COMMENTS ON CONCRETE MATHEMATICS (2E) BINOMIAL COEFFICIENTS

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Contents

1. Conventions

- Use variable z that indicates complex value in generating functions.
- Give particular names to binomial identities, for example *absorption identity*
- Give particular names to generating functions to remember them easily

- Use subscript indices for generating functions that are powers of some value t, for clarity. Example: $A_t(z) = (1+z)^t$ for binomial coefficients.

2. Important binomial identities

Identities from Concrete Mathematics [1, p. 174]

Identity 2.1. *Factorial expansion:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \textit{integers } n \geq k \geq 0.$$

Identity 2.2. *Symmetry:*

$$\binom{n}{k} = \binom{n}{n-k}, \quad \textit{integer } n \geq 0, \ \textit{integer } k.$$

Identity 2.3. *Sign even exponent simplification:*

$$(-1)^{t+2s}\binom{n}{k} = (-1)^t\binom{n}{k}$$

More generally, for every even k

$$(-1)^{t+k}\binom{n}{k} = (-1)^t\binom{n}{k}$$

Identity 2.4. *Sign exponent alternation:*

$$(-1)^{t+k}\binom{n}{k} = (-1)^{t-k}\binom{n}{k}$$

Identity 2.5. *Absorption/extraction:*

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, \quad \textit{integer } k \neq 0.$$

$$k\binom{r}{k} = r\binom{r-1}{k-1}$$

$$(r-k)\binom{r}{k} = (r-k)\binom{r}{r-k} = r\binom{r-1}{r-k-1} = r\binom{r-1}{k}.$$

Identity 2.6. *Addition/induction:*

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}, \quad \textit{integer } k.$$

Identity 2.7. *Upper negation:*

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}, \quad \textit{integer } k.$$

Let $r = \ell - 1 - t$ *and* $k = \ell - t - m + s$*, then*

$$\binom{\ell-1-t}{\ell-t-m+s} = (-1)^{\ell-t-m+s} \binom{m+s}{\ell-t-m+s}.$$

Also

$$\begin{aligned}
(-1)^{t+s}\binom{\ell-1-t}{\ell-t-m+s} &= (-1)^{\ell-m+2s}\binom{m+s}{\ell-t-m+s} \\
&= (-1)^{\ell-m}\binom{m+s}{\ell-t-m+s} \\
&= (-1)^{\ell+m}\binom{m+s}{\ell-t-m+s}
\end{aligned}$$

Because

$$\begin{aligned}
(-1)^{k+2s}\binom{n}{t} &= (-1)^{k}\binom{n}{t} \\
(-1)^{k-s}\binom{n}{t} &= (-1)^{k+s}\binom{n}{t}
\end{aligned}$$

Identity 2.8. *Trinomial revision:*

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}, \quad \textit{integers } m, k.$$

Identity 2.9. *Binomial theorem:*

$$\sum_k \binom{r}{k} x^k y^{r-k} = (x+y)^r, \quad \textit{integer } r \geq 0, \ \textit{or } |x/y| < 1.$$

Identity 2.10. *Parallel summation:*

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}, \quad \textit{integer } n.$$

Identity 2.11. *Upper summation:*

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}, \quad \textit{integers } m, n \geq 0.$$

Identity 2.12. *Vandermonde convolution:*

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}, \quad \textit{integer } n.$$

3. Important generating functions

Identity 3.1. *Cauchy product rule of two generating functions* $A(z)$, $B(z)$

$$A(z) \cdot B(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

Identity 3.2. *Cauchy product rule for* $(1+z)^{r+s}$

$$(1+z)^{r+s} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k}\right) z^n$$

Identity 3.3. *Shift selected coefficient of generating function*

$$[z^{p-q}]A(z) = [z^p]z^q A(z)$$

$$[z^{p+q}]A(z) = [z^p]\frac{1}{z^q}A(z)$$

Identity 3.4. *Binomial coefficient, fixed* r

$$\binom{r}{n} = [z^n](1+z)^r$$

Identity 3.5. *Shifted binomial coefficient, fixed* m, r

$$\binom{r}{m+n} = [z^n]\frac{(1+z)^r}{z^m}$$

$$\binom{r}{n-m} = [z^n](1+z)^r z^m$$

Identity 3.6. *Binomial coefficient of multiset* [2, eq. 8], *fixed* k

$$A_k(z) = \sum_{n=0}^{\infty}\binom{n}{k}z^n = \frac{z^k}{(1-z)^{k+1}}$$

Then

$$\binom{t}{k} = [z^t]\frac{z^k}{(1-z)^{k+1}}$$

So that

$$[z^t]\sum_{m=0}^{\infty}\binom{m}{k}z^m = [z^t]\frac{z^k}{(1-z)^{k+1}}$$

Identity 3.7. *Shifted Binomial coefficient of multiset, fixed k*

$$\binom{t}{k+r} = [z^t]\frac{z^{k+r}}{(1-z)^{k+r+1}}$$

Identity 3.8. *Shifted Binomial coefficient of multiset in two variables* [2, eq. 15]

$$\sum_{n=0}^{\infty}(1+x)^n y^n = \frac{1}{1-(1+x)y}$$

Identity 3.9. *Shifted Binomial coefficient of multiset in two variables (negated)*

$$\sum_{n=0}^{\infty}(1+x)^n y^n (-1)^n = \frac{1}{1+(1+x)y}$$

Identity 3.10. *Binomial coefficients row summation East-West*

$$\begin{aligned}\sum_k \binom{r}{j-k} &= \sum_k [z^{j-k}](1+z)^r = [z^j]\sum_k z^k(1+z)^r \\ &= (1+z)^r[z^j]\sum_k z^k\end{aligned}$$

Because $[z^{p-q}]A(z) = [z^p]z^q A(z)$.

Identity 3.11.

$$\frac{1}{(1-z)^r} = \sum_{k=0}^{\infty} C(r+k-1,k)z^k = 1 + C(r,1)z + C(r+1,2)z^2 + \cdots$$

$$\frac{1}{(1-z)^r} = \sum_{k=0}^{\infty}\binom{r+k-1}{k}z^k = 1 + \binom{r}{1}z + \binom{r+1}{2}z^2 + \binom{r+2}{3}z^3 + \cdots$$

Therefore,

$$\binom{r+n-1}{n} = [z^n]\frac{1}{(1-z)^r} = [z^n]\sum_{k=0}^{\infty}\binom{r+k-1}{k}z^k$$

$$\binom{r+n-1}{r-1} = [z^n]\frac{1}{(1-z)^r} = [z^n]\sum_{k=0}^{\infty}\binom{r+k-1}{k}z^k$$

4. Important binomial sums

Identities from Concrete Mathematics [1, p. 169]

Identity 4.1.

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}, \quad \textit{integers } m, n.$$

Identity 4.2.

$$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}, \quad \textit{integer } l \geq 0, \textit{integers } m, n.$$

Identity 4.3.

$$\sum_k \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}, \quad \textit{integer } l \geq 0, \textit{integers } m, n.$$

Identity 4.4.

$$\sum_{k\leq l} \binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n}, \quad \textit{integers } l, m, n \geq 0.$$

Identity 4.5.

$$\sum_{0\leq k\leq l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}, \quad \textit{integers } l, m \geq 0, \textit{integers } n \geq q \geq 0.$$

5. Problem 1: Prove that $\sum_{k=0}^{t}\binom{t-k}{r}\binom{k}{s}=\binom{t+1}{r+s+1}$

Prove that

$$\sum_{k=0}^{t}\binom{t-k}{r}\binom{k}{s}=\binom{t+1}{r+s+1} \tag{5.1}$$

We can see that iteration in the left-hand side of the equation (5.1) is running over the upper index of binomial coefficients, so let's figure out proper generating function for it. We know that

$$\sum_{n=0}^{\infty}\binom{n}{k}y^n=\frac{y^k}{(1-y)^{k+1}} \tag{5.2}$$

Now we keep attention to the lower index of binomial coefficient in right-hand side of the equation (5.1) which is $r+s+1$. We have to match some generating function to reach the $\sum_{n=0}^{\infty}\binom{n}{r+s+1}x^n$ Let be generating functions

$$A_r(x)=\sum_{l=0}^{\infty}\binom{l}{r}x^l$$
$$B_s(x)=\sum_{k=0}^{\infty}\binom{k}{s}x^k$$

Now let's match our generating function

$$\begin{aligned} xA_r(x)B_s(x) &= x\cdot\left(\sum_{l=0}^{\infty}\binom{l}{r}x^l\right)\cdot\left(\sum_{k=0}^{\infty}\binom{k}{s}x^k\right)\\ &= x\cdot\frac{x^r}{(1-x)^{r+1}}\cdot\frac{x^s}{(1-x)^{s+1}}=\frac{x^{r+s+1}}{(1-x)^{r+s+2}}\\ &= \sum_{n=0}^{\infty}\binom{n}{r+s+1}x^n \end{aligned}$$

Because of the equation (5.2). Actually, we could simply substitute $k=r+s+1$ to the equation (5.2) to reach desired generating function, anyway.

The coefficient of x^{t+1} in the $x\cdot\sum_{l=0}^{\infty}\binom{l}{r}x^l\cdot\sum_{k=0}^{\infty}\binom{k}{s}x^k$ is

$$[x^{t+1}]xA_r(x)B_s(x)=\sum_{k=0}^{t}a_kb_{t-k}=\sum_{k=0}^{t}\binom{t-k}{r}\binom{k}{s}$$

Note that upper summation bound is t while coefficient is $[z^{t+1}]$ it is because of the x factor in the generating function. General rule for that is

$$[z^{p-q}]A(z) = [z^p]z^q A(z)$$

While

$$[x^{t+1}]\sum_{n=0}^{\infty}\binom{n}{r+s+1}x^n = \binom{t+1}{r+s+1}$$

Thus

$$\binom{t+1}{r+s+1} = \sum_{k=0}^{t}\binom{t-k}{r}\binom{k}{s}$$

This approach is based on generating functions by http://math.arizona.edu/~faris/combinatoricsweb/generate.pdf

5.1. Flow of the solution.

- First, keep your attention to the left part of the problem (5.1) to see over what index iteration is running, there can be three cases: lower, upper, or both. If both find identity to simplify it to run over either lower or upper index. If is upper index use generating function (5.2). If lower index use binomial theorem as generating function.
- Then keep your attention to the right part of the problem (5.1), precisely the lower index of binomial coefficient, as it defines the generating function we're looking for.
- Then seeing the sum over binomial coefficients multiplication, it means that it should be expressed in terms of convolution of two generating functions, for our case is $[x^{t+1}]xA_r(x)B_s(x)$.
- Match the coefficient given by the convolution of two generating functions and the generating function.

6. Problem 2: Prove that $\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$

It is from Concrete mathematics equation (5.22). Use generating functions to prove that

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n} \tag{6.1}$$

Okay, first let's review the summation boundary such that terms are non-zero. Summation is done over k so that binomial coefficient $\binom{s}{n-k}$ fixes k to be less or equal to n. Rewrite the statement of the problem

$$\sum_{k=0}^{n} \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n} \tag{6.2}$$

Left-hand side of it reminds me sequence convolution of two generating functions. Let be two generating functions for such left-hand side summation:

$$A_r(z); \quad B_s(z)$$

Multiplying those generating functions yields

$$C(x) = \left(\sum_{m=0}^{\infty} a_m x^m\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_m b_{k-m}\right) x^k$$

Then

$$A_r(z) \cdot B_s(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

We can notice the similar structure as we have in our problem (6.2). So let's find the generating function for the binomial coefficient $\binom{r+s}{m+n}$. We know that generating function for the binomial coefficient $\binom{n}{k}$ is

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

If we want to have $m+k$ as lower index, then

$$(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{m+k} z^{m+k}$$

$$(1+z)^r = z^m \sum_{k=0}^{\infty} \binom{r}{m+k} z^{k}$$

$$\frac{(1+z)^r}{z^m} = \sum_{k=0}^{\infty} \binom{r}{m+k} z^{k}$$

Thus, the coefficient of z^n in $\frac{(1+z)^r}{z^m}$ is

$$[z^n]\frac{(1+z)^r}{z^m} = \binom{r}{m+n}$$

So that our first generating function is

$$A_r(z) = \frac{(1+z)^r}{z^m}$$

The second generating function is

$$B_s(z) = (1+z)^s$$

Multiplying them

$$A_r(z) \cdot B_s(z) = \frac{(1+z)^r}{z^m} \cdot (1+z)^s = \frac{(1+z)^{r+s}}{z^m}$$

Convolution form is

$$A_r(z) \cdot B_s(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{r}{m+k} \binom{s}{n-k} \right) x^n$$

Coefficient of z^n in $\frac{(1+z)^{r+s}}{z^m}$ is

$$[z^n]\frac{(1+z)^{r+s}}{z^m} = \binom{r+s}{m+n}$$

Coefficient of z^n in $\sum_{k=0}^{\infty} \left(\sum_{k=0}^{n} \binom{r}{m+k} \binom{s}{n-k} \right) x^n$ is

$$[z^n]A_r(z) \cdot B_s(z) = \sum_{k=0}^{n} \binom{r}{m+k} \binom{s}{n-k}$$

7. Problem 3: Prove that $\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$

This is from Concrete mathematics equation (5.23). Use generating functions to prove that

$$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n} \tag{7.1}$$

The first problem is that it is not clear what summation bounds should be to keep non-zero terms. Let's reverse the coefficient $\binom{l}{m+k}$ to see exact summation bounds

$$\sum_{k=0}^{l-m} \binom{l}{l-m-k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$$

So now it is clear that we are hunting for the coefficient of z^{l-m} in the generating function. Let's rearrange the sum above

$$\sum_{k=0}^{(l-m)} \binom{s}{n+k}\binom{l}{(l-m)-k} = \binom{l+s}{n+(l-m)}$$

So we can see that left-hand side of the equation above matches the convolution of two generating functions.

$$A(z) \cdot B(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n$$

having $n = l - m$ so that basically our identity is taking the coefficient of z^{l-m} in the convolution of two generating functions. Let be $n = t$ for the sake of simplicity

$$\sum_{k=0}^{(l-m)} \binom{s}{t+k}\binom{l}{(l-m)-k} = \binom{l+s}{t+(l-m)} \tag{7.2}$$

So we have to match two generating functions $A_s(z)$, $B_l(z)$ for the binomial coefficients: $\binom{s}{t+k}$ and $\binom{l}{(l-m)-k}$. So that

$$A_s(z) = \frac{(1+z)^s}{z^t}$$

$$B_l(z) = (1+z)^l$$

So that product of them

$$A_s(z) \cdot B_l(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{s}{t+k} \binom{l}{n-k} \right) z^n = \frac{(1+z)^s}{z^t} \cdot (1+z)^l = \frac{(1+z)^{l+s}}{z^t}$$

Right side of the (7.2) is the coefficient of z^{l-m} in $\frac{(1+z)^{l+s}}{z^t}$

$$[z^{l-m}]\frac{(1+z)^{l+s}}{z^t} = \binom{l+s}{l-m+t}$$

Left side of the (7.2) is the coefficient of z^{l-m} in the convolution of two generating functions

$$[z^{l-m}]A_s(z) \cdot B_l(z) = \sum_{k=0}^{(l-m)} \binom{s}{t+k} \binom{l}{(l-m)-k}$$

Thus,

$$\sum_{k=0}^{(l-m)} \binom{s}{t+k} \binom{l}{(l-m)-k} = \binom{l+s}{l-m+t}$$

8. Problem 4: Prove that $\sum_j \binom{n}{j}^2 = \binom{2n}{n}$

This is from Generating functionology by Herbert S. Wilf. Using generating function prove that

$$\sum_j \binom{n}{j}^2 = \binom{2n}{n}$$

9. Problem 5: Prove that $\sum_{k\le n}\binom{r+k}{k}=\binom{r+n+1}{n}$

This is from Concrete Mathematics equation (5.9). Using generating function prove that

$$\sum_{k=0}^{n}\binom{r+k}{k}=\binom{r+n+1}{n}=\binom{r+n+1}{r+1} \tag{9.1}$$

So we are looking for the coefficient of z^n in multiplication of two yet-unspecified generating functions A, B because of upper index of summation n.

9.1. Left-hand side of the identity.

Rewrite our identity to get rid of k as lower index of binomial coefficient, that makes easier to guess required generating function. By symmetry of binomial coefficients we have

$$\sum_{k=0}^{n}\binom{r+k}{r}=\binom{r+n+1}{n}$$

Multiply the sum by $\binom{n-k}{j}$ to match the Cauchy product rule

$$\sum_{k=0}^{n}\binom{r+k}{r}\binom{n-k}{j}$$

As iteration goes over upper index of binomial coefficients, let be $r=l$ in lower index of $\binom{r+k}{r}$. So for fixed r, l, j we have general form of the problem

$$\sum_{k=0}^{n}\binom{r+k}{l}\binom{n-k}{j} \tag{9.2}$$

Looking to the upper index of summation n we can conclude that we are looking for the coefficient of z^n in the convolution of two generating functions in the Cauchy product rule of two generating functions $A(z),\ B(z)$

$$A(z)\cdot B(z)=\left(\sum_{n=0}^{\infty}a_n z^n\right)\left(\sum_{n=0}^{\infty}b_n z^n\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k b_{n-k}\right)z^n$$

As summation runs over upper index of binomial coefficients, our basic generating function is

$$\binom{t}{k}=[z^t]\frac{z^k}{(1-z)^{k+1}}$$

So that

$$[z^t]\sum_{m=0}^{\infty}\binom{m}{k}z^m = [z^t]\frac{z^k}{(1-z)^{k+1}}$$

Consider the binomial coefficient $\binom{r+n}{l}$ and its generating function. Note that r is fixed value there and iteration goes over n. Therefore,

$$[z^t]\sum_{m=0}^{\infty}\binom{r+m}{k}z^{r+m} = [z^t]\frac{z^k}{(1-z)^{k+1}}$$

$$[z^t]\sum_{m=0}^{\infty}\binom{r+m}{k}z^{m} = [z^t]\frac{z^k}{(1-z)^{k+1}\cdot z^r}$$

Denote above generating function as $A_{r,k}(z)$.

$$A_{r,k}(z) = \sum_{m=0}^{\infty}\binom{r+m}{k}z^m = \frac{z^k}{(1-z)^{k+1}\cdot z^r}$$

Consider the second binomial coefficient $\binom{n-k}{j}$ where j is fixed value. Its generating function $B_j(z)$ is

$$B_j(z) = \frac{z^j}{(1-z)^{j+1}}$$

So considering base form of the problem (9.2) we have the following generating functions. For $\binom{r+k}{l}$ having fixed r, l

$$A_{r,l}(z) = \sum_{m=0}^{\infty}\binom{r+m}{l}z^m = \frac{z^l}{(1-z)^{l+1}\cdot z^r}$$

for $\binom{n-k}{j}$ having fixed j

$$B_j(z) = \frac{z^j}{(1-z)^{j+1}}$$

Multiplication yields

$$\begin{aligned}A_{r,l}(z)\cdot B_j(z) &= \frac{z^l}{(1-z)^{l+1}\cdot z^r}\cdot\frac{z^j}{(1-z)^{j+1}} = \frac{z^{l+j}}{(1-z)^{l+j+2}\cdot z^r}\\ &= \frac{z^{l+j-r}}{(1-z)^{l+j+2}}\end{aligned}$$

General Cauchy form yields

$$A_{r,l}(z) \cdot B_j(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{r+k}{l} \binom{n-k}{j} \right) z^n$$

Therefore, we have an identity

$$\sum_{k=0}^{n} \binom{r+k}{l} \binom{n-k}{j} = [z^n] \frac{z^{l+j-r}}{(1-z)^{l+j+2}}$$

Having $l = r$ and $j = 0$

$$\sum_{k=0}^{n} \binom{r+k}{r} \binom{n-k}{0} = [z^n] \frac{1}{(1-z)^{r+2}}$$

Which evaluates

$$\binom{r+n-1}{n} = [z^n] \frac{1}{(1-z)^r} = [z^n] \sum_{k=0}^{\infty} \binom{r+k-1}{k} z^k$$
$$\binom{r+n-1}{r-1} = [z^n] \frac{1}{(1-z)^r} = [z^n] \sum_{k=0}^{\infty} \binom{r+k-1}{k} z^k$$

So that

$$[z^n] \frac{1}{(1-z)^{r+2}} = \binom{r+n+1}{r+1}$$

Therefore, desired result holds

$$\sum_{k=0}^{n} \binom{r+k}{r} = \binom{r+n+1}{n}$$

9.2. **Right-hand side of the identity.** Consider the binomial coefficient $\binom{r+n+1}{r+1}$ from RHS of the problem (9.1) and its generating function. First, let's do not forget about symmetry rule

$$\binom{r+n+1}{r+1} = \binom{r+n+1}{n}$$

So it is the coefficient of z^n in the generating function

$$\binom{(r+1)+n}{(r+1)} = [z^n] \sum_{k=0}^{\infty} \binom{(r+1)+k}{(r+1)} z^k$$

Setting $c = r + 1$ and $u = r + 1$ we have, fixed c, u

$$\binom{c+n}{u} = [z^n] \sum_{k=0}^{\infty} \binom{c+k}{u} z^k$$

Then working generating function is

$$F_{c,u}(z) = \sum_{m=0}^{\infty} \binom{c+m}{u} z^m = \frac{z^u}{(1-z)^{u+1} \cdot z^c}$$

So the right-hand side in terms of F is

$$\begin{aligned}\binom{(r+1)+n}{(r+1)} &= [z^n] F_{r+1,r+1}(z) = [z^n] \frac{z^{r+1}}{(1-z)^{r+1+1} \cdot z^{r+1}} \\ &= [z^n] \frac{1}{(1-z)^{r+2}}\end{aligned}$$

10. Problem 6: Prove that $\sum_{k\le n}\binom{k}{m}=\binom{n+1}{m+1}$

This is from Concrete Mathematics equation (5.10). Using generating function prove that

$$\sum_{k=0}^{n}\binom{k}{m}=\binom{n+1}{m+1} \tag{10.1}$$

Now it is clear that we are looking for the coefficient of z^n in the convolution of two generating functions. Rewrite problem to match the Cauchy product rule

$$\sum_{k=0}^{n}\binom{k}{m}\binom{n-k}{j}=\binom{n+1}{m+1} \tag{10.2}$$

Generating function for the binomial coefficient $\binom{k}{m}$ is given by

$$A_m(z)=\sum_{k=0}^{\infty}\binom{k}{m}z^k=\frac{z^m}{(1-z)^{m+1}}$$

Generating function for the binomial coefficient $\binom{n-k}{j}$ is given by

$$B_j(z)=\sum_{k=0}^{\infty}\binom{k}{j}z^k=\frac{z^j}{(1-z)^{j+1}}$$

Multiplication of them yields

$$\begin{aligned}A_m(z)\cdot B_j(z)&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{k}{m}\binom{n-k}{j}\right)z^n=\frac{z^m}{(1-z)^{m+1}}\cdot\frac{z^j}{(1-z)^{j+1}}\\&=\frac{z^{m+j}}{(1-z)^{m+j+2}}\end{aligned}$$

Let $j=0$ then

$$A_m(z)\cdot B_0(z)=\frac{z^m}{(1-z)^{m+1}}\cdot\frac{1}{(1-z)}=\frac{z^m}{(1-z)^{m+2}}$$

Note that generating function above does not match the generating function of binomial coefficient $\binom{k}{n}$, fixed n. Multiplying by z gives

$$\begin{aligned}zA_m(z)\cdot B_0(z)&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{k}{m}\binom{n-k}{0}\right)z^{n+1}=\frac{z^m}{(1-z)^{m+1}}\cdot\frac{1}{(1-z)}\\&=\frac{z^{m+1}}{(1-z)^{m+2}}\end{aligned}$$

Now it matches. Taking the coefficient of z^{n+1} yields

$$[z^{n+1}]zA_m(z) \cdot B_0(z) = \binom{n+1}{m+1}$$

Therefore, it is indeed true that

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

11. Problem 7: Prove that $\sum_{k\le m}\binom{r}{k}(-1)^k = (-1)^m\binom{r-1}{m}$

References

[1] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. https://archive.org/details/concrete-mathematics.

[2] Faris, William G. Generating Functions Notes for Math 447, 2011. https://math.arizona.edu/~faris/combinatoricsweb/generate.pdf.

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