

# SURPRISING POLYNOMIAL IDENTITIES ARISING FROM A CLASSICAL INTERPOLATION PROBLEM

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ABSTRACT.

## 1. THE PROBLEM OF INTERPOLATION OF CUBES

Once there was a curious student, curious in general and especially in mathematics. He was not an educated mathematician, but he was strongly captured by the mathematical beauty and aesthetics.

One day this student was having fun with tables of finite differences, precisely finite differences of cubes. By observing the table

$n$	$n^3$	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

**Table 1.** Table of finite differences of  $n^3$ .

The first question that triggered his mind was

**Question 1.1.** *How to reconstruct the value of  $n^3$  from its finite differences?*

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The question is quite old, it can be traced back to ancient Babylonian and Greek times, several centuries BC and first centuries AD [citation]. The process is called interpolation which is a process of finding new data points based on the range of a discrete set of known data points. Interpolation we know nowadays was developed in 1674–1684 by Issac Newton in his works referenced as foundation of classical interpolation theory [citation].

Great! But there is one thing, the student who has risen the question (1.1) had no clue about interpolation theory at all. What he decided then? Exactly, he decided to try to re-invent interpolation formula himself, fueled by the purest feeling of mystery. His mind was occupied by only a single thought: *All mathematical truths exist timelessly, we only reveal and describe them.* That mindset inspired our student to start his own mathematical journey.

By observing the table of finite differences (1) we can notice that the first order finite difference of cubes may be expressed in terms of its third order finite difference  $\Delta^3(n^3) = 6$ , as follows

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

$$\vdots$$

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6n$$

By using sigma notation for sums we get

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^n k$$

However, there is a more beautiful way to prove that  $\Delta(n^3) = 1 + 6 \sum_{k=0}^n k$ . We refer to one of the finest articles in the area of polynomials and power sums, that is *Johann Faulhaber and sums of powers* written by Donald Knuth [?]. Indeed, this article is a great source to

reach piece of mind in mathematics. We now focus on the following odd power identities shown on p.9 of [?]

$$\begin{aligned}n^1 &= \binom{n}{1} \\n^3 &= 6\binom{n+1}{3} + \binom{n}{1} \\n^5 &= 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}\end{aligned}$$

It is particularly interesting that well-known identity that connects triangular numbers  $\binom{n+1}{2}$  and finite differences of cubes becomes clear and obvious

$$\Delta n^3 = (n+1)^3 - n^3 = 6\binom{n+1}{2} + \binom{n}{0}$$

It easy to see that

$$\Delta n^3 = \left[6\binom{n+2}{3} + \binom{n+1}{1}\right] - \left[6\binom{n+1}{3} + \binom{n}{1}\right] = 6\binom{n+1}{2} + \binom{n}{0}$$

because  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ .

Moreover, the concept above allows to reach  $N$ -fold power sums  $\sum^N k^{2m+1}$  or finite differences  $\Delta^N k^{2m+1}$  of odd powers by simply altering binomial coefficients indexes. Quite strong and impressive.

We can observe that triangular numbers  $\binom{n+1}{2}$  are equivalent to

$$\binom{n+1}{2} = \sum_{k=0}^n k$$

because  $\binom{n+1}{m+1} = \sum_{k=0}^n \binom{k}{m}$ . This leads to the identity in finite differences of cubes

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6 \sum_{k=0}^n k$$

An experienced mathematician would immediately notice a spot to apply Faulhaber's formula [?] to expand the term  $\sum_{k=0}^n k$

$$\sum_{k=0}^n k = \frac{1}{2}(n + n^2)$$

Thus, the finite difference  $\Delta(n^3)$  takes a well-known form such that matches Binomial theorem [?]

$$\Delta(n^3) = 1 + 6 \left[ \frac{1}{2}(n + n^2) \right] = 1 + 3n + 3n^2 = \sum_{k=0}^2 \binom{3}{k} n^k$$

And... that could be the end of the story, isn't it? Because all what remains is to say that

$$n^3 = \sum_{k=0}^{n-1} (k+1)^3 - k^3 = \sum_{k=0}^{n-1} \left( 1 + 6 \sum_{t=0}^k t \right) = \sum_{k=0}^{n-1} 1 + 3k + 3k^2$$

Thus, the polynomial  $n^3$  is interpolated successfully, and thus, the question (1.1) is answered positively. Because we have successfully found the function that matches  $n^3$  from the values of its finite differences from the table (1).

However, not this time. Luckily enough (say), the student who has stated the question (1.1) wasn't really aware of the approaches above. Probably, that's exactly the case when unawareness leads to a fresh sight to century-old questions, leading to unexpected results and new insights.