ROW SUMS CONJECTURE IN ITERATED RASCAL TRIANGLES

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Abstract. In [1], Gregory et al. provide the following conjecture for row sums of iterated rascal triangles. For every i

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

where $\binom{n}{k}_i$ is an iterated rascal number.

Contents

1. Introduction

In [1], Gregory et al. provide the following conjecture for row sums of iterated rascal triangles.

Conjecture 1.1. *For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

where $\binom{n}{k}_i$ is an iterated rascal number. Define the iterated rascal number

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Sources: https://github.com/kolosovpetro/RowSumsConjectureInRascalTriangle

Definition 1.2. *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m}$$

Note that iterated rascal numbers are closely related to Vandermonde convolution $\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m}\binom{b}{r-m}$

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m}$$

While

$$\binom{n}{k} = \sum_{m=0}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

It is straightforward to see that

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

In particular, above sum is zero for $k \leq i$, that means

$$\binom{n}{k} = \binom{n}{k}_i, \quad 0 \leq k \leq i$$

To prove the conjecture (1.1) we utilize above relations in terms of binomial coefficients and iterated rascal numbers. Recall the row sums property of binomial coefficients

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k} = 2^{4i+3}$$

If conjecture (1.1) is true, then it is also true that

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k} - \sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

because $2^{4i+3} - 2^{4i+2} = 2^{4i+2}$. Expanding both sums we get

$$2^{4i+2} = \sum_{k=0}^{4i+3}\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{k=0}^{4i+3}\sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m}$$

$$2^{4i+2} = \sum_{k=0}^{4i+3}\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m}$$

Note that $\binom{n}{k} \geq \binom{n}{k}_i$ for each n, k, i. Now we have three possible relation between i, k: $k < i$, $k = i$, $k > i$.

If $k < i$ then inner sums turn into

$$\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m} = 0$$

Because $\binom{k}{k-m}$ in the sum over i is zero for all $m > k$.

If $k = i$ obviously

$$\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m} = 0$$

If $k > i$ then

$$\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m} = \sum_{m=i+1}^{k} \binom{4i+3-k}{m}\binom{k}{k-m}$$

Thus, we have to prove that

$$2^{4i+2} = \sum_{k} \sum_{m=i+1}^{k} \binom{4i+3-k}{m}\binom{k}{k-m}$$

Let m to iterate from 0

$$2^{4i+2} = \sum_{k} \sum_{m=0}^{k} \binom{4i+3-k}{i+1+m}\binom{k}{i+1+m}$$

Although, above equation almost exactly matches Vandermonde identity, it cannot be applied directly. Even it were applied, the result would disprove the main conjecture giving 2^{4i+3} as row sums. My validations show that indeed conjecture true for $i \leq 100$. Therefore, propose the following conjecture

Conjecture 1.3. *For every i*

$$2^{4i+2} = \sum_{k} \sum_{m=0}^{k} \binom{4i+3-k}{i+m}\binom{k}{i+m}$$

Above conjecture validated up to $i = 100$.

References

[1] Gregory, Jena and Kronholm, Brandt and White, Jacob. Iterated rascal triangles. *Aequationes mathematicae*, pages 1–18, 2023. https://doi.org/10.1007/s00010-023-00987-6.

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