SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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Abstract. We obtain formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. In addition, we note that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind. Thus, we express formulas for sums of powers in terms of Eulerian numbers, and Stirling numbers of the second kind.

Contents

1. Introduction and main results

In this manuscript, we obtain formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. The idea to derive sums of powers

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using difference operator and Newton's series is quite generic, thus, formulas for sums of powers using forward and central differences can be found in the works [1, 2].

Define multifold sums of powers in Knuth's [3] notation

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

The book Interpolation by Steffensen [4, chapter 2, eq. (19)] gives Newton's formula for backward differences evaluated in zero $f(x) = \sum_{k=0}^{n} \binom{x+k-1}{k} \nabla^k f(0)$.

In general,

Proposition 1.1 (Newton formula via backward differences)**.**

$$f(x) = \sum_{k=0}^{n} \binom{x-a+k-1}{k} \nabla^k f(a)$$

where $\nabla^k f(a) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(a-j)$.

Thus, by setting $f(n) = n^m$

$$n^m = \sum_{j=0}^{m} \binom{n-t+j-1}{j} \nabla^j t^m,$$

where $\nabla^j t^m = \sum_{k=0}^{j} (-1)^k \binom{j}{k} (t-k)^m$. Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^{m} \nabla^j t^m \sum_{k=1}^{n} \binom{k-t+j-1}{j}$$

We notice that the sum $\sum_{k=1}^{n} \binom{k-t+j-1}{j}$ is a good candidate for hockey stick identity for binomial coefficients $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$. Thus, by setting $a = j-t$ and $b = j-t-1+n$, we get

$$\sum_{k=1}^{n} \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Thus,

$$\sum_{k=1}^{n} \binom{-t+j-1+k}{j} = \binom{j-t+n}{j+1} - \binom{j-t}{j+1}$$

Because,

Lemma 1.2 (Generalized hockey stick identity)**.**

$$\sum_{m=a}^{b} \binom{m}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}$$

Applying the identity for binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, we obtain

Proposition 1.3 (Ordinary sums of powers via backward differences)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right]$$

For example, by setting $t = 2$ and $m = 1, 2, 3, 4$, we get formulas for sums of cubes

$$\Sigma^1 n^1 = 2\left[-\binom{2}{1} + \binom{n-2}{1} \right] + 1\left[\binom{2}{2} + \binom{n-1}{2} \right],$$

$$\begin{aligned}\Sigma^1 n^2 = 4&\left[-\binom{2}{1} + \binom{n-2}{1} \right] + 3\left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &+ 2\left[-\binom{2}{3} + \binom{n}{3} \right].\end{aligned}$$

$$\begin{aligned}\Sigma^1 n^3 = 8&\left[-\binom{2}{1} + \binom{n-2}{1} \right] + 7\left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &+ 6\left[-\binom{2}{3} + \binom{n}{3} \right] + 6\left[\binom{2}{4} + \binom{n+1}{4} \right].\end{aligned}$$

$$\begin{aligned}\Sigma^1 n^4 = 16&\left[-\binom{2}{1} + \binom{n-2}{1} \right] + 15\left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &+ 14\left[-\binom{2}{3} + \binom{n}{3} \right] + 12\left[\binom{2}{4} + \binom{n+1}{4} \right] \\ &+ 24\left[-\binom{2}{5} + \binom{n+2}{5} \right].\end{aligned}$$

- For $t = 0$ the coefficients are $1, 0, 1, 0, -1, 2, 0, 1, -6, 6, \ldots$ and registered in the OEIS [5] as A278075.
- For $t = 1$ the coefficients are $1, 1, 1, 1, 1, 2, 1, 1, 0, 6, \ldots$ and registered in the OEIS [5] as A389570.
- For $t = 2$ the coefficients are $1, 2, 1, 4, 3, 2, 8, 7, 6, 6, \ldots$ and registered in the OEIS [5] as A391068.
- For $t = 3$ the coefficients are $1, 3, 1, 9, 5, 2, 27, 19, 12, 6, \ldots$ and registered in the OEIS [5] as A391210.

Lemma 1.4 (Backward differences in Eulerian numbers)**.**

$$\Delta^j t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{t+k-j}{m-j}$$

Proof. By Worpitzky identity $t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{t+k}{m}$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, see [6]. □

Thus, let be a formula for ordinary sums of powers in terms of Eulerian numbers $\left\langle {m \atop k} \right\rangle$

Proposition 1.5 (Ordinary sums of powers in Eulerian numbers)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \left\langle {m \atop k} \right\rangle \binom{t+k-j}{m-j}$$

Remarkable that having $t = 0$ formula for sums of powers turns into double binomial view

Proposition 1.6.

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \binom{j+n}{j+1} \binom{t+k-j}{m-j} \left\langle {m \atop k} \right\rangle$$

Lemma 1.7 (Backward differences in Stirling numbers)**.**

$$\nabla^j t^m = \sum_{k=j}^{m} \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

Proof. By the identity $t^m = \sum_{k=0}^{m} \binom{t}{k} \left\{ {m \atop k} \right\} k!$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. $\square$

Thus, let be a formula for ordinary sums of powers in terms of Stirling numbers $\left\{ {m \atop k} \right\}$

Proposition 1.8 (Ordinary sums of powers in Stirling numbers)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

By setting $t = 0$ yields

Proposition 1.9.

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \binom{j+n}{j+1} \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

Formula for double sums of powers can be derived in a similar manner, by applying summation to the proposition (1.3), which in turn implies generalized hockey-stick identity, thus

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \binom{j-t+k}{j+1} \right]$$

By applying generalized hockey stick identity (1.2), we obtain

$$\sum_{k=1}^{n} \binom{j-t+k}{j+1} = \sum_{k=j-t+1}^{j-t+n} \binom{k}{j+1} = \binom{j-t+n+1}{j+2} - \binom{j-t+1}{j+2}$$

Thus,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} n + \left(\binom{j-t+n+1}{j+2} - \binom{j-t+1}{j+2} \right) \right]$$

By applying the identity for negative binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, we get

$$\binom{-(t-j-1)}{j+2} = (-1)^{j+2} \binom{t}{j+2}$$

Hence,

Proposition 1.10 (Double sums of powers via backward differences)**.**

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} n + (-1)^{j+1} \binom{t}{j+2} n^0 + \binom{j-t+n+1}{j+2} \right]$$

2. Conclusions

In this manuscript, we derived formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. In addition, we noticed that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind. Thus, we express formulas for sums of powers in terms of Eulerian numbers, and Stirling numbers of the second kind. All the results are validated using `Mathematica` programs, see dedicated section below.

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Mathematica programs

Use the *Mathematica* package [7] to validate the results

Mathematica Function	**Validates / Prints**
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r n^m$
`ValidateOrdinarySumsOfPowersViaBackwardDifferences[20]`	Validates Proposition (1.3)
`ValidateBackwardDifferencesInEulerianNumbers[20]`	Validates Lemma (1.4)
`ValidateOrdinarySumsOfPowersInEulerianNumbers[10]`	Validates Proposition (1.5)
`ValidateBackwardDifferencesInStirlingNumbers[20]`	Validates Lemma (1.7)
`ValidateOrdinarySumsOfPowersInStirlingNumbers[20]`	Validates Proposition (1.8)
`ValidateDoubleSumsOfPowersViaBackwardDifferences[10]`	Validates Proposition (1.10)

Version: Local-0.1.0

Sources: github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences

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