Prove that

$$\sum_{k=0}^{t} \binom{t-k}{r}\binom{k}{s} = \binom{t+1}{r+s+1} \tag{1}$$

We can see that iteration in the left-hand side of the equation (1) is running over the upper index of binomial coefficients, so let's figure out proper generating function for it. We know that

$$\sum_{n=0}^{\infty} \binom{n}{k} y^n = \frac{y^k}{(1-y)^{k+1}} \tag{2}$$

Now we keep attention to the lower index of binomial coefficient in right-hand side of the equation (1) which is $r+s+1$. We have to match some generating function to reach the $\sum_{n=0}^{\infty} \binom{n}{r+s+1} x^n$ Let be generating functions

$$A_r(x) = \sum_{l=0}^{\infty} \binom{l}{r} x^l$$
$$B_s(x) = \sum_{k=0}^{\infty} \binom{k}{s} x^k$$

Now let's match our generating function

$$\begin{aligned} xA_r(x)B_s(x) &= x \cdot \left(\sum_{l=0}^{\infty} \binom{l}{r} x^l\right) \cdot \left(\sum_{k=0}^{\infty} \binom{k}{s} x^k\right) \\ &= x \cdot \frac{x^r}{(1-x)^{r+1}} \cdot \frac{x^s}{(1-x)^{s+1}} = \frac{x^{r+s+1}}{(1-x)^{r+s+2}} \\ &= \sum_{n=0}^{\infty} \binom{n}{r+s+1} x^n \end{aligned}$$

Because of the equation (2). Actually, we could simply substitute $k = r+s$ to the equation (2) to reach desired generating function, anyway.

The coefficient of $x^{t+1}$ in the $x \cdot \sum_{l=0}^{\infty} \binom{l}{r} x^l \cdot \sum_{k=0}^{\infty} \binom{k}{s} x^k$ is

$$[x^{t+1}]xA_r(x)B_s(x) = \sum_{k=0}^{t} a_k b_{t-k} = \sum_{k=0}^{t} \binom{t-k}{r}\binom{k}{s}$$

Note that upper summation bound is $t$ while coefficient is $[z^{t+1}]$ it is because of the $x$ factor in the generating function. General rule for that is

$$[z^{p-q}]A(z) = [z^p]z^q A(z)$$

While

$$[z^{t+1}] \sum_{n=0}^{\infty} \binom{n}{r+s+1} x^n = \binom{t+1}{r+s+1}$$

Thus

$$\binom{t+1}{r+s+1} = \sum_{k=0}^{t} \binom{t-k}{r}\binom{k}{s}$$

This approach is based on generating functions by `http://math.arizona.edu/~faris/combinatoricsweb/generate.pdf`