1 The problem

Use generating functions to prove that

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n} \tag{1}$$

Okay, first let's review the summation boundary such that terms are non-zero. Summation is done over k so that binomial coefficient $\binom{s}{n-k}$ fixes k to be less or equal to n. Rewrite the statement of the problem

$$\sum_{k=0}^{n} \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n} \tag{2}$$

Left-hand side of it reminds me sequence convolution of two generating functions. Let be two generating functions for such left-hand side summation:

$$A_r(z); \quad B_s(z)$$

Multiplying those generating functions yields

$$C(x) = \left(\sum_{m=0}^{\infty} a_m x^m\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_m b_{k-m}\right) x^k$$

Then

$$A_r(z) \cdot B_s(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

We can notice the similar structure as we have in our problem (2). So let's find the generating function for the binomial coefficient $\binom{r+s}{m+n}$. We know that generating function for the binomial coefficient $\binom{n}{k}$ is

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

If we want to have $m+k$ as lower index, then

$$(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{m+k} z^{m+k}$$

$$(1+z)^r = z^m \sum_{k=0}^{\infty} \binom{r}{m+k} z^k$$

$$\frac{(1+z)^r}{z^m} = \sum_{k=0}^{\infty} \binom{r}{m+k} z^k$$

Thus, the coefficient of z^n in $\frac{(1+z)^r}{z^m}$ is

$$[z^n]\frac{(1+z)^r}{z^m} = \binom{r}{m+n}$$

So that our first generating function is

$$A_r(z) = \frac{(1+z)^r}{z^m}$$

The second generating function is

$$B_s(z) = (1+z)^s$$

Multiplying them

$$A_r(z) \cdot B_s(z) = \frac{(1+z)^r}{z^m} \cdot (1+z)^s = \frac{(1+z)^{r+s}}{z^m}$$

Convolution form is

$$\begin{aligned} A_r(z) \cdot B_s(z) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{r}{m+k} \binom{s}{n-k} \right) x^n \end{aligned}$$

Coefficient of z^n in $\frac{(1+z)^{r+s}}{z^m}$ is

$$[z^n]\frac{(1+z)^{r+s}}{z^m} = \binom{r+s}{m+n}$$

Coefficient of z^n in $\sum_{k=0}^{\infty} \left(\sum_{k=0}^{n} \binom{r}{m+k} \binom{s}{n-k} \right) x^n$ is

$$[z^n]A_r(z) \cdot B_s(z) = \sum_{k=0}^{n} \binom{r}{m+k} \binom{s}{n-k}$$