POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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Contents

1. Formulae

Consider the polynomial relation

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \tag{1.1}$$

Expanding the $(n-k)^r$ part via Binomial theorem we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \end{aligned} \tag{1.2}$$

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Explicitly (**??**) is

$$
\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} \left[\sum_{t=0}^{0} (-1)^t \binom{0}{t} n^{0-t} \sum_{k=1}^{n} k^{t+0} \right] + \mathbf{A}_{m,1} \left[\sum_{t=0}^{1} (-1)^t \binom{1}{t} n^{1-t} \sum_{k=1}^{n} k^{t+1} \right] \\
&+ \mathbf{A}_{m,2} \left[\sum_{t=0}^{2} (-1)^t \binom{2}{t} n^{2-t} \sum_{k=1}^{n} k^{t+2} \right] + \mathbf{A}_{m,3} \left[\sum_{t=0}^{3} (-1)^t \binom{3}{t} n^{3-t} \sum_{k=1}^{n} k^{t+3} \right] + \cdots
\end{aligned}
\tag{1.3}
$$

Moreover,

$$
\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n \\
&+ \mathbf{A}_{m,1} \left[n^1 \sum_{k=1}^{n} k^1 - n^0 \sum_{k=1}^{n} k^2 \right] \\
&+ \mathbf{A}_{m,2} \left[n^2 \sum_{k=1}^{n} k^2 - 2n^1 \sum_{k=1}^{n} k^3 + n^0 \sum_{k=1}^{n} k^4 \right] \\
&+ \mathbf{A}_{m,3} \left[n^3 \sum_{k=1}^{n} k^3 - 3n^2 \sum_{k=1}^{n} k^4 + 3n^1 \sum_{k=1}^{n} k^5 - n^0 \sum_{k=1}^{n} k^6 \right] \\
&+ \mathbf{A}_{m,4} \left[n^4 \sum_{k=1}^{n} k^4 - 4n^3 \sum_{k=1}^{n} k^5 + 6n^2 \sum_{k=1}^{n} k^6 - 4n^1 \sum_{k=1}^{n} k^7 + n^0 \sum_{k=1}^{n} k^8 \right] + \cdots
\end{aligned}
\tag{1.4}
$$

For arbitrary m we have

$$\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5) \right] + \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n+7n^3+3n^7) \right] \\
&+ \mathbf{A}_{m,4} \left[\frac{1}{630}(-21n+20n^3+n^9) \right] + \mathbf{A}_{m,5} \left[\frac{1}{2772}(-210n+231n^3-22n^5+n^{11}) \right] \\
&+ \mathbf{A}_{m,6} \left[\frac{1}{60060}(-15202n+18200n^3-3003n^5+5n^{13}) \right] \\
&+ \mathbf{A}_{m,7} \left[\frac{1}{51480}(-60060n+76010n^3-16380n^5+429n^7+n^{15}) \right] \\
&+ \mathbf{A}_{m,8} \left[\frac{1}{218790}(-1551693n+2042040n^3-516868n^5+26520n^7+n^{17}) \right] + \cdots
\end{aligned} \tag{1.5}$$

Expanding previous we get

$$\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n \\
&+ \mathbf{A}_{m,1} n^1 \sum_{k=1}^{n} k^1 - \mathbf{A}_{m,1} n^0 \sum_{k=1}^{n} k^2 \\
&+ \mathbf{A}_{m,2} n^2 \sum_{k=1}^{n} k^2 - \mathbf{A}_{m,2} 2n^1 \sum_{k=1}^{n} k^3 + \mathbf{A}_{m,2} n^0 \sum_{k=1}^{n} k^4 \\
&+ \mathbf{A}_{m,3} n^3 \sum_{k=1}^{n} k^3 - \mathbf{A}_{m,3} 3n^2 \sum_{k=1}^{n} k^4 + \mathbf{A}_{m,3} 3n^1 \sum_{k=1}^{n} k^5 - \mathbf{A}_{m,3} n^0 \sum_{k=1}^{n} k^6 \\
&+ \mathbf{A}_{m,4} n^4 \sum_{k=1}^{n} k^4 - \mathbf{A}_{m,4} 4n^3 \sum_{k=1}^{n} k^5 + \mathbf{A}_{m,4} 6n^2 \sum_{k=1}^{n} k^6 - \mathbf{A}_{m,4} 4n^1 \sum_{k=1}^{n} k^7 + \mathbf{A}_{m,4} n^0 \sum_{k=1}^{n} k^8 \\
&+ \cdots
\end{aligned} \tag{1.6}$$

Rearranging sum we get for $m = 4$

$$
\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n \\
&+ \sum_{k=1}^{n} k^1 \mathbf{A}_{m,1} n^1 \\
&+ \sum_{k=1}^{n} k^2 \left[-\mathbf{A}_{m,1} n^0 + \mathbf{A}_{m,2} n^2 \right] \\
&+ \sum_{k=1}^{n} k^3 \left[-\mathbf{A}_{m,2} 2n + \mathbf{A}_{m,3} n^3 \right] \\
&+ \sum_{k=1}^{n} k^4 \left[\mathbf{A}_{m,2} - \mathbf{A}_{m,3} 3n^2 + \mathbf{A}_{m,4} n^4 \right] \\
&+ \sum_{k=1}^{n} k^5 \left[\mathbf{A}_{m,3} 3n - \mathbf{A}_{m,4} 4n^3 \right] \\
&+ \sum_{k=1}^{n} k^6 \left[-\mathbf{A}_{m,3} + \mathbf{A}_{m,4} 6n^2 \right] \\
&+ \sum_{k=1}^{n} k^7 \left[-\mathbf{A}_{m,4} 4n \right] \\
&+ \sum_{k=1}^{n} k^8 \left[-\mathbf{A}_{m,4} \right]
\end{aligned}
\tag{1.7}
$$

Rearranging sum we get for $m = 4$

$$
\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n \binom{0}{0} \\
&+ \sum_{k=1}^{n} k^1 \mathbf{A}_{m,1} n^1 \binom{1}{0} \\
&+ \sum_{k=1}^{n} k^2 \left[-\mathbf{A}_{m,1} n^0 \binom{1}{1} + \mathbf{A}_{m,2} n^2 \binom{2}{0} \right] \\
&+ \sum_{k=1}^{n} k^3 \left[-\mathbf{A}_{m,2} n \binom{2}{1} + \mathbf{A}_{m,3} n^3 \binom{3}{0} \right] \\
&+ \sum_{k=1}^{n} k^4 \left[\mathbf{A}_{m,2} \binom{2}{2} - \mathbf{A}_{m,3} n^2 \binom{3}{1} + \mathbf{A}_{m,4} n^4 \binom{4}{0} \right] \\
&+ \sum_{k=1}^{n} k^5 \left[\mathbf{A}_{m,3} n \binom{3}{2} - \mathbf{A}_{m,4} n^3 \binom{4}{1} \right] \\
&+ \sum_{k=1}^{n} k^6 \left[-\mathbf{A}_{m,3} \binom{3}{3} + \mathbf{A}_{m,4} n^2 \binom{4}{2} \right] \\
&+ \sum_{k=1}^{n} k^7 \left[-\mathbf{A}_{m,4} n \binom{4}{3} \right] \\
&+ \sum_{k=1}^{n} k^8 \left[-\mathbf{A}_{m,4} \binom{4}{4} \right]
\end{aligned}
\tag{1.8}
$$

Polynomial identities

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k) + 1$$

$$n^9 = \sum_{k=1}^{n} 630k^4(n-k)^4 - 120k(n-k) + 1$$

$$n^{11} = \sum_{k=1}^{n} 2772k^5(n-k)^5 + 660k^2(n-k)^2 - 1386k(n-k) + 1$$

$$n^{13} = \sum_{k=1}^{n} 51480k^7(n-k)^7 - 60060k^3(n-k)^3 + 491400k^2(n-k)^2 - 450054k(n-k) + 1$$

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