PROOF OF KNUTH BINOMIAL IDENTITY (5.43)

1. Introduction

Proposition 1.1 (Knuth binomial identity)**.**

$$\sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^k = s^n$$

Proposition 1.2 (Falling binomial identity)**.**

$$\binom{n}{k} = \frac{(n)_k}{k!}$$

Proposition 1.3 (Binomial coefficients in stirling form)**.**

$$\begin{aligned}\binom{n}{k} = \frac{(n)_k}{k!} &= \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} n^j \\ &= \frac{1}{k!} \left(\begin{bmatrix} k \\ 0 \end{bmatrix} n^0 - \begin{bmatrix} k \\ 1 \end{bmatrix} n^1 + \begin{bmatrix} k \\ 2 \end{bmatrix} n^2 + \cdots + (-1)^k \begin{bmatrix} k \\ k \end{bmatrix} n^k \right)\end{aligned}$$

Corollary 1.4.

$$\binom{r-sk}{n} = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} (r-sk)^j$$

Proposition 1.5 (Binomial theorem)**.**

$$\begin{aligned}(r-sk)^j &= \sum_{t=0}^{j} (-1)^{j-t} \binom{j}{t} r^j sk^{j-t} \\ &= (-1)^j (sk)^j + \sum_{t=1}^{j} (-1)^{j-t} \binom{j}{t} r^j (sk)^{j-t}\end{aligned}$$