DERIVATION OF $\mathbf{A}_{m,r}$ COEFFICIENTS

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Abstract. Derivation of $\mathbf{A}_{m,r}$ in a simple and explicit manner.

Contents

1. Introduction and Main Results

In this section the MathOverflow answer mathoverflow.net/a/297916 is considered and analyzed precisely. This section is copy-paste of original MO answer with in-place clarifications of some parts where formulae derivation is not so obvious. Few part are not clear for me at all, so that I kept original text there. In general, this section is motivated to provide original answer and only after that there will be questions provided in ongoing sections. To clarify and simplify navigation over the document please find table of contents. All equations in this document are numbered (regardless if they are referenced or not), so that if you have some comment it is simpler to reference particular formula. So, let's begin. Consider the polynomial relation

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \tag{1.1}$$

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Expanding the $(n-k)^r$ part via Binomial theorem we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \right] \end{aligned} \tag{1.2}$$

Consider the Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} \tag{1.3}$$

it is very important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot skip summation bounds unless we do some trick as

$$\begin{aligned} \sum_{k=1}^{n} k^p &= \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} \\ &= \left[\frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \\ &= \left[\frac{1}{p+1} \sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \end{aligned} \tag{1.4}$$

Using Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}$ we get

$$\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r}\\
&= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right]\\
&= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right]\\
&= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\\
&= \sum_j\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\\
&= \sum_j B_j n^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}
\end{aligned} \tag{1.5}$$

Now, we notice that

$$\sum_t\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases}\frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j=0;\\ \frac{(-1)^r}{j}\binom{r}{2r-j+1}, & \text{if } j>0.\end{cases} \tag{1.6}$$

In particular, the last sum is zero for $0 < t \le j$. So taking $j=0$ we have

$$\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} &+ \left[\sum_{j\ge1}B_j n^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right]\\
&- \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]
\end{aligned} \tag{1.7}$$

Now let's simplify the double summation

$$\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} &+ \underbrace{\left[\sum_{j\ge1}\frac{(-1)^r}{j}\binom{r}{2r-j+1}B_j n^{2r+1-j}\right]}_{(\star)}\\
&- \underbrace{\left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]}_{(\diamond)}
\end{aligned} \tag{1.8}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$, we get

$$\begin{aligned}\sum_{k=1}^{n} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\end{aligned} \tag{1.9}$$

Using the definition of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \tag{1.10}$$

Replacing odd ℓ by d we get

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r} \sum_{d} \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

$$\sum_r \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_r \mathbf{A}_{m,r} \left[\sum_{d} \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0. \tag{1.11}$$

Taking the coefficient of n^{2m+1} in (1.11), we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \tag{1.12}$$

and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0 \tag{1.13}$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \leq d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2\underbrace{(2m+1)\binom{2m}{m}}_{\mathbf{A}_{m,m}} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0, \tag{1.14}$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d} \tag{1.15}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \leq r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} \tag{1.16}$$

Finally, the coefficient $\mathbf{A}_{m,r}$ is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{1.17}$$

where B_t are Bernoulli numbers. It is assumed that $B_1 = \frac{1}{2}$.

2. Questions

(1) Any proof or reference to the relation (1.6)?

(2) What is the motivation to use (1.4) version of Faulhaber's formula?

(3) Why is there twice odd ℓ in (1.9)?

(4) Why $\mathbf{A}_{m,m}$ equals $(2m+1)\binom{2m}{m}$?

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