

SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. We obtain formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. In addition, we note that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind. Thus, we express formulas for sums of powers in terms of Eulerian numbers, and Stirling numbers of the second kind.

CONTENTS

Abstract	1
1. Introduction and main results	2
2. Conclusions	5
References	5
Mathematica programs	5

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1. INTRODUCTION AND MAIN RESULTS

Define multifold sums of powers in Knuth's [1] notation

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

The book Interpolation by Steffensen [2, chapter 2, eq. (19)] gives Newton's formula for backward differences evaluated in zero $f(x) = \sum_{k=0}^n \binom{x+k-1}{k} \nabla^k f(0)$.

In general,

Proposition 1.1 (Newton formula via backward differences).

$$f(x) = \sum_{k=0}^n \binom{x-a+k-1}{k} \nabla^k f(a)$$

where $\nabla^k f(a) = \sum_{j=0}^k (-1)^j \binom{k}{j} f(a-j)$.

Thus, by setting $f(n) = n^m$

$$n^m = \sum_{j=0}^m \binom{n-t+j-1}{j} \nabla^j t^m,$$

where $\nabla^j t^m = \sum_{k=0}^j (-1)^k \binom{j}{k} (t-k)^m$. Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \sum_{k=1}^n \binom{k-t+j-1}{j}$$

We notice that the sum $\sum_{k=1}^n \binom{k-t+j-1}{j}$ is a good candidate for hockey stick identity for binomial coefficients $\sum_{k=0}^n \binom{k}{j} = \binom{n+1}{j+1}$. Thus, by setting $a = j-t$ and $b = j-t-1+n$, we get

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Thus,

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \binom{j-t+n}{j+1} - \binom{j-t}{j+1}$$

Because,

$$\sum_{m=a}^b \binom{m}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}$$

Applying the identity for binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, we obtain

Proposition 1.2 (Ordinary sums of powers via backward differences). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right]$$

For example, by setting $t = 2$ and $m = 1, 2, 3, 4$, we get formulas for sums of cubes

$$\Sigma^1 n^1 = 2 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 1 \left[\binom{2}{2} + \binom{n-1}{2} \right],$$

$$\begin{aligned} \Sigma^1 n^2 &= 4 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 3 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 2 \left[-\binom{2}{3} + \binom{n}{3} \right]. \end{aligned}$$

$$\begin{aligned} \Sigma^1 n^3 &= 8 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 7 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 6 \left[-\binom{2}{3} + \binom{n}{3} \right] + 6 \left[\binom{2}{4} + \binom{n+1}{4} \right]. \end{aligned}$$

$$\begin{aligned} \Sigma^1 n^4 &= 16 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 15 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 14 \left[-\binom{2}{3} + \binom{n}{3} \right] + 12 \left[\binom{2}{4} + \binom{n+1}{4} \right] \\ &\quad + 24 \left[-\binom{2}{5} + \binom{n+2}{5} \right]. \end{aligned}$$

- For $t = 0$ the coefficients are $1, 0, 1, 0, -1, 2, 0, 1, -6, 6, \dots$ and registered in the OEIS [3] as [A278075](#).
- For $t = 1$ the coefficients are $1, 1, 1, 1, 1, 2, 1, 1, 0, 6, \dots$ and registered in the OEIS [3] as [A389570](#).

- For $t = 2$ the coefficients are 1, 2, 1, 4, 3, 2, 8, 7, 6, 6, ... and registered in the OEIS [3] as A391068.
- For $t = 3$ the coefficients are 1, 3, 1, 9, 5, 2, 27, 19, 12, 6, ... and registered in the OEIS [3] as A391210.

Lemma 1.3 (Backward differences in Eulerian numbers).

$$\Delta^j t^m = \sum_{k=0}^m \left\langle m \atop k \right\rangle \binom{t+k-j}{m-j}$$

Proof. By Worpitzky identity $t^m = \sum_{k=0}^m \left\langle m \atop k \right\rangle \binom{t+k}{m}$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, see [4]. \square

Thus, let be a formula for ordinary sums of powers in terms of Eulerian numbers $\left\langle m \atop k \right\rangle$

Proposition 1.4 (Ordinary sums of powers in Eulerian numbers). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^m \sum_{k=0}^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \left\langle m \atop k \right\rangle \binom{t+k-j}{m-j}$$

Lemma 1.5 (Backward differences in Stirling numbers).

$$\nabla^j t^m = \sum_{k=j}^m \binom{t-j}{k-j} \left\{ m \atop k \right\} k!$$

Proof. By the identity $t^m = \sum_{k=0}^m \binom{t}{k} \left\{ m \atop k \right\} k!$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. \square

Thus, let be a formula for ordinary sums of powers in terms of Stirling numbers $\left\{ m \atop k \right\}$

Proposition 1.6 (Ordinary sums of powers in Stirling numbers). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^m \sum_{k=j}^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t-j}{k-j} \left\{ m \atop k \right\} k!$$

2. CONCLUSIONS

In this manuscript, we derived formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. In addition, we noticed that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind. Thus, we express formulas for sums of powers in terms of Eulerian numbers, and Stirling numbers of the second kind. All the results are validated using `Mathematica` programs, see dedicated section below.

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MATHEMATICA PROGRAMS

Use the *Mathematica* package [5] to validate the results

Mathematica Function	Validates / Prints
MultifoldSumOfPowersRecurrence[r, n, m]	Computes $\sum^r n^m$
ValidateOrdinarySumsOfPowersViaBackwardDifferences[20]	Validates Proposition (1.2)
ValidateBackwardDifferencesInEulerianNumbers[20]	Validates Lemma (1.3)
ValidateOrdinarySumsOfPowersInEulerianNumbers[10]	Validates Proposition (1.4)
ValidateBackwardDifferencesInStirlingNumbers[20]	Validates Lemma (1.5)
ValidateOrdinarySumsOfPowersInStirlingNumbers[20]	Validates Proposition (1.6)

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Sources: github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences

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