

# SUMS OF POWERS VIA CENTRAL FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT.

## 1. INTRODUCTION AND MAIN RESULTS

**Proposition 1.1** (Newton's series in central differences).

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k f(0)$$

where  $\delta^k f(0) = \sum_{j=0}^k (-1)^j \binom{k}{j} f\left(\frac{k}{2} - j\right)$  is central finite difference in zero, and  $x^{[k]} = n\left(n + \frac{k}{2} - 1\right)\left(n + \frac{k}{2} - 2\right) \cdots \left(n - \frac{k}{2} + 1\right)$  is central factorial.

**Lemma 1.2** (Central factorial).

$$n^{[k]} = n \left(n + \frac{k}{2} - 1\right) \left(n + \frac{k}{2} - 2\right) \cdots \left(n - \frac{k}{2} + 1\right) = n \prod_{j=1}^{k-1} \left(n + \frac{k}{2} - j\right)$$

We observe that central factorials are closely related to falling factorials  $(x)_n = x(x-1)(x-2)(x-3) \cdots (x-n+1) = \prod_{k=0}^{n-1} (x-k)$ . Therefore,

$$n^{[k]} = n \left(n + \frac{k}{2} - 1\right)_{k-1}$$

To derive formula for multifold sums of powers, we follow the strategy to express the Newton's formula [eq ref] in terms of binomial coefficients, then to reach closed forms of column sum of binomial coefficients by means of hockey stick pattern. Therefore,

**Proposition 1.3.** For  $k \geq 1$

$$\frac{n^{[k]}}{k!} = \frac{n}{k!} \left(n + \frac{k}{2} - 1\right)_{k-1} = \frac{n}{k(k-1)!} \left(n + \frac{k}{2} - 1\right)_{k-1} = \frac{n}{k} \binom{n + \frac{k}{2} - 1}{k-1}$$

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*Proof.* The identity above is true because  $\frac{(x)_n}{n!} = \binom{x}{n}$ .  $\square$

Which yields Newton's formula for powers, in terms of central differences. For positive integers  $n \geq 1$  and  $m \geq 1$

$$n^m = \sum_{k=1}^m \frac{n}{k} \binom{n + \frac{k}{2} - 1}{k-1} \delta^k 0^m$$

In the proposition above, we start the summation from  $k = 1$  to avoid division by zero in  $\frac{n}{k}$ . It is a valid trick, because the central difference  $\delta^k 0^n$  is zero for all  $n \geq 1$  and  $k = 0$ .

By factoring out and simplifying the term  $n$ , we get

$$n^{m-1} = \sum_{k=1}^m \frac{1}{k} \binom{n + \frac{k}{2} - 1}{k-1} \delta^k 0^m$$

By reindexing the sum yields

**Proposition 1.4** (Newton's series for power in zero).

$$n^m = \sum_{k=0}^m \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1}{k} \delta^{k+1} 0^{m+1}$$

Thus, formula for ordinary sums of powers follows

**Proposition 1.5** (Ordinary sums of powers).

$$\Sigma^1 n^m = \sum_{k=0}^m \frac{1}{k+1} \binom{n + \frac{k+1}{2}}{k+1} \delta^{k+1} 0^{m+1}$$

*Proof.* We have  $\Sigma^1 n^m = \sum_{k=0}^m \frac{1}{k+1} \delta^{k+1} 0^{m+1} \sum_{j=1}^n \binom{j + \frac{k+1}{2} - 1}{k}$ . By hockey stick identity  $\sum_{j=1}^n \binom{j + \frac{k+1}{2} - 1}{k} = \binom{n + \frac{k+1}{2}}{k+1}$ . Thus, the claim follows.  $\square$

Continuing similarly, we get formula for multifold sums of powers

**Theorem 1.6** (Multifold sums of powers).

$$\Sigma^r n^m = \sum_{k=0}^m \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1 + r}{k+r} \delta^{k+1} 0^{m+1}$$

Additionally, the formula for multifold sums of powers can be expressed in terms of central factorial numbers of the second kind [references]. In Riordan notation,

**Lemma 1.7** (Central factorial numbers).

$$T(n, k) = \frac{\delta^k 0^n}{k!}$$

Note that central factorial numbers of the second kind  $T(n, k)$  are non-zero only for pairs  $(n, k)$  such that  $n - k$  is even. Meaning that  $T(2n, 2k)$  is always non-zero. The triangle of central factorial numbers  $T(2n, 2k)$  is the sequence [ID] in the OEIS [?].

**Proposition 1.8** (Multifold sums of powers via central factorial numbers).

$$\sum^r n^m = \sum_{k=0}^m k! \binom{n + \frac{k+1}{2} - 1 + r}{k + r} T(m + 1, k + 1)$$

## 2. PROOF OF KNUTH'S FORMULA

**Proposition 2.1** (Knuth's formula for Multifold sums of odd powers).

$$\sum^r n^{2m-1} = \sum_{k=1}^m (2k-1)! \binom{n + k - 1 + r}{2k - 1 + r} T(2m, 2k)$$

where  $T(n, k)$  is central factorial number  $T(n, k) = \frac{1}{k!} \delta^k 0^n$  and  $\delta^k 0^n$  is the central finite difference of power in zero.

*Proof.* Consider the Riordan's power identity, see [cite]

**Lemma 2.2** (Riordan's power identity).

$$n^m = \sum_{k=1}^m T(m, k) n^{[k]}$$

where  $n^{[k]}$  is central factorial [eq ref].

It is easy to see that Riordan's power identity is a direct consequence of Newton's series for central difference [eq ref], with  $T(m, k) = \frac{1}{k!} \delta^k 0^m$ , where  $\delta^k 0^m$  is central difference of power in zero. Now we notice that  $T(m, k) = 0$  whether  $m - k$  is odd. Thus,

$$n^{2m} = \sum_{k=1}^{2m} T(2m, k) n^{[k]}$$

We allow  $k$  to run over the integers  $k = 2, 4, 6, \dots, 2m$  to maintain the condition  $T(2m, k) \neq 0$ , hence

$$n^{2m} = \sum_{k=1}^m T(2m, 2k) n^{[2k]}$$

By expressing the central factorials  $n^{[2k]}$  in terms of falling factorials

$$n^{[2k]} = n(n+k-1)_{2k-1}$$

Yields

$$n^{2m} = \sum_{k=1}^m T(2m, 2k) n(n+k-1)_{2k-1}$$

By dividing by  $n$  and applying the identity  $\frac{(x)_n}{n!} = \binom{x}{n}$ , we get

$$n^{2m-1} = \sum_{k=1}^m (2k-1)! \binom{n+k-1}{2k-1} T(2m, 2k)$$

which is the base identity for odd powers. Now, the ordinary sum of odd powers is

$$\Sigma^1 n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \sum_{j=1}^n \binom{j+k-1}{2k-1}$$

By hockey-stick identity  $\sum_{j=1}^n \binom{j+k-1}{2k-1} = \binom{n+k}{2k}$

$$\Sigma^1 n^{2m-1} = \sum_{k=1}^m (2k-1)! T(2m, 2k) \binom{n+k}{2k}$$

By induction yields

$$\Sigma^r n^{2m-1} = \sum_{k=1}^m (2k-1)! \binom{n+k-1+r}{2k-1+r} T(2m, 2k)$$

□

We can compare the Knuth's formula and special case of [eqref] for  $m \rightarrow 2m-1$ , that is

**Corollary 2.3.**

$$\Sigma^r n^{2m-1} = \sum_{k=0}^m k! \binom{n + \frac{k+1}{2} - 1 + r}{k+r} T(2m, 2k)$$

## 3. SUMS OF POWERS IN STIRLING AND EULERIAN NUMBERS

By the identity

$$\delta^k x^m = \Delta^k \left( x - \frac{k}{2} \right)^m$$

Therefore,

**Proposition 3.1** (Multifold sums of powers).

$$\sum_{k=0}^r n^m = \sum_{k=0}^m \frac{1}{k+1} \left( n + \frac{k+1}{2} - 1 + r \right) \Delta^k \left( -\frac{k}{2} \right)^m$$

**Lemma 3.2** (Finite difference via Eulerian numbers at a half-shift). *For non-negative integers  $k, m$*

$$\Delta^k \left( -\frac{k}{2} \right)^m = \sum_{r=0}^m \left\langle m \right\rangle_r \left( r - \frac{k}{2} \right)_{m-k}$$

**Lemma 3.3** (Finite differences via Stirling numbers at a half-shift). *For non-negative integers  $k, m$*

$$\Delta^k \left( -\frac{k}{2} \right)^m = \sum_{r=0}^m \binom{-\frac{k}{2}}{r} \left\{ m \right\}_{k+r} (k+r)!$$

Therefore,

## 4. CONCLUSIONS

## 5. ACKNOWLEDGEMENTS

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**Sources:** [github.com/kolosovpetro/SumsOfPowersViaCentralDifferences](https://github.com/kolosovpetro/SumsOfPowersViaCentralDifferences)

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