POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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Contents

1. Approach via recursion

Consider the Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} \tag{1.1}$$

it is very important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot skip summation bounds unless we do some trick as

$$\begin{aligned}\sum_{k=1}^{n} k^p &= \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} \\ &= \left[\frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1} \\ &= \left[\frac{1}{p+1} \sum_{j} \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}\end{aligned} \tag{1.2}$$

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Using Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}$ we get

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r}\\ &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_j\binom{t+r+1}{j}B_jn^{t+r+1-j} - B_{t+r+1}\right]\\ &= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_jn^{2r+1-j} - B_{t+r+1}n^{r-t}\right]\\ &= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_jn^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\\ &= \sum_j\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_jn^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\\ &= \sum_j B_jn^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\end{aligned} \tag{1.3}$$

Now, we notice that

$$\sum_t\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases}\frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j=0;\\ \frac{(-1)^r}{j}\binom{r}{2r-j+1}, & \text{if } j>0.\end{cases} \tag{1.4}$$

In particular, the last sum is zero for $0 < t \leq j$. So taking $j=0$ we have

$$\begin{aligned}\sum_{k=1}^{n}k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} &+ \left[\sum_{j\geq1}B_jn^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right]\\ &- \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]\end{aligned} \tag{1.5}$$

Now let's simplify the double summation

$$\begin{aligned}\sum_{k=1}^{n}k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} &+ \underbrace{\left[\sum_{j\geq1}\frac{(-1)^r}{j}\binom{r}{2r-j+1}B_jn^{2r+1-j}\right]}_{(\star)}\\ &- \underbrace{\left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]}_{(\diamond)}\end{aligned} \tag{1.6}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$, we get

$$\begin{aligned}\sum_{k=1}^{n} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell}\frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\end{aligned} \tag{1.7}$$

Using the definition of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \tag{1.8}$$

Replacing odd ℓ by d we get

$$\begin{aligned}&\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r} \sum_d \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1} \\ &\sum_r \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_r \mathbf{A}_{m,r} \left[\sum_d \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0 \\ &\sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{d=0}^{(r-1)/2} \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0\end{aligned} \tag{1.9}$$

Taking the coefficient of n^{2m+1} in (**??**), we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \tag{1.10}$$

and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \le d < m$, we get

$$\mathbf{A}_{m,d} = 0 \tag{1.11}$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \le d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2\underbrace{(2m+1)\binom{2m}{m}}_{\mathbf{A}_{m,m}} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0, \tag{1.12}$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d} \tag{1.13}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \leq r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} \tag{1.14}$$

Finally, the coefficient $\mathbf{A}_{m,r}$ is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{1.15}$$

where B_t are Bernoulli numbers. It is assumed that $B_1 = \frac{1}{2}$.

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