1 The problem

Use generating functions to prove that

$$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n} \tag{1}$$

The first problem is that it is not clear what summation bounds should be to keep non-zero terms. Let's reverse the coefficient $\binom{l}{m+k}$ to see exact summation bounds

$$\sum_{k=0}^{l-m} \binom{l}{l-m-k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$$

So now it is clear that we are hunting for the coefficient of z^{l-m} in the generating function. Let's rearrange the sum above

$$\sum_{k=0}^{(l-m)} \binom{s}{n+k}\binom{l}{(l-m)-k} = \binom{l+s}{n+(l-m)}$$

So we can see that left-hand side of the equation above matches the convolution of two generating functions.

$$A(z) \cdot B(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

having $n = l - m$ so that basically our identity is taking the coefficient of z^{l-m} in the convolution of two generating functions. Let be $n = t$ for the sake of simplicity

$$\sum_{k=0}^{(l-m)} \binom{s}{t+k}\binom{l}{(l-m)-k} = \binom{l+s}{t+(l-m)} \tag{2}$$

So we have to match two generating functions $A_s(z)$, $B_l(z)$ for the binomial coefficients: $\binom{s}{t+k}$ and $\binom{l}{(l-m)-k}$. So that

$$A_s(z) = \frac{(1+z)^s}{z^t}$$
$$B_l(z) = (1+z)^l$$

So that product of them

$$A_s(z) \cdot B_l(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{s}{t+k}\binom{l}{n-k}\right) z^n = \frac{(1+z)^s}{z^t} \cdot (1+z)^l = \frac{(1+z)^{l+s}}{z^t}$$

Right side of the (2) is the coefficient of z^{l-m} in $\frac{(1+z)^{l+s}}{z^t}$

$$[z^{l-m}]\frac{(1+z)^{l+s}}{z^t} = \binom{l+s}{l-m+t}$$

Left side of the (2) is the coefficient of z^{l-m} in the convolution of two generating functions

$$[z^{l-m}]A_s(z) \cdot B_l(z) = \sum_{k=0}^{(l-m)} \binom{s}{t+k}\binom{l}{(l-m)-k}$$

Thus,

$$\sum_{k=0}^{(l-m)} \binom{s}{t+k}\binom{l}{(l-m)-k} = \binom{l+s}{l-m+t}$$