SURPRISING POLYNOMIAL IDENTITIES ARISING FROM A CLASSICAL INTERPOLATION PROBLEM

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Abstract.

1. The problem of interpolation of cubes

Back then, in 2016 being a student at the faculty of mechanical engineering, I remember myself playing with finite differences of the polynomial n^3 over the domain of natural numbers $n \in \mathbb{N}$ having at most $0 \leq n \leq 20$ values. Looking to the values in my finite difference tables, the first and very naive question that came to my mind was

Question 1.1. *Is it possible to re-assemble the value of the polynomial n^3 backwards having its finite differences?*

The answer to this question is certainly *Yes*, by utilizing interpolation methods. Interpolation is a process of finding new data points based on the range of a discrete set of known data points. It has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

At that time, in 2016, I was a first-year mechanical engineering undergraduate. Therefore, due to lack of knowledge in mathematics, I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. *All mathematical laws and relations exist from the very beginning, but we only reveal and describe them*, I thought. That mindset truly inspired me so that my own mathematical journey began.

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Let's start considering the table of finite differences of the polynomial n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

First and foremost, we can observe that finite difference $\Delta(n^3)$ of the polynomial n^3 can be expressed through summation over n, e.g

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 \\
&\vdots
\end{aligned} \tag{1}$$

Finally reaching its generic form

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k \tag{2}$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term $\sum_{k=0}^{n} k$ reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Then our relation (2) immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n+n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k \tag{3}$$

However, as was said, I was not the experienced one mathematician back then, so that I reviewed the relation (2) from a little bit different perspective. Not following the convenient solution (3), I have introduced the explicit formula for cubes, using (1)

$$\begin{aligned} n^3 &= [1 + 6\cdot 0] + [1 + 6\cdot 0 + 6\cdot 1] + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2] + \cdots \\ &\quad + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + \cdots + 6\cdot (n-1)] \end{aligned} \tag{4}$$

Then, rearranging the terms in equation (4) so that it turns into summation in terms of $k(n-k)$

$$\begin{aligned} n^3 &= n + [(n-0)\cdot 6\cdot 0] + [(n-1)\cdot 6\cdot 1] + [(n-2)\cdot 6\cdot 2] + \cdots \\ &\quad \cdots + [(n-k)\cdot 6\cdot k] + \cdots + [1\cdot 6\cdot (n-1)] \end{aligned}$$

By applying compact sigma notation and moving n under summation because there is exactly n iteration, yields

$$n^3 = n + \sum_{k=1}^{n} 6k(n-k); \qquad n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{5}$$

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $6k(n-k)+1$. See the OEIS entry: A287326 [4].

Therefore, we have reached our base case by successfully interpolating the polynomial n^3. Fairly enough that the next curiosity would be

Question 1.2. *Well, if the relation (5) true for the polynomial n^3, then is it true that (5) can be generalized for higher powers, e.g. for n^4 or n^5 similarly?*

That was the next question, however without any expectation of the final form of generalized formula. Long story short, the answer to this question is also *Yes*, by utilizing certain approaches in terms of systems of linear equations or recurrence formula, which is discussed.

Let us begin from the background and history overview of systems of linear equations approach.

2. System of linear equations approach

Soon enough my question (1.2) got attention from other people. In 2018, Albert Tkaczyk has published two of his works [5, 6] showing the cases for polynomials n^5, n^7 and n^9 that were obtained similarly as (5). In short, it appears that relation (5) could be generalized for any non-negative odd power $2m+1$ solving a system of linear equations. It was proposed that the case for n^5 has explicit form

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C \right]$$

where A, B, C are yet-unknown coefficients. Denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ to reach the form of a compact double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

Observing the equation above, the potential form of generalized odd-power identity becomes more obvious. To evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ it is necessary construct and solve a system of linear equations following the process

$$\begin{aligned}
n^5 &= \sum_{r=0}^{2} \mathbf{A}_{2,r} \sum_{k=1}^{n} k^r (n-k)^r \\
&= \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2
\end{aligned}$$

Expand the terms $\sum_{k=1}^{n} k^r (n-k)^r$ applying the Faulhaber's formula [2] to get the equation

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5) \right] - n^5 = 0$$

Multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0} n + 5\mathbf{A}_{2,1}(-n+n^3) + \mathbf{A}_{2,2}(-n+n^5) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} n + 5\mathbf{A}_{2,1} n^3 - \mathbf{A}_{2,2} n + \mathbf{A}_{2,2} n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1} n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases}
30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\
\mathbf{A}_{2,1} = 0 \\
\mathbf{A}_{2,2} - 30 = 0
\end{cases}$$

Solving it, we get

$$\begin{cases}
\mathbf{A}_{2,2} = 30 \\
\mathbf{A}_{2,1} = 0 \\
\mathbf{A}_{2,0} = 1
\end{cases}$$

So that the odd-power identity holds

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true by arranging the terms $30k^2(n-k)^2+1$ over $0 \leq k \leq n$ as tabular. See the OEIS sequence [7]

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $30k^2(n-k)^2+1$. See the OEIS entry A300656 [7].

Now we can see that the relation (5) we got via interpolation of cubes can be generalized for all non-negative odd-powers $2m+1$ by constructing and solving a certain system of linear equations. Therefore, the generalized form of odd-power identity has the form

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \tag{6}$$

where $\mathbf{A}_{m,r}$ are real coefficients. In more details, the identity (6) is discussed separately in [8, 9].

As one more example, let be $m = 3$ so that we have the following relation defined by (6)

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n+n^5)\right] + \mathbf{A}_{m,3}\left[\frac{1}{420}(-10n+7n^3+3n^7)\right] - n^7 = 0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{2,1}(-n+n^3) + 14\mathbf{A}_{2,2}(-n+n^5) + \mathbf{A}_{3,3}(-10n+7n^3+3n^7) - 420n^7 = 0$$

Expanding brackets and rearranging the terms gives

$$\begin{aligned} 420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5 \\ - 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0 \end{aligned}$$

Combining the common terms yields

$$n(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3})$$

$$+ n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5 14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} = 0 \\ 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} = 0 \\ \mathbf{A}_{3,2} - 30 = 0 \\ 3\mathbf{A}_{3,3} - 420 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \mathbf{A}_{3,2} = 0 \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1} + 10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

So that odd-power identity (6) holds

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms $140k^3(n-k)^3 - 14k(n-k) + 1$ over $0 \leq k \leq n$ as the OEIS sequence A300785 [10] shows.

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	127	1					
3	1	1093	1093	1				
4	1	3739	8905	3739	1			
5	1	8905	30157	30157	8905	1		
6	1	17431	71569	101935	71569	17431	1	
7	1	30157	139861	241753	241753	139861	30157	1

Table 4. Values of $140k^3(n-k)^3 - 14k(n-k) + 1$. See the OEIS entry A300785 [10].

However, constructing and solving a system of linear equations for every odd-power $2m+1$ requires a huge effort

Assumption 2.1. *There must be a formula that generates a set of real coefficients* $\mathbf{A}_{m,0}, \mathbf{A}_{m,0}, \dots \mathbf{A}_{m,m}$ *for each fixed* m *such that*

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r$$

I thought.

3. Recurrence relation approach

As it turned out, that assumption was correct. So that I reached MathOverflow community in search of answers that arrived quite shortly.

In [11], Dr. Max Alekseyev has provided a complete and comprehensive formula to calculate coefficient $\mathbf{A}_{m,r}$ for each natural m, r such that $m \geq 0$ and $0 \leq r \leq m$. The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate the $\mathbf{A}_{m,r}$ recursively, taking the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ by using backtracking, continuing similarly up to $\mathbf{A}_{m,0}$.

Before we consider the derivation of recurrent formula for coefficients $\mathbf{A}_{m,r}$, a few aspects regarding the Faulhaber's formula [2] should be discussed

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$$

it is important to notice that iteration step j is bounded by the value of power p, while the upper index of the binomial coefficient $\binom{p+1}{j}$ is $p+1$. Therefore, we can omit summation bounds letting j run over infinity by applying the following on the Faulhaber's formula.

$$\begin{aligned}\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} &= \left[\frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1} \\ &= \left[\frac{1}{p+1} \sum_{j} \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}\end{aligned}$$

Now we are good to go through the entire derivation of the recurrent formula for coefficients $\mathbf{A}_{m,r}$.

By applying Binomial theorem $(n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t$ and Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1} \sum_j \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}$, we get

$$\begin{aligned}&\sum_{k=1}^{n} k^r (n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \\ &= \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \left[\frac{1}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{t+r+1-j} - B_{t+r+1}\right] \\ &= \sum_{t=0}^{r} \binom{r}{t} \left[\frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t}\right] \\ &= \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right] \\ &= \left[\sum_{j,t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j}\right] - \left[\sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]\end{aligned}$$

Rearranging terms yields

$$\left[\sum_j B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right] - \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right] \tag{7}$$

We can notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j = 0 \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1} & \text{if } j > 0 \end{cases} \tag{8}$$

An elegant proof of the binomial identity (8) is presented in [12].

In particular, equation (8) is zero for $0 < t \leq j$. In order to apply (8), we have to move $j = 0$ out of summation in (7) to avoid division by zero in $\frac{(-1)^r}{j}$, which yields

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j \geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

Now we do not care about division by zero in $\frac{(-1)^r}{j}$ so that simplifying above equation by using (8) yields

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j \geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1} \right]}_{(\star)} - \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms across two sums

$$\begin{aligned} \sum_{k=1}^{n} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by (6), we obtain the following relation for polynomials in n

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r,\, \text{odd } \ell} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by k we get

$$\sum_{r} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r,\, k} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1}$$

Taking the coefficient of n^{2m+1} we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

because $\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1$.

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0$$

because we focus on sum $2\sum_{r,\, k} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1}$, in particular on n^{2k+1} and binomial coefficient $\binom{r}{2k+1}$. For instance, if we have to get coefficient of n^{2d+1} in range $\frac{m}{2} \leq d < m$, we set $d = m-1$, thus we have to get coefficient of $m-1$ in $2\sum_{r,\, k} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1}$. Therefore, we set $k = m-1$ and $r = m-1$ which leads that $\binom{r}{2k+1} = \binom{m-1}{2m-1} = 0$, so that $\mathbf{A}_{m,m-1} \frac{1}{(2m-1)\binom{2m-2}{m-1}} n^{2m-1} = 0$. Same applies for every d in the range $\frac{m}{2} \leq d < m$, because $r = \frac{m}{2}$ and $k = \frac{m}{2}$ means that $\binom{r}{2k+1} = \binom{\frac{m}{2}}{m+1} = 0$.

To summarize, the value of k should be in range $k \leq \frac{d-1}{2}$ so that binomial coefficient $\binom{d}{2k+1}$ is non-zero.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}$, iterating consecutively over $s = 1, 2, \ldots$ by using previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient $\mathbf{A}_{m,r}$

Definition 3.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r}\sum_{d\geq 2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{9}$$

where B_t are Bernoulli numbers [13]. It is assumed that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 5. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [14, 15].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

Let be a theorem

Theorem 3.2. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively by (9).

Finally, we got our road to definition of $2m+1$-degree polynomial $\mathbf{P}_b^m(x)$. Introducing the parameter b to the upper summation bound of the equation (3.2), we have the definition

Definition 3.3. *(Polynomial $\mathbf{P}_b^m(x)$ of degree $2m+1$.)*

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{10}$$

where $\mathbf{A}_{m,r}$ are real coefficients (9). A comprehensive discussion on the polynomial $\mathbf{P}_b^m(x)$ as well as its properties can be found at [16]. For example,

$$\mathbf{P}_b^0(x) = b,$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x,$$

$$\begin{aligned}\mathbf{P}_b^2(x) &= 10b^3 - 15b^4 + 6b^5 - 15b^2x + 30b^3x - 15b^4x \\ &\quad + 5bx^2 - 15b^2x^2 + 10b^3x^2\end{aligned}$$

$$\begin{aligned}\mathbf{P}_b^3(x) &= -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x \\ &\quad + 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 + 35b^2x^3 - 70b^3x^3 + 35b^4x^3\end{aligned}$$

In 2023, Albert Tkaczyk yet again extended the theorem (3.2) to the so-called three dimension case so that it gives polynomials of the form n^{3l+2} at [17].

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