

Prove that

$$\sum_{k=0}^t \binom{t-k}{r} \binom{k}{s} = \binom{t+1}{r+s+1}$$

This approach is based on generating functions.

By <http://math.arizona.edu/~faris/combinatoricsweb/generate.pdf>

We know that

$$\sum_{n=0}^{\infty} \binom{n}{k} y^n = \frac{y^k}{(1-y)^{k+1}}$$

Then

$$\begin{aligned} x \cdot \sum_{l=0}^{\infty} \binom{l}{r} x^l \cdot \sum_{k=0}^{\infty} \binom{k}{s} x^k &= x \cdot \frac{x^r}{(1-x)^{r+1}} \cdot \frac{x^s}{(1-x)^{s+1}} = \frac{x^{r+s+1}}{(1-x)^{r+s+2}} \\ &= \sum_{n=0}^{\infty} \binom{n}{r+s+1} x^n \end{aligned}$$

The coefficient of  $x^{t+1}$  of the  $x \cdot \sum_{l=0}^{\infty} \binom{l}{r} x^l \cdot \sum_{k=0}^{\infty} \binom{k}{s} x^k$  is

$$[z^{t+1}] A(z) B(z) = \sum_{k=0}^t a_k b_{t-k} = \sum_{k=0}^t \binom{t-k}{r} \binom{k}{s}$$

Note that upper summation bound is  $t$  while coefficient is  $[z^{t+1}]$  it is because of the  $x$  factor in the generating function. While

$$[z^{t+1}] \sum_{n=0}^{\infty} \binom{n}{r+s+1} x^n = \binom{t+1}{r+s+1}$$

Thus

$$\binom{t+1}{r+s+1} = \sum_{k=0}^t \binom{t-k}{r} \binom{k}{s}$$