1 The problem

Use generating functions to prove that

$$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n} \tag{1}$$

The first problem is that it is not clear what summation bounds should be to keep non-zero terms. Let's reverse the coefficient $\binom{l}{m+k}$ to see exact summation bounds

$$\sum_{k=0}^{l-m} \binom{l}{l-m-k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$$

So now it is clear that we are hunting for the coefficient of z^{l-m} in the generating function.

$$\sum_{k=0}^{l-m} \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{n+l-m}$$