expansion $\sum_{j=0}^{k}\binom{k}{j}(-1)^{n-k}m^j$ instead $(m-1)^k$ we receive desired result

$$\begin{aligned} m^n &= \sum_{k=0}^{n}\binom{n}{k}\cdot\underbrace{(m-1)^k}_{\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}m^j} = \sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}m^j \qquad (1.9) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}(-1)^{k-j}m^j \end{aligned}$$

This completes the proof. □

Lemma 1.2. *Number of elements $k-$face elements $\mathscr{E}_k(\mathbf{Y}_n^p)$ of Generalized Hypercube $\mathbf{Y}_n^p$ equals to*

$$\mathscr{E}_k(\mathbf{Y}_n^p) = \sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}(-1)^{k-j}(p-1)^j \qquad (1.10)$$

Expression (1.10) has Multinomial analog

$$\sum_{a_1+a_2+\cdots+a_k=n}\binom{n}{a_1,a_2,\ldots,a_k} = k^n \qquad (1.11)$$

It could be found at .txt file in the last line here.

2. Conclusions

Conclusions of your manuscript.

References

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