COMMENTS ON CONCRETE MATHEMATICS (2E) BINOMIAL COEFFICIENTS

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Contents

1. Conventions

- Use variable z that indicates complex value in generating functions.
- Give particular names to binomial identities, for example *absorption identity*
- Give particular names to generating functions to remember them easily

2. Important binomial identities

2.1. **Generating functions.** Generating function is a power series that generates an infinite sequence of numbers $\{a_0, a_1, a_2, a_3, \dots\}$

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots = \sum_{k=0}^{\infty} a_k z^k$$

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Coefficient of z^n in $A(z)$ denoted as

$$[z^n]A(z) = a_n$$

is the n-th term of the sequence. For example, generating function for the sequence of binomial coefficients is

$$(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{k} z^k$$

Let be a product of two generating functions $A(z)$ and $B(z)$, then c_n in such sequence is a sum

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Above sum is called the convolution of two sequences $\{a_0, a_1, a_2, a_3, \dots\}$ and $\{b_0, b_1, b_2, b_3, \dots\}$. So that

$$[z^n]A(z)B(z) = c_n$$

Example for Vandermonde convolution, let be $A(z) = (1+z)^r$ and $B(z) = (1+z)^s$, then multiplying them

$$A(z)B(z) = (1+z)^r(1+z)^s = (1+z)^{r+s}$$

Then the coefficient of z^n in $(1+z)^{r+s}$ is

$$[z^n]A(z)B(z) = [z^n](1+z)^{r+s} = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

3. Important generating functions

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