SUMS OF POWERS VIA CENTRAL FINITE DIFFERENCES AND NEWTON'S FORMULA

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Abstract.

1. Introduction

Theorem 1.1 (Newton series via central difference)**.**

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k f(0)$$

Theorem 1.2 (Newton series for power via central difference)**.**

$$x^n = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k 0^n$$

Where

Corollary 1.3 (Central difference of power in zero)**.**

$$\delta^k 0^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^n$$

Lemma 1.4 (Central factorial)**.**

$$n^{[k]} = n\left(n + \frac{k}{2} - 1\right)\left(n + \frac{k}{2} - 2\right) \cdots \left(n - \frac{k}{2} + 1\right) = n \prod_{j=1}^{k-1} \left(n + \frac{k}{2} - j\right)$$

Such that

Date: December 29, 2025.

2010 *Mathematics Subject Classification.* 05A19, 05A10, 41A15, 11B68, 11B73, 11B83.

Key words and phrases. Sums of powers, Newton's interpolation formula, Finite differences, Binomial coefficients, Faulhaber's formula, Bernoulli numbers, Bernoulli polynomials, Interpolation, Discrete convolution, Combinatorics, Polynomial identities, Central factorial numbers, Stirling numbers, Eulerian numbers, Worpitzky identity, Pascal's triangle, OEIS.

Proposition 1.5.

$$n^{[k]} = n\left(n + \frac{k}{2} - 1\right)_{k-1}$$

Proposition 1.6. *For* $k \geq 1$

$$\frac{n^{[k]}}{k!} = \frac{1}{k!}n\left(n + \frac{k}{2} - 1\right)_{k-1} = \frac{n}{k(k-1)!}\left(n + \frac{k}{2} - 1\right)_{k-1}$$
$$= \frac{n}{k}\binom{n + \frac{k}{2} - 1}{k-1}$$

Because

Proposition 1.7.

$$\frac{(x)_n}{n!} = \binom{x}{n}$$

Thus

Proposition 1.8.

$$x^n = \sum_{k=1}^{n} \frac{x}{k}\binom{x + \frac{k}{2} - 1}{k-1}\delta^k 0^n$$

Thus

Proposition 1.9.

$$x^{n-1} = \sum_{k=1}^{n} \frac{1}{k}\binom{x + \frac{k}{2} - 1}{k-1}\delta^k 0^n$$

Thus

Proposition 1.10.

$$x^n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{x + \frac{k+1}{2} - 1}{k}\delta^{k+1} 0^{n+1}$$

Thus

Proposition 1.11.

$$\Sigma^r n^m = \sum_{k=0}^{m} \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1 + r}{k+r} \delta^{k+1} 0^{m+1}$$

Proof. By hockey stick identity. □

Lemma 1.12 (Central factorial numbers)**.**

$$\delta^k 0^n = k! T(n,k)$$

Thus

Proposition 1.13.

$$\Sigma^r n^m = \sum_{k=0}^{m} \frac{1}{k+1} \binom{n + \frac{k+1}{2} - 1 + r}{k+r} (k+1)! T(m+1, k+1)$$

Thus

Proposition 1.14.

$$\Sigma^r n^{m-1} = \sum_{k=0}^{m} k! \binom{n + \frac{k+1}{2} - 1 + r}{k+r} T(m, k+1)$$

2. Conclusions

3. Acknowledgements

The author is grateful to [Full Name] for his valuable contribution [contribution] about the fact that [interesting claim].

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Version: `Local-0.1.0`

Sources: github.com/kolosovpetro/SumsOfPowersViaCentralFiniteDifferencesAndNewtonFormula

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