ROW SUMS CONJECTURE IN ITERATED RASCAL TRIANGLES

1. Introduction

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were challenged to provide the next row for the following number triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & & & \cdots & & & \end{array}$$

Teacher's expected answer was the one that matches Pascal's triangle, e.g "1 4 6 4 1". However, Anggoro, Liu, and Tulloch suggested "1 4 5 4 1" instead. They devised this new row via what they called diamond formula

$$\mathbf{South} = \frac{\mathbf{East} \cdot \mathbf{West} + 1}{\mathbf{North}}$$

So they obtained the following triangle

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	5	4	1			
5	1	5	7	7	5	1		
6	1	6	9	10	9	6	1	
7	1	7	11	13	13	11	7	1

Table 1. Rascal triangle.

Indeed, the forth row is “1 4 5 4 1” because $4 = \frac{1\cdot 3+1}{1}$ and $5 = \frac{3\cdot 3+1}{2}$. Since then, a lot of work has been done over the topic of rascal triangles. In this article we stick our attention to the one of rascal triangles generalizations, namely iterated rascal triangles [2]. Iterated rascal number is defined via a sum of binomial coefficients multiplication

Definition 1.1. *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} \tag{1}$$

Thus, the rascal triangle (1) is the triangle generated by $\binom{n}{k}_1$.

2. Row sums conjecture

As we see iterated rascal triangles are indeed of Pascal-like triangles family. If rascal triangles are of Pascal-like triangles family, then similar properties must hold. I believe that is how the authors of [2] were thinking proposing the row sums conjecture for iterated rascal triangles.

Conjecture 2.1. *(Conjecture 7.5 in [2].) For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

where $\binom{n}{k}_i$ is an iterated rascal number.

Proof. Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^{i} \sum_{k} \binom{4i+3-k}{m}\binom{k}{m} = 2^{4i+2} \tag{2}$$

In Concrete mathematics [[3], p. 169, eq (5.26)], Knuth et al. provide the identity for the column sum of binomial coefficients multiplication

$$\sum_{k=0}^{l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1} \tag{3}$$

We can observe this pattern in the equation (2), thus the sum $\sum_k \binom{4i+3-k}{m}\binom{k}{m}$ equals to

$$\sum_k \binom{4i+3-k}{m}\binom{k}{m} = \binom{4i+4}{2m+1}$$

Therefore, conjecture (2.1) is equivalent to

$$\sum_{m=0}^{i} \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+3}$$

So that

$$\frac{1}{2}\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^{i} \binom{4i+4}{2m+1} = 2^{4i+2}$$

This completes the proof. □

Therefore, the row sums conjecture for iterated rascal triangles is true for every row $n = 4i+3$, $i \geq 0$.

Proposition 2.2. *For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

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References

[1] Anggoro, Alif and Liu, Eddy and Tulloch, Angus. The Rascal Triangle. *The College Mathematics Journal*, 41(5):393–395, 2010. https://doi.org/10.4169/074683410X521991.

[2] Gregory, Jena and Kronholm, Brandt and White, Jacob. Iterated rascal triangles. *Aequationes mathematicae*, pages 1–18, 2023. https://doi.org/10.1007/s00010-023-00987-6.

[3] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: a foundation for computer science*. Pearson Education India, 1994.