

# SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. We develop formula for sums of powers using Newton's interpolation formula in terms of backward finite differences of powers.

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## 1. INTRODUCTION AND MAIN RESULTS

Define multifold sums of powers in Knuth's [1] notation

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

The book Interpolation by Steffensen [2, chapter 2, eq. (19)] gives Newton's formula for backward differences evaluated in zero  $f(x) = \sum_{k=0}^{\infty} \binom{x+k-1}{k} \nabla^k f(0)$ .

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In general,

**Proposition 1.1** (Newton formula via backward differences).

$$f(x) = \sum_{k=0}^{\infty} \binom{x-a+k-1}{k} \nabla^k f(a)$$

where  $\nabla^k f(a) = \sum_{j=0}^k (-1)^j \binom{k}{j} f(a-j)$ .

Thus, by setting  $f(n) = n^m$

$$n^m = \sum_{j=0}^m \binom{n-t+j-1}{j} \nabla^j t^m,$$

where  $\nabla^j t^m = \sum_{k=0}^j (-1)^k \binom{j}{k} (t-k)^m$ . Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \sum_{k=1}^n \binom{k-t+j-1}{j}$$

We notice that the sum  $\sum_{k=1}^n \binom{k-t+j-1}{j}$  is a good candidate for hockey stick identity for binomial coefficients  $\sum_{k=0}^n \binom{k}{j} = \binom{n+1}{j+1}$ . Thus, by setting  $a = j-t$  and  $b = j-t-1+n$ , we get

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Thus,

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \binom{j-t+n}{j+1} - \binom{j-t}{j+1}$$

Because,

$$\sum_{m=a}^b \binom{m}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}$$

Applying the identity for binomial coefficients  $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$ , we obtain

**Proposition 1.2** (Ordinary sums of powers via backward differences). *For non-negative integers  $n, m$  and an arbitrary integer  $t$*

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \left[ (-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right]$$

For example, by setting  $t = 2$  and  $m = 1, 2, 3, 4$ , we get formulas for sums of cubes

$$\Sigma^1 n^1 = 2 \left[ -\binom{2}{1} + \binom{n-2}{1} \right] + 1 \left[ \binom{2}{2} + \binom{n-1}{2} \right],$$

$$\begin{aligned} \Sigma^1 n^2 &= 4 \left[ -\binom{2}{1} + \binom{n-2}{1} \right] + 3 \left[ \binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 2 \left[ -\binom{2}{3} + \binom{n}{3} \right]. \end{aligned}$$

$$\begin{aligned} \Sigma^1 n^3 &= 8 \left[ -\binom{2}{1} + \binom{n-2}{1} \right] + 7 \left[ \binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 6 \left[ -\binom{2}{3} + \binom{n}{3} \right] + 6 \left[ \binom{2}{4} + \binom{n+1}{4} \right]. \end{aligned}$$

$$\begin{aligned} \Sigma^1 n^4 &= 16 \left[ -\binom{2}{1} + \binom{n-2}{1} \right] + 15 \left[ \binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 14 \left[ -\binom{2}{3} + \binom{n}{3} \right] + 12 \left[ \binom{2}{4} + \binom{n+1}{4} \right] \\ &\quad + 24 \left[ -\binom{2}{5} + \binom{n+2}{5} \right]. \end{aligned}$$

The coefficients  $1, 2, 1, 4, 3, 2, 8, 7, 6, 6, \dots$  for  $t = 2$  is the sequence [A391068](#) in the OEIS [3].

For  $t = 0$  the coefficients are  $1, 0, 1, 0, -1, 2, 0, 1, -6, 6, \dots$  and registered in the OEIS as [A278075](#). For  $t = 1$  the coefficients are  $1, 1, 1, 1, 1, 2, 1, 1, 0, 6, \dots$  and registered in the OEIS as [A389570](#). For  $t = 3$  the coefficients are  $1, 3, 1, 9, 5, 2, 27, 19, 12, 6, \dots$  and registered in the OEIS as [A391210](#).

## REFERENCES

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- [3] Sloane, Neil J.A. and others. The on-line encyclopedia of integer sequences, 2003. <https://oeis.org/>.

- [4] Petro Kolosov. Mathematica programs for backward finite differences and newton's formula. <https://github.com/kolosovpetro/SumsOfPowersViaBackwardFiniteDifferencesAndNewtonFormula/tree/main/mathematica>, 2026. GitHub repository, Mathematica source files.

### MATHEMATICA PROGRAMS

Use the *Mathematica* package [4] to validate the results

Mathematica Function	Validates / Prints
<code>MultifoldSumOfPowersRecurrence[r, n, m]</code>	Computes $\sum^r n^m$
<code>ValidateOrdinarySumsOfPowersViaBackwardDifferences[20]</code>	Validates Proposition (1.2)

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**Sources:** [github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences](https://github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences)

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