COMMENTS ON CONCRETE MATHEMATICS (2E) BINOMIAL COEFFICIENTS

PETRO KOLOSOV

Contents

1. Conventions

- Use variable z that indicates complex value in generating functions.
- Give particular names to binomial identities, for example *absorption identity*
- Give particular names to generating functions to remember them easily
- Use subscript indices for generating functions that are powers of some value t, for clarity. Example: $A_t(z) = (1+z)^t$ for binomial coefficients.

2. Important binomial identities

Identities from Concrete Mathematics [1, p. 174]

Identity 2.1. *Factorial expansion:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \textit{integers } n \geq k \geq 0.$$

Identity 2.2. *Symmetry:*

$$\binom{n}{k} = \binom{n}{n-k}, \quad \textit{integer } n \geq 0, \ \textit{integer } k.$$

Identity 2.3. *Absorption/extraction:*

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, \quad \textit{integer } k \neq 0.$$

$$k\binom{r}{k} = r\binom{r-1}{k-1}$$

$$(r-k)\binom{r}{k} = (r-k)\binom{r}{r-k} = r\binom{r-1}{r-k-1} = r\binom{r-1}{k}.$$

Identity 2.4. *Addition/induction:*

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}, \quad \textit{integer } k.$$

Identity 2.5. *Upper negation:*

$$\binom{r}{k} = (-1)^k\binom{k-r-1}{k}, \quad \textit{integer } k.$$

Let $r = \ell - 1 - t$ *and* $k = \ell - t - m + s$*, then*

$$\binom{\ell-1-t}{\ell-t-m+s} = (-1)^{\ell-t-m+s}\binom{m+s}{\ell-t-m+s}.$$

Also

$$\begin{aligned}(-1)^{t+s}\binom{\ell-1-t}{\ell-t-m+s} &= (-1)^{\ell-m+2s}\binom{m+s}{\ell-t-m+s}\\ &= (-1)^{\ell-m}\binom{m+s}{\ell-t-m+s}\\ &= (-1)^{\ell+m}\binom{m+s}{\ell-t-m+s}\end{aligned}$$

Because

$$(-1)^{k+2s}\binom{n}{t} = (-1)^k\binom{n}{t}$$

$$(-1)^{k-s}\binom{n}{t} = (-1)^{k+s}\binom{n}{t}$$

Identity 2.6. *Trinomial revision:*

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}, \quad \textit{integers } m, k.$$

Identity 2.7. *Binomial theorem:*

$$\sum_k \binom{r}{k} x^k y^{r-k} = (x+y)^r, \quad \textit{integer } r \geq 0, \ \textit{or } |x/y| < 1.$$

Identity 2.8. *Parallel summation:*

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}, \quad \textit{integer } n.$$

Identity 2.9. *Upper summation:*

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}, \quad \textit{integers } m, n \geq 0.$$

Identity 2.10. *Vandermonde convolution:*

$$\sum_k \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n}, \quad \textit{integer } n.$$

3. Important generating functions

Identity 3.1. *Cauchy product rule of two generating functions $A(z),\ B(z)$*

$$A(z) \cdot B(z) = \left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{n=0}^{\infty} b_n z^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n$$

Identity 3.2. *Cauchy product rule for $(1+z)^{r+s}$*

$$(1+z)^{r+s} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k} \right) z^n$$

Identity 3.3. *Shift selected coefficient of generating function*

$$[z^{p-q}]A(z) = [z^p] z^q A(z)$$

$$[z^{p+q}]A(z) = [z^p] \frac{1}{z^q} A(z)$$

Identity 3.4. *Binomial coefficient, fixed r*

$$\binom{r}{n} = [z]^n (1+z)^r$$

Identity 3.5. *Shifted binomial coefficient, fixed m, r*

$$\binom{r}{m+n} = [z]^n \frac{(1+z)^r}{z^m}$$

$$\binom{r}{n-m} = [z]^n (1+z)^r z^m$$

Identity 3.6. *Binomial coefficient of multiset* [2, eq. 8]*, fixed k*

$$A_k(z) = \sum_{n=0}^{\infty} \binom{n}{k} z^n = \frac{z^k}{(1-z)^{k+1}}$$

Then

$$\binom{t}{k} = [z]^t \frac{z^k}{(1-z)^{k+1}}$$

So that iteration goes over upper index of binomial coefficient.

Identity 3.7. *Shifted Binomial coefficient of multiset, fixed k*

$$\binom{t}{k+r} = [z]^t \frac{z^{k+r}}{(1-z)^{k+r+1}}$$

Identity 3.8. *Shifted Binomial coefficient of multiset in two variables* [2, eq. 15]

$$\sum_{n=0}^{\infty}(1+x)^n y^n = \frac{1}{1-(1+x)y}$$

Identity 3.9. *Shifted Binomial coefficient of multiset in two variables (negated)*

$$\sum_{n=0}^{\infty}(1+x)^n y^n (-1)^n = \frac{1}{1+(1+x)y}$$

Identity 3.10. *Binomial coefficients row summation East-West*

$$\sum_k \binom{r}{j-k} = \sum_k [z]^{j-k}(1+z)^r = [z]^j \sum_k z^k (1+z)^r$$
$$= (1+z)^r [z]^j \sum_k z^k$$

Because $[z^{p-q}]A(z) = [z^p]z^q A(z)$

4. Important binomial sums

Identities from Concrete Mathematics [1, p. 169]

Identity 4.1.

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}, \quad \textit{integers } m, n.$$

Identity 4.2.

$$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}, \quad \textit{integer } l \geq 0, \textit{integers } m, n.$$

Identity 4.3.

$$\sum_k \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}, \quad \textit{integer } l \geq 0, \textit{integers } m, n.$$

Identity 4.4.

$$\sum_{k \leq l} \binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n}, \quad \textit{integers } l, m, n \geq 0.$$

Identity 4.5.

$$\sum_{0 \leq k \leq l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}, \quad \textit{integers } l, m \geq 0, \textit{integers } n \geq q \geq 0.$$

References

[1] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. `https://archive.org/details/concrete-mathematics`.

[2] Faris, William G. Generating Functions Notes for Math 447, 2011. `https://math.arizona.edu/~faris/combinatoricsweb/generate.pdf`.

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