

POLYNOMIAL IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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ABSTRACT. In this manuscript we show that for every $n \geq 1$, $n, m \in \mathbb{N}$ there are coefficients

$\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \dots + \mathbf{A}_{m,m} k^m (n-k)^m$$

CONTENTS

1. Introduction	1
2. Approach via the system of linear equations	3
3. Approach via recursion	8
4. Approach via recursion: Examples	11
5. Conclusions	13

1. INTRODUCTION

Considering the table of forward finite differences of the polynomial n^3

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n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3 .

We can observe easily that finite differences of the polynomial n^3 may be expressed according to the following relation, via rearrangement of the terms

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

$$\vdots$$

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n$$

Furthermore, the polynomial n^3 is identical to

$$\begin{aligned} n^3 &= [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\ &\quad + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

Rearranging the above equation, we get

$$n^3 = n + (n-0) \cdot 6 \cdot 0 + (n-1) \cdot 6 \cdot 1 + (n-2) \cdot 6 \cdot 2 + \cdots + 1 \cdot 6 \cdot (n-1)$$

Therefore, we can consider n^3 as

$$n^3 = \sum_{k=1}^n 6k(n-k) + 1 \quad (1.1)$$

Assume that equation (1.1) has an implicit form as follows

$$n^3 = \sum_{k=1}^n \mathbf{A}_{1,1} k^1 (n-k)^1 + \mathbf{A}_{1,0} k^0 (n-k)^0, \quad (1.2)$$

where $\mathbf{A}_{1,1} = 6$ and $\mathbf{A}_{1,0} = 1$, respectively. So could the relation (1.2) be generalized for all positive odd powers? Therefore, let be a conjecture

Conjecture 1.1. *For every $n \geq 1$, $n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that*

$$n^{2m+1} = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \dots + \mathbf{A}_{m,m} k^m (n-k)^m$$

2. APPROACH VIA THE SYSTEM OF LINEAR EQUATIONS

One approach to prove the conjecture was proposed by Albert Tkaczyk in his series of preprints (two references). The main idea is to construct and solve a system of linear equations. Such a system of linear equations is constructed via expanding the definition of the coefficients $\mathbf{A}_{m,r}$ (reference) applying Binomial theorem and Faulhaber's formula. Consider the definition of the coefficients $\mathbf{A}_{m,r}$

$$n^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=1}^n k^r (n-k)^r \quad (2.1)$$

Expanding the $(n-k)^r$ part via Binomial theorem we get

$$\begin{aligned} n^{2m+1} &= \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=1}^n k^r (n-k)^r \\ &= \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=1}^n k^r \left[\sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} k^t \right] \\ &= \sum_{r=0}^m \mathbf{A}_{m,r} \left[\sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^n k^{t+r} \right] \end{aligned} \quad (2.2)$$

Applying the Faulhaber's formula to $\sum_{k=1}^n k^{t+r}$ we get

$$\begin{aligned}
 n^{2m+1} &= \sum_{r=0}^m \mathbf{A}_{m,r} \left[\sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^n k^{t+r} \right] \\
 &= \mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n + n^5) \right] + \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n + 7n^3 + 3n^7) \right] \\
 &\quad + \mathbf{A}_{m,4} \left[\frac{1}{630}(-21n + 20n^3 + n^9) \right] + \mathbf{A}_{m,5} \left[\frac{1}{2772}(-210n + 231n^3 - 22n^5 + n^{11}) \right] \\
 &\quad + \mathbf{A}_{m,6} \left[\frac{1}{60060}(-15202n + 18200n^3 - 3003n^5 + 5n^{13}) \right] \\
 &\quad + \mathbf{A}_{m,7} \left[\frac{1}{51480}(-60060n + 76010n^3 - 16380n^5 + 429n^7 + n^{15}) \right] \\
 &\quad + \mathbf{A}_{m,8} \left[\frac{1}{218790}(-1551693n + 2042040n^3 - 516868n^5 + 26520n^7 + n^{17}) \right] + \dots
 \end{aligned} \tag{2.3}$$

Given fixed m , the coefficients $\mathbf{A}_{m,r}$ can be determined via a system of linear equations.

Consider an example

Example 2.1. Let be $m = 1$ so that we have the following relation defined by (eqref)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n + n^3) \right] - n^3 = 0$$

Multiplying by 6 right-hand side and left-hand side, we get

$$6\mathbf{A}_{1,0}n + \mathbf{A}_{1,1}(-n + n^3) - 6n^3 = 0$$

Opening brackets and rearranging the terms gives

$$6\mathbf{A}_{1,0} - \mathbf{A}_{1,1}n + \mathbf{A}_{1,1}n^3 - 6n^3 = 0$$

Combining the common terms yields

$$n(6\mathbf{A}_{1,0} - \mathbf{A}_{1,1}) + n^3(\mathbf{A}_{1,1} - 6) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 6\mathbf{A}_{1,0} - \mathbf{A}_{1,1} = 0 \\ \mathbf{A}_{1,1} - 6 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{1,1} = 6 \\ \mathbf{A}_{1,0} = 1 \end{cases}$$

Example 2.2. Let be $m = 2$ so that we have the following relation defined by (eqref)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n + n^5) \right] - n^5 = 0$$

Multiplying by 30 right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(-n + n^3) + \mathbf{A}_{2,2}(-n + n^5) - 30n^5 = 0$$

Opening brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

Example 2.3. Let be $m = 3$ so that we have the following relation defined by (eqref)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n + n^5) \right] + \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n + 7n^3 + 3n^7) \right] - n^7 = 0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{3,1}(-n + n^3) + 14\mathbf{A}_{3,2}(-n + n^5) + \mathbf{A}_{3,3}(-10n + 7n^3 + 3n^7) - 420n^7 = 0$$

Opening brackets and rearranging the terms gives

$$420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5 - 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0$$

Combining the common terms yields

$$N(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3}) + n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5(14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420)) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} = 0 \\ 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} = 0 \\ \mathbf{A}_{3,2} - 30 = 0 \\ 3\mathbf{A}_{3,3} - 420 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \mathbf{A}_{3,2} = 0 \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1} + 10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

Example 2.4. Let be $m = 4$ so that we have the following relation defined by (eqref)

$$\begin{aligned} &\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n + n^5) \right] + \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n + 7n^3 + 3n^7) \right] \\ &+ \mathbf{A}_{m,4} \left[\frac{1}{630}(-21n + 20n^3 + n^9) \right] - n^9 = 0 \end{aligned}$$

Multiplying by 630 right-hand side and left-hand side, we get

$$630\mathbf{A}_{4,0}n + 105\mathbf{A}_{4,1}(-n + n^3) + 21\mathbf{A}_{4,2}(-n + n^5) \\ + \frac{3}{2}\mathbf{A}_{4,3}(-10n + 7n^3 + 3n^7) + \mathbf{A}_{4,4}(-21n + 20n^3 + n^9) - 630n^9 = 0$$

Opening brackets and rearranging the terms gives

$$630\mathbf{A}_{4,0}n - 105\mathbf{A}_{4,1}n + 105\mathbf{A}_{4,1}n^3 - 21\mathbf{A}_{4,2}n + 21\mathbf{A}_{4,2}n^5 \\ - \frac{3}{2}\mathbf{A}_{4,3} \cdot 10n + \frac{3}{2}\mathbf{A}_{4,3} \cdot 7n^3 + \frac{3}{2}\mathbf{A}_{4,3} \cdot 3n^7 \\ - 21\mathbf{A}_{4,4}n + 20\mathbf{A}_{4,4}n^3 + \mathbf{A}_{4,4}n^9 - 630n^9 = 0$$

Combining the common terms yields

$$n(630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4}) \\ + n^3 \left(105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4} \right) + n^5(21\mathbf{A}_{4,2}) \\ + n^7 \left(\frac{9}{2}\mathbf{A}_{4,3} \right) + n^9(\mathbf{A}_{4,4} - 630) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4} = 0 \\ 105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4} = 0 \\ \mathbf{A}_{4,2} = 0 \\ \mathbf{A}_{4,3} = 0 \\ \mathbf{A}_{4,4} - 630 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{4,4} = 630 \\ \mathbf{A}_{4,3} = 0 \\ \mathbf{A}_{4,2} = 0 \\ \mathbf{A}_{4,1} = -\frac{20}{105}\mathbf{A}_{4,4} = -120 \\ \mathbf{A}_{4,0} = \frac{105\mathbf{A}_{4,1} + 21\mathbf{A}_{4,4}}{630} = 1 \end{cases}$$

3. APPROACH VIA RECURSION

Consider the Faulhaber's formula

$$\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j} \quad (3.1)$$

it is very important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot skip summation bounds unless we do some trick as

$$\begin{aligned} \sum_{k=1}^n k^p &= \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j} \\ &= \left[\frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \\ &= \left[\frac{1}{p+1} \sum_j \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \end{aligned} \quad (3.2)$$

Using the Faulhaber's formula $\sum_{k=1}^n k^p = \left[\frac{1}{p+1} \sum_j \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1}$ we get

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^n k^{t+r} \\ &= \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \left[\frac{1}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{t+r+1-j} - B_{t+r+1} \right] \\ &= \sum_{t=0}^r \binom{r}{t} \left[\frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t} \right] \\ &= \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \\ &= \sum_j \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} - \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \\ &= \sum_j B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} - \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \end{aligned} \quad (3.3)$$

Now, we notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j = 0; \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1}, & \text{if } j > 0. \end{cases} \quad (3.4)$$

In particular, the last sum is zero for $0 < t \leq j$. So taking $j = 0$ we have

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j \geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] \\ &\quad - \left[\sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \end{aligned} \quad (3.5)$$

Now let's simplify the double summation

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j \geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r+1-j} \right]}_{(\star)} \\ &\quad - \underbrace{\left[\sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)} \end{aligned} \quad (3.6)$$

Hence, introducing $\ell = 2r - j + 1$ to (\star) and $\ell = r - t$ to (\diamond) , we get

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned} \quad (3.7)$$

Using the definition of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_r \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1} \quad (3.8)$$

Replacing odd ℓ by d we get

$$\begin{aligned} & \sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_r \mathbf{A}_{m,r} \sum_d \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1} \\ & \sum_r \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} \right] + 2 \sum_r \mathbf{A}_{m,r} \left[\sum_d \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \right] - n^{2m+1} = 0 \\ & \sum_{r=0}^m \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} \right] + 2 \sum_{r=0}^m \mathbf{A}_{m,r} \left[\sum_{d=0}^{(r-1)/2} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \right] - n^{2m+1} = 0 \end{aligned} \quad (3.9)$$

Taking the coefficient of n^{2m+1} in (3.9), we get

$$\mathbf{A}_{m,m} = (2m+1) \binom{2m}{m} \quad (3.10)$$

and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0 \quad (3.11)$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \leq d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2 \underbrace{(2m+1) \binom{2m}{m}}_{\mathbf{A}_{m,m}} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0, \quad (3.12)$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d} \quad (3.13)$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \leq r < m/2^s$ (iterating consecutively $s = 1, 2, \dots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} \quad (3.14)$$

Finally, the coefficient $\mathbf{A}_{m,r}$ is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1) \binom{2r}{r}, & \text{if } r = m; \\ (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \quad (3.15)$$

where B_t are Bernoulli numbers. It is assumed that $B_1 = \frac{1}{2}$.

4. APPROACH VIA RECURSION: EXAMPLES

Consider the coefficients $\mathbf{A}_{m,r}$ definition (eqref), it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1) \binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \underbrace{(2r+1) \binom{2r}{r} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition for the polynomial $T(d, r)$

Definition 4.1.

$$T(d, r) = (2r+1) \binom{2r}{r} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Example 4.2. Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$

$$\mathbf{A}_{2,2} = 5 \binom{4}{2} = 30$$

Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the coefficient $\mathbf{A}_{2,0}$ is

$$\begin{aligned} \mathbf{A}_{2,0} &= \sum_{d \geq 1}^2 \mathbf{A}_{2,d} \cdot T(d, 0) = \mathbf{A}_{2,1} \cdot T(1, 0) + \mathbf{A}_{2,2} \cdot T(2, 0) \\ &= 30 \cdot \frac{1}{30} = 1 \end{aligned}$$

Example 4.3. Let be $m = 3$ so that first we get $\mathbf{A}_{3,3}$

$$\mathbf{A}_{3,3} = 7 \binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^3 \mathbf{A}_{3,d} \cdot T(d, 1) = \mathbf{A}_{3,3} \cdot T(3, 1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally, the coefficient $\mathbf{A}_{3,0}$ is

$$\begin{aligned}\mathbf{A}_{3,0} &= \sum_{d \geq 1}^3 \mathbf{A}_{3,d} \cdot T(d, 0) = \mathbf{A}_{3,1} \cdot T(1, 0) + \mathbf{A}_{3,2} \cdot T(2, 0) + \mathbf{A}_{3,3} \cdot T(3, 0) \\ &= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1\end{aligned}$$

Example 4.4. Let be $m = 4$ so that first we get $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9 \binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ and $\mathbf{A}_{4,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 4$. The value of the coefficient $\mathbf{A}_{4,1}$ is non-zero and calculated as

$$\mathbf{A}_{4,1} = \sum_{d \geq 3}^4 \mathbf{A}_{4,d} \cdot T(d, 1) = \mathbf{A}_{4,3} \cdot T(3, 1) + \mathbf{A}_{4,4} \cdot T(4, 1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ is

$$\mathbf{A}_{4,0} = \sum_{d \geq 1}^4 \mathbf{A}_{4,d} \cdot T(d, 0) = \mathbf{A}_{4,1} \cdot T(1, 0) + \mathbf{A}_{4,4} \cdot T(4, 0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 4.5. Let be $m = 5$ so that first we get $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11 \binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ and $\mathbf{A}_{5,3} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $3 \leq d < 5$. The value of the coefficient $\mathbf{A}_{5,2}$ is non-zero and calculated as

$$\mathbf{A}_{5,2} = \sum_{d \geq 5}^5 \mathbf{A}_{5,d} \cdot T(d, 2) = \mathbf{A}_{5,5} \cdot T(5, 2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ is non-zero and calculated as

$$\begin{aligned}\mathbf{A}_{5,1} &= \sum_{d \geq 3}^5 \mathbf{A}_{5,d} \cdot T(d, 1) = \mathbf{A}_{5,3} \cdot T(3, 1) + \mathbf{A}_{5,4} \cdot T(4, 1) + \mathbf{A}_{5,5} \cdot T(5, 1) \\ &= 2772 \cdot \left(-\frac{1}{2}\right) = -1386\end{aligned}$$

Finally, the coefficient $\mathbf{A}_{5,0}$ is

$$\begin{aligned}\mathbf{A}_{5,0} &= \sum_{d \geq 1}^5 \mathbf{A}_{5,d} \cdot T(d, 0) = \mathbf{A}_{5,1} \cdot T(1, 0) + \mathbf{A}_{5,2} \cdot T(2, 0) + \mathbf{A}_{5,5} \cdot T(5, 0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1\end{aligned}$$

5. CONCLUSIONS

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