SURPRISING POLYNOMIAL IDENTITIES ARISING FROM A CLASSICAL INTERPOLATION PROBLEM

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Abstract.

1. The problem of interpolation of cubes

Once there was a curious student, curious in general and especially in mathematics. He was not an educated mathematician, but he was strongly captured by the mathematical beauty and aesthetics.

One day this student was having fun with tables of finite differences, precisely finite differences of cubes. By observing the table

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of n^3.

The first question that triggered his mind was

Question 1.1. *How to reconstruct the value of n^3 from its finite differences?*

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The question is quite old, it can be traced back to ancient Babylonian and Greek times, several centuries BC and first centuries AD [citation]. The process is called interpolation which is a process of finding new data points based on the range of a discrete set of known data points. Interpolation we know nowadays was developed in 1674–1684 by Issac Newton in his works referenced as foundation of classical interpolation theory [citation].

Great! But there is one thing, the one who rised the question (1.1) had no clue about all this. What he decided then? Exactly, he decided to try to re-invent interpolation formula himself, fueled by the purest feeling of mystery. His mind was occupied by only a single thought: *All mathematical truths exist timelessly, we only reveal and describe them.* That mindset inspired our student to start his own mathematical journey.

By observing the table of finite differences (1) we can notice that the first order finite difference of cubes may be expressed in terms of its third order finite difference $\Delta^3(n^3) = 6$, as follows

$$\begin{aligned}
\Delta(0^3) &= 1 + 6\cdot 0 \\
\Delta(1^3) &= 1 + 6\cdot 0 + 6\cdot 1 \\
\Delta(2^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 \\
\Delta(3^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 \\
&\vdots \\
\Delta(n^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + \cdots + 6n
\end{aligned} \tag{1.1}$$

By using sigma notation for sums we get

$$\Delta(n^3) = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + \cdots + 6\cdot n = 1 + 6\sum_{k=0}^{n} k \tag{1.2}$$

An experienced mathematician would immediately notice a spot to apply Faulhaber's formula [?] to expand the term $\sum_{k=0}^{n} k$

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Thus, the identity $\Delta(n^3) = 1 + 6\sum_{k=0}^{n} k$ takes a well-known view that matches Binomial theorem [?] so that

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n+n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2}\binom{3}{k}n^k \tag{1.3}$$

There is a more beautiful way to express finite difference of cubes $\Delta(n^3)$. We refer to one of the prominent articles in area of polynomials, power sums etc., that is *Johann Faulhaber and sums of powers* written by Donald Knuth [?]. Indeed, this article is a great source to reach piece of mind in mathematics. The thing that occupied my attention was the odd power identity in terms of Binomial coefficients and Central factorial numbers, which can be found on page 9

$$n^1 = \binom{n}{1}$$
$$n^3 = 6\binom{n+1}{3} + \binom{n}{1}$$
$$n^5 = 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}$$

It is particularly interesting that well-known identity in terms of triangular numbers [citation] and finite differences of cubes becomes clear and obvious

$$\Delta n^3 = (n+1)^3 - n^3 = 6\binom{n+1}{2} + \binom{n}{0}$$

where $\binom{n+1}{2}$ are triangular numbers. It is true that

$$\Delta n^3 = \left[6\binom{n+2}{3} + \binom{n+1}{1}\right] - \left[6\binom{n+1}{3} + \binom{n}{1}\right] = 6\binom{n+1}{2} + \binom{n}{0}$$

by means of binomial coefficients' recurrence. Moreover, the concept above allows to reach N-order power sum $\sum^N k^{2m+1}$ or finite difference $\Delta^N k^{2m+1}$ of odd powers simply by changing binomial coefficients indexes. Quite strong and impressive.