

# POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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ABSTRACT. In this manuscript we show that for every  $n \geq 1$ ,  $n, m \in \mathbb{N}$  there are coefficients

$\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$  such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \dots + \mathbf{A}_{m,m} k^m (n-k)^m$$

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## 1. INTRODUCTION

Considering the table of forward finite differences of the polynomial  $n^3$

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$n$	$n^3$	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

**Table 1.** Table of finite differences of the polynomial  $n^3$ .

We can observe easily that finite differences of the polynomial  $n^3$  may be expressed according to the following relation, via rearrangement of the terms

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

$$\vdots$$

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n$$

Furthermore, the polynomial  $n^3$  is identical to

$$\begin{aligned} n^3 &= [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\ &\quad + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

Rearranging the above equation, we get

$$n^3 = n + (n-0) \cdot 6 \cdot 0 + (n-1) \cdot 6 \cdot 1 + (n-2) \cdot 6 \cdot 2 + \cdots + 1 \cdot 6 \cdot (n-1)$$

Therefore, we can consider  $n^3$  as

$$n^3 = \sum_{k=1}^n 6k(n-k) + 1 \quad (1.1)$$

Assume that equation (1.1) has an implicit form as follows

$$n^3 = \sum_{k=1}^n \mathbf{A}_{1,1} k^1 (n-k)^1 + \mathbf{A}_{1,0} k^0 (n-k)^0, \quad (1.2)$$

where  $\mathbf{A}_{1,1} = 6$  and  $\mathbf{A}_{1,0} = 1$ , respectively. So is there a generalization of the relation (1.2) for all positive odd powers? Therefore, let be a conjecture

**Conjecture 1.1.** *For every  $n \geq 1$ ,  $n, m \in \mathbb{N}$  there are coefficients  $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$  such that*

$$n^{2m+1} = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \dots + \mathbf{A}_{m,m} k^m (n-k)^m$$

## 2. APPROACH VIA A SYSTEM OF LINEAR EQUATIONS

One approach to prove the conjecture was proposed by Albert Tkaczyk in his series of the preprints [1, 2] and extended further at [3]. The main idea is to construct and solve a system of linear equations. Such a system of linear equations is constructed via expanding the definition of the coefficients  $\mathbf{A}_{m,r}$  applying Binomial theorem [4] and Faulhaber's formula [5]. Consider the definition of the coefficients  $\mathbf{A}_{m,r}$

$$n^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=1}^n k^r (n-k)^r \quad (2.1)$$

Expanding the  $(n-k)^r$  part via Binomial theorem we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=1}^n k^r (n-k)^r \\ &= \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=1}^n k^r \left[ \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\ &= \sum_{r=0}^m \mathbf{A}_{m,r} \left[ \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^n k^{t+r} \right] \end{aligned}$$

Applying the Faulhaber's formula to the sum  $\sum_{k=1}^n k^{t+r}$  we get

$$\begin{aligned}
n^{2m+1} &= \sum_{r=0}^m \mathbf{A}_{m,r} \left[ \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^n k^{t+r} \right] \\
&= \mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[ \frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[ \frac{1}{30}(-n + n^5) \right] \\
&+ \mathbf{A}_{m,3} \left[ \frac{1}{420}(-10n + 7n^3 + 3n^7) \right] + \mathbf{A}_{m,4} \left[ \frac{1}{630}(-21n + 20n^3 + n^9) \right] \\
&+ \mathbf{A}_{m,5} \left[ \frac{1}{2772}(-210n + 231n^3 - 22n^5 + n^{11}) \right] \\
&+ \mathbf{A}_{m,6} \left[ \frac{1}{60060}(-15202n + 18200n^3 - 3003n^5 + 5n^{13}) \right] \\
&+ \mathbf{A}_{m,7} \left[ \frac{1}{51480}(-60060n + 76010n^3 - 16380n^5 + 429n^7 + n^{15}) \right] \\
&+ \mathbf{A}_{m,8} \left[ \frac{1}{218790}(-1551693n + 2042040n^3 - 516868n^5 + 26520n^7 + n^{17}) \right] + \dots
\end{aligned}$$

Given fixed  $m$ , the coefficients  $\mathbf{A}_{m,r}$  can be determined via a system of linear equations.

Consider an example

**Example 2.1.** Let be  $m = 1$  so that we have the following relation defined by (eqref)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[ \frac{1}{6}(-n + n^3) \right] - n^3 = 0$$

Multiplying by 6 right-hand side and left-hand side, we get

$$6\mathbf{A}_{1,0}n + \mathbf{A}_{1,1}(-n + n^3) - 6n^3 = 0$$

Opening brackets and rearranging the terms gives

$$6\mathbf{A}_{1,0} - \mathbf{A}_{1,1}n + \mathbf{A}_{1,1}n^3 - 6n^3 = 0$$

Combining the common terms yields

$$n(6\mathbf{A}_{1,0} - \mathbf{A}_{1,1}) + n^3(\mathbf{A}_{1,1} - 6) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 6\mathbf{A}_{1,0} - \mathbf{A}_{1,1} = 0 \\ \mathbf{A}_{1,1} - 6 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{1,1} = 6 \\ \mathbf{A}_{1,0} = 1 \end{cases}$$

So that odd-power identity (2.1) holds

$$n^3 = \sum_{k=1}^n 6k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms  $6k(n-k) + 1$  over  $0 \leq k \leq n$  as it is shown at [6].

**Example 2.2.** Let be  $m = 2$  so that we have the following relation defined by (eqref)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[ \frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[ \frac{1}{30}(-n + n^5) \right] - n^5 = 0$$

Multiplying by 30 right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(-n + n^3) + \mathbf{A}_{2,2}(-n + n^5) - 30n^5 = 0$$

Opening brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that odd-power identity (2.1) holds

$$n^5 = \sum_{k=1}^n 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true evaluating the terms  $30k^2(n-k)^2 + 1$  over  $0 \leq k \leq n$  as it is shown at [7].

**Example 2.3.** Let be  $m = 3$  so that we have the following relation defined by (eqref)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[ \frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[ \frac{1}{30}(-n + n^5) \right] + \mathbf{A}_{m,3} \left[ \frac{1}{420}(-10n + 7n^3 + 3n^7) \right] - n^7 = 0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{2,1}(-n + n^3) + 14\mathbf{A}_{2,2}(-n + n^5) + \mathbf{A}_{3,3}(-10n + 7n^3 + 3n^7) - 420n^7 = 0$$

Opening brackets and rearranging the terms gives

$$420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5 - 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0$$

Combining the common terms yields

$$N(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3}) + n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5 14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} = 0 \\ 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} = 0 \\ \mathbf{A}_{3,2} - 30 = 0 \\ 3\mathbf{A}_{3,3} - 420 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \mathbf{A}_{3,2} = 0 \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1}+10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

So that odd-power identity (2.1) holds

$$n^7 = \sum_{k=1}^n 140k^3(n-k)^3 - 14k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms  $140k^3(n-k)^3 - 14k(n-k) + 1$  over  $0 \leq k \leq n$  as it is shown at [8].

**Example 2.4.** Let be  $m = 4$  so that we have the following relation defined by (eqref)

$$\begin{aligned} & \mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[ \frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[ \frac{1}{30}(-n + n^5) \right] + \mathbf{A}_{m,3} \left[ \frac{1}{420}(-10n + 7n^3 + 3n^7) \right] \\ & + \mathbf{A}_{m,4} \left[ \frac{1}{630}(-21n + 20n^3 + n^9) \right] - n^9 = 0 \end{aligned}$$

Multiplying by 630 right-hand side and left-hand side, we get

$$\begin{aligned} & 630\mathbf{A}_{4,0}n + 105\mathbf{A}_{4,1}(-n + n^3) + 21\mathbf{A}_{4,2}(-n + n^5) \\ & + \frac{3}{2}\mathbf{A}_{4,3}(-10n + 7n^3 + 3n^7) + \mathbf{A}_{4,4}(-21n + 20n^3 + n^9) - 630n^9 = 0 \end{aligned}$$

Opening brackets and rearranging the terms gives

$$\begin{aligned} & 630\mathbf{A}_{4,0}n - 105\mathbf{A}_{4,1}n + 105\mathbf{A}_{4,1}n^3 - 21\mathbf{A}_{4,2}n + 21\mathbf{A}_{4,2}n^5 \\ & - \frac{3}{2}\mathbf{A}_{4,3} \cdot 10n + \frac{3}{2}\mathbf{A}_{4,3} \cdot 7n^3 + \frac{3}{2}\mathbf{A}_{4,3} \cdot 3n^7 \\ & - 21\mathbf{A}_{4,4}n + 20\mathbf{A}_{4,4}n^3 + \mathbf{A}_{4,4}n^9 - 630n^9 = 0 \end{aligned}$$

Combining the common terms yields

$$\begin{aligned}
& n(630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4}) \\
& + n^3 \left( 105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4} \right) + n^5(21\mathbf{A}_{4,2}) \\
& + n^7 \left( \frac{9}{2}\mathbf{A}_{4,3} \right) + n^9(\mathbf{A}_{4,4} - 630) = 0
\end{aligned}$$

Therefore, the system of linear equations follows

$$\begin{cases}
630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4} = 0 \\
105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4} = 0 \\
\mathbf{A}_{4,2} = 0 \\
\mathbf{A}_{4,3} = 0 \\
\mathbf{A}_{4,4} - 630 = 0
\end{cases}$$

Solving it, we get

$$\begin{cases}
\mathbf{A}_{4,4} = 630 \\
\mathbf{A}_{4,3} = 0 \\
\mathbf{A}_{4,2} = 0 \\
\mathbf{A}_{4,1} = -\frac{20}{105}\mathbf{A}_{4,4} = -120 \\
\mathbf{A}_{4,0} = \frac{105\mathbf{A}_{4,1} + 21\mathbf{A}_{4,4}}{630} = 1
\end{cases}$$

So that odd-power identity (2.1) holds

$$n^9 = \sum_{k=1}^n 630k^4(n-k)^4 - 120k(n-k) + 1$$

### 3. APPROACH VIA RECURSION

Another approach to determine the coefficients  $\mathbf{A}_{m,r}$  was provided by Dr. Max Alekseyev in MathOverflow discussion [9]. Generally, the idea was to determine the coefficients  $\mathbf{A}_{m,r}$  recursively starting from the base case  $\mathbf{A}_{m,m}$  up to  $\mathbf{A}_{m,r-1}, \dots, \mathbf{A}_{m,0}$  via previously determined



values. Consider the Faulhaber's formula

$$\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j}$$

it is very important to note that summation bound is  $p$  while binomial coefficient upper bound is  $p+1$ . It means that we cannot skip summation bounds unless we do some trick as

$$\begin{aligned} \sum_{k=1}^n k^p &= \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j} = \left[ \frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \\ &= \left[ \frac{1}{p+1} \sum_j \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \end{aligned}$$

Using the Faulhaber's formula  $\sum_{k=1}^n k^p = \left[ \frac{1}{p+1} \sum_j \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1}$  we get

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^n k^{t+r} \\ &= \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \left[ \frac{1}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{t+r+1-j} - B_{t+r+1} \right] \\ &= \sum_{t=0}^r \binom{r}{t} \left[ \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t} \right] \\ &= \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \\ &= \sum_j \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} - \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \\ &= \sum_j B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} - \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \end{aligned}$$

Now, we notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1) \binom{2r}{r}}, & \text{if } j = 0; \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1}, & \text{if } j > 0. \end{cases} \quad (3.1)$$

An elegant proof of the above binomial identity is provided at [10]. In particular, the equation (3.1) is zero for  $0 < t \leq j$ . So that taking  $j = 0$  we have

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[ \sum_{j \geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] \\ &\quad - \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \end{aligned}$$

Now let's simplify the double summation applying the identity (3.1)

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[ \sum_{j \geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r+1-j} \right]}_{(\star)} \\ &\quad - \underbrace{\left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)} \end{aligned}$$

Hence, introducing  $\ell = 2r - j + 1$  to  $(\star)$  and  $\ell = r - t$  to  $(\diamond)$  we collapse the common terms of the above equation so that we get

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[ \sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &\quad - \left[ \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned}$$

Using the definition of  $\mathbf{A}_{m,r}$ , we obtain the following identity for polynomials in  $n$

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_r \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd  $\ell$  by  $d$  we get

$$\begin{aligned}
& \sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_r \mathbf{A}_{m,r} \sum_d \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1} \\
& \sum_r \mathbf{A}_{m,r} \left[ \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} \right] + 2 \sum_r \mathbf{A}_{m,r} \left[ \sum_d \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \right] - n^{2m+1} = 0 \\
& \sum_{r=0}^m \mathbf{A}_{m,r} \left[ \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} \right] + 2 \sum_{r=0}^m \mathbf{A}_{m,r} \left[ \sum_{d=0}^{(r-1)/2} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \right] - n^{2m+1} = 0
\end{aligned} \tag{3.2}$$

Taking the coefficient of  $n^{2m+1}$  in (3.2), we get

$$\mathbf{A}_{m,m} = (2m+1) \binom{2m}{m}$$

and taking the coefficient of  $n^{2d+1}$  for an integer  $d$  in the range  $m/2 \leq d < m$ , we get

$$\mathbf{A}_{m,d} = 0$$

Taking the coefficient of  $n^{2d+1}$  for  $d$  in the range  $m/4 \leq d < m/2$  we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1) \binom{2m}{m} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can express  $\mathbf{A}_{m,r}$  for each integer  $r$  in range  $m/2^{s+1} \leq r < m/2^s$  (iterating consecutively  $s = 1, 2, \dots$ ) via previously determined values of  $\mathbf{A}_{m,d}$  as follows

$$\mathbf{A}_{m,r} = (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Finally, the coefficient  $\mathbf{A}_{m,r}$  is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1) \binom{2r}{r}, & \text{if } r = m; \\ (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{3.3}$$

where  $B_t$  are Bernoulli numbers [11]. It is assumed that  $B_1 = \frac{1}{2}$ . For example,

$m/r$	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

**Table 2.** Coefficients  $\mathbf{A}_{m,r}$ .

The coefficients  $\mathbf{A}_{m,r}$  are also registered in the OEIS [12, 13].

#### 4. APPROACH VIA RECURSION: EXAMPLES

Consider the coefficients  $\mathbf{A}_{m,r}$  definition (eqref), it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1) \binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^m \underbrace{\mathbf{A}_{m,d} (2r+1) \binom{2r}{r} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition for the polynomial  $T(d, r)$

**Definition 4.1.**

$$T(d, r) = (2r+1) \binom{2r}{r} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

**Example 4.2.** Let be  $m = 2$  so first we get  $\mathbf{A}_{2,2}$

$$\mathbf{A}_{2,2} = 5 \binom{4}{2} = 30$$

Then  $\mathbf{A}_{2,1} = 0$  because  $\mathbf{A}_{m,d}$  is zero in the range  $m/2 \leq d < m$  means that zero for  $d$  in  $1 \leq d < 2$ . Finally, the coefficient  $\mathbf{A}_{2,0}$  is

$$\begin{aligned}\mathbf{A}_{2,0} &= \sum_{d \geq 1}^2 \mathbf{A}_{2,d} \cdot T(d, 0) = \mathbf{A}_{2,1} \cdot T(1, 0) + \mathbf{A}_{2,2} \cdot T(2, 0) \\ &= 30 \cdot \frac{1}{30} = 1\end{aligned}$$

**Example 4.3.** Let be  $m = 3$  so that first we get  $\mathbf{A}_{3,3}$

$$\mathbf{A}_{3,3} = 7 \binom{6}{3} = 140$$

Then  $\mathbf{A}_{3,2} = 0$  because  $\mathbf{A}_{m,d}$  is zero in the range  $m/2 \leq d < m$  means that zero for  $d$  in  $2 \leq d < 3$ . The  $\mathbf{A}_{3,1}$  coefficient is non-zero and calculated as

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^3 \mathbf{A}_{3,d} \cdot T(d, 1) = \mathbf{A}_{3,3} \cdot T(3, 1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally, the coefficient  $\mathbf{A}_{3,0}$  is

$$\begin{aligned}\mathbf{A}_{3,0} &= \sum_{d \geq 1}^3 \mathbf{A}_{3,d} \cdot T(d, 0) = \mathbf{A}_{3,1} \cdot T(1, 0) + \mathbf{A}_{3,2} \cdot T(2, 0) + \mathbf{A}_{3,3} \cdot T(3, 0) \\ &= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1\end{aligned}$$

**Example 4.4.** Let be  $m = 4$  so that first we get  $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9 \binom{8}{4} = 630$$

Then  $\mathbf{A}_{4,3} = 0$  and  $\mathbf{A}_{4,2} = 0$  because  $\mathbf{A}_{m,d}$  is zero in the range  $m/2 \leq d < m$  means that zero for  $d$  in  $2 \leq d < 4$ . The value of the coefficient  $\mathbf{A}_{4,1}$  is non-zero and calculated as

$$\mathbf{A}_{4,1} = \sum_{d \geq 3}^4 \mathbf{A}_{4,d} \cdot T(d, 1) = \mathbf{A}_{4,3} \cdot T(3, 1) + \mathbf{A}_{4,4} \cdot T(4, 1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient  $\mathbf{A}_{4,0}$  is

$$\mathbf{A}_{4,0} = \sum_{d \geq 1}^4 \mathbf{A}_{4,d} \cdot T(d, 0) = \mathbf{A}_{4,1} \cdot T(1, 0) + \mathbf{A}_{4,4} \cdot T(4, 0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

**Example 4.5.** Let be  $m = 5$  so that first we get  $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11 \binom{10}{5} = 2772$$

Then  $\mathbf{A}_{5,4} = 0$  and  $\mathbf{A}_{5,3} = 0$  because  $\mathbf{A}_{m,d}$  is zero in the range  $m/2 \leq d < m$  means that zero for  $d$  in  $3 \leq d < 5$ . The value of the coefficient  $\mathbf{A}_{5,2}$  is non-zero and calculated as

$$\mathbf{A}_{5,2} = \sum_{d \geq 5}^5 \mathbf{A}_{5,d} \cdot T(d, 2) = \mathbf{A}_{5,5} \cdot T(5, 2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient  $\mathbf{A}_{5,1}$  is non-zero and calculated as

$$\begin{aligned} \mathbf{A}_{5,1} &= \sum_{d \geq 3}^5 \mathbf{A}_{5,d} \cdot T(d, 1) = \mathbf{A}_{5,3} \cdot T(3, 1) + \mathbf{A}_{5,4} \cdot T(4, 1) + \mathbf{A}_{5,5} \cdot T(5, 1) \\ &= 2772 \cdot \left(-\frac{1}{2}\right) = -1386 \end{aligned}$$

Finally, the coefficient  $\mathbf{A}_{5,0}$  is

$$\begin{aligned} \mathbf{A}_{5,0} &= \sum_{d \geq 1}^5 \mathbf{A}_{5,d} \cdot T(d, 0) = \mathbf{A}_{5,1} \cdot T(1, 0) + \mathbf{A}_{5,2} \cdot T(2, 0) + \mathbf{A}_{5,5} \cdot T(5, 0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1 \end{aligned}$$

## 5. CONCLUSIONS

In this manuscript, we have shown that for every  $n \geq 1$ ,  $n, m \in \mathbb{N}$  there are coefficients  $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$  such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \dots + \mathbf{A}_{m,m} k^m (n-k)^m$$

In particular, the coefficients  $\mathbf{A}_{m,r}$  may be evaluated both ways, by constructing and solving a system of linear equations or applying recurrence relations; all these approaches are explained in the sections (1,2 refs), respectively, including detailed examples.

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