

# COMMENTS ON CONCRETE MATHEMATICS (2E) BINOMIAL COEFFICIENTS

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## 1. CONVENTIONS

- Use variable  $z$  that indicates complex value in generating functions.
- Give particular names to binomial identities, for example *absorption identity*
- Give particular names to generating functions to remember them easily
- Use subscript indices for generating functions that are powers of some value  $t$ , for clarity. Example:  $A_t(z) = (1 + z)^t$  for binomial coefficients.

## 2. IMPORTANT BINOMIAL IDENTITIES

Identities from Concrete Mathematics [1, p. 174]

**Identity 2.1.** *Factorial expansion:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{integers } n \geq k \geq 0.$$

**Identity 2.2.** *Symmetry:*

$$\binom{n}{k} = \binom{n}{n-k}, \quad \text{integer } n \geq 0, \text{ integer } k.$$

**Identity 2.3.** *Absorption/extraction:*

$$\begin{aligned}\binom{r}{k} &= \frac{r}{k} \binom{r-1}{k-1}, \quad \text{integer } k \neq 0. \\ k \binom{r}{k} &= r \binom{r-1}{k-1} \\ (r-k) \binom{r}{k} &= (r-k) \binom{r}{r-k} = r \binom{r-1}{r-k-1} = r \binom{r-1}{k}.\end{aligned}$$

**Identity 2.4.** *Addition/induction:*

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}, \quad \text{integer } k.$$

**Identity 2.5.** *Upper negation:*

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}, \quad \text{integer } k.$$

**Identity 2.6.** *Trinomial revision:*

$$\binom{r}{m} \binom{m}{k} = \binom{r}{k} \binom{r-k}{m-k}, \quad \text{integers } m, k.$$

**Identity 2.7.** *Binomial theorem:*

$$\sum_k \binom{r}{k} x^k y^{r-k} = (x+y)^r, \quad \text{integer } r \geq 0, \text{ or } |x/y| < 1.$$

**Identity 2.8.** *Parallel summation:*

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}, \quad \text{integer } n.$$

**Identity 2.9.** *Upper summation:*

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}, \quad \text{integers } m, n \geq 0.$$

**Identity 2.10.** *Vandermonde convolution:*

$$\sum_k \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n}, \quad \text{integer } n.$$

## 3. IMPORTANT GENERATING FUNCTIONS

**Identity 3.1.** *Cauchy product rule of two generating functions  $A(z)$ ,  $B(z)$*

$$A(z) \cdot B(z) = \left( \sum_{n=0}^{\infty} a_n z^n \right) \left( \sum_{n=0}^{\infty} b_n z^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) z^n$$

**Identity 3.2.** *Cauchy product rule for  $(1+z)^{r+s}$*

$$(1+z)^{r+s} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} \right) z^n$$

**Identity 3.3.** *Shift selected coefficient of generating function*

$$[z^{p-q}]A(z) = [z^p]z^q A(z)$$

$$[z^{p+q}]A(z) = [z^p] \frac{1}{z^q} A(z)$$

**Identity 3.4.** *Binomial coefficient, fixed  $r$*

$$\binom{r}{n} = [z]^n (1+z)^r$$

**Identity 3.5.** *Shifted binomial coefficient, fixed  $m, r$*

$$\binom{r}{m+n} = [z]^n \frac{(1+z)^r}{z^m}$$

$$\binom{r}{n-m} = [z]^n (1+z)^r z^m$$

**Identity 3.6.** *Binomial coefficient of multiset [2, eq. 8], fixed  $k$*

$$A_k(z) = \sum_{n=0}^{\infty} \binom{n}{k} z^n = \frac{z^k}{(1-z)^{k+1}}$$

*Then*

$$\binom{t}{k} = [z]^t \frac{z^k}{(1-z)^{k+1}}$$

*So that iteration goes over upper index of binomial coefficient.*

**Identity 3.7.** *Shifted Binomial coefficient of multiset, fixed  $k$*

$$\binom{t}{k+r} = [z]^t \frac{z^{k+r}}{(1-z)^{k+r+1}}$$

**Identity 3.8.** *Shifted Binomial coefficient of multiset in two variables [2, eq. 15]*

$$B(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} x^k y^n = \sum_{n=0}^{\infty} (1+x)^n y^n = \frac{1}{1 - (1+x)y}$$

**Identity 3.9.** *Shifted Binomial coefficient of multiset in two variables (negated)*

$$\sum_{n=0}^{\infty} (1+x)^n y^n (-1)^n = \frac{1}{1 + (1+x)y}$$

**Identity 3.10.** *Binomial coefficients row summation East-West*

$$\begin{aligned} \sum_k \binom{r}{j-k} &= \sum_k [z]^{j-k} (1+z)^r = [z]^j \sum_k z^k (1+z)^r \\ &= (1+z)^r [z]^j \sum_k z^k \end{aligned}$$

Because  $[z^{p-q}]A(z) = [z^p]z^q A(z)$

## 4. IMPORTANT BINOMIAL SUMS

Identities from Concrete Mathematics [1, p. 169]

**Identity 4.1.**

$$\sum_k \binom{r}{m+k} \binom{s}{n-k} = \binom{r+s}{m+n}, \quad \text{integers } m, n.$$

**Identity 4.2.**

$$\sum_k \binom{l}{m+k} \binom{s}{n+k} = \binom{l+s}{l-m+n}, \quad \text{integer } l \geq 0, \text{ integers } m, n.$$

**Identity 4.3.**

$$\sum_k \binom{l}{m+k} \binom{s+k}{n} (-1)^k = (-1)^{l+m} \binom{s-m}{n-l}, \quad \text{integer } l \geq 0, \text{ integers } m, n.$$

**Identity 4.4.**

$$\sum_{k \leq l} \binom{l-k}{m} \binom{s}{k-n} (-1)^k = (-1)^{l+m} \binom{s-m-1}{l-m-n}, \quad \text{integers } l, m, n \geq 0.$$

**Identity 4.5.**

$$\sum_{0 \leq k \leq l} \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1}, \quad \text{integers } l, m \geq 0, \text{ integers } n \geq q \geq 0.$$

## REFERENCES

- [1] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. <https://archive.org/details/concrete-mathematics>.
- [2] Faris, William G. Generating Functions Notes for Math 447, 2011. <https://math.arizona.edu/~faris/combinatoricsweb/generate.pdf>.

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