

# SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. We develop formula for sums of powers using Newton's interpolation formula in terms of backward finite differences of powers.

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## 1. INTRODUCTION AND MAIN RESULTS

Define multifold sums of powers in Knuth's [1] notation

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

The book Interpolation by Steffensen [2, chapter 2, eq. (19)] gives Newton's formula for backward differences evaluated in zero  $f(x) = \sum_{k=0}^n \binom{x+k-1}{k} \nabla^k f(0)$ .

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*Date:* January 2, 2026.

2010 *Mathematics Subject Classification.* 05A19, 05A10, 11B68, 11B73, 11B83.

*Key words and phrases.* Sums of powers, Newton's interpolation formula, Finite differences, Binomial coefficients, Faulhaber's formula, Bernoulli numbers, Bernoulli polynomials, Interpolation, Discrete convolution, Combinatorics, Polynomial identities, Central factorial numbers, Stirling numbers, Eulerian numbers, Worpitzky identity, Pascal's triangle, OEIS.

In general,

**Proposition 1.1** (Newton formula via backward differences).

$$f(x) = \sum_{k=0}^n \binom{x-a+k-1}{k} \nabla^k f(a)$$

where  $\nabla^k f(a) = \sum_{j=0}^k (-1)^j \binom{k}{j} f(a-j)$ .

Thus, by setting  $f(n) = n^m$

$$n^m = \sum_{j=0}^m \binom{n-t+j-1}{j} \nabla^j t^m,$$

where  $\nabla^j t^m = \sum_{k=0}^j (-1)^k \binom{j}{k} (t-k)^m$ . Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \sum_{k=1}^n \binom{k-t+j-1}{j}$$

We notice that the sum  $\sum_{k=1}^n \binom{k-t+j-1}{j}$  is a good candidate for hockey stick identity for binomial coefficients  $\sum_{k=0}^n \binom{k}{j} = \binom{n+1}{j+1}$ . Thus, by setting  $a = j-t$  and  $b = j-t-1+n$ , we get

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Thus,

$$\sum_{k=1}^n \binom{-t+j-1+k}{j} = \binom{j-t+n}{j+1} - \binom{j-t}{j+1}$$

Because,

$$\sum_{m=a}^b \binom{m}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}$$

Applying the identity for binomial coefficients  $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$ , we obtain

**Proposition 1.2** (Ordinary sums of powers via backward differences). *For non-negative integers  $n, m$  and an arbitrary integer  $t$*

$$\Sigma^1 n^m = \sum_{j=0}^m \nabla^j t^m \left[ (-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right]$$

For example, by setting  $t = 2$  and  $m = 1, 2, 3, 4$ , we get formulas for sums of cubes

$$\Sigma^1 n^1 = 2 \left[ -\binom{2}{1} + \binom{n-2}{1} \right] + 1 \left[ \binom{2}{2} + \binom{n-1}{2} \right],$$

$$\begin{aligned} \Sigma^1 n^2 &= 4 \left[ -\binom{2}{1} + \binom{n-2}{1} \right] + 3 \left[ \binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 2 \left[ -\binom{2}{3} + \binom{n}{3} \right]. \end{aligned}$$

$$\begin{aligned} \Sigma^1 n^3 &= 8 \left[ -\binom{2}{1} + \binom{n-2}{1} \right] + 7 \left[ \binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 6 \left[ -\binom{2}{3} + \binom{n}{3} \right] + 6 \left[ \binom{2}{4} + \binom{n+1}{4} \right]. \end{aligned}$$

$$\begin{aligned} \Sigma^1 n^4 &= 16 \left[ -\binom{2}{1} + \binom{n-2}{1} \right] + 15 \left[ \binom{2}{2} + \binom{n-1}{2} \right] \\ &\quad + 14 \left[ -\binom{2}{3} + \binom{n}{3} \right] + 12 \left[ \binom{2}{4} + \binom{n+1}{4} \right] \\ &\quad + 24 \left[ -\binom{2}{5} + \binom{n+2}{5} \right]. \end{aligned}$$

The coefficients  $1, 2, 1, 4, 3, 2, 8, 7, 6, 6, \dots$  for  $t = 2$  is the sequence [A391068](#) in the OEIS [3].

For  $t = 0$  the coefficients are  $1, 0, 1, 0, -1, 2, 0, 1, -6, 6, \dots$  and registered in the OEIS as [A278075](#). For  $t = 1$  the coefficients are  $1, 1, 1, 1, 1, 2, 1, 1, 0, 6, \dots$  and registered in the OEIS as [A389570](#). For  $t = 3$  the coefficients are  $1, 3, 1, 9, 5, 2, 27, 19, 12, 6, \dots$  and registered in the OEIS as [A391210](#).

**Lemma 1.3** (Backward differences in Eulerian numbers).

$$\Delta^j t^m = \sum_{k=0}^m \langle m \rangle \binom{t+k-j}{m-j}$$

*Proof.* By Worpitzky identity  $t^m = \sum_{k=0}^m \langle m \rangle \binom{t+k}{m}$  and binomial recurrence  $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ , see [4].  $\square$

Thus, let be a formula for ordinary sums of powers in terms of Eulerian numbers  $\langle m \rangle$

**Proposition 1.4** (Ordinary sums of powers in Eulerian numbers). *For non-negative integers  $n, m$  and an arbitrary integer  $t$*

$$\Sigma^1 n^m = \sum_{j=0}^m \sum_{k=0}^m \left[ (-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \begin{Bmatrix} m \\ k \end{Bmatrix} \binom{t+k-j}{m-j}$$

**Lemma 1.5** (Backward differences in Stirling numbers).

$$\nabla^j t^m = \sum_{k=j}^m \binom{t-j}{k-j} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$$

*Proof.* By the identity  $t^m = \sum_{k=0}^m \binom{t}{k} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$  and binomial recurrence  $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ .  $\square$

Thus, let be a formula for ordinary sums of powers in terms of Stirling numbers  $\begin{Bmatrix} m \\ k \end{Bmatrix}$

**Proposition 1.6** (Ordinary sums of powers in Stirling numbers). *For non-negative integers  $n, m$  and an arbitrary integer  $t$*

$$\Sigma^1 n^m = \sum_{j=0}^m \sum_{k=j}^m \left[ (-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t-j}{k-j} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$$

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## MATHEMATICA PROGRAMS

Use the *Mathematica* package [5] to validate the results

Mathematica Function	Validates / Prints
MultifoldSumOfPowersRecurrence[r, n, m]	Computes $\sum^r n^m$
ValidateOrdinarySumsOfPowersViaBackwardDifferences[20]	Validates Proposition (1.2)
ValidateBackwardDifferencesInEulerianNumbers[20]	Validates Lemma (1.3)
ValidateOrdinarySumsOfPowersInEulerianNumbers[10]	Validates Proposition (1.4)
ValidateBackwardDifferencesInStirlingNumbers[20]	Validates Lemma (1.5)
ValidateOrdinarySumsOfPowersInStirlingNumbers[20]	Validates Proposition (1.6)

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**Version:** Local-0.1.0

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**Sources:** [github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences](https://github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences)

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**DOI:** [10.5281/zenodo.18118011](https://doi.org/10.5281/zenodo.18118011)

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