

POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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1. APPROACH VIA RECURSION EXAMPLES

Consider the coefficients $\mathbf{A}_{m,r}$ definition (eqref), it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1) \binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \underbrace{(2r+1) \binom{2r}{r} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition for the polynomial $T(d, r)$

Definition 1.1.

$$T(d, r) = (2r+1) \binom{2r}{r} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Example 1.2. Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$

$$\mathbf{A}_{2,2} = 5 \binom{4}{2} = 30$$

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Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the $\mathbf{A}_{2,0}$ is

$$\begin{aligned}\mathbf{A}_{2,0} &= \sum_{d \geq 1}^2 \mathbf{A}_{2,d} \cdot T(d, 0) = \mathbf{A}_{2,1} \cdot T(1, 0) + \mathbf{A}_{2,2} \cdot T(2, 0) \\ &= 30 \cdot \frac{1}{30} = 1\end{aligned}$$

Example 1.3. Let be $m = 3$ so that first we get $\mathbf{A}_{3,3}$

$$\mathbf{A}_{3,3} = 7 \binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^3 \mathbf{A}_{3,d} \cdot T(d, 1) = \mathbf{A}_{3,3} \cdot T(3, 1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally $\mathbf{A}_{3,0}$ coefficient is

$$\begin{aligned}\mathbf{A}_{3,0} &= \sum_{d \geq 1}^3 \mathbf{A}_{3,d} \cdot T(d, 0) = \mathbf{A}_{3,1} \cdot T(1, 0) + \mathbf{A}_{3,2} \cdot T(2, 0) + \mathbf{A}_{3,3} \cdot T(3, 0) \\ &= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1\end{aligned}$$

Example 1.4. Let be $m = 4$ so that first we get $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9 \binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ and $\mathbf{A}_{4,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 4$. The value of $\mathbf{A}_{4,1}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{4,1} = \sum_{d \geq 3}^4 \mathbf{A}_{4,d} \cdot T(d, 1) = \mathbf{A}_{4,3} \cdot T(3, 1) + \mathbf{A}_{4,4} \cdot T(4, 1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally $\mathbf{A}_{4,0}$ coefficient is

$$\mathbf{A}_{4,0} = \sum_{d \geq 1}^4 \mathbf{A}_{4,d} \cdot T(d, 0) = \mathbf{A}_{4,1} \cdot T(1, 0) + \mathbf{A}_{4,4} \cdot T(4, 0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 1.5. *Let be $m = 5$ so that first we get $\mathbf{A}_{5,5}$*

$$\mathbf{A}_{5,5} = 11 \binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ and $\mathbf{A}_{5,3} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $3 \leq d < 5$. The value of $\mathbf{A}_{5,2}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{5,2} = \sum_{d \geq 5}^5 \mathbf{A}_{5,d} \cdot T(d, 2) = \mathbf{A}_{5,5} \cdot T(5, 2) = 2772 \cdot \frac{5}{21} = 660$$

The value of $\mathbf{A}_{5,1}$ coefficient is non-zero and calculated as

$$\begin{aligned} \mathbf{A}_{5,1} &= \sum_{d \geq 3}^5 \mathbf{A}_{5,d} \cdot T(d, 1) = \mathbf{A}_{5,3} \cdot T(3, 1) + \mathbf{A}_{5,4} \cdot T(4, 1) + \mathbf{A}_{5,5} \cdot T(5, 1) \\ &= 2772 \cdot \left(-\frac{1}{2}\right) = -1386 \end{aligned}$$

Finally the $\mathbf{A}_{5,0}$ coefficient is

$$\begin{aligned} \mathbf{A}_{5,0} &= \sum_{d \geq 1}^5 \mathbf{A}_{5,d} \cdot T(d, 0) = \mathbf{A}_{5,1} \cdot T(1, 0) + \mathbf{A}_{5,2} \cdot T(2, 0) + \mathbf{A}_{5,5} \cdot T(5, 0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1 \end{aligned}$$

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