# ROW SUMS CONJECTURE IN ITERATED RASCAL TRIANGLES

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ABSTRACT. In [1], Gregory et al. provide the following conjecture for row sums of iterated rascal triangles. For every $i$

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

where $\binom{n}{k}_i$ is an iterated rascal number.

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## 1. Introduction

In [1], Gregory et al. provide the following conjecture for row sums of iterated rascal triangles.

**Conjecture 1.1.** *For every $i$*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

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Sources: `https://github.com/kolosovpetro/RowSumsConjectureInRascalTriangle` where $\binom{n}{k}_i$ is an iterated rascal number. Define the iterated rascal number

**Definition 1.2.** *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m}$$

Note that iterated rascal numbers are closely related to Vandermonde convolution $\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m}\binom{b}{r-m}$

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m}$$

While

$$\binom{n}{k} = \sum_{m=0}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

It is straightforward to see that

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

In particular, above sum is zero for $k \le i$, that means

$$\binom{n}{k} = \binom{n}{k}_i, \quad 0 \le k \le i$$

To prove the conjecture (1.1) we utilize above relations in terms of binomial coefficients and iterated rascal numbers. Recall the row sums property of binomial coefficients

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k} = 2^{4i+3}$$

If conjecture (1.1) is true, then it is also true that

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k} - \sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

because $2^{4i+3} - 2^{4i+2} = 2^{4i+2}$. Expanding both sums we get

$$2^{4i+2} = \sum_{k=0}^{4i+3}\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{k=0}^{4i+3}\sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m}$$

$$2^{4i+2} = \sum_{k=0}^{4i+3}\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m}$$

Note that $\binom{n}{k} \geq \binom{n}{k}_i$ for each $n, k, i$. Now we have three possible relation between $i, k$: $k < i$, $k = i$, $k > i$.

If $k < i$ then inner sums turn into

$$\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m} = 0$$

Because $\binom{k}{k-m}$ in the sum over $i$ is zero for all $m > k$.

If $k = i$ obviously

$$\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m} = 0$$

If $k > i$ then

$$\sum_{m=0}^{k} \binom{4i+3-k}{m}\binom{k}{k-m} - \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{k-m} = \sum_{m=i+1}^{k} \binom{4i+3-k}{m}\binom{k}{k-m}$$

Thus, we have to prove that

$$2^{4i+2} = \sum_{k} \sum_{m=i+1}^{k} \binom{4i+3-k}{m}\binom{k}{k-m}$$

Let $m$ to iterate from 0

$$2^{4i+2} = \sum_{k} \sum_{m=0}^{k} \binom{4i+3-k}{i+1+m}\binom{k}{i+1+m}$$

Although, above equation almost exactly matches Vandermonde identity, it cannot be applied directly. Even it were applied, the result would disprove the main conjecture giving $2^{4i+3}$ as row sums. My validations show that indeed conjecture true for $i \leq 100$. Therefore, propose the following conjecture

**Conjecture 1.3.** *For every $i$*

$$2^{4i+2} = \sum_{k} \sum_{m=0}^{k} \binom{4i+3-k}{i+1+m}\binom{k}{i+1+m}$$

Above conjecture validated up to $i = 100$.

## 2. Draft proof

Let $n = i$. We need to find the following sum:

$$\sum_{m=0}^{n} \sum_{k} \binom{4n+3-k}{m} \binom{k}{m} \quad (1.0)$$

Note that we can assume that $k$ goes over all integers without bounds. We can rewrite the inner sum as

$$[x^{4n+3}] \sum_{i,j} \binom{i}{m} x^i \binom{j}{m} x^j = [x^{4n+3}] \frac{x^m}{(1-x)^{m+1}} \frac{x^m}{(1-x)^{m+1}} \quad (2.0)$$

$$= [x^{4n+3}] \frac{x^{2m}}{(1-x)^{2m+2}}$$

which evaluates to

$$\binom{4n+4}{2m+1} \quad (3.0)$$

meaning that the sum is same as $S = \sum\limits_{m=0}^{n} \binom{4n+4}{2m+1}$. Note that

$$\binom{4n+4}{2m+1} = \binom{4n+4}{4n+4-(2m+1)} = \binom{4n+4}{2(2n+1-m)+1}$$

Therefore, by substitution $m \to 2n+1-m$, we have

$$S = \sum_{m=0}^{n} \binom{4n+4}{2(2n+1-m)+1} = \sum_{m=n+1}^{2n+1} \binom{4n+4}{2m+1}$$

But this means that $2S = \sum\limits_{m=0}^{2n+1} \binom{4n+4}{2m+1} = 2^{4n+3}$, which implies $S = 2^{4n+2}$. $\square$

P.S. Note that, more generally, it means that $\sum\limits_{k} \binom{t-k}{m}\binom{k}{m} = \binom{t+1}{2m+1}$, so we have

$$\sum_{k=0}^{t} \binom{t}{k}_n = \sum_{m=0}^{n} \binom{t+1}{2m+1}$$

P.S.S. Also see [this question](https://math.stackexchange.com/questions/73015) for the identity above.

## References

[1] Gregory, Jena and Kronholm, Brandt and White, Jacob. Iterated rascal triangles. *Aequationes mathematicae*, pages 1–18, 2023. https://doi.org/10.1007/s00010-023-00987-6.

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