

POLYNOMIAL IDENTITIES INVOLVING PASCAL'S TRIANGLE ROWS

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ABSTRACT. In this short report we consider the famous binomial identity

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Based on it, the following binomial identities are derived

$$m^n = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j,$$
$$m^n = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{j} \binom{n-j}{k-j} (-1)^{k-j} m^j,$$

where $\binom{n}{k}$ are binomial coefficients and (m, n) are non-negative integers.

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1. INTRODUCTION

We start from the famous relation about row sums of the Pascal triangle, that is

$$\sum_{k=0}^n \binom{n}{k} = 2^n, \tag{1.1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are binomial coefficients [GKPL89]. Identity (1.1) is straightforward because the Pascal's triangle is

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n/k	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	21	8	1

Table 1. Pascal's triangle [CG96]. Each k -th term of n -th row is $\binom{n}{k} \cdot 1^k$. Sequence A007318 in OEIS [Slo64].

Consider a generating function such as $f_2(n, k) = \binom{n}{k} \cdot 2^k$. The function $f_2(n, k)$ generates the following Pascal-like triangle

n/k	0	1	2	3	4	5	6	7	8
0	1								
1	1	2							
2	1	4	4						
3	1	6	12	8					
4	1	8	24	32	16				
5	1	10	40	80	80	32			
6	1	12	60	160	240	192	64		
7	1	14	84	280	560	672	448	128	
8	1	16	112	448	1120	1792	1792	1024	256

Table 2. Triangle generated by the function $\binom{n}{k} \cdot 2^k$. Can be reproduced using Mathematica function `GeneratePascalLikeTriangle[2, 8]` at [Kol22]. Sequence A013609 in OEIS [Slo64].

Now we can notice that

$$\sum_{k=0}^n \binom{n}{k} \cdot 2^k = 3^n \quad (1.2)$$

Continue similarly we can generalize the equations (1.1), (1.2) as follows

$$\begin{aligned}
 2^n &= \sum_{k=0}^n \binom{n}{k} \cdot 1^k \\
 3^n &= \sum_{k=0}^n \binom{n}{k} \cdot 2^k \\
 4^n &= \sum_{k=0}^n \binom{n}{k} \cdot 3^k \\
 &\dots \\
 m^n &= \sum_{k=0}^n \binom{n}{k} \cdot (m-1)^k
 \end{aligned}$$

Therefore, let be a theorem

Theorem 1.1. *The following identity involving polynomial m^n holds*

$$m^n = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \quad (1.3)$$

where (m, n) are non-negative integers.

Proof. Recall the induction over m , let be a base case $m = 2$, hereby

$$2^n = \sum_{k=0}^n \binom{n}{k} (2-1)^k \quad (1.4)$$

Reviewing an equation (1.4) we can see that

$$\underbrace{(2+1)}_{m=3}^n = \sum_{k=0}^n \binom{n}{k} \cdot \underbrace{((2-1)+1)}_{m-1}^k \quad (1.5)$$

Continue similarly it is straightforward that $m^n = \sum_{k=0}^n \binom{n}{k} \cdot (m-1)^k$. However, we are able to expand the part $(m-1)^k$ by means of Binomial theorem [AS72], that is

$$(m-1)^k = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} m^j$$

So that now we are able to merge both results $m^n = \sum_{k=0}^n \binom{n}{k} \cdot (m-1)^k$ and $(m-1)^k = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} m^j$ to receive

$$\begin{aligned} m^n &= \sum_{k=0}^n \binom{n}{k} \cdot \underbrace{(m-1)^k}_{\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} m^j} = \sum_{k=0}^n \binom{n}{k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} m^j \\ &= \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \end{aligned}$$

Theorem (1.3) may be verified using Mathematica command `PolynomialIdentity[m, n]` at [Kol22]. This completes the proof. \square

Moreover, by means of the binomial identity [[Gro16], Chapter 4]

$$\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}$$

The polynomial m^n is identical to

$$m^n = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{j} \binom{n-j}{k-j} (-1)^{k-j} m^j$$

2. CONCLUSIONS

The following binomial identities are derived

$$\begin{aligned} m^n &= \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j, \\ m^n &= \sum_{k=0}^n \sum_{j=0}^k \binom{n}{j} \binom{n-j}{k-j} (-1)^{k-j} m^j. \end{aligned}$$

Moreover, above results are verified by means of specified Mathematica scripts available at github.com/kolosovpetro/PolynomialIdentitiesInvolvingPascalsTriangleRows.

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