POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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Abstract. In this manuscript we show that for every $n \geq 1,\ n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

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1. Introduction

Back then in 2016 I was playing with numbers and have noticed the pattern in terms of finite differences of cubes n^3. Considering the table of forward finite differences of the polynomial n^3

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n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

We can observe easily that finite differences of the polynomial n^3 may be expressed according to the following relation, via rearrangement of the terms

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 \\
&\vdots \\
\Delta(n^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n
\end{aligned}$$

Furthermore, the polynomial n^3 is identical to

$$\begin{aligned}
n^3 = &[1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\
&+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)]
\end{aligned}$$

Rearranging above equation we get

$$n^3 = n + (n-0) \cdot 6 \cdot 0 + (n-1) \cdot 6 \cdot 1 + (n-2) \cdot 6 \cdot 2 + \cdots + 1 \cdot 6 \cdot (n-1)$$

Therefore, we can consider n^3 as

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.1}$$

Assume that equation (1.1) has an implicit form as follows

$$n^3 = \sum_{k=1}^{n} \mathbf{A}_{1,1} k^1 (n-k)^1 + \mathbf{A}_{1,0} k^0 (n-k)^0, \tag{1.2}$$

where $\mathbf{A}_{1,1} = 6$ and $\mathbf{A}_{1,0} = 1$, respectively. So could the relation (1.2) be generalised for all positive odd powers? Therefore, let be a conjecture

Conjecture 1.1. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that*

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

2. Approach via system of linear equations

One approach to prove the conjecture was proposed by Albert Tkaczyk in his series of preprints (two references). The essence of the approach lays in construction and solving of the particular system of linear equations. Such system linear equations is constructed using Binomial theorem and Faulhaber's formula that allows us to find closed froms of power sums as part of identity (reference to equation). Consider the polynomial relation

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \tag{2.1}$$

Expanding the $(n-k)^r$ part via Binomial theorem we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \end{aligned} \tag{2.2}$$

For arbitrary m we have

$$
\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5) \right] + \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n+7n^3+3n^7) \right] \\
&+ \mathbf{A}_{m,4} \left[\frac{1}{630}(-21n+20n^3+n^9) \right] + \mathbf{A}_{m,5} \left[\frac{1}{2772}(-210n+231n^3-22n^5+n^{11}) \right] \\
&+ \mathbf{A}_{m,6} \left[\frac{1}{60060}(-15202n+18200n^3-3003n^5+5n^{13}) \right] \\
&+ \mathbf{A}_{m,7} \left[\frac{1}{51480}(-60060n+76010n^3-16380n^5+429n^7+n^{15}) \right] \\
&+ \mathbf{A}_{m,8} \left[\frac{1}{218790}(-1551693n+2042040n^3-516868n^5+26520n^7+n^{17}) \right] + \cdots
\end{aligned} \tag{2.3}
$$

Example 2.1. *Let be fixed $m = 1$ so that we have the following relation defined by (eqref)*

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] - n^3 = 0$$

Multiplying by 6 both, right hand side and left hand side, we get

$$6\mathbf{A}_{1,0} n + \mathbf{A}_{1,1}(-n+n^3) - 6n^3 = 0$$

Opening brackets and rearranging the terms gives

$$6\mathbf{A}_{1,0} - \mathbf{A}_{1,1} n + \mathbf{A}_{1,1} n^3 - 6n^3 = 0$$

Combining the terms yields

$$n(6\mathbf{A}_{1,0} - \mathbf{A}_{1,1}) + n^3(\mathbf{A}_{1,1} - 6) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 6\mathbf{A}_{1,0} - \mathbf{A}_{1,1} = 0 \\ \mathbf{A}_{1,1} - 6 = 0 \end{cases}$$

Solving it we get

$$\begin{cases} \mathbf{A}_{1,1} = 6 \\ \mathbf{A}_{1,0} = 1 \end{cases}$$

Example 2.2. *Let be fixed $m = 2$ so that we have the following relation defined by (eqref)*

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n+n^5)\right] - n^5 = 0$$

Multiplying by 30 both, right hand side and left hand side, we get

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(-n+n^3) + \mathbf{A}_{2,2}(-n+n^5) - 30n^5 = 0$$

Opening brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

Combining the terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

Example 2.3. *Let be fixed $m = 3$ so that we have the following relation defined by (eqref)*

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n+n^5)\right] + \mathbf{A}_{m,3}\left[\frac{1}{420}(-10n+7n^3+3n^7)\right] - n^7 = 0$$

Multiplying by 420 both, right hand side and left hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{2,1}(-n + n^3) + 14\mathbf{A}_{2,2}(-n + n^5) + \mathbf{A}_{3,3}(-10n + 7n^3 + 3n^7) - 420n^7 = 0$$

Opening brackets and rearranging the terms gives

$$420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5 - 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0$$

Combining the terms yields

$$n(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3}) + n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5 14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} = 0 \\ 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} = 0 \\ \mathbf{A}_{3,2} - 30 = 0 \\ 3\mathbf{A}_{3,3} - 420 = 0 \end{cases}$$

Solving it we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \mathbf{A}_{3,2} = 0 \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1} + 10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

Example 2.4. *Let be fixed $m = 4$ so that we have the following relation defined by (eqref)*

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n + n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n + n^5)\right] + \mathbf{A}_{m,3}\left[\frac{1}{420}(-10n + 7n^3 + 3n^7)\right] \\ + \mathbf{A}_{m,4}\left[\frac{1}{630}(-21n + 20n^3 + n^9)\right] - n^9 = 0$$

Multiplying by 630 *both, right hand side and left hand side, we get*

$$630\mathbf{A}_{4,0}n + 105\mathbf{A}_{4,1}(-n+n^3) + 21\mathbf{A}_{4,2}(-n+n^5) \\ + \frac{3}{2}\mathbf{A}_{4,3}(-10n+7n^3+3n^7) + \mathbf{A}_{4,4}(-21n+20n^3+n^9) - 630n^9 = 0$$

Opening brackets and rearranging the terms gives

$$630\mathbf{A}_{4,0}n - 105\mathbf{A}_{4,1}n + 105\mathbf{A}_{4,1}n^3 - 21\mathbf{A}_{4,2}n + 21\mathbf{A}_{4,2}n^5 \\ - \frac{3}{2}\mathbf{A}_{4,3}\cdot 10n + \frac{3}{2}\mathbf{A}_{4,3}\cdot 7n^3 + \frac{3}{2}\mathbf{A}_{4,3}\cdot 3n^7 \\ - 21\mathbf{A}_{4,4}n + 20\mathbf{A}_{4,4}n^3 + \mathbf{A}_{4,4}n^9 - 630n^9 = 0$$

Combining the terms yields

$$n(630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4}) \\ + n^3\left(105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4}\right) + n^5(21\mathbf{A}_{4,2}) \\ + n^7\left(\frac{9}{2}\mathbf{A}_{4,3}\right) + n^9(\mathbf{A}_{4,4} - 630) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4} = 0 \\ 105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4} = 0 \\ \mathbf{A}_{4,2} = 0 \\ \mathbf{A}_{4,3} = 0 \\ \mathbf{A}_{4,4} - 630 = 0 \end{cases}$$

Solving it we get

$$\begin{cases} \mathbf{A}_{4,4} = 630 \\ \mathbf{A}_{4,3} = 0 \\ \mathbf{A}_{4,2} = 0 \\ \mathbf{A}_{4,1} = -\frac{20}{105}\mathbf{A}_{4,4} = -120 \\ \mathbf{A}_{4,0} = \frac{105\mathbf{A}_{4,1}+21\mathbf{A}_{4,4}}{630} = 1 \end{cases}$$

3. Approach via recursion

Consider the Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} \tag{3.1}$$

it is very important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot skip summation bounds unless we do some trick as

$$\begin{aligned}\sum_{k=1}^{n} k^p &= \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}\\ &= \left[\frac{1}{p+1}\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}\\ &= \left[\frac{1}{p+1}\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}\end{aligned} \tag{3.2}$$

Using Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}$ we get

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r}\\ &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right]\\ &= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right]\\ &= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\\ &= \sum_{j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\\ &= \sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\end{aligned} \tag{3.3}$$

Now, we notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j = 0; \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1}, & \text{if } j > 0. \end{cases} \tag{3.4}$$

In particular, the last sum is zero for $0 < t \leq j$. So taking $j = 0$ we have

$$\begin{aligned} \sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} &+ \left[\sum_{j \geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] \\ &- \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \end{aligned} \tag{3.5}$$

Now let's simplify the double summation

$$\begin{aligned} \sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} &+ \underbrace{\left[\sum_{j \geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r+1-j} \right]}_{(\star)} \\ &- \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)} \end{aligned} \tag{3.6}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$, we get

$$\begin{aligned} \sum_{k=1}^{n} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned} \tag{3.7}$$

Using the definition of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_r \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \tag{3.8}$$

Replacing odd ℓ by d we get

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r} \sum_d \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

$$\sum_r \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_r \mathbf{A}_{m,r} \left[\sum_d \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0$$

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{d=0}^{(r-1)/2} \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0 \tag{3.9}$$

Taking the coefficient of n^{2m+1} in (3.9), we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \tag{3.10}$$

and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0 \tag{3.11}$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \leq d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2\underbrace{(2m+1)\binom{2m}{m}}_{\mathbf{A}_{m,m}} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0, \tag{3.12}$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d} \tag{3.13}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \leq r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} \tag{3.14}$$

Finally, the coefficient $\mathbf{A}_{m,r}$ is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{3.15}$$

where B_t are Bernoulli numbers. It is assumed that $B_1 = \frac{1}{2}$.

4. Approach via recursion: Examples

Consider the coefficients $\mathbf{A}_{m,r}$ definition (eqref), it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ \sum_{d\geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition for the polynomial $T(d, r)$

Definition 4.1.

$$T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Example 4.2. *Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$*

$$\mathbf{A}_{2,2} = 5\binom{4}{2} = 30$$

Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the $\mathbf{A}_{2,0}$ is

$$\begin{aligned} \mathbf{A}_{2,0} = \sum_{d\geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) &= \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0) \\ &= 30 \cdot \frac{1}{30} = 1 \end{aligned}$$

Example 4.3. *Let be $m = 3$ so that first we get $\mathbf{A}_{3,3}$*

$$\mathbf{A}_{3,3} = 7\binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{3,1} = \sum_{d\geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally $\mathbf{A}_{3,0}$ *coefficient is*

$$\mathbf{A}_{3,0} = \sum_{d\geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 4.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 4$*. The value of* $\mathbf{A}_{4,1}$ *coefficient is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d\geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally $\mathbf{A}_{4,0}$ *coefficient is*

$$\mathbf{A}_{4,0} = \sum_{d\geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 4.5. *Let be* $m = 5$ *so that first we get* $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11\binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ *and* $\mathbf{A}_{5,3} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $3 \leq d < 5$*. The value of* $\mathbf{A}_{5,2}$ *coefficient is non-zero and calculated as*

$$\mathbf{A}_{5,2} = \sum_{d\geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of $\mathbf{A}_{5,1}$ *coefficient is non-zero and calculated as*

$$\mathbf{A}_{5,1} = \sum_{d\geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally the $\mathbf{A}_{5,0}$ coefficient is

$$\begin{aligned}\mathbf{A}_{5,0} &= \sum_{d\geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1\end{aligned}$$

5. Conclusions

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