POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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Contents

1. Formulae

In this section the MathOverflow answer mathoverflow.net/a/297916 is considered and analyzed precisely. This section is copy-paste of original MO answer with in-place clarifications of some parts where formulae derivation is not so obvious. Few part are not clear for me at all, so that I kept original text there. In general, this section is motivated to provide original answer and only after that there will be questions provided in ongoing sections. To clarify and simplify navigation over the document please find table of contents. All equations in this document are numbered (regardless if they are referenced or not), so that if you have some comment it is simpler to reference particular formula. So, let's begin. Consider the polynomial relation

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \tag{1.1}$$

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Expanding the $(n-k)^r$ part via Binomial theorem we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \end{aligned} \tag{1.2}$$

Explicitly (**??**) is

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\ &= \mathbf{A}_{m,0} \left[\sum_{t=0}^{0} (-1)^t \binom{0}{t} n^{0-t} \sum_{k=1}^{n} k^{t+0} \right] + \mathbf{A}_{m,1} \left[\sum_{t=0}^{1} (-1)^t \binom{1}{t} n^{1-t} \sum_{k=1}^{n} k^{t+1} \right] \\ &+ \mathbf{A}_{m,2} \left[\sum_{t=0}^{2} (-1)^t \binom{2}{t} n^{2-t} \sum_{k=1}^{n} k^{t+2} \right] + \mathbf{A}_{m,3} \left[\sum_{t=0}^{3} (-1)^t \binom{3}{t} n^{3-t} \sum_{k=1}^{n} k^{t+3} \right] + \cdots \end{aligned} \tag{1.3}$$

Moreover,

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\ &= \mathbf{A}_{m,0} n \\ &+ \mathbf{A}_{m,1} \left[n^1 \sum_{k=1}^{n} k^1 - n^0 \sum_{k=1}^{n} k^2 \right] \\ &+ \mathbf{A}_{m,2} \left[n^2 \sum_{k=1}^{n} k^2 - 2n^1 \sum_{k=1}^{n} k^3 + n^0 \sum_{k=1}^{n} k^4 \right] \\ &+ \mathbf{A}_{m,3} \left[n^3 \sum_{k=1}^{n} k^3 - 3n^2 \sum_{k=1}^{n} k^4 + 3n^1 \sum_{k=1}^{n} k^5 - n^0 \sum_{k=1}^{n} k^6 \right] \\ &+ \mathbf{A}_{m,4} \left[n^4 \sum_{k=1}^{n} k^4 - 4n^3 \sum_{k=1}^{n} k^5 + 6n^2 \sum_{k=1}^{n} k^6 - 4n^1 \sum_{k=1}^{n} k^7 + n^0 \sum_{k=1}^{n} k^8 \right] + \cdots \end{aligned} \tag{1.4}$$

For arbitrary m we have

$$
\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5) \right] + \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n+7n^3+3n^7) \right] \\
&+ \mathbf{A}_{m,4} \left[\frac{1}{630}(-21n+20n^3+n^9) \right] + \mathbf{A}_{m,5} \left[\frac{1}{2772}(-210n+231n^3-22n^5+n^{11}) \right] \\
&+ \mathbf{A}_{m,6} \left[\frac{1}{60060}(-15202n+18200n^3-3003n^5+5n^{13}) \right] \\
&+ \mathbf{A}_{m,7} \left[\frac{1}{51480}(-60060n+76010n^3-16380n^5+429n^7+n^{15}) \right] \\
&+ \mathbf{A}_{m,8} \left[\frac{1}{218790}(-1551693n+2042040n^3-516868n^5+26520n^7+n^{17}) \right] + \cdots
\end{aligned} \tag{1.5}
$$

Expanding previous we get

$$
\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n \\
&+ \mathbf{A}_{m,1} n^1 \sum_{k=1}^{n} k^1 - \mathbf{A}_{m,1} n^0 \sum_{k=1}^{n} k^2 \\
&+ \mathbf{A}_{m,2} n^2 \sum_{k=1}^{n} k^2 - \mathbf{A}_{m,2} 2n^1 \sum_{k=1}^{n} k^3 + \mathbf{A}_{m,2} n^0 \sum_{k=1}^{n} k^4 \\
&+ \mathbf{A}_{m,3} n^3 \sum_{k=1}^{n} k^3 - \mathbf{A}_{m,3} 3n^2 \sum_{k=1}^{n} k^4 + \mathbf{A}_{m,3} 3n^1 \sum_{k=1}^{n} k^5 - \mathbf{A}_{m,3} n^0 \sum_{k=1}^{n} k^6 \\
&+ \mathbf{A}_{m,4} n^4 \sum_{k=1}^{n} k^4 - \mathbf{A}_{m,4} 4n^3 \sum_{k=1}^{n} k^5 + \mathbf{A}_{m,4} 6n^2 \sum_{k=1}^{n} k^6 - \mathbf{A}_{m,4} 4n^1 \sum_{k=1}^{n} k^7 + \mathbf{A}_{m,4} n^0 \sum_{k=1}^{n} k^8 \\
&+ \cdots
\end{aligned} \tag{1.6}
$$

Rearranging sum we get for $m = 4$

$$
\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n \\
&+ \sum_{k=1}^{n} k^1 \mathbf{A}_{m,1} n^1 \\
&+ \sum_{k=1}^{n} k^2 \left[-\mathbf{A}_{m,1} n^0 + \mathbf{A}_{m,2} n^2 \right] \\
&+ \sum_{k=1}^{n} k^3 \left[-\mathbf{A}_{m,2} 2n + \mathbf{A}_{m,3} n^3 \right] \\
&+ \sum_{k=1}^{n} k^4 \left[\mathbf{A}_{m,2} - \mathbf{A}_{m,3} 3n^2 + \mathbf{A}_{m,4} n^4 \right] \\
&+ \sum_{k=1}^{n} k^5 \left[\mathbf{A}_{m,3} 3n - \mathbf{A}_{m,4} 4n^3 \right] \\
&+ \sum_{k=1}^{n} k^6 \left[-\mathbf{A}_{m,3} + \mathbf{A}_{m,4} 6n^2 \right] \\
&+ \sum_{k=1}^{n} k^7 \left[-\mathbf{A}_{m,4} 4n \right] \\
&+ \sum_{k=1}^{n} k^8 \left[-\mathbf{A}_{m,4} \right]
\end{aligned} \tag{1.7}
$$

Rearranging sum we get for $m = 4$

$$\begin{aligned}
n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\
&= \mathbf{A}_{m,0} n \binom{0}{0} \\
&+ \sum_{k=1}^{n} k^1 \mathbf{A}_{m,1} n^1 \binom{1}{0} \\
&+ \sum_{k=1}^{n} k^2 \left[-\mathbf{A}_{m,1} n^0 \binom{1}{1} + \mathbf{A}_{m,2} n^2 \binom{2}{0} \right] \\
&+ \sum_{k=1}^{n} k^3 \left[-\mathbf{A}_{m,2} n \binom{2}{1} + \mathbf{A}_{m,3} n^3 \binom{3}{0} \right] \\
&+ \sum_{k=1}^{n} k^4 \left[\mathbf{A}_{m,2} \binom{2}{2} - \mathbf{A}_{m,3} n^2 \binom{3}{1} + \mathbf{A}_{m,4} n^4 \binom{4}{0} \right] \\
&+ \sum_{k=1}^{n} k^5 \left[\mathbf{A}_{m,3} n \binom{3}{2} - \mathbf{A}_{m,4} n^3 \binom{4}{1} \right] \\
&+ \sum_{k=1}^{n} k^6 \left[-\mathbf{A}_{m,3} \binom{3}{3} + \mathbf{A}_{m,4} n^2 \binom{4}{2} \right] \\
&+ \sum_{k=1}^{n} k^7 \left[-\mathbf{A}_{m,4} n \binom{4}{3} \right] \\
&+ \sum_{k=1}^{n} k^8 \left[-\mathbf{A}_{m,4} \binom{4}{4} \right]
\end{aligned} \tag{1.8}$$

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