PROOF OF KNUTH BINOMIAL IDENTITY (5.43)

1. Preliminaries

Proposition 1.1 (Falling binomial identity)**.**

$$\binom{n}{k} = \frac{(n)_k}{k!}$$

Proposition 1.2 (Binomial-Stirling form)**.**

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} n^j$$

Thus, the explicit form

Corollary 1.3 (Explicit Binomial-Stirling form)**.**

$$\binom{n}{k} = \frac{1}{k!}\left(\begin{bmatrix} k \\ 0 \end{bmatrix} n^0 - \begin{bmatrix} k \\ 1 \end{bmatrix} n^1 + \begin{bmatrix} k \\ 2 \end{bmatrix} n^2 + \cdots + (-1)^{k-1}\begin{bmatrix} k \\ k-1 \end{bmatrix} n^{k-1} + (-1)^k \begin{bmatrix} k \\ k \end{bmatrix} n^k\right)$$

By changing summation order yields

Corollary 1.4 (Reversed Binomial-Stirling form)**.**

$$\binom{n}{k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j} \begin{bmatrix} k \\ k-j \end{bmatrix} n^{k-j}$$

2. Knuth identity

Proposition 2.1 (Knuth binomial identity)**.**

$$\sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^k = s^n$$

Proposition 2.2 (Forward finite differences)**.**

$$\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k)$$

Proposition 2.3 (Binomial function)**.**

$$F(x) = \binom{r-sx}{n}$$

Corollary 2.4 (Forward difference of Binomial function)**.**

$$\Delta^t F(x) = \sum_{k=0}^{t} \binom{t}{k} (-1)^{n-k} \binom{r-s(x+k)}{n}$$

By setting $x = 0$ in equation above yields

Corollary 2.5 (Forward difference of Binomial function)**.**

$$\Delta^t F(x) = \sum_{k=0}^{t} \binom{t}{k} \binom{r-sk}{n} (-1)^{n-k}$$

By setting $t = n$ gives the structure of (2.1), such that

Corollary 2.6 (Forward difference of Binomial function)**.**

$$\Delta^n F(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^{n-k}$$

Still, it is not quite clear why $\Delta^n F(x) = (-1)^n s^n$, until we refer to relation given by Knuth in Concrete mathematics, see (5.42)

$$\sum_{k} \binom{n}{k} (-1)^k \left[a_0 + a_1 k^1 + a_2 k^2 + \cdots + a_n k^n\right] = (-1)^n n! a_n$$

meaning that n-th difference of a polynomial of degree n in k equals to the coefficient of k^n times $(-1)^n n!$.

Now we notice that $F(x) = \binom{r-sx}{n}$ is a polynomial of degree n in $r - sk$

Corollary 2.7.

$$\binom{r-sk}{n} = \sum_{j=0}^{n} \frac{(-1)^j}{n!} \begin{bmatrix} n \\ j \end{bmatrix} (r-sk)^j$$
$$= \frac{(-1)^0}{n!} \begin{bmatrix} n \\ 0 \end{bmatrix} (r-sk)^0 + \frac{(-1)^1}{n!} \begin{bmatrix} n \\ 1 \end{bmatrix} (r-sk)^1 + \cdots + \frac{(-1)^n}{n!} \begin{bmatrix} n \\ n \end{bmatrix} (r-sk)^n$$

Thus, the coefficient of k^n is

$$a_n = [k^n] \frac{(-1)^n}{n!} \begin{bmatrix} n \\ n \end{bmatrix} (r-sk)^n$$

By binomial theorem

$$(r-sk)^n = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} r^m s^{n-m} k^{n-m}$$

Thus,

$$[k^n](r-sk)^n = (-1)^{n-0} \binom{n}{0} r^0 s^{n-0} = (-1)^n s^n$$

Thus,

$$a_n = \frac{(-1)^n}{n!} \begin{bmatrix} n \\ n \end{bmatrix} (-1)^n s^n = \frac{s^n}{n!}$$

Thus,

$$\Delta^n F(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^{n-k} = (-1)^n n! a_n = (-1)^n n! \frac{s^n}{n!}$$

Hence, Knuth identity is indeed true

$$s^n = (-1)^n \Delta^n F(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^k$$