SUMS OF POWERS VIA CENTRAL FINITE DIFFERENCES AND NEWTON'S FORMULA

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Abstract.

1. Introduction and main results

Proposition 1.1 (Newton's series in central differences)**.**

$$f(x)=\sum_{k=0}^{\infty}\frac{x^{[k]}}{k!}\delta^k f(0)$$

where $\delta^k f(0) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}f\left(\frac{k}{2}-j\right)$ *is central finite difference in zero, and* $x^{[k]} = n\left(n+\frac{k}{2}-1\right)\left(n+\frac{k}{2}-2\right)\cdots\left(n-\frac{k}{2}+1\right)$ *is central factorial.*

Lemma 1.2 (Central factorial)**.**

$$n^{[k]}=n\left(n+\frac{k}{2}-1\right)\left(n+\frac{k}{2}-2\right)\cdots\left(n-\frac{k}{2}+1\right)=n\prod_{j=1}^{k-1}\left(n+\frac{k}{2}-j\right)$$

We observe that central factorials are closely related to falling factorials $(x)_n = x(x-1)(x-2)(x-3)\cdots(x-n+1)$. Therefore,

$$n^{[k]}=n\left(n+\frac{k}{2}-1\right)_{k-1}$$

To derive formula for multifold sums of powers, we follow the strategy to express the Newton's formula [eq ref] in terms of binomial coefficients, then to reach closed forms of column sum of binomial coefficients by means of hockey stick pattern. Therefore,

Proposition 1.3. *For* $k\geq 1$

$$\frac{n^{[k]}}{k!}=\frac{n}{k!}\left(n+\frac{k}{2}-1\right)_{k-1}=\frac{n}{k(k-1)!}\left(n+\frac{k}{2}-1\right)_{k-1}=\frac{n}{k}\binom{n+\frac{k}{2}-1}{k-1}$$

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Proof. The identity above is true because $\frac{(x)_n}{n!} = \binom{x}{n}$. □

Which yields Newton's formula for powers, in terms of central differences. For positive integers $n \geq 1$ and $m \geq 1$

$$n^m = \sum_{k=1}^{m} \frac{n}{k} \binom{n + \frac{k}{2} - 1}{k - 1} \delta^k 0^m$$

In the proposition above, we start the summation from $k = 1$ to avoid division by zero in $\frac{n}{k}$. It is a valid trick, because the central difference $\delta^k 0^n$ is zero for all $n \geq 1$ and $k = 0$.

By factoring out and simplifying the term n, we get

$$n^{m-1} = \sum_{k=1}^{m} \frac{1}{k} \binom{n + \frac{k}{2} - 1}{k - 1} \delta^k 0^m$$

By reindexing the sum yields

Proposition 1.4 (Newton's series for power in zero)**.**

$$n^m = \sum_{k=0}^{m} \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1}{k} \delta^{k+1} 0^{m+1}$$

Thus, formula for ordinary sums of powers follows

Proposition 1.5 (Ordinary sums of powers)**.**

$$\Sigma^1 \, n^m = \sum_{k=0}^{m} \frac{1}{k+1} \binom{n + \frac{k+1}{2}}{k + 1} \delta^{k+1} 0^{m+1}$$

Proof. We have $\Sigma^1 \, n^m = \sum_{k=0}^{m} \frac{1}{k+1} \delta^{k+1} 0^{m+1} \sum_{j=1}^{n} \binom{j + \frac{k+1}{2} - 1}{k}$. By hockey stick identity $\sum_{j=1}^{n} \binom{j + \frac{k+1}{2} - 1}{k} = \binom{n + \frac{k+1}{2}}{k+1}$. Thus, the claim follows. □

Continuing similarly, we get formula for multifold sums of powers

Theorem 1.6 (Multifold sums of powers)**.**

$$\Sigma^r \, n^m = \sum_{k=0}^{m} \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1 + r}{k + r} \delta^{k+1} 0^{m+1}$$

Additionally, the formula for multifold sums of powers can be expressed in terms of central factorial numbers of the second kind [references]. In Riordan notation,

Lemma 1.7 (Central factorial numbers).

$$T(n,k) = \frac{\delta^k 0^n}{k!}$$

Note that central factorial numbers of the second kind $T(n,k)$ are non-zero only for pairs (n,k) such that $n-k$ is even. Meaning that $T(2n,2k)$ is always non-zero. The triangle of central factorial numbers $T(2n,2k)$ is the sequence [ID] in the OEIS [**?**].

Proposition 1.8 (Multifold sums of powers via central factorial numbers).

$$\Sigma^r n^m = \sum_{k=0}^{m} k! \binom{n + \frac{k+1}{2} - 1 + r}{k+r} T(m+1, k+1)$$

2. Proof of Knuth's formula

Proposition 2.1 (Knuth's formula for Multifold sums of odd powers).

$$\Sigma^r n^{2m-1} = \sum_{k=1}^{m} (2k-1)! \binom{n+k-1+r}{2k-1+r} T(2m,2k)$$

where $T(n,k)$ is central factorial number $T(n,k) = \frac{1}{k!}\delta^k 0^n$ and $\delta^k 0^n$ is the central finite difference of power in zero.

Proof. Consider the Riordan's power identity, see [cite]

Lemma 2.2 (Riordan's power identity).

$$n^m = \sum_{k=1}^{m} T(m,k) n^{[k]}$$

where $n^{[k]}$ is central factorial [eq ref].

It is easy to see that Riordan's power identity is a direct consequence of Newton's series for central difference [eq ref], with $T(m,k) = \frac{1}{k!}\delta^k 0^m$, where $\delta^k 0^m$ is central difference of power in zero. Now we notice that $T(m,k) = 0$ whether $m-k$ is odd. Thus,

$$n^{2m} = \sum_{k=1}^{2m} T(2m,k) n^{[k]}$$

We allow k to run over the integers $k = 2, 4, 6, \ldots, 2m$ to maintain the condition $T(2m, k) \neq 0$, hence

$$n^{2m} = \sum_{k=1}^{m} T(2m, 2k) n^{[2k]}$$

By expressing the central factorials $n^{[2k]}$ in terms of falling factorials

$$n^{[2k]} = n\,(n + k - 1)_{2k-1}$$

Yields

$$n^{2m} = \sum_{k=1}^{m} T(2m, 2k) n\,(n + k - 1)_{2k-1}$$

By dividing by n an applying the identity $\frac{(x)_n}{n!} = \binom{x}{n}$, we get

$$n^{2m-1} = \sum_{k=1}^{m} (2k-1)! \binom{n+k-1}{2k-1} T(2m, 2k)$$

which is the base identity for odd powers. Now, the ordinary sum of odd powers is

$$\Sigma^1\, n^{2m-1} = \sum_{k=1}^{m} (2k-1)! T(2m, 2k) \sum_{j=1}^{n} \binom{j+k-1}{2k-1}$$

By hockey-stick identity $\sum_{j=1}^{n} \binom{j+k-1}{2k-1} = \binom{n+k}{2k}$

$$\Sigma^1\, n^{2m-1} = \sum_{k=1}^{m} (2k-1)! T(2m, 2k) \binom{n+k}{2k}$$

By induction yields

$$\Sigma^r\, n^{2m-1} = \sum_{k=1}^{m} (2k-1)! \binom{n+k-1+r}{2k-1+r} T(2m, 2k)$$

□

We can compare the Knuth's formula and special case of [eqref] for $m \to 2m-1$, that is

Corollary 2.3.

$$\Sigma^r\, n^{2m-1} = \sum_{k=0}^{m} k! \binom{n + \frac{k+1}{2} - 1 + r}{k+r} T(2m, 2k)$$

3. Sums of powers in Stirling and Eulerian numbers

By the identity

$$\delta^k x^m = \Delta^k \left(x - \frac{k}{2}\right)^m$$

Therefore,

Proposition 3.1 (Multifold sums of powers)**.**

$$\Sigma^r\, n^m = \sum_{k=0}^{m} \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1 + r}{k+r} \Delta^k \left(-\frac{k}{2}\right)^m$$

Lemma 3.2 (Finite difference via Eulerian numbers at a half-shift)**.** *For non-negative integers k, m*

$$\Delta^k \left(-\frac{k}{2}\right)^m = \sum_{r=0}^{m} \left\langle {m \atop r} \right\rangle \binom{r - \frac{k}{2}}{m-k}$$

Lemma 3.3 (Finite differences via Stirling numbers at a half-shift)**.** *For non-negative integers k, m*

$$\Delta^k \left(-\frac{k}{2}\right)^m = \sum_{r=0}^{m} \binom{-\frac{k}{2}}{r} \left\{ {m \atop k+r} \right\} (k+r)!$$

Therefore,

4. Conclusions

5. Acknowledgements

The author is grateful to [Full Name] for his valuable contribution [contribution] about the fact that [interesting claim].

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Sources: github.com/kolosovpetro/SumsOfPowersViaCentralDifferences

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