SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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Abstract. We obtain formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. In addition, we note that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind. Thus, we express formulas for sums of powers in terms of Eulerian numbers, and Stirling numbers of the second kind.

Contents

1. Introduction and main results

In this manuscript, we obtain formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. The idea to derive sums of powers

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using difference operator and Newton's series is quite generic, thus, formulas for sums of powers using forward and central differences can be found in the works [1, 2].

Define multifold sums of powers in Knuth's [3] notation

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

The book Interpolation by Steffensen [4, chapter 2, eq. (19)] gives Newton's formula for backward differences evaluated in zero $f(x) = \sum_{k=0}^{n} \binom{x+k-1}{k} \nabla^k f(0)$.

In general,

Proposition 1.1 (Newton formula via backward differences)**.**

$$f(x) = \sum_{k=0}^{n} \binom{x-a+k-1}{k} \nabla^k f(a)$$

where $\nabla^k f(a) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(a-j)$.

Thus, by setting $f(n) = n^m$

$$n^m = \sum_{j=0}^{m} \binom{n-t+j-1}{j} \nabla^j t^m,$$

where $\nabla^j t^m = \sum_{k=0}^{j} (-1)^k \binom{j}{k} (t-k)^m$. Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^{m} \nabla^j t^m \sum_{k=1}^{n} \binom{k-t+j-1}{j}$$

We notice that the sum $\sum_{k=1}^{n} \binom{k-t+j-1}{j}$ is a good candidate for hockey stick identity for binomial coefficients $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$. Thus, by setting $a = j-t$ and $b = j-t-1+n$, we get

$$\sum_{k=1}^{n} \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Thus,

$$\sum_{k=1}^{n}\binom{-t+j-1+k}{j}=\binom{j-t+n}{j+1}-\binom{j-t}{j+1}$$

Because,

Lemma 1.2 (Generalized hockey stick identity)**.**

$$\sum_{m=a}^{b}\binom{m}{j}=\binom{b+1}{j+1}-\binom{a}{j+1}$$

Applying the identity for binomial coefficients $\binom{-k}{j}=(-1)^j\binom{j+k-1}{j}$, we obtain

Proposition 1.3 (Ordinary sums of powers via backward differences)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m=\sum_{j=0}^{m}\nabla^j t^m\left[(-1)^j\binom{t}{j+1}+\binom{j-t+n}{j+1}\right]$$

For example, by setting $t=2$ and $m=1,2,3,4$, we get formulas for sums of cubes

$$\Sigma^1 n^1=2\left[-\binom{2}{1}+\binom{n-2}{1}\right]+1\left[\binom{2}{2}+\binom{n-1}{2}\right],$$

$$\begin{aligned}\Sigma^1 n^2=&\,4\left[-\binom{2}{1}+\binom{n-2}{1}\right]+3\left[\binom{2}{2}+\binom{n-1}{2}\right]\\&+2\left[-\binom{2}{3}+\binom{n}{3}\right].\end{aligned}$$

$$\begin{aligned}\Sigma^1 n^3=&\,8\left[-\binom{2}{1}+\binom{n-2}{1}\right]+7\left[\binom{2}{2}+\binom{n-1}{2}\right]\\&+6\left[-\binom{2}{3}+\binom{n}{3}\right]+6\left[\binom{2}{4}+\binom{n+1}{4}\right].\end{aligned}$$

$$\begin{aligned}\Sigma^1 n^4=&\,16\left[-\binom{2}{1}+\binom{n-2}{1}\right]+15\left[\binom{2}{2}+\binom{n-1}{2}\right]\\&+14\left[-\binom{2}{3}+\binom{n}{3}\right]+12\left[\binom{2}{4}+\binom{n+1}{4}\right]\\&+24\left[-\binom{2}{5}+\binom{n+2}{5}\right].\end{aligned}$$

- For $t = 0$ the coefficients are $1, 0, 1, 0, -1, 2, 0, 1, -6, 6, \ldots$ and registered in the OEIS [5] as A278075.
- For $t = 1$ the coefficients are $1, 1, 1, 1, 1, 2, 1, 1, 0, 6, \ldots$ and registered in the OEIS [5] as A389570.
- For $t = 2$ the coefficients are $1, 2, 1, 4, 3, 2, 8, 7, 6, 6, \ldots$ and registered in the OEIS [5] as A391068.
- For $t = 3$ the coefficients are $1, 3, 1, 9, 5, 2, 27, 19, 12, 6, \ldots$ and registered in the OEIS [5] as A391210.

Lemma 1.4 (Backward differences in Eulerian numbers)**.**

$$\Delta^j t^m = \sum_{k=0}^{m} \binom{t+k-j}{m-j} \left\langle {m \atop k} \right\rangle$$

Proof. By Worpitzky identity $t^m = \sum_{k=0}^{m} \binom{t+k}{m} \left\langle {m \atop k} \right\rangle$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, see [6]. □

Thus, let be a formula for ordinary sums of powers in terms of Eulerian numbers $\left\langle {m \atop k} \right\rangle$

Proposition 1.5 (Ordinary sums of powers in Eulerian numbers)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t+k-j}{m-j} \left\langle {m \atop k} \right\rangle$$

Remarkable that having $t = 0$ formula for sums of powers turns into double binomial view

Proposition 1.6 (Ordinary Eulerian sums of powers in zero)**.** *For non-negative integers n, m*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \binom{j+n}{j+1} \binom{k-j}{m-j} \left\langle {m \atop k} \right\rangle$$

Lemma 1.7 (Backward differences in Stirling numbers)**.**

$$\nabla^j t^m = \sum_{k=j}^{m} \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

Proof. By the identity $t^m = \sum_{k=0}^{m} \binom{t}{k} {m \brace k} k!$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. $\square$

Thus, let be a formula for ordinary sums of powers in terms of Stirling numbers ${m \brace k}$

Proposition 1.8 (Ordinary sums of powers in Stirling numbers). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t-j}{k-j} {m \brace k} k!$$

By setting $t = 0$ yields

Proposition 1.9 (Ordinary Stirling sums of powers in zero). *For non-negative integers n, m*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \binom{j+n}{j+1} \binom{-j}{k-j} {m \brace k} k!$$

Formula for double sums of powers can be derived in a similar manner, by applying summation to the proposition (1.3), which in turn implies generalized hockey-stick identity, thus

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \binom{j-t+k}{j+1} \right]$$

By applying generalized hockey stick identity (1.2), we obtain

$$\sum_{k=1}^{n} \binom{j-t+k}{j+1} = \sum_{k=j-t+1}^{j-t+n} \binom{k}{j+1} = \binom{j-t+n+1}{j+2} - \binom{j-t+1}{j+2}$$

Therefore,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} n + \left(\binom{j-t+n+1}{j+2} - \binom{j-t+1}{j+2} \right) \right]$$

By applying the identity for negative binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, we get

$$\binom{-(t-j-1)}{j+2} = (-1)^{j+2} \binom{t}{j+2}$$

Hence,

Proposition 1.10 (Double sums of powers via backward differences). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} n + (-1)^{j+1} \binom{t}{j+2} n^0 + \binom{j-t+n+1}{j+2}\right]$$

In general,

Theorem 1.11 (Multifold sums of powers via backward difference). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \nabla^j t^m \left[\binom{j-t+n+r-1}{j+r} + \sum_{s=0}^{r-1} (-1)^{j+s} \binom{t}{j+s+1} \Sigma^{r-1-s} n^0\right]$$

We may observe that

Proposition 1.12 (Multifold sum of zero powers). *For integers r and n*

$$\Sigma^r n^0 = \binom{r+n-1}{r}$$

Proof. By hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. □

By $\Sigma^{r-1-s} n^0 = \binom{r-s+n-2}{r-s-1}$, we get

Proposition 1.13 (Multifold sums of powers binomial form). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \nabla^j t^m \left[\binom{j-t+n+r-1}{j+r} + \sum_{s=0}^{r-1} (-1)^{j+s} \binom{t}{j+s+1} \binom{r-s+n-2}{r-s-1}\right]$$

By setting $r \to r+1$

Corollary 1.14 (Multifold sums of powers binomial form shifted). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^{r+1} n^m = \sum_{j=0}^{m} \nabla^j t^m \left[\binom{j-t+n+r}{j+r+1} + \sum_{s=0}^{r} (-1)^{j+s} \binom{t}{j+s+1} \binom{r-s+n-1}{r-s}\right]$$

By lemma (1.7), we get formula for multifold sums of powers in terms of Stirling numbers of the second kind

Proposition 1.15 (Multifold sums of powers in Stirling numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^{r+1} n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \left[\binom{j-t+n+r}{j+r+1} + \sum_{s=0}^{r} (-1)^{j+s} \binom{t}{j+s+1} \binom{r-s+n-1}{r-s} \right] \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

By lemma (1.4), we get formula for multifold sums of powers in terms of Eulerian numbers

Proposition 1.16 (Multifold sums of powers in Eulerian numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^{r+1} n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{j-t+n+r}{j+r+1} + \sum_{s=0}^{r} (-1)^{j+s} \binom{t}{j+s+1} \binom{r-s+n-1}{r-s} \right] \binom{t+k-j}{m-j} \left\langle {m \atop k} \right\rangle$$

2. Conclusions

In this manuscript, we derived formula for sums of powers (1.11) via Newton's interpolation formula based on backward finite differences of powers. In addition, we noticed that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind. Thus, we expressed formulas for sums of powers in terms of Eulerian numbers (1.16), and Stirling numbers of the second kind (1.15).

Future research directions are discussed and proposed at [1], which includes development of generalized algorithm for sums of powers using interpolation formulas combined with hockey-stick family identities for binomial coefficients.

All the results are validated using `Mathematica` programs, see section (3).

References

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[7] Petro Kolosov. Mathematica programs for backward finite differences and newton's formula. https://github.com/kolosovpetro/SumsOfPowersViaBackwardFiniteDifferencesAndNewtonFormula/tree/main/mathematica, 2026. GitHub repository, Mathematica source files.

3. Mathematica programs

Use the *Mathematica* package [7] to validate the results

Mathematica Function	**Validates / Prints**
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r n^m$
`ValidateOrdinarySumsOfPowersViaBackwardDifferences[20]`	Validates Proposition (1.3)
`ValidateBackwardDifferencesInEulerianNumbers[20]`	Validates Lemma (1.4)
`ValidateOrdinarySumsOfPowersInEulerianNumbers[10]`	Validates Proposition (1.5)
`ValidateBackwardDifferencesInStirlingNumbers[20]`	Validates Lemma (1.7)
`ValidateOrdinarySumsOfPowersInStirlingNumbers[20]`	Validates Proposition (1.8)
`ValidateDoubleSumsOfPowersViaBackwardDifferences[10]`	Validates Proposition (1.10)
`ValidateMultifoldSumsOfPowersViaBackwardDifference[5]`	Validates Theorem (1.11)
`ValidateMultifoldSumsOfPowersBackwardDiffBinomialForm[5]`	Validates Proposition (1.13)

Version: Local-0.1.0

Sources: github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences

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