SURPRISING POLYNOMIAL IDENTITIES ARISING FROM A CLASSICAL INTERPOLATION PROBLEM

PETRO KOLOSOV

ABSTRACT.

1. THE PROBLEM OF INTERPOLATION OF CUBES

Back then, in 2016 being a student at the faculty of mechanical engineering, I remember myself playing with finite differences of the polynomial n^3 over the domain of natural numbers $n \in \mathbb{N}$ having at most $0 \leq n \leq 20$ values. Looking to the values in my finite difference tables, the first and very naive question that came to my mind was

Question 1.1. *Is it possible to re-assemble the value of the polynomial n^3 backwards having its finite differences?*

The answer to this question is certainly *Yes*, by utilizing interpolation methods. Interpolation is a process of finding new data points based on the range of a discrete set of known data points. It has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

At that time, in 2016, I was a first-year mechanical engineering undergraduate. Therefore, due to lack of knowledge in mathematics, I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. *All mathematical laws and relations exist from the very beginning, but we only reveal and describe them*, I thought. That mindset truly inspired me so that my own mathematical journey began.

Let's start considering the table of finite differences of the polynomial n^3

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n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

First and foremost, we can observe that finite difference $\Delta(n^3)$ of the polynomial n^3 can be expressed through summation over n, e.g

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \tag{1.1}$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

$$\vdots$$

Finally reaching its generic form

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k \tag{1.2}$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term $\sum_{k=0}^{n} k$ reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Then our relation (1.2) immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k \tag{1.3}$$

However, as was said, I was not the experienced one mathematician back then, so that I reviewed the relation (1.2) from a little bit different perspective. Not following the convenient solution (1.3), I have introduced the explicit formula for cubes, using (1.1)

$$n^3 = [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots$$

$$+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \tag{1.4}$$

Then, rearranging the terms in equation (1.4) so that it turns into summation in terms of $k(n-k)$

$$n^3 = n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] + \cdots$$

$$\cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)]$$

By applying compact sigma notation and moving n under summation because there is exactly n iteration, yields

$$n^3 = n + \sum_{k=1}^{n} 6k(n-k); \qquad n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.5}$$

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $6k(n-k) + 1$. See the OEIS entry: A287326 [4].

Therefore, we have reached our base case by successfully interpolating the polynomial n^3. Fairly enough that the next curiosity would be

Question 1.2. *Well, if the relation* (1.5) *true for the polynomial n^3, then is it true that* (1.5) *can be generalized for higher powers, e.g. for n^4 or n^5 similarly?*

That was the next question, however without any expectation of the final form of generalized formula. Long story short, the answer to this question is also *Yes*, by utilizing certain approaches in terms of systems of linear equations or recurrence formula, which is discussed.

Let us begin from the background and history overview of systems of linear equations approach.

References

[1] Meijering, Erik. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, 2002. https://infoscience.epfl.ch/record/63085/files/meijering0201.pdf.

[2] Alan F. Beardon. Sums of powers of integers. *The American mathematical monthly*, 103(3):201–213, 1996. https://les-mathematiques.net/vanilla/uploads/editor/5w/3wm5emwr0i7p.pdf.

[3] Abramowitz, Milton and Stegun, Irene A. and Romer, Robert H. Handbook of mathematical functions with formulas, graphs, and mathematical tables, 1988.

[4] Petro Kolosov. Numerical triangle, row sums give third power, Entry A287326 in The On-Line Encyclopedia of Integer Sequences, 2017. https://oeis.org/A287326.

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