DERIVATION OF A COEFFICIENTS

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Abstract. Derivation of $\mathbf{A}_{m,r}$ in a simple and explicit manner.

Contents

1. Introduction and Main Results

Consider a polynomial relation

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r$$

Expanding the $(n-k)^r$ part via Binomial theorem we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \end{aligned}$$

Consider the Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$$

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it is very important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot skip summation bounds unless we do some trick as

$$\begin{aligned}\sum_{k=1}^{n} k^p &= \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}\\ &= \left[\frac{1}{p+1}\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}\\ &= \left[\frac{1}{p+1}\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}\end{aligned}$$

Using Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$ we get

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r}\\ &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right]\\ &= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right]\\ &= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\\ &= \sum_{j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\\ &= \sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\end{aligned}$$

Now, we notice that

$$\sum_{t}\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases}\frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j=0;\\ \frac{(-1)^r}{j}\binom{r}{2r-j+1}, & \text{if } j>0.\end{cases}$$

In particular, the last sum is zero for $0 < t \leq j$. So taking $j = 0$ we have

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j \geq 1} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

Now let's simplify the double summation

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{t} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r+1-j} \right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

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