SUMS OF POWERS VIA CENTRAL FINITE DIFFERENCES AND NEWTON'S FORMULA

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Abstract.

1. Introduction and main results

Theorem 1.1 (Newton series via central difference)**.**

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k f(0)$$

Theorem 1.2 (Newton series for power via central difference)**.**

$$x^n = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k 0^n$$

Where

Corollary 1.3 (Central difference of power in zero)**.**

$$\delta^k 0^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^n$$

Lemma 1.4 (Central factorial)**.**

$$n^{[k]} = n\left(n + \frac{k}{2} - 1\right)\left(n + \frac{k}{2} - 2\right) \cdots \left(n - \frac{k}{2} + 1\right) = n \prod_{j=1}^{k-1} \left(n + \frac{k}{2} - j\right)$$

Such that

Proposition 1.5.

$$n^{[k]} = n\left(n + \frac{k}{2} - 1\right)_{k-1}$$

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Proposition 1.6. *For $k \geq 1$*

$$\frac{n^{[k]}}{k!} = \frac{1}{k!} n \left(n + \frac{k}{2} - 1\right)_{k-1} = \frac{n}{k(k-1)!} \left(n + \frac{k}{2} - 1\right)_{k-1}$$
$$= \frac{n}{k} \binom{n + \frac{k}{2} - 1}{k-1}$$

Because

Proposition 1.7.

$$\frac{(x)_n}{n!} = \binom{x}{n}$$

Thus

Proposition 1.8.

$$x^n = \sum_{k=1}^{n} \frac{x}{k} \binom{x + \frac{k}{2} - 1}{k-1} \delta^k 0^n$$

We start from $k = 1$ there because for all $k = 0$ the central difference $\delta^k 0^n$ is zero.

Thus

Proposition 1.9.

$$x^{n-1} = \sum_{k=1}^{n} \frac{1}{k} \binom{x + \frac{k}{2} - 1}{k-1} \delta^k 0^n$$

Thus

Proposition 1.10.

$$x^n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{x + \frac{k+1}{2} - 1}{k} \delta^{k+1} 0^{n+1}$$

Thus

Proposition 1.11.

$$\Sigma^r \, n^m = \sum_{k=0}^{m} \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1 + r}{k + r} \delta^{k+1} 0^{m+1}$$

Proof. By hockey stick identity. □

Lemma 1.12 (Central factorial numbers)**.**

$$\delta^k 0^n = k! T(n, k)$$

Thus

Proposition 1.13.

$$\Sigma^r n^m = \sum_{k=0}^{m} \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1 + r}{k+r} (k+1)! T(m+1, k+1)$$

Thus

Proposition 1.14.

$$\Sigma^r n^{m-1} = \sum_{k=0}^{m} k! \binom{n + \frac{k+1}{2} - 1 + r}{k+r} T(m, k+1)$$

2. Proof of Knuth's formula

3. Conclusions

4. Acknowledgements

The author is grateful to [Full Name] for his valuable contribution [contribution] about the fact that [interesting claim].

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Sources: github.com/kolosovpetro/SumsOfPowersViaCentralFiniteDifferencesAndNewtonFormula

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