PROOF OF KNUTH BINOMIAL IDENTITY (5.43)

1. Preliminaries

Proposition 1.1 (Falling binomial identity)**.**

$$\binom{n}{k} = \frac{(n)_k}{k!}$$

Proposition 1.2 (Binomial-Stirling form)**.**

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \begin{bmatrix}k\\j\end{bmatrix} n^j$$

Thus, the explicit form

Corollary 1.3 (Explicit Binomial-Stirling form)**.**

$$\binom{n}{k} = \frac{1}{k!}\left(\begin{bmatrix}k\\0\end{bmatrix}n^0 - \begin{bmatrix}k\\1\end{bmatrix}n^1 + \begin{bmatrix}k\\2\end{bmatrix}n^2 + \cdots + (-1)^{k-1}\begin{bmatrix}k\\k-1\end{bmatrix}n^{k-1} + (-1)^k\begin{bmatrix}k\\k\end{bmatrix}n^k\right)$$

By changing summation order yields

Corollary 1.4 (Reversed Binomial-Stirling form)**.**

$$\binom{n}{k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\begin{bmatrix}k\\k-j\end{bmatrix}n^{k-j}$$

Proposition 1.5 (Binomial theorem)**.**

$$\begin{aligned}(r-sk)^j &= \sum_{t=0}^{j}(-1)^{j-t}\binom{j}{t}r^j sk^{j-t}\\ &= (-1)^j(sk)^j + \sum_{t=1}^{j}(-1)^{j-t}\binom{j}{t}r^j(sk)^{j-t}\end{aligned}$$

2. Knuth identity

Proposition 2.1 (Knuth binomial identity)**.**

$$\sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^k = s^n$$

Proposition 2.2 (Forward finite differences)**.**

$$\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k)$$

Proposition 2.3 (Binomial function)**.**

$$F(x) = \binom{r-sx}{n}$$

Corollary 2.4 (Forward difference of Binomial function)**.**

$$\Delta^t F(x) = \sum_{k=0}^{t} \binom{t}{k} (-1)^{n-k} \binom{r-s(x+k)}{n}$$