POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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ABSTRACT. In this manuscript we show that for every $n \geq 1$, $n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

CONTENTS

1. INTRODUCTION

Considering the table of forward finite differences of the polynomial n^3

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n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

We can observe easily that finite differences of the polynomial n^3 may be expressed according to the following relation, via rearrangement of the terms

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

$$\vdots$$

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n$$

Furthermore, the polynomial n^3 is identical to

$$n^3 = [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots$$

$$+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)]$$

Rearranging the above equation, we get

$$n^3 = n + (n-0) \cdot 6 \cdot 0 + (n-1) \cdot 6 \cdot 1 + (n-2) \cdot 6 \cdot 2 + \cdots + 1 \cdot 6 \cdot (n-1)$$

Therefore, we can consider n^3 as

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.1}$$

Assume that equation (1.1) has an implicit form as follows

$$n^3 = \sum_{k=1}^{n} \mathbf{A}_{1,1}k^1(n-k)^1 + \mathbf{A}_{1,0}k^0(n-k)^0, \tag{1.2}$$

where $\mathbf{A}_{1,1} = 6$ and $\mathbf{A}_{1,0} = 1$, respectively. So is there a generalization of the relation (1.2) for all positive odd powers? Therefore, let be a conjecture

Conjecture 1.1. *For every $n \geq 1$, $n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that*

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0}k^0(n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m}k^m(n-k)^m$$

2. Approach via a system of linear equations

One approach to prove the conjecture was proposed by Albert Tkaczyk in his series of the preprints [1, 2] and extended further at [3]. The main idea is to construct and solve a system of linear equations. Such a system of linear equations is constructed via expanding the definition of the coefficients $\mathbf{A}_{m,r}$ applying Binomial theorem [4] and Faulhaber's formula [5]. Consider the definition of the coefficients $\mathbf{A}_{m,r}$

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r(n-k)^r \tag{2.1}$$

Expanding the $(n-k)^r$ part via Binomial theorem we get

$$\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r(n-k)^r \\
&= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\
&= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right]
\end{aligned} \tag{2.2}$$

Applying the Faulhaber's formula to $\sum_{k=1}^{n} k^{t+r}$ we get

$$
\begin{aligned}
n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} & \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
= & \mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n + n^5) \right] + \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n + 7n^3 + 3n^7) \right] \\
& + \mathbf{A}_{m,4} \left[\frac{1}{630}(-21n + 20n^3 + n^9) \right] + \mathbf{A}_{m,5} \left[\frac{1}{2772}(-210n + 231n^3 - 22n^5 + n^{11}) \right] \\
& + \mathbf{A}_{m,6} \left[\frac{1}{60060}(-15202n + 18200n^3 - 3003n^5 + 5n^{13}) \right] \\
& + \mathbf{A}_{m,7} \left[\frac{1}{51480}(-60060n + 76010n^3 - 16380n^5 + 429n^7 + n^{15}) \right] \\
& + \mathbf{A}_{m,8} \left[\frac{1}{218790}(-1551693n + 2042040n^3 - 516868n^5 + 26520n^7 + n^{17}) \right] + \cdots
\end{aligned}
$$

$$(2.3)$$

Given fixed m, the coefficients $\mathbf{A}_{m,r}$ can be determined via a system of linear equations. Consider an example

Example 2.1. *Let be $m = 1$ so that we have the following relation defined by (eqref)*

$$
\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n + n^3) \right] - n^3 = 0
$$

Multiplying by 6 right-hand side and left-hand side, we get

$$
6\mathbf{A}_{1,0} n + \mathbf{A}_{1,1}(-n + n^3) - 6n^3 = 0
$$

Opening brackets and rearranging the terms gives

$$
6\mathbf{A}_{1,0} - \mathbf{A}_{1,1} n + \mathbf{A}_{1,1} n^3 - 6n^3 = 0
$$

Combining the common terms yields

$$
n(6\mathbf{A}_{1,0} - \mathbf{A}_{1,1}) + n^3(\mathbf{A}_{1,1} - 6) = 0
$$

Therefore, the system of linear equations follows

$$\begin{cases} 6\mathbf{A}_{1,0} - \mathbf{A}_{1,1} = 0 \\ \mathbf{A}_{1,1} - 6 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{1,1} = 6 \\ \mathbf{A}_{1,0} = 1 \end{cases}$$

Example 2.2. *Let be $m = 2$ so that we have the following relation defined by (eqref)*

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n + n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n + n^5)\right] - n^5 = 0$$

Multiplying by 30 right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(-n + n^3) + \mathbf{A}_{2,2}(-n + n^5) - 30n^5 = 0$$

Opening brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

Example 2.3. *Let be $m = 3$ so that we have the following relation defined by (eqref)*

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n + n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n + n^5)\right] + \mathbf{A}_{m,3}\left[\frac{1}{420}(-10n + 7n^3 + 3n^7)\right] - n^7 = 0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{2,1}(-n + n^3) + 14\mathbf{A}_{2,2}(-n + n^5) + \mathbf{A}_{3,3}(-10n + 7n^3 + 3n^7) - 420n^7 = 0$$

Opening brackets and rearranging the terms gives

$$420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5 - 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0$$

Combining the common terms yields

$$N(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3}) + n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5 14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} = 0 \\ \\ 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} = 0 \\ \\ \mathbf{A}_{3,2} - 30 = 0 \\ \\ 3\mathbf{A}_{3,3} - 420 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \\ \mathbf{A}_{3,2} = 0 \\ \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1} + 10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

Example 2.4. *Let be $m = 4$ so that we have the following relation defined by (eqref)*

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n + n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n + n^5)\right] + \mathbf{A}_{m,3}\left[\frac{1}{420}(-10n + 7n^3 + 3n^7)\right]$$

$$+ \mathbf{A}_{m,4}\left[\frac{1}{630}(-21n + 20n^3 + n^9)\right] - n^9 = 0$$

Multiplying by 630 right-hand side and left-hand side, we get

$$630\mathbf{A}_{4,0}n + 105\mathbf{A}_{4,1}(-n + n^3) + 21\mathbf{A}_{4,2}(-n + n^5)$$

$$+ \frac{3}{2}\mathbf{A}_{4,3}(-10n + 7n^3 + 3n^7) + \mathbf{A}_{4,4}(-21n + 20n^3 + n^9) - 630n^9 = 0$$

Opening brackets and rearranging the terms gives

$$630\mathbf{A}_{4,0}n - 105\mathbf{A}_{4,1}n + 105\mathbf{A}_{4,1}n^3 - 21\mathbf{A}_{4,2}n + 21\mathbf{A}_{4,2}n^5$$

$$- \frac{3}{2}\mathbf{A}_{4,3} \cdot 10n + \frac{3}{2}\mathbf{A}_{4,3} \cdot 7n^3 + \frac{3}{2}\mathbf{A}_{4,3} \cdot 3n^7$$

$$- 21\mathbf{A}_{4,4}n + 20\mathbf{A}_{4,4}n^3 + \mathbf{A}_{4,4}n^9 - 630n^9 = 0$$

Combining the common terms yields

$$n(630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4})$$

$$+ n^3\left(105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4}\right) + n^5(21\mathbf{A}_{4,2})$$

$$+ n^7\left(\frac{9}{2}\mathbf{A}_{4,3}\right) + n^9(\mathbf{A}_{4,4} - 630) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4} = 0 \\[2mm] 105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4} = 0 \\[2mm] \mathbf{A}_{4,2} = 0 \\[2mm] \mathbf{A}_{4,3} = 0 \\[2mm] \mathbf{A}_{4,4} - 630 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{4,4} = 630 \\[2mm] \mathbf{A}_{4,3} = 0 \\[2mm] \mathbf{A}_{4,2} = 0 \\[2mm] \mathbf{A}_{4,1} = -\frac{20}{105}\mathbf{A}_{4,4} = -120 \\[2mm] \mathbf{A}_{4,0} = \frac{105\mathbf{A}_{4,1} + 21\mathbf{A}_{4,4}}{630} = 1 \end{cases}$$

3. Approach via recursion

Consider the Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} \tag{3.1}$$

it is very important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot skip summation bounds unless we do some trick as

$$
\begin{aligned}
\sum_{k=1}^{n} k^p &= \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} \\
&= \left[\frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \\
&= \left[\frac{1}{p+1} \sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1}
\end{aligned}
\tag{3.2}
$$

Using the Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1} \sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1}$ we get

$$
\begin{aligned}
\sum_{k=1}^{n} k^r (n-k)^r &= \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \\
&= \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \left[\frac{1}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{t+r+1-j} - B_{t+r+1} \right] \\
&= \sum_{t=0}^{r} \binom{r}{t} \left[\frac{(-1)^t}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t} \right] \\
&= \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} - \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \\
&= \sum_{j} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} - \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \\
&= \sum_{j} B_j n^{2r+1-j} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} - \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}
\end{aligned}
\tag{3.3}
$$

Now, we notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j = 0; \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1}, & \text{if } j > 0. \end{cases} \tag{3.4}$$

An elegant proof of the above binomial identity is provided at [6]. In particular, the equation (3.4) is zero for $0 < t \le j$. So that taking $j = 0$ we have

$$\sum_{k=1}^n k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j \ge 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] \\ - \left[\sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \tag{3.5}$$

Now let's simplify the double summation

$$\sum_{k=1}^n k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j \ge 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r+1-j} \right]}_{(\star)} \\ - \underbrace{\left[\sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)} \tag{3.6}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$, we get

$$\sum_{k=1}^n k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_\ell \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^\ell \right] \\ - \left[\sum_\ell \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^\ell \right] \tag{3.7} \\ = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^\ell$$

Using the definition of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_r \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^\ell \equiv n^{2m+1} \tag{3.8}$$

Replacing odd ℓ by d we get

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r} \sum_d \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

$$\sum_r \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_r \mathbf{A}_{m,r} \left[\sum_d \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0$$

$$\sum_{r=0}^m \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_{r=0}^m \mathbf{A}_{m,r} \left[\sum_{d=0}^{(r-1)/2} \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0$$

$$(3.9)$$

Taking the coefficient of n^{2m+1} in (3.9), we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \tag{3.10}$$

and taking the coefficient of n^{2d+1} for an integer d in the range $m/2 \le d < m$, we get

$$\mathbf{A}_{m,d} = 0 \tag{3.11}$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \le d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2\underbrace{(2m+1)\binom{2m}{m}}_{\mathbf{A}_{m,m}}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d} B_{2m-2d} = 0, \tag{3.12}$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d} B_{2m-2d} \tag{3.13}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \le r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d \ge 2r+1}^m \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r} B_{2d-2r} \tag{3.14}$$

Finally, the coefficient $\mathbf{A}_{m,r}$ is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\[2ex] (2r+1)\binom{2r}{r}\sum_{d\ge 2r+1}^m \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \le r < m; \\[2ex] 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{3.15}$$

where B_t are Bernoulli numbers [7]. It is assumed that $B_1 = \frac{1}{2}$.

4. APPROACH VIA RECURSION: EXAMPLES

Consider the coefficients $\mathbf{A}_{m,r}$ definition (eqref), it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition for the polynomial $T(d,r)$

Definition 4.1.

$$T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Example 4.2. *Let be* $m = 2$ *so first we get* $\mathbf{A}_{2,2}$

$$\mathbf{A}_{2,2} = 5\binom{4}{2} = 30$$

Then $\mathbf{A}_{2,1} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $1 \leq d < 2$. *Finally, the coefficient* $\mathbf{A}_{2,0}$ *is*

$$\mathbf{A}_{2,0} = \sum_{d \geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) = \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0)$$

$$= 30 \cdot \frac{1}{30} = 1$$

Example 4.3. *Let be* $m = 3$ *so that first we get* $\mathbf{A}_{3,3}$

$$\mathbf{A}_{3,3} = 7\binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 3$. *The* $\mathbf{A}_{3,1}$ *coefficient is non-zero and calculated as*

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally, the coefficient $\mathbf{A}_{3,0}$ is

$$\mathbf{A}_{3,0} = \sum_{d \geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 4.4. *Let be $m = 4$ so that first we get $\mathbf{A}_{4,4}$*

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ and $\mathbf{A}_{4,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 4$. The value of the coefficient $\mathbf{A}_{4,1}$ is non-zero and calculated as

$$\mathbf{A}_{4,1} = \sum_{d \geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ is

$$\mathbf{A}_{4,0} = \sum_{d \geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 4.5. *Let be $m = 5$ so that first we get $\mathbf{A}_{5,5}$*

$$\mathbf{A}_{5,5} = 11\binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ and $\mathbf{A}_{5,3} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $3 \leq d < 5$. The value of the coefficient $\mathbf{A}_{5,2}$ is non-zero and calculated as

$$\mathbf{A}_{5,2} = \sum_{d \geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ is non-zero and calculated as

$$\mathbf{A}_{5,1} = \sum_{d \geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally, the coefficient $\mathbf{A}_{5,0}$ *is*

$$\mathbf{A}_{5,0} = \sum_{d \geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0)$$

$$= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1$$

5. Conclusions

In this manuscript, we have shown that for every $n \geq 1$, $n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0}k^0(n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m}k^m(n-k)^m$$

In particular, the coefficients $\mathbf{A}_{m,r}$ may be evaluated both ways, by constructing and solving a system of linear equations or applying recurrence relations; all these approaches are explained in the sections (1,2 refs), respectively, including detailed examples.

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