POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

PETRO KOLOSOV

Contents

1. Approach via recursion examples

Consider the coefficients $\mathbf{A}_{m,r}$ definition (eqref), it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r} B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition for the polynomial $T(d,r)$

Definition 1.1.

$$T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Example 1.2. *Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$*

$$\mathbf{A}_{2,2} = 5\binom{4}{2} = 30$$

Date: July 14, 2023.

2010 *Mathematics Subject Classification.* 26E70, 05A30.

Key words and phrases. Binomial theorem, Polynomial identities, Binomial coefficients, Bernoulli numbers, Pascal's triangle, Faulhaber's formula .

Then $\mathbf{A}_{2,1} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $1 \leq d < 2$*. Finally, the* $\mathbf{A}_{2,0}$ *is*

$$\begin{aligned}\mathbf{A}_{2,0} = \sum_{d \geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) &= \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0) \\ &= 30 \cdot \frac{1}{30} = 1\end{aligned}$$

Example 1.3. *Let be* $m = 3$ *so that first we get* $\mathbf{A}_{3,3}$

$$\mathbf{A}_{3,3} = 7\binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 3$*. The* $\mathbf{A}_{3,1}$ *coefficient is non-zero and calculated as*

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally $\mathbf{A}_{3,0}$ *coefficient is*

$$\begin{aligned}\mathbf{A}_{3,0} = \sum_{d \geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) &= \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0) \\ &= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1\end{aligned}$$

Example 1.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 4$*. The value of* $\mathbf{A}_{4,1}$ *coefficient is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d \geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally $\mathbf{A}_{4,0}$ *coefficient is*

$$\mathbf{A}_{4,0} = \sum_{d \geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Email address: `kolosovp94@gmail.com`

URL: `https://kolosovpetro.github.io`