

SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. We develop formula for sums of powers using Newton's interpolation formula in terms of backward finite differences of powers.

1. INTRODUCTION AND MAIN RESULTS

Define multifold sums of powers in Knuth's [1] notation

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

The book Interpolation by Steffensen [2] gives Newton's formula in terms of backward finite differences

Proposition 1.1 (Newton formula via backward differences).

$$f(x) = \sum_{k=0}^{\infty} \binom{x - a + k - 1}{k} \nabla^k f(a)$$

where $\nabla^k f(a) = \sum_{j=0}^k (-1)^j \binom{k}{j} f(a - j)$.

Date: January 1, 2026.

2010 *Mathematics Subject Classification.* 05A19, 05A10, 11B68, 11B73, 11B83.

Key words and phrases. Sums of powers, Newton's interpolation formula, Finite differences, Binomial coefficients, Faulhaber's formula, Bernoulli numbers, Bernoulli polynomials, Interpolation, Discrete convolution, Combinatorics, Polynomial identities, Central factorial numbers, Stirling numbers, Eulerian numbers, Worpitzky identity, Pascal's triangle, OEIS.

Thus, by setting $f(n) = n^m$

$$n^m = \sum_{j=0}^m \binom{n-t+j-1}{j} \nabla^j t^m$$

Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \sum_{k=1}^n \binom{k-t+j-1}{j}$$

We notice that the sum $\sum_{k=1}^n \binom{k-t+j-1}{j}$ is a good candidate for hockey stick identity for binomial coefficients $\sum_{k=0}^n \binom{k}{j} = \binom{n+1}{j+1}$. Thus, by setting $a = j - t$ and $b = j - t - 1 + n$, we get

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Because,

$$\sum_{m=a}^b \binom{m}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}$$

Thus,

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \binom{j-t+n}{j+1} - \binom{j-t}{j+1}$$

Applying the identity for binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, we obtain

Proposition 1.2 (Ordinary sums of powers via backward differences).

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right]$$

For example, by setting $t = 2$, $m = 1, 2, 3, 4$, we get formulas for sums of cubes

$$\Sigma^1 n^1 = 2 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 1 \left[\binom{2}{2} + \binom{n-1}{2} \right],$$

$$\begin{aligned} \Sigma^1 n^2 &= 4 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 3 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 2 \left[-\binom{2}{3} + \binom{n}{3} \right]. \end{aligned}$$

$$\begin{aligned}\Sigma^1 n^3 &= 8 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 7 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 6 \left[-\binom{2}{3} + \binom{n}{3} \right] + 6 \left[\binom{2}{4} + \binom{n+1}{4} \right].\end{aligned}$$

$$\begin{aligned}\Sigma^1 n^4 &= 16 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 15 \left[\binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 14 \left[-\binom{2}{3} + \binom{n}{3} \right] + 12 \left[\binom{2}{4} + \binom{n+1}{4} \right] \\ &\quad + 24 \left[-\binom{2}{5} + \binom{n+2}{5} \right].\end{aligned}$$

The coefficients $1, 2, 1, 4, 3, 2, 8, 7, 6, 6, \dots$ for $t = 2$ is the sequence [A391068](#) in the OEIS [3].

For $t = 0$ the coefficients are $1, 0, 1, 0, -1, 2, 0, 1, -6, 6, \dots$ and registered in the OEIS as [A278075](#). For $t = 1$ the coefficients are $1, 1, 1, 1, 1, 2, 1, 1, 0, 6, \dots$ and registered in the OEIS as [A389570](#). For $t = 3$ the coefficients are $1, 3, 1, 9, 5, 2, 27, 19, 12, 6, \dots$ and registered in the OEIS as [A391210](#).

REFERENCES

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Version: Local-0.1.0

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Sources: github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences

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