

SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. We develop formula for sums of powers using Newton's interpolation formula in terms of backward finite differences of powers.

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1. INTRODUCTION AND MAIN RESULTS

Define multifold sums of powers in Knuth's [1] notation

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

The book Interpolation by Steffensen [2, chapter 2, eq. (19)] gives Newton's formula for backward differences evaluated in zero $f(x) = \sum_{k=0}^n \binom{x+k-1}{k} \nabla^k f(0)$.

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In general,

Proposition 1.1 (Newton formula via backward differences).

$$f(x) = \sum_{k=0}^n \binom{x-a+k-1}{k} \nabla^k f(a)$$

where $\nabla^k f(a) = \sum_{j=0}^k (-1)^j \binom{k}{j} f(a-j)$.

Thus, by setting $f(n) = n^m$

$$n^m = \sum_{j=0}^m \binom{n-t+j-1}{j} \nabla^j t^m,$$

where $\nabla^j t^m = \sum_{k=0}^j (-1)^k \binom{j}{k} (t-k)^m$. Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \sum_{k=1}^n \binom{k-t+j-1}{j}$$

We notice that the sum $\sum_{k=1}^n \binom{k-t+j-1}{j}$ is a good candidate for hockey stick identity for binomial coefficients $\sum_{k=0}^n \binom{k}{j} = \binom{n+1}{j+1}$. Thus, by setting $a = j-t$ and $b = j-t-1+n$, we get

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Thus,

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \binom{j-t+n}{j+1} - \binom{j-t}{j+1}$$

Because,

$$\sum_{m=a}^b \binom{m}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}$$

Applying the identity for binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, we obtain

Proposition 1.2 (Ordinary sums of powers via backward differences). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right]$$

For example, by setting $t = 2$ and $m = 1, 2, 3, 4$, we get formulas for sums of cubes

$$\Sigma^1 n^1 = 2 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 1 \left[\binom{2}{2} + \binom{n-1}{2} \right],$$

$$\begin{aligned} \Sigma^1 n^2 &= 4 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 3 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 2 \left[-\binom{2}{3} + \binom{n}{3} \right]. \end{aligned}$$

$$\begin{aligned} \Sigma^1 n^3 &= 8 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 7 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 6 \left[-\binom{2}{3} + \binom{n}{3} \right] + 6 \left[\binom{2}{4} + \binom{n+1}{4} \right]. \end{aligned}$$

$$\begin{aligned} \Sigma^1 n^4 &= 16 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 15 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 14 \left[-\binom{2}{3} + \binom{n}{3} \right] + 12 \left[\binom{2}{4} + \binom{n+1}{4} \right] \\ &\quad + 24 \left[-\binom{2}{5} + \binom{n+2}{5} \right]. \end{aligned}$$

The coefficients $1, 2, 1, 4, 3, 2, 8, 7, 6, 6, \dots$ for $t = 2$ is the sequence [A391068](#) in the OEIS [3].

For $t = 0$ the coefficients are $1, 0, 1, 0, -1, 2, 0, 1, -6, 6, \dots$ and registered in the OEIS as [A278075](#). For $t = 1$ the coefficients are $1, 1, 1, 1, 1, 2, 1, 1, 0, 6, \dots$ and registered in the OEIS as [A389570](#). For $t = 3$ the coefficients are $1, 3, 1, 9, 5, 2, 27, 19, 12, 6, \dots$ and registered in the OEIS as [A391210](#).

Lemma 1.3 (Backward differences in Eulerian numbers).

$$\Delta^j t^m = \sum_{k=0}^m \langle m \rangle \binom{t+k-j}{m-j}$$

Proof. By Worpitzky identity $t^m = \sum_{k=0}^m \langle m \rangle \binom{t+k}{m}$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, see [4]. \square

Thus, let be a formula for ordinary sums of powers in terms of Eulerian numbers $\langle m \rangle$

Proposition 1.4 (Ordinary sums of powers in Eulerian numbers). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^m \sum_{k=0}^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \begin{Bmatrix} m \\ k \end{Bmatrix} \binom{t+k-j}{m-j}$$

Lemma 1.5 (Backward differences in Stirling numbers).

$$\nabla^j t^m = \sum_{k=j}^m \binom{t-j}{k-j} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$$

Proof. By the identity $t^m = \sum_{k=0}^m \binom{t}{k} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. \square

Thus, let be a formula for ordinary sums of powers in terms of Stirling numbers $\begin{Bmatrix} m \\ k \end{Bmatrix}$

Proposition 1.6 (Ordinary sums of powers in Stirling numbers). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^m \sum_{k=j}^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t-j}{k-j} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$$

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MATHEMATICA PROGRAMS

Use the *Mathematica* package [5] to validate the results

Mathematica Function	Validates / Prints
MultifoldSumOfPowersRecurrence[r, n, m]	Computes $\sum^r n^m$
ValidateOrdinarySumsOfPowersViaBackwardDifferences[20]	Validates Proposition (1.2)
ValidateBackwardDifferencesInEulerianNumbers[20]	Validates Lemma (1.3)
ValidateOrdinarySumsOfPowersInEulerianNumbers[10]	Validates Proposition (1.4)
ValidateBackwardDifferencesInStirlingNumbers[20]	Validates Lemma (1.5)

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Sources: github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences

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