COMMENTS ON CONCRETE MATHEMATICS (2E) BINOMIAL COEFFICIENTS

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Contents

1. Conventions

- Use variable z that indicates complex value in generating functions.
- Give particular names to binomial identities, for example *absorption identity*
- Give particular names to generating functions to remember them easily
- Use subscript indices for generating functions that are powers of some value t, for clarity. Example: $A_t(z) = (1+z)^t$ for binomial coefficients.

2. Important binomial identities

3. Important generating functions

Identity 3.1. *Cauchy product rule of two generating functions* $A(z)$, $B(z)$

$$A(z) \cdot B(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

Identity 3.2. *Cauchy product rule for* $(1+z)^{r+s}$

$$(1+z)^{r+s} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k} \right) z^n$$

Identity 3.3. *Shift selected coefficient of generating function*

$$[z^{p-q}]A(z) = [z^p]z^q A(z)$$

$$[z^{p+q}]A(z) = [z^p]\frac{1}{z^q}A(z)$$

Identity 3.4. *Binomial coefficient, fixed* r

$$\binom{r}{n} = [z]^n(1+z)^r$$

Identity 3.5. *Shifted binomial coefficient, fixed* m, r

$$\binom{r}{m+n} = [z]^n \frac{(1+z)^r}{z^m}$$

Identity 3.6. *Binomial coefficient of multiset* [1, eq. 8]*, fixed* k

$$A_k(z) = \sum_{n=0}^{\infty} \binom{n}{k} z^n = \frac{z^k}{(1-z)^{k+1}}$$

Then

$$\binom{t}{k} = [z]^t \frac{z^k}{(1-z)^{k+1}}$$

So that iteration goes over upper index of binomial coefficient.

Identity 3.7. *Shifted Binomial coefficient of multiset, fixed* k

$$\binom{t}{k+r} = [z]^t \frac{z^{k+r}}{(1-z)^{k+r+1}}$$

Identity 3.8. *Shifted Binomial coefficient of multiset in two variables* [1, eq. 15]

$$B(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} x^k y^n = \sum_{n=0}^{\infty} (1+x)^n y^n = \frac{1}{1-(1+x)y}$$

Identity 3.9. *Shifted Binomial coefficient of multiset in two variables (negated)*

$$\sum_{n=0}^{\infty} (1+x)^n y^n (-1)^n = \frac{1}{1+(1+x)y}$$

4. Important binomial sums

References

[1] Faris, William G. Generating Functions Notes for Math 447, 2011. https://math.arizona.edu/~faris/combinatoricsweb/generate.pdf.

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