POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

PETRO KOLOSOV

Abstract. In this manuscript, we have shown that for every $n \geq 1,\ n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

In particular, the coefficients $\mathbf{A}_{m,r}$ can be evaluated in both ways, by constructing and solving a certain system of linear equations or by deriving a recurrence relation; all these approaches are examined providing examples. To validate the results, there are supplementary Mathematica programs available.

Contents

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Sources: https://github.com/kolosovpetro/PolynomialIdentityInvolvingBTandFaulhaber

1. Introduction

Considering the table of forward finite differences of the polynomial n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

We can easily observe that finite differences [1] of the polynomial n^3 may be expressed according to the following relation, via rearrangement of the terms

$$\begin{aligned}\Delta(0^3) &= 1 + 6\cdot 0\\ \Delta(1^3) &= 1 + 6\cdot 0 + 6\cdot 1\\ \Delta(2^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2\\ \Delta(3^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3\\ &\vdots\\ \Delta(n^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + \cdots + 6\cdot n\end{aligned} \tag{1.1}$$

[1] One may assume that it is possible to reach the form $n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$ simply taking finite differences of the odd-powered polynomial n^{2m+1} up to order of $2m+1$ and interpolating it backwards similarly as it is shown in the equation (1.1). However, my observations do not provide any evidence that such assumption is correct. Interestingly enough is that we could have been arrived to the pure differential approach of the relation (1.4) then.

Furthermore, the polynomial n^3 is equivalent to

$$n^3 = [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots$$
$$+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)]$$

Rearranging the above equation, we get

$$n^3 = n + (n-0) \cdot 6 \cdot 0 + (n-1) \cdot 6 \cdot 1 + (n-2) \cdot 6 \cdot 2 + \cdots + 1 \cdot 6 \cdot (n-1)$$

Therefore, we can consider the polynomial n^3 as

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.2}$$

Assume that equation (1.2) has the following implicit form

$$n^3 = \sum_{k=1}^{n} \mathbf{A}_{1,1} k^1 (n-k)^1 + \mathbf{A}_{1,0} k^0 (n-k)^0, \tag{1.3}$$

where $\mathbf{A}_{1,1} = 6$ and $\mathbf{A}_{1,0} = 1$, respectively. Note that here the power of 3 is actually defined by $2m+1$ where $m = 1$. So, is there a generalization of the relation (1.3) for all positive odd powers $2m+1,\ m = 0, 1, 2, \ldots$? Therefore, let us propose a conjecture

Conjecture 1.1. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that*

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m \tag{1.4}$$

2. Approach via a system of linear equations

One approach to proving the conjecture was proposed by Albert Tkaczyk in his series of the preprints [1, 2] and extended further at [3]. The main idea is to construct and solve a system of linear equations. Such a system of linear equations is constructed by expanding the definition of the coefficients $\mathbf{A}_{m,r}$ applying Binomial theorem [4] and Faulhaber's formula [5]. Consider the definition of the coefficients $\mathbf{A}_{m,r}$

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \tag{2.1}$$

Expanding the $(n-k)^r$ part via Binomial theorem, we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \end{aligned}$$

Applying the Faulhaber's formula to the sum $\sum_{k=1}^{n} k^{t+r}$ we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \\ &= \mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5) \right] \\ &+ \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n+7n^3+3n^7) \right] + \mathbf{A}_{m,4} \left[\frac{1}{630}(-21n+20n^3+n^9) \right] \\ &+ \mathbf{A}_{m,5} \left[\frac{1}{2772}(-210n+231n^3-22n^5+n^{11}) \right] \\ &+ \mathbf{A}_{m,6} \left[\frac{1}{60060}(-15202n+18200n^3-3003n^5+5n^{13}) \right] \\ &+ \mathbf{A}_{m,7} \left[\frac{1}{51480}(-60060n+76010n^3-16380n^5+429n^7+n^{15}) \right] \\ &+ \mathbf{A}_{m,8} \left[\frac{1}{218790}(-1551693n+2042040n^3-516868n^5+26520n^7+n^{17}) \right] + \cdots \end{aligned} \tag{2.2}$$

Given a fixed integer m, the coefficients $\mathbf{A}_{m,r}$ can be determined via a system of linear equations. Consider an example

Example 2.1. *Let be $m=1$ so that we have the following relation defined by (2.2)*

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] - n^3 = 0$$

Multiplying by 6 right-hand side and left-hand side, we get

$$6\mathbf{A}_{1,0} n + \mathbf{A}_{1,1}(-n+n^3) - 6n^3 = 0$$

Opening brackets and rearranging the terms gives

$$6\mathbf{A}_{1,0} - \mathbf{A}_{1,1}n + \mathbf{A}_{1,1}n^3 - 6n^3 = 0$$

Combining the common terms yields

$$n(6\mathbf{A}_{1,0} - \mathbf{A}_{1,1}) + n^3(\mathbf{A}_{1,1} - 6) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 6\mathbf{A}_{1,0} - \mathbf{A}_{1,1} = 0 \\ \mathbf{A}_{1,1} - 6 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{1,1} = 6 \\ \mathbf{A}_{1,0} = 1 \end{cases}$$

So that odd-power identity (2.1) holds

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms $6k(n-k)+1$ over $0 \leq k \leq n$ as the following table shows

n/k	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
0	*1*							
1	*1*	*1*						
2	*1*	*7*	*1*					
3	*1*	*13*	*13*	*1*				
4	*1*	*19*	*25*	*19*	*1*			
5	*1*	*25*	*37*	*37*	*25*	*1*		
6	*1*	*31*	*49*	*55*	*49*	*31*	*1*	
7	*1*	*37*	*61*	*73*	*73*	*61*	*37*	*1*

Table 2. Values of $6k(n-k)+1$. See the OEIS entry: A287326 [6].

Example 2.2. *Let be $m = 2$ so that we have the following relation defined by* (2.2)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n+n^5)\right] - n^5 = 0$$

Multiplying by 30 *right-hand side and left-hand side, we get*

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(-n+n^3) + \mathbf{A}_{2,2}(-n+n^5) - 30n^5 = 0$$

Opening brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that odd-power identity (2.1) *holds*

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true evaluating the terms $30k^2(n-k)^2+1$ *over* $0 \le k \le n$ *as the following table shows*

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $30k^2(n-k)^2+1$. See the OEIS entry A300656 [7].

Example 2.3. *Let be $m=3$ so that we have the following relation defined by (2.2)*

$$\mathbf{A}_{m,0}n+\mathbf{A}_{m,1}\left[\frac{1}{6}(-n+n^3)\right]+\mathbf{A}_{m,2}\left[\frac{1}{30}(-n+n^5)\right]+\mathbf{A}_{m,3}\left[\frac{1}{420}(-10n+7n^3+3n^7)\right]-n^7=0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n+70\mathbf{A}_{2,1}(-n+n^3)+14\mathbf{A}_{2,2}(-n+n^5)+\mathbf{A}_{3,3}(-10n+7n^3+3n^7)-420n^7=0$$

Opening brackets and rearranging the terms gives

$$\begin{aligned}420\mathbf{A}_{3,0}n-70\mathbf{A}_{3,1}+70\mathbf{A}_{3,1}n^3-14\mathbf{A}_{3,2}n+14\mathbf{A}_{3,2}n^5\\-10\mathbf{A}_{3,3}n+7\mathbf{A}_{3,3}n^3+3\mathbf{A}_{3,3}n^7-420n^7=0\end{aligned}$$

Combining the common terms yields

$$\begin{aligned}&n(420\mathbf{A}_{3,0}-70\mathbf{A}_{3,1}-14\mathbf{A}_{3,2}-10\mathbf{A}_{3,3})\\&+n^3(70\mathbf{A}_{3,1}+7\mathbf{A}_{3,3})+n^5 14\mathbf{A}_{3,2}+n^7(3\mathbf{A}_{3,3}-420)=0\end{aligned}$$

Therefore, the system of linear equations follows

$$\begin{cases}420\mathbf{A}_{3,0}-70\mathbf{A}_{3,1}-14\mathbf{A}_{3,2}-10\mathbf{A}_{3,3}=0\\70\mathbf{A}_{3,1}+7\mathbf{A}_{3,3}=0\\\mathbf{A}_{3,2}-30=0\\3\mathbf{A}_{3,3}-420=0\end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \mathbf{A}_{3,2} = 0 \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1}+10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

So that odd-power identity (2.1) *holds*

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms $140k^3(n-k)^3 - 14k(n-k) + 1$ *over* $0 \le k \le n$ *as the OEIS sequence A300785* [8] *shows*

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	127	1					
3	1	1093	1093	1				
4	1	3739	8905	3739	1			
5	1	8905	30157	30157	8905	1		
6	1	17431	71569	101935	71569	17431	1	
7	1	30157	139861	241753	241753	139861	30157	1

Table 4. Values of $140k^3(n-k)^3 - 14k(n-k) + 1$. See the OEIS entry A300785 [8].

Example 2.4. *Let be* $m = 4$ *so that we have the following relation defined by* (2.2)

$$\begin{aligned} \mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n+n^5)\right] \\ + \mathbf{A}_{m,3}\left[\frac{1}{420}(-10n+7n^3+3n^7)\right] \\ + \mathbf{A}_{m,4}\left[\frac{1}{630}(-21n+20n^3+n^9)\right] - n^9 = 0 \end{aligned}$$

Multiplying by 630 *right-hand side and left-hand side, we get*

$$\begin{aligned}630\mathbf{A}_{4,0}n + 105\mathbf{A}_{4,1}(-n+n^3) + 21\mathbf{A}_{4,2}(-n+n^5)\\ + \frac{3}{2}\mathbf{A}_{4,3}(-10n+7n^3+3n^7)\\ + \mathbf{A}_{4,4}(-21n+20n^3+n^9) - 630n^9 = 0\end{aligned}$$

Opening brackets and rearranging the terms gives

$$\begin{aligned}630\mathbf{A}_{4,0}n - 105\mathbf{A}_{4,1}n + 105\mathbf{A}_{4,1}n^3 - 21\mathbf{A}_{4,2}n + 21\mathbf{A}_{4,2}n^5\\ - \frac{3}{2}\mathbf{A}_{4,3}\cdot 10n + \frac{3}{2}\mathbf{A}_{4,3}\cdot 7n^3 + \frac{3}{2}\mathbf{A}_{4,3}\cdot 3n^7\\ - 21\mathbf{A}_{4,4}n + 20\mathbf{A}_{4,4}n^3 + \mathbf{A}_{4,4}n^9 - 630n^9 = 0\end{aligned}$$

Combining the common terms yields

$$\begin{aligned}&n(630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4})\\ &+ n^3\left(105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4}\right) + n^5(21\mathbf{A}_{4,2})\\ &+ n^7\left(\frac{9}{2}\mathbf{A}_{4,3}\right) + n^9(\mathbf{A}_{4,4} - 630) = 0\end{aligned}$$

Therefore, the system of linear equations follows

$$\begin{cases}630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4} = 0\\ 105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4} = 0\\ \mathbf{A}_{4,2} = 0\\ \mathbf{A}_{4,3} = 0\\ \mathbf{A}_{4,4} - 630 = 0\end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{4,4} = 630 \\ \mathbf{A}_{4,3} = 0 \\ \mathbf{A}_{4,2} = 0 \\ \mathbf{A}_{4,1} = -\frac{20}{105}\mathbf{A}_{4,4} = -120 \\ \mathbf{A}_{4,0} = \frac{105\mathbf{A}_{4,1}+21\mathbf{A}_{4,4}}{630} = 1 \end{cases}$$

So that odd-power identity (2.1) *holds*

$$n^9 = \sum_{k=1}^{n} 630k^4(n-k)^4 - 120k(n-k) + 1$$

3. Finding a recurrence relation

Another approach to determine the coefficients $\mathbf{A}_{m,r}$ was proposed by Dr. Max Alekseyev in MathOverflow discussion [9]. Generally, the idea was to determine the coefficients $\mathbf{A}_{m,r}$ recursively starting from the base case $\mathbf{A}_{m,m}$ up to $\mathbf{A}_{m,r-1}, \ldots, \mathbf{A}_{m,0}$ via previously determined values.

Before we consider the derivation of recurrent formula for coefficients $\mathbf{A}_{m,r}$, a few aspects regarding the Faulhaber's formula [5] should be discussed

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$$

it is important to notice that iteration step j is bounded by the value of power p, while the upper index of the binomial coefficient $\binom{p+1}{j}$ is $p+1$. Therefore, we can omit summation bounds letting j run over infinity by applying the following on the Faulhaber's formula.

$$\begin{aligned} \sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} &= \left[\frac{1}{p+1}\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1} \\ &= \left[\frac{1}{p+1}\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1} \end{aligned}$$

Now we are good to go through the entire derivation of the recurrent formula for coefficients $\mathbf{A}_{m,r}$.

By applying Binomial theorem $(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}k^t$ and Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}$, we get

$$
\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r} \\
&= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right] \\
&= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right] \\
&= \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \\
&= \left[\sum_{j,t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]
\end{aligned}
$$

Rearranging terms yields

$$
\left[\sum_j B_j n^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] - \left[\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \tag{3.1}
$$

We can notice that

$$
\sum_t\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j=0 \\ \frac{(-1)^r}{j}\binom{r}{2r-j+1} & \text{if } j>0 \end{cases} \tag{3.2}
$$

An elegant proof of the binomial identity (3.2) is presented in [10].

In particular, equation (3.2) is zero for $0 < t \le j$. In order to apply (3.2), we have to move $j=0$ out of summation in (3.1) to avoid division by zero in $\frac{(-1)^r}{j}$, which yields

$$
\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} &+ \left[\sum_{j\ge 1} B_j n^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] \\
&- \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]
\end{aligned}
$$

Now we do not care about division by zero in $\frac{(-1)^r}{j}$ so that simplifying above equation by using (3.2) yields

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \underbrace{\left[\sum_{j\geq 1}\frac{(-1)^r}{j}\binom{r}{2r-j+1}B_j n^{2r-j+1}\right]}_{(\star)}$$
$$- \underbrace{\left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r-j+1$ to $(\star)$ and $\ell = r-t$ to $(\diamond)$ we collapse the common terms across two sums

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \left[\sum_{\ell}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^{\ell}\right] \\ &\quad - \left[\sum_{\ell}\binom{r}{\ell}\frac{(-1)^{r-\ell}}{2r+1-\ell}B_{2r+1-\ell}n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{\text{odd } \ell}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^{\ell}\end{aligned}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by (2.1), we obtain the following relation for polynomials in n

$$\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\text{ odd }\ell}\mathbf{A}_{m,r}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by k we get

$$\sum_{r}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\, k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1} \equiv n^{2m+1}$$

Taking the coefficient of n^{2m+1} we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

because $\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} = 1$.

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0$$

because we focus on sum $2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}$, in particular on n^{2k+1} and binomial coefficient $\binom{r}{2k+1}$. For instance, if we have to get coefficient of n^{2d+1} in range $\frac{m}{2}\leq d<m$, we set $d=m-1$, thus we have to get coefficient of $m-1$ in $2\sum_{r,\,k}\mathbf{A}_{m,r}\frac{(-1)^r}{2r-2k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}$. Therefore, we set $k=m-1$ and $r=m-1$ which leads that $\binom{r}{2k+1}=\binom{m-1}{2m-1}=0$, so that $\mathbf{A}_{m,m-1}\frac{1}{(2m-1)\binom{2m-2}{m-1}}n^{2m-1}=0$. Same applies for every d in the range $\frac{m}{2}\leq d<m$, because $r=\frac{m}{2}$ and $k=\frac{m}{2}$ means that $\binom{r}{2k+1}=\binom{\frac{m}{2}}{m+1}=0$.

To summarize, the value of k should be in range $k\leq\frac{d-1}{2}$ so that binomial coefficient $\binom{d}{2k+1}$ is non-zero.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4}\leq d<\frac{m}{2}$ we get

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}}+2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d}=0$$

i.e

$$\mathbf{A}_{m,d}=(-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $\frac{m}{2^{s+1}}\leq r<\frac{m}{2^s}$, iterating consecutively over $s=1,2,\ldots$ by using previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r}=(2r+1)\binom{2r}{r}\sum_{d\geq 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient $\mathbf{A}_{m,r}$

Definition 3.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r}=\begin{cases}(2r+1)\binom{2r}{r} & \text{if } r=m\\ (2r+1)\binom{2r}{r}\sum_{d\geq 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0\leq r<m\\ 0 & \text{if } r<0 \text{ or } r>m\end{cases}\tag{3.3}$$

where B_t are Bernoulli numbers [11]. It is assumed that $B_1=\frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 5. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [12, 13].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

Let be a theorem

Theorem 3.2. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively by (3.3).

4. Recurrence relation: examples

Consider the definition (3.3) of the coefficients $\mathbf{A}_{m,r}$, it can be written as

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ \sum_{d\geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition of the real coefficient $T(d,r)$

Definition 4.1. *Real coefficient $T(d,r)$*

$$T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Example 4.2. *Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$*

$$\mathbf{A}_{2,2} = 5\binom{4}{2} = 30$$

Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the coefficient $\mathbf{A}_{2,0}$ is

$$\begin{aligned} \mathbf{A}_{2,0} = \sum_{d\geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) &= \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0) \\ &= 30 \cdot \frac{1}{30} = 1 \end{aligned}$$

Example 4.3. *Let be $m = 3$ so that first we get $\mathbf{A}_{3,3}$*

$$\mathbf{A}_{3,3} = 7\binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{3,1} = \sum_{d\geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally, the coefficient $\mathbf{A}_{3,0}$ *is*

$$\mathbf{A}_{3,0} = \sum_{d\geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 4.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 4$. *The value of the coefficient* $\mathbf{A}_{4,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d\geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ *is*

$$\mathbf{A}_{4,0} = \sum_{d\geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 4.5. *Let be* $m = 5$ *so that first we get* $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11\binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ *and* $\mathbf{A}_{5,3} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $3 \leq d < 5$. *The value of the coefficient* $\mathbf{A}_{5,2}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,2} = \sum_{d\geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,1} = \sum_{d\geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally, the coefficient $\mathbf{A}_{5,0}$ *is*

$$\begin{aligned}\mathbf{A}_{5,0} &= \sum_{d\geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1\end{aligned}$$

As expected.

5. Conclusions

In this manuscript, we have shown that for every $n \geq 1,\ n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

In particular, the coefficients $\mathbf{A}_{m,r}$ can be evaluated in both ways, by constructing and solving a certain system of linear equations or by deriving a recurrence relation; all these approaches are examined providing examples in the sections (2) and (3). Moreover, to validate the results, supplementary Mathematica programs are available at [14].

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Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io