UNEXPECTED POLYNOMIAL IDENTITIES ARISING FROM A CLASSICAL INTERPOLATION PROBLEM

PETRO KOLOSOV

Abstract. This paper originates from a classical interpolation problem: how to reconstruct a cubic polynomial from its finite differences. The investigation leads to an unexpected identity expressing n^3 as a sum involving products of linear terms, revealing a hidden structure. This identity serves as a base case of a more general pattern. By solving systems of linear equations, a family of odd power identities n^{2m+1} involving special numbers $\mathbf{A}_{m,r}$ and symmetric bivariate sums. A recurrence relation for these coefficients $\mathbf{A}_{m,r}$ is obtained via a specific generating function, enabling recursive construction of the coefficients. The main results include several odd power identities, an identity for binomial version, and identities for sums of odd powers.

Contents

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1. Discussion on interpolation of cubes

This is the story of unexpected results, obtained by an amateur student with a deep curiosity for mathematics. Although, not a specialist in mathematics, our young explorer always possessed a strong sense of mathematical beauty and aesthetics. The mathematical knowledge of the individual was limited by undergraduate level course, which includes the basics of matrix operations, basic calculus, and elementary linear algebra. One day, our student found himself observing the tables of finite differences, precisely finite differences of cubes

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of n^3.

The first question that visited curious mind was

Question 1.1. *How to reconstruct the value of n^3 from the values of its finite differences?*

Precisely, the inquiry is to find a way to reconstruct the values of the sequence $\{0, 1, 8, 27, 64, \ldots\}$ given the values of finite differences in the table (1), which refers to interpolation.

In its essence, the problem is so old that it can be traced back to ancient Babylonian and Greek times, several centuries BC and first centuries AD [1]. The process of finding new data points based on the range of a discrete set of known data points is called interpolation. Interpolation, as we know it today, was developed in 1674–1684 by Isaac Newton in his works referenced as foundation of classical interpolation theory [2]. For instance, Newton's interpolation formula addresses the question (1.1) immediately, because

$$27 = 6\binom{3}{3} + 6\binom{3}{2} + 1\binom{3}{1} + 0\binom{3}{0} = 6 + 18 + 3$$

where $6, 6, 1, 0$ is the first row in table (1). In general,

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + 1\binom{n}{1} + 0\binom{n}{0}$$

because $f(x) = \sum_{k=0}^{d} \Delta^{d-k} f(0)\binom{x}{d-k}$, see [3, p. 190].

Great! But there is one thing, the student who has risen the question (1.1) had no clue about interpolation theory at all. What was decided then? Exactly, our inquirer has decided to find a solution himself, being driven by the pure feeling of mystery. His mind was occupied by only a single thought: *All mathematical truths exist timelessly, we only reveal and describe them.* That mindset inspired our student to start his own mathematical journey.

By observing the table of finite differences (1) we can notice that the first finite difference of cubes may be expressed in terms of its third finite differences $\Delta^3(n^3) = 6$, as follows

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 \\
&\vdots \\
\Delta(n^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6n
\end{aligned}$$

By using sigma notation, we get

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k$$

However, there is a more beautiful way to prove that $\Delta(n^3) = 1 + 6 \sum_{k=0}^{n} k$. We refer to one of the finest articles in the area of polynomials and power sums, that is *Johann Faulhaber and sums of powers* written by Donald Knuth [4]. Indeed, this article is a great mean to reach piece of mind in mathematics. We now focus on the odd power identities shown at [4, p. 9]

$$\begin{aligned}
n^1 &= \binom{n}{1} \\
n^3 &= 6\binom{n+1}{3} + \binom{n}{1} \\
n^5 &= 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}
\end{aligned}$$

It is quite interesting that the identity in terms of triangular numbers $\binom{n+1}{2}$ and finite differences of cubes becomes more obvious

$$\Delta n^3 = (n+1)^3 - n^3 = 6\binom{n+1}{2} + \binom{n}{0}$$

It easy to see that

$$\Delta n^3 = \left[6\binom{n+2}{3} + \binom{n+1}{1}\right] - \left[6\binom{n+1}{3} + \binom{n}{1}\right] = 6\binom{n+1}{2} + \binom{n}{0}$$

because $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Moreover, the concept above allows us to reach N-fold power sums $\sum^N k^{2m+1}$ or finite differences $\Delta^N k^{2m+1}$ by simply altering binomial coefficients indexes. Quite strong and impressive.

We can observe that triangular numbers $\binom{n+1}{2}$ are equivalent to

$$\binom{n+1}{2} = \sum_{k=0}^{n} \binom{k}{1} = \sum_{k=0}^{n} k$$

because $\binom{n+1}{m+1} = \sum_{k=0}^{n} \binom{k}{m}$. This leads us to the fact that

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6\sum_{k=0}^{n} k$$

An experienced mathematician would immediately notice a spot to apply Faulhaber's formula [5] to get the closed form of the sum $\sum_{k=0}^{n} k$

$$\sum_{k=0}^{n} k = \frac{1}{2}(n+n^2)$$

Thus, the finite difference $\Delta(n^3)$ takes the well-known form, which matches Binomial theorem [6]

$$\Delta(n^3) = 1 + 6\left[\frac{1}{2}(n+n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k$$

And...that could be the end of the story, isn't it? Because all what remains is to say that

$$n^3 = \sum_{k=0}^{n-1} (k+1)^3 - k^3 = \sum_{k=0}^{n-1} \left(1 + 6\sum_{t=0}^{k} t\right) = \sum_{k=0}^{n-1} 1 + 3k + 3k^2$$

Thus, the formula for polynomial n^3 is derived successfully, and thus, our protégée's question (1.1) is answered positively. Because we have successfully found the function that matches n^3 from the values of its finite differences from the table (1).

However, not this time. Luckily enough (say), the student who has stated the question (1.1) wasn't really aware of the approaches above neither. What a lazy student! Probably, that's exactly the case when unawareness leads to a fresh sight to classical questions, leading to unexpected results and new insights. Instead, our investigator spotted a little bit different pattern in $\Delta n^3 = 6\binom{n+1}{2} + \binom{n}{0}$.

Consider the polynomial n^3 as sum of its finite differences

$$\begin{aligned} n^3 = {} & [1 + 6 \cdot 0] \\ & + [1 + 6 \cdot 0 + 6 \cdot 1] \\ & + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\ & + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

We can observe that the term 1 appears n times, the item $6 \cdot 0$ appears $n - 0$ times, the item $6 \cdot 1$ appears $n - 1$ times and so on. By rearranging recurring common terms

$$\begin{aligned} n^3 = n & + [(n-0) \cdot 6 \cdot 0] \\ & + [(n-1) \cdot 6 \cdot 1] \\ & + [(n-2) \cdot 6 \cdot 2] + \cdots \\ & + [(n-k) \cdot 6 \cdot k] + \cdots \\ & + [1 \cdot 6 \cdot (n-1)] \end{aligned}$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

We can freely move the term n under the summation, because there are exactly n iterations. Therefore,

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$$

By inspecting the expression $6k(n-k)+1$, we can notice that it is symmetric over k. Let be $T_1(n,k) = 6k(n-k)+1$ then

$$T_1(n,k) = T_1(n,n-k)$$

This symmetry allows us to alter summation bounds easily. Hence,

$$n^3 = \sum_{k=1}^{n} 6k(n-k)+1$$

By arranging the values of $T_1(n,k)$ as a triangular array, we see that cube identities indeed are true

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $T_1(n,k) = 6k(n-k)+1$. See the sequence A287326 in OEIS [7].

The following recurrence holds for $T_1(n,k)$

$$T_1(n,k) = 2T_1(n-1,k) - T_1(n-2,k)$$

Which is indeed true, because

$$T_1(5,2) = 2 \cdot 25 - 13 = 37$$

Finally, our curious learner has reached the first milestone, by finding his own answer to the question (1.1) and the answer was positive. What an excitement it was! However, it wouldn't take long. Indeed, curiosity is not something that can be fulfilled completely, and thus new questions arise. Somehow, the inquirer got a strong feeling that something bigger, something even more general hides behind the identity $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$. That was quite intuitive. Fair enough that the next question was

Question 1.2. *Given that the identity $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$ holds for the polynomial n^3, can it be extended or generalized to higher-degree powers, such as n^4 or n^5, in a similar manner?*

However, this time it was not so easy for the young explorer to find identity for n^4 or n^5 by simply observing the tables of finite differences. The previous approach to express the difference of cubes Δn^3 in terms of $\Delta^3 n^3 = 6$ and then express the cubes as $n^3 = \sum_k 6k(n-k)+1$ — was not successful. Moreover, it wasn't even clear what is the generic form of an identity our student was looking for, a lot of concerns came from a simple interpolation task. Thus, the question (1.2) was shared with the mathematical community. And there was an answer.

2. A system of linear equations

In 2018, Albert Tkaczyk published two papers [8, 9] presenting analogous identities for polynomials n^5, n^7 and n^9 derived in a manner similar to $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$. Further, these results were polished and published in *Mathematical gazette* [10]. Tkaczyk assumed that the identity for n^5 takes the following explicit form

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C\right]$$

where A, B, C are yet-unknown coefficients. We denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ to reach the compact form of double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

By observing the equation above, the potential form of generalized odd-power identity becomes more obvious. One important note to add here, we define $0^x = 1$ for all x, see [3, p.

162]. This is because when $k = n$ and $r = 0$ the term $k^r(n-k)^r = n^0 \cdot 0^0$, thus we define $0^x = 1$ for all x.

To evaluate the set of coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ we construct and solve a certain system of linear equations, which is built as follows

$$n^5 = \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0(n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1(n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2(n-k)^2$$

By expanding the sums $\sum_{k=1}^{n} k^r(n-k)^r$ using Faulhaber's formula [5], we get an equation

$$\mathbf{A}_{2,0}n + \mathbf{A}_{2,1}\left[\frac{1}{6}(n^3-n)\right] + \mathbf{A}_{2,2}\left[\frac{1}{30}(n^5-n)\right] - n^5 = 0$$

By multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(n^3-n) + \mathbf{A}_{2,2}(n^5-n) - 30n^5 = 0$$

By expanding the brackets and rearranging the terms

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

By combining the common terms, we obtain

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore,

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} & = 0 \\ \mathbf{A}_{2,1} & = 0 \\ \mathbf{A}_{2,2} - 30 & = 0 \end{cases}$$

By solving the system above, we evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$

$$\begin{cases} \mathbf{A}_{2,2} & = 30 \\ \mathbf{A}_{2,1} & = 0 \\ \mathbf{A}_{2,0} & = 1 \end{cases}$$

Thus, the identity for n^5

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

Again, the terms $30k^2(n-k)^2 + 1$ are symmetric over k. Let be $T_2(n,k) = 30k^2(n-k)^2 + 1$ then

$$T_2(n,k) = T_2(n, n-k)$$

By arranging the values of $T_2(n,k)$ as a triangular array, we see that the identity for n^5 is indeed true

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $T_2(n,k) = 30k^2(n-k)^2 + 1$. See the sequence A300656 in OEIS [11].

The following recurrence holds for $T_2(n,k)$

$$T_2(n,k) = 3T_2(n-1,k) - 3T_2(n-2,k) + T_2(n-3,k)$$

Which is indeed true because

$$T_2(6,2) = 3 \cdot 1081 - 3 \cdot 481 + 271 = 1921$$

Thus, our curious learner who stated the question (1.2) got the answer, yet again green light was lit. This time, the answer contained even more than methodology to find a set of coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \ldots, \mathbf{A}_{2,2}$ — it contained a generic form of odd power identity n^{2m+1}

for any natural m. Hence, the part of questions-answers we discuss so far ends here, it is time to state a conjecture.

Conjecture 2.1. *There is a set of coefficients* $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ *such that*

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r$$

We already know that to identify the coefficients we have to build and solve a certain system of linear equations, however, we cannot perform any kind of induction on that, hence conjecture cannot be proven just by building and solving endless systems of linear equations. There must be a formula that evaluates the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ for every non-negative integer m — our young investigator thought.

3. Recurrence relation

In 2018, the recurrence relation [12] that evaluates the coefficients $\mathbf{A}_{m,r}$ for non-negative integer m was provided by Max Alekseyev, George Washington University. The main idea of Alekseyev's approach was to utilize a generating function to evaluate the set of coefficients $\mathbf{A}_{m,r}$ starting from the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ recursively, and so on up to $\mathbf{A}_{m,0}$. We utilize Binomial theorem $(n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t$ and a specific version of Faulhaber's formula [5] with upper bound $p+1$

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} = \frac{1}{p+1} \left[\sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1}$$

The reason we use the Faulhaber's formula above is because we tend to omit summation bounds, for simplicity. This helps us to collapse the common terms across complex sums, because now can extend the sum over all integers j, while only finitely many terms $\binom{p+1}{j}$ are non-zero, see also [13, p. 2]. Hence,

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \left[\sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1} \tag{1}$$

Now we expand the sum $\sum_{k=1}^{n} k^r(n-k)^r$ using Binomial theorem

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r}$$

By applying Faulhaber's formula (1), we obtain

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\left(\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j}\right) - \frac{B_{t+r+1}}{t+r+1}\right]$$

By moving the common term $\frac{(-1)^t}{t+r+1}$ out of brackets

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\left[\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right]$$

By expanding the brackets

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r = &\left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] \\ &-\left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]\end{aligned}$$

By moving the sum in j and omitting summation bounds in t

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_{j,t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

By rearranging the sums we obtain

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r = &\left[\sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] \\ &-\left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]\end{aligned} \tag{2}$$

We can notice that

Lemma 3.1. *For integers r, j*

$$\sum_{t}\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases}\frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j=0\\ \frac{(-1)^r}{j}\binom{r}{2r-j+1} & \text{if } j>0\end{cases}$$

Proof. An elegant proof is done by Markus Scheuer in [14]. □

In particular, the sum above is zero for $0 < j \le r$. To simplify (2) using lemma (3.1), we have to move $j = 0$ out of the sum Σ in (2) to avoid division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j=1}^{\infty} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right] - \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

Hence, we simplify the equation (2) by using lemma (3.1)

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j=1}^{\infty} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1}\right] - \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

By setting $\ell = 2r - j + 1$ to the sum $\sum_{j=1}^{\infty}$, and $\ell = r - t$ to the sum $\sum_t$, we collapse the common terms across two sums, thus

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\end{aligned}$$

By replacing odd ℓ by $\ell = 2k+1$, and by simplifying 2's, we get

Proposition 3.2 (Bivariate Faulhaber's Formula)**.**

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{k=0}^{\infty} \frac{(-1)^r}{r-k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by the odd-power identity in conjecture (2.1), we obtain the following relation for polynomials in n

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{r=0}^{m} \sum_{k=0}^{\infty} \mathbf{A}_{m,r} \frac{(-1)^r}{r-k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1} \tag{3}$$

Basically, the relation (3) is the generating function we utilize to evaluate the values of $\mathbf{A}_{m,m}, \mathbf{A}_{m,m-1}, \ldots, \mathbf{A}_{m,0}$. We now fix the unused values of $\mathbf{A}_{m,r}$ so that $\mathbf{A}_{m,r} = 0$ for every $r < 0$ or $r > m$.

Taking the coefficient of n^{2m+1} in (3) yields

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

because $\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} = 1$.

That's may not be immediately clear why the coefficient of n^{2m+1} is $(2m+1)\binom{2m}{m}$. To extract the coefficient of n^{2m+1} from the expression (3), we isolate the relevant terms by setting $r = m$ in the first sum, and $k = m$ in the second sum. This gives

$$\begin{aligned}
&[n^{2m+1}]\left(\sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \sum_{r=0}^{m}\sum_{k=0}^{\infty} \mathbf{A}_{m,r}\frac{(-1)^r}{r-k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1} - n^{2m+1}\right) \\
&\qquad = \mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} + \sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{(-1)^r}{r-m}\binom{r}{2m+1}B_{2r-2m} - 1
\end{aligned}$$

We observe that the sum

$$\sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{(-1)^r}{r-m}\binom{r}{2m+1}B_{2r-2m}$$

does not contribute to the determination of the coefficients $\mathbf{A}_{m,r}$, because the binomial coefficient $\binom{r}{2m+1}$ vanishes for all $r \leq m$. Consequently, all terms in the sum are zero. Thus,

$$\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0 \implies \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d \leq m-1$ in (3) gives

$$\begin{aligned}
&[n^{2d+1}]\left(\sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \sum_{r=0}^{m}\sum_{k=0}^{\infty} \mathbf{A}_{m,r}\frac{(-1)^r}{r-k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1} - n^{2m+1}\right) \\
&\qquad = \mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + \sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{(-1)^r}{r-d}\binom{r}{2d+1}B_{2r-2d}.
\end{aligned}$$

For every $\frac{m}{2} \le d$, the binomial coefficient $\binom{r}{2d+1}$ vanishes, because for all $r \le m$ holds $r < 2d+1$. As a particular example, when $r = m$ and $d = \frac{m}{2}$, we have

$$\binom{m}{m+1} = 0.$$

Therefore, the entire sum involving $\binom{r}{2d+1}$ vanishes, and we conclude

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} = 0 \implies \mathbf{A}_{m,d} = 0.$$

Hence, for all integers d such that $\frac{m}{2} \le d \le m-1$, the coefficient $\mathbf{A}_{m,d} = 0$. In contrast, for values $d \le \frac{m}{2} - 1$, the binomial coefficient $\binom{r}{2d+1}$ can be nonzero; for instance, if $r = m$ and $d = \frac{m}{2} - 1$, then

$$\binom{m}{m-1} \neq 0,$$

allowing the corresponding terms to contribute to the determination of $\mathbf{A}_{m,d}$.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \le d < \frac{m}{2}$ in (3), we obtain

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d} B_{2m-2d} = 0.$$

Solving for $\mathbf{A}_{m,d}$ yields

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!\,d!\,m!\,(m-2d-1)!} \cdot \frac{1}{m-d} B_{2m-2d}.$$

Proceeding recursively, we can compute each coefficient $\mathbf{A}_{m,r}$ for integers r in the ranges $\frac{m}{2^{s+1}} \le r < \frac{m}{2^s}$, for $s = 1, 2, \ldots$, by using previously computed values $\mathbf{A}_{m,d}$ for $d > r$, via the relation

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}.$$

Finally, we define the following recurrence relation for coefficients $\mathbf{A}_{m,r}$

Proposition 3.3. *For integers m and r*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \le r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

where B_t are Bernoulli numbers [15]. It is assumed that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 4. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [16, 17].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$.
- $\mathbf{A}_{m,r} = 0$ for $r < 0$ and $r > m$.
- $\mathbf{A}_{m,r} = 0$ for $m < 0$.
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \le r < m$.
- $\mathbf{A}_{m,0} = 1$ for $m \ge 0$.
- $\mathbf{A}_{m,r}$ are all integers up to row $m = 11$.
- Row sums: $\sum_{r=0}^{m}\mathbf{A}_{m,r} = 2^{2m+1} - 1$.

For instance,

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k) + 1$$

$$n^9 = \sum_{k=1}^{n} 630k^4(n-k)^4 - 120k(n-k) + 1$$

$$n^{11} = \sum_{k=1}^{n} 2772k^5(n-k)^5 + 660k^2(n-k)^2 - 1386k(n-k) + 1$$

$$n^{13} = \sum_{k=1}^{n} 51480k^7(n-k)^7 - 60060k^3(n-k)^3 + 491400k^2(n-k)^2 - 450054k(n-k) + 1$$

4. Main results

Thus, the conjecture (2.1) is true

Theorem 4.1 (Odd power identity). *There is a set of coefficients* $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ *such that*

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r$$

Definition 4.2 (Bivariate sum T_m). *For integers* n, k *and* $m \geq 0$

$$T_m(n,k) = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

Proposition 4.3 (Symmetry of T_m). *For integers* n *and* k

$$T_m(n,k) = T_m(n, n-k)$$

Proposition 4.4 (Backward Recurrence for T_m). *For non-negative integers* m *and* n

$$T_m(n,k) = \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n-t, k)$$

Proof. The polynomial $T_m(n,k)$ is a polynomial of degree m in n. Thus, the backward difference with respect to n is $\nabla^{m+1} T_m(n,k) = \sum_{t=0}^{m+1} (-1)^t \binom{m+1}{t} T_m(n-t,k) = 0$. By isolating $(-1)^0 \binom{m+1}{0} T_m(n-0,k)$ yields $T_m(n,k) = (-1) \sum_{t=1}^{m+1} (-1)^t \binom{m+1}{t} T_m(n-t,k)$. □

Proposition 4.5 (Odd power backward decomposition)**.** *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n-t,k)$$

Proof. Direct consequence of (4.1) and backward recurrence (4.4). □

Proposition 4.6 (Odd power backward decomposition shifted)**.** *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n-t,k)$$

Proof. Direct consequence of (4.1), backward recurrence (4.4), and symmetry (4.3). □

Corollary 4.7 (Odd power backward decomposition $m-1$)**.**

$$n^{2m-1} = \sum_{k=1}^{n} \sum_{t=1}^{m} (-1)^{t+1} \binom{m}{t} T_{m-1}(n-t,k)$$

Proof. By setting $m \to m-1$ to (4.5). □

Proposition 4.8 (Forward Recurrence for T_m)**.**

$$T_m(n,k) = \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n+t,k)$$

Proof. The polynomial $T_m(n,k)$ is a polynomial of degree m in n. Thus, the forward difference with respect to n is $\Delta^{m+1} T_m(n,k) = \sum_{t=0}^{m+1} (-1)^t \binom{m+1}{t} T_m(n+t,k) = 0$. By isolating $(-1)^0 \binom{m+1}{0} T_m(n-0,k)$ yields $T_m(n,k) = (-1) \sum_{t=1}^{m+1} (-1)^t \binom{m+1}{t} T_m(n+t,k)$. □

Proposition 4.9 (Odd power forward decomposition)**.** *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n+t,k)$$

Proof. Direct consequence of (4.1) and forward recurrence (4.8). □

Proposition 4.10 (Odd power forward decomposition shifted)**.** *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n+t, k)$$

Proof. Direct consequence of (4.1), forward recurrence (4.8), and symmetry (4.3). □

Proposition 4.11 (Central Recurrence for T_m)**.**

$$T_m\left(n + \frac{m}{2}, k\right) = \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m\left(n + \frac{m}{2} - t, k\right)$$

Proof. The polynomial $T_m(n,k)$ is a polynomial of degree m in n. Thus, the central difference with respect to n is $\delta^{m+1} T_m(n,k) = \sum_{t=0}^{m+1} (-1)^t \binom{m+1}{t} T_m\left(n + \frac{m}{2} - t, k\right) = 0$. By isolating $(-1)^0 \binom{m+1}{0} T_m\left(n + \frac{m}{2} - 0, k\right)$ yields $T_m\left(n + \frac{m}{2} - 0, k\right) = (-1) \sum_{t=1}^{m+1} (-1)^t \binom{m+1}{t} T_m(n+t, k)$. □

Proposition 4.12 (Odd power binomial form)**.** *For integers n and a such that $n - 2a \geq 0$*

$$(n-2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=a+1}^{n-a} \mathbf{A}_{m,r} (k-a)^r (n-k-a)^r$$

Proof. By observing the summation limits we can see that k runs as $k = a+1, a+2, a+3, \ldots, a+n-a$, which implies that $(k-a) = 1, 2, 3, \ldots, n$. By observing the term $(n-k-a)$ we see that $(n-k-a) = n-1, n-2, n-3, \ldots, 0$. Thus, by reindexing the sum $(n-2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n-2a} \mathbf{A}_{m,r} (a+k-a)^r (n-(a+k)-a)^r$ the statement (4.12) is equivalent to (4.1) with setting $n \to n - 2a$. □

Corollary 4.13 (Odd power binomial form shifted)**.** *For integers n and a such that $n - 2a \geq 0$*

$$(n-2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=a}^{n-a-1} \mathbf{A}_{m,r} (k-a)^r (n-k-a)^r$$

Proposition 4.14 (Sum of odd powers)**.**

$$\sum_{t=1}^{n} t^{2m+1} = \sum_{t=1}^{n} \sum_{k=1}^{t} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

Proposition 4.15 (Sum of odd powers forward decomposition)**.** *For non-negative integers m and n*

$$\sum_{n=1}^{p} n^{2m+1} = \sum_{n=1}^{p} \sum_{k=1}^{n} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n+t,k)$$

Proposition 4.16 (Sum of odd powers backward decomposition)**.** *For non-negative integers m and n*

$$\sum_{n=1}^{p} n^{2m+1} = \sum_{n=1}^{p} \sum_{k=1}^{n} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n-t,k)$$

5. Related research

5.1. **Spline approximation for power function.** The paper [18] describes a remarkable result that follows from the odd power identity (4.1). As revealed, by introducing an additional parameter for upper summation bound in k to (4.1), the resulting family of polynomials approximate the odd power in some neighborhood of a fixed point.

$$P(m,X,N) = \sum_{r=0}^{m} \sum_{k=1}^{N} \mathbf{A}_{m,r} k^r (X-k)^r$$

For example,

$$P(2,X,0) = 0$$

$$P(2,X,1) = 30X^2 - 60X + 31$$

$$P(2,X,2) = 150X^2 - 540X + 512$$

$$P(2,X,3) = 420X^2 - 2160X + 2943$$

$$P(2,X,4) = 900X^2 - 6000X + 10624$$

The following image demonstrates the approximation of fifth power X^5 by $P(2,X,4) = 900X^2 - 6000X + 10624$

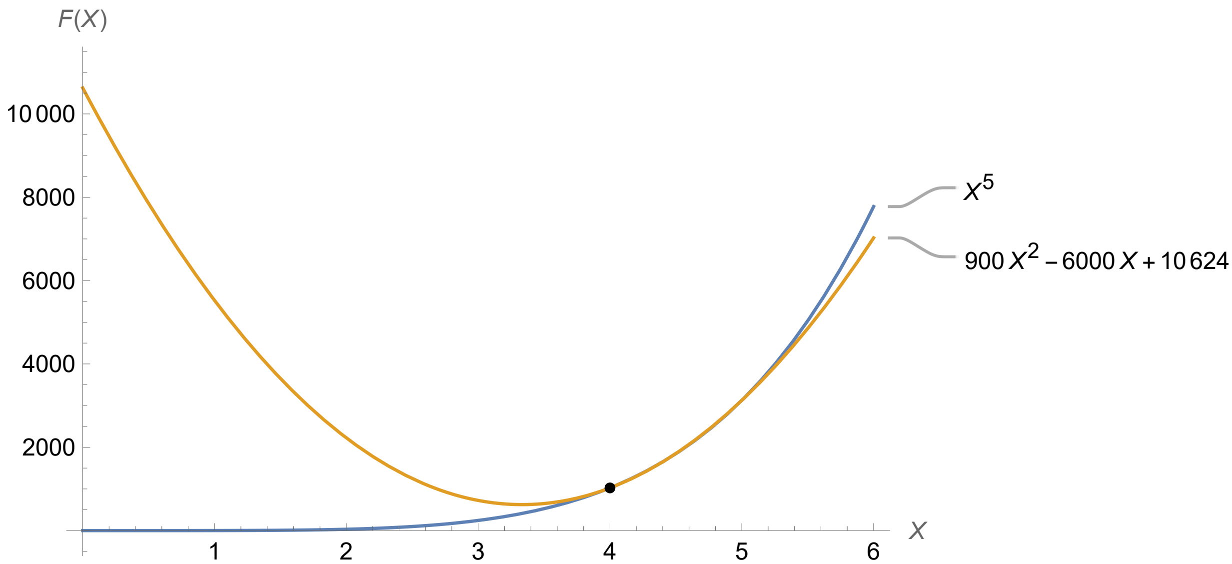

Figure 1. Approximation of fifth power X^5 by $P(2, X, 4)$. Convergence interval is $4.0 \leq X \leq 5.1$ with a percentage error $E < 1\%$.

Another remarkable observation is One more interesting observation arises by increasing the value of N in $P(m, X, N)$ while keeping m fixed. As N increases, the length of the convergence interval with the odd-power X^{2m+1} also increases. For instance,

- For $P(2, X, 4)$ and X^5, the convergence interval with a percentage error less than 1% is $4.0 \leq X \leq 5.1$, with a length $L = 1.1$
- For $P(2, X, 20)$ and X^5, the convergence interval with a percentage error less than 1% is $18.7 \leq X \leq 22.9$, with a length $L = 4.2$
- For $P(2, X, 120)$ and X^5, the convergence interval with a percentage error less than 1% is $110.0 \leq X \leq 134.7$, with a length $L = 24.7$

5.2. **Two-sided Faulhaber's formulas.** The paper [19] generalizes the proposition (3.2) to a new family of polynomials, namely two-sided Faulhaber-like formulas involving Bernoulli polynomials.

5.3. **Derivatives.** The paper [20] reveals a connection between ordinary derivatives of odd power and partial derivatives of the function

$$f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r$$

Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx} g_v(u)$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)'_x(u,u)$ evaluate in point (u,u) plus partial derivative $(f_v)'_z(u,u)$ evaluate in point (u,u)

$$\frac{d}{dx} g_v(u) = (f_v)'_x(u,u) + (f_v)'_z(u,u)$$

6. Future research

Several promising directions emerge from the findings of this manuscript:

- **Integration into mathematical literature.** The identities presented in this work do not appear in standard mathematical references, despite their elementary nature and apparent classical flavor. Notably, related sequences are absent from major repositories such as the OEIS. Future work should investigate the originality of these results and aim to contextualize them within the broader mathematical framework.
- **Extension of approximation methods.** The approximation technique developed in [18] is generalizable to a broader class of polynomials. In particular, by leveraging the symmetry property (4.3), one could explore alternative summation domains for the polynomials $P(m, X, N)$.
- **Combinatorial interpretation of $T_m(n,k)$.** The polynomial family $T_m(n,k)$, introduced in (4.2), currently lacks a clear combinatorial interpretation. Understanding its structural or enumerative significance would deepen insight into the algebraic identities presented.

- **Connection with finite differences and derivatives.** The binomial form of the odd power identity (4.12) offers a mechanism to express both finite differences and classical derivatives of odd powers in terms of the coefficients $\mathbf{A}_{m,r}$.
- **q-derivative representation.** The general identity (4.1) also suggests a natural expression for q-derivatives via the coefficients $\mathbf{A}_{m,r}$, potentially leading to a generalized notion of differentiation through limiting procedures.

7. Conclusions

This work began with a seemingly elementary interpolation problem and evolved into a broader investigation of polynomial identities involving odd powers. Starting from the finite differences of the cubic function, we uncovered a nontrivial identity that served as a base case for a family of structured decompositions of n^{2m+1}. These identities were expressed in terms of symmetric bivariate sums with recursively defined coefficients. By employing systems of linear equations and a generating function approach, we derived both closed-form expressions and recurrence relations for these identities. The results were further extended to include binomial forms of odd power identities and formulas for the sums of odd powers. Computational experiments in *Mathematica* confirmed all theoretical claims and provided a toolkit for further exploration. These findings not only contribute novel results to the theory of polynomial identities but also open pathways to related domains, such as approximation theory and calculus.

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Version: Local-0.1.0

Sources: github.com/kolosovpetro/surprising-polynomial-identities

Application 1: Mathematica programs

We support our theoretical findings with Wolfram Mathematica programs that verify the main results of this manuscript. All source code and computational notebooks are available in the `GitHub repository`. The repository includes the following files:

- `unexpected-polynomial-identities-classical-interpolation.m` — the package file where all Mathematica functions are defined. Load it into your session using `filename.m` or by evaluating the file with `Shift+Enter`.
- `unexpected-polynomial-identities-classical-interpolation.nb` — a working notebook that demonstrates the usage of these functions to validate the manuscript's results.

Below we list the Mathematica functions with their corresponding mathematical statements:

Mathematica Function	**Validates / Prints**
`A[m, r]`	Coefficient $\mathbf{A}_{m,r}$ (Definition 3.3)
`OddPowerIdentity[n, m]`	Theorem 4.1
`OddPowerIdentitySimplified[n, m]`	Expanded form of Theorem 4.1
`PrintTriangleA[m]`	Triangle of $A_{n,k}$ values
`BivariateSumT[m, n, k]`	Definition 4.2
`BackwardRecurrenceForT[m, n, k]`	Proposition 4.4
`ForwardRecurrenceForT[m, n, k]`	Proposition 4.8
`CentralRecurrenceForT[m, n, k]`	Proposition 4.11
`OddPowerBackwardDecomposition[n, m]`	Proposition 4.5
`OddPowerBackwardDecompositionMMinus1[n, m]`	Corollary 4.7
`OddPowerForwardDecomposition[n, m]`	Proposition 4.9
`OddPowerForwardDecompositionShifted[n, m]`	Corollary 4.10
`OddPowerBinomialForm[m, n, a]`	Proposition 4.12
`OddPowerBinomialFormShifted[m, n, a]`	Corollary 4.13
`TableFormBackwardRecurrenceForT[m, rows]`	Triangle view of Proposition 4.4
`TableFormForwardRecurrenceForT[m, rows]`	Triangle view of Proposition 4.8
`TableFormCentralRecurrenceForT[m, rows]`	Triangle view of Proposition 4.11
`TableFormBivariateSumT[m, rows]`	Triangle view of $T_m(n, k)$

To test and experiment with these identities computationally, load the package and call any of the functions listed above with appropriate parameters.

Application 2: Examples of coefficients A

Consider the proposition (3.3) of the coefficients $\mathbf{A}_{m,r}$, it can be written as

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Let be the definition of the polynomial $T(d,r)$

Definition 7.1.

$$T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Example 7.2. *Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$*

$$\mathbf{A}_{2,2} = 5\binom{4}{2} = 30$$

Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the coefficient $\mathbf{A}_{2,0}$ is

$$\mathbf{A}_{2,0} = \sum_{d \geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) = \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0)$$

$$= 30 \cdot \frac{1}{30} = 1$$

Example 7.3. *Let be $m = 3$ so that first we get $\mathbf{A}_{3,3}$*

$$\mathbf{A}_{3,3} = 7\binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally, the coefficient $\mathbf{A}_{3,0}$ *is*

$$\mathbf{A}_{3,0} = \sum_{d\geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 7.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 4$. *The value of the coefficient* $\mathbf{A}_{4,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d\geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ *is*

$$\mathbf{A}_{4,0} = \sum_{d\geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 7.5. *Let be* $m = 5$ *so that first we get* $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11\binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ *and* $\mathbf{A}_{5,3} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $3 \leq d < 5$. *The value of the coefficient* $\mathbf{A}_{5,2}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,2} = \sum_{d\geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,1} = \sum_{d\geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally, the coefficient $\mathbf{A}_{5,0}$ *is*

$$\begin{aligned}\mathbf{A}_{5,0} &= \sum_{d\geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1\end{aligned}$$

Email address: kolosovp94@gmail.com

Software Developer, DevOps Engineer

URL: https://kolosovpetro.github.io