SURPRISING POLYNOMIAL IDENTITIES ARISING FROM A CLASSICAL INTERPOLATION PROBLEM

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Abstract.

1. The problem of interpolation of cubes

Once there was a curious student, especially in mathematics. He was not an educated mathematician, instead he was strongly captured by the mathematical beauty and aesthetics. His mathematical knowledge was limited by bachelor level course, which includes very basics like matrix multiplication, partial derivatives, integration by parts etc. What he liked most is to question widely-known facts, trying to reach more deep understanding. He was obsessed by the process of change, *What are the proofs that you even existed, if nothing changes* — he thought.

One day this student was having fun with tables of finite differences, precisely finite differences of cubes. By observing the table

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of n^3.

Date: July 19, 2025.

The first question that triggered his mind was

Question 1.1. *How to reconstruct the value of n^3 from its finite differences?*

Precisely, the inquiry was to find a way to reconstruct the values of $\{0, 1, 8, 27, 64, \ldots\}$ having values of finite differences in the table.

In its essence, the problem is so old that it can be traced back to ancient Babylonian and Greek times, several centuries BC and first centuries AD [**?**]. The process of finding new data points based on the range of a discrete set of known data points is called interpolation. Interpolation, as we know it today, was developed in 1674–1684 by Isaac Newton in his works referenced as foundation of classical interpolation theory [**?**]. For instance, Newton's series for n^3 is

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + 1\binom{n}{1} + 0\binom{n}{0}$$

because $f(x) = \sum_{k=0}^{d} \Delta^{d-k} f(0)\binom{x}{d-k}$, see [**?**, p. 190].

Great! But there is one thing, the student who has risen the question (1.1) had no clue about interpolation theory at all. What he decided then? Exactly, he decided to try to re-invent interpolation formula himself, fueled by the purest feeling of mystery. His mind was occupied by only a single thought: *All mathematical truths exist timelessly, we only reveal and describe them.* That mindset inspired our student to start his own mathematical journey.

By observing the table of finite differences (1) we can notice that the first order finite difference of cubes may be expressed in terms of its third order finite difference $\Delta^3(n^3) = 6$, as follows

$$\begin{aligned}
\Delta(0^3) &= 1 + 6\cdot 0\\
\Delta(1^3) &= 1 + 6\cdot 0 + 6\cdot 1\\
\Delta(2^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2\\
\Delta(3^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3\\
&\vdots\\
\Delta(n^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + \cdots + 6n
\end{aligned}$$

By using sigma notation, we get

$$\Delta(n^3) = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + \cdots + 6\cdot n = 1 + 6\sum_{k=0}^{n} k$$

However, there is a more beautiful way to prove that $\Delta(n^3) = 1 + 6\sum_{k=0}^{n} k$. We refer to one of the finest articles in the area of polynomials and power sums, that is *Johann Faulhaber and sums of powers* written by Donald Knuth [**?**]. Indeed, this article is a great source to reach piece of mind in mathematics. We now focus on the following odd power identities shown at [**?**, p. 9]

$$\begin{aligned}
n^1 &= \binom{n}{1}\\
n^3 &= 6\binom{n+1}{3} + \binom{n}{1}\\
n^5 &= 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}
\end{aligned}$$

It is in particular interesting that well-known identity which connects triangular numbers $\binom{n+1}{2}$ and finite differences of cubes becomes clear and obvious

$$\Delta n^3 = (n+1)^3 - n^3 = 6\binom{n+1}{2} + \binom{n}{0}$$

It easy to see that

$$\Delta n^3 = \left[6\binom{n+2}{3} + \binom{n+1}{1}\right] - \left[6\binom{n+1}{3} + \binom{n}{1}\right] = 6\binom{n+1}{2} + \binom{n}{0}$$

because $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Moreover, the concept above allows to reach N-fold power sums $\sum^N k^{2m+1}$ or finite differences $\Delta^N k^{2m+1}$ of odd powers by simply altering binomial coefficients indexes. Quite strong and impressive.

We can observe that triangular numbers $\binom{n+1}{2}$ are equivalent to

$$\binom{n+1}{2} = \sum_{k=0}^{n} k$$

because $\binom{n+1}{m+1} = \sum_{k=0}^{n} \binom{k}{m}$. This leads to the identity in finite differences of cubes

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6 \sum_{k=0}^{n} k$$

An experienced mathematician would immediately notice a spot to apply Faulhaber's formula [**?**] to expand the term $\sum_{k=0}^{n} k$

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Thus, the finite difference $\Delta(n^3)$ takes a well-known form such that matches Binomial theorem [**?**]

$$\Delta(n^3) = 1 + 6 \left[\frac{1}{2}(n + n^2) \right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k$$

And…that could be the end of the story, isn't it? Because all what remains is to say that

$$n^3 = \sum_{k=0}^{n-1} (k+1)^3 - k^3 = \sum_{k=0}^{n-1} \left(1 + 6 \sum_{t=0}^{k} t \right) = \sum_{k=0}^{n-1} 1 + 3k + 3k^2$$

Thus, the polynomial n^3 is interpolated successfully, and thus, the question (1.1) is answered positively. Because we have successfully found the function that matches n^3 from the values of its finite differences from the table (1).

However, not this time. Luckily enough (say), the student who has stated the question (1.1) wasn't really aware of the approaches above neither. What a lazy student! Probably, that's exactly the case when unawareness leads to a fresh sight to century-old questions, leading to unexpected results and new insights. Instead, our student spotted a little bit different pattern in $\Delta n^3 = 6\binom{n+1}{2} + \binom{n}{0}$.

Consider the polynomial n^3 as sum of its finite differences

$$
\begin{aligned}
n^3 = &[1 + 6 \cdot 0] \\
&+ [1 + 6 \cdot 0 + 6 \cdot 1] \\
&+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\
&+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)]
\end{aligned}
$$

We can observe that the term 1 appears n times, the item $6 \cdot 0$ appears $n - 0$ times, the item $6 \cdot 1$ appears $n - 1$ times and so on. By rearranging recurring common terms

$$
\begin{aligned}
n^3 = n &+ [(n-0) \cdot 6 \cdot 0] \\
&+ [(n-1) \cdot 6 \cdot 1] \\
&+ [(n-2) \cdot 6 \cdot 2] + \cdots \\
&+ [(n-k) \cdot 6 \cdot k] + \cdots \\
&+ [1 \cdot 6 \cdot (n-1)]
\end{aligned}
$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

We can freely move the term n under the summation because there are exactly n iterations. Therefore,

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$$

By inspecting the expression $6k(n-k) + 1$, we can notice that it is symmetric over k. Let be $T_1(n,k) = 6k(n-k) + 1$ then

$$T_1(n,k) = T_1(n, n-k)$$

This symmetry allows us to alter summation bounds easily. Hence,

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

By arranging the values of $T_1(n,k)$ as a triangular array, we see that cube identities indeed are true

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $T_1(n,k) = 6k(n-k)+1$. See the sequence A287326 in OEIS [?].

The following recurrence holds for $T_1(n,k)$

$$T_1(n,k) = 2T_1(n-1,k) - T_1(n-2,k)$$

Which is indeed true, because

$$T_1(5,2) = 2 \cdot 25 - 13 = 37$$

Finally, our curious student has reached the first milestone, by finding his own answer to the question (1.1) and the answer was positive. What an excitement it was! However, it wouldn't take long. Indeed, curiosity is not something that can be fulfilled completely, and thus new questions arise. Somehow, he got a strong feeling that something bigger, something even more general hides behind the identity $n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$. Quite intuitive he was. Fair enough that the next question was

Question 1.2. *Given that the identity $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$ holds for the polynomial n^3, can it be extended or generalized to higher-degree powers, such as n^4 or n^5, in a similar manner?*

However, this time it was not so easy to find identity for n^4 or n^5 by simply observing the tables of finite differences. The previous approach to express the difference of cubes Δn^3 in terms of $\Delta^3 n^3 = 6$ and then express the cubes as $n^3 = \sum_k 6k(n-k)+1$ — was not successful. Moreover, it wasn't even clear what is the generic form of an identity out student was looking for, a lot of concerns came from a simple interpolation task. Thus, he reached mathematical community in search of advice. And there was a response.

2. A system of linear equations

In 2018, Albert Tkaczyk published two papers [**?**, **?**] presenting analogous identities for polynomials n^5, n^7 and n^9 derived in a manner similar to $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$. Tkaczyk assumed that the identity for n^5 takes the following explicit form

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C\right]$$

where A, B, C are yet-unknown coefficients. We denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ to reach the compact form of double sum

$$n^5 = \sum_{k=1}^{n}\sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

By observing the previous equation, the potential form of generalized odd-power identity becomes more obvious. To evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ we construct and solve a certain system of linear equations, which is built as follows

$$n^5 = \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2$$

By expanding the sums $\sum_{k=1}^{n} k^r (n-k)^r$ using Faulhaber's formula [**?**], we get an equation

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5)\right] - n^5 = 0$$

By multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(-n + n^3) + \mathbf{A}_{2,2}(-n + n^5) - 30n^5 = 0$$

By expanding the brackets and rearranging the terms

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

By combining the common terms, we obtain

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore,

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

By solving the system above, we evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

Thus, the identity for n^5

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

Again, the terms $30k^2(n-k)^2 + 1$ are symmetric over k. Let be $T_2(n,k) = 30k^2(n-k)^2 + 1$ then

$$T_2(n,k) = T_2(n,n-k)$$

By arranging the values of $T_2(n,k)$ as a triangular array, we see that the identity for n^5 is indeed true

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $T_2(n,k) = 30k^2(n-k)^2+1$. See the sequence A300656 in OEIS [**?**].

Now we can see that the relation (**??**) we got via interpolation of cubes can be generalized for all non-negative odd-powers $2m+1$ by constructing and solving a certain system of linear equations. Therefore, the generalized form of odd-power identity has the form

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ are real coefficients. In more details, the identity (**??**) is discussed separately in [**?**, **?**].

As one more example, let be $m=3$ so that we have the following relation defined by (**??**)

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5)\right] + \mathbf{A}_{m,3} \left[\frac{1}{420}(-10n+7n^3+3n^7)\right] - n^7 = 0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{2,1}(-n+n^3) + 14\mathbf{A}_{2,2}(-n+n^5) + \mathbf{A}_{3,3}(-10n+7n^3+3n^7) - 420n^7 = 0$$

Expanding brackets and rearranging the terms gives

$$\begin{aligned} 420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5 \\ - 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0 \end{aligned}$$

Combining the common terms yields

$$n(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3})$$

$$+ n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5 14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} = 0 \\ 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} = 0 \\ \mathbf{A}_{3,2} - 30 = 0 \\ 3\mathbf{A}_{3,3} - 420 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \mathbf{A}_{3,2} = 0 \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1} + 10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

So that odd-power identity (**??**) holds

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms $140k^3(n-k)^3 - 14k(n-k) + 1$ over $0 \le k \le n$ as the OEIS sequence A300785 [**?**] shows.

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	127	1					
3	1	1093	1093	1				
4	1	3739	8905	3739	1			
5	1	8905	30157	30157	8905	1		
6	1	17431	71569	101935	71569	17431	1	
7	1	30157	139861	241753	241753	139861	30157	1

Table 4. Values of $140k^3(n-k)^3-14k(n-k)+1$. See the OEIS entry A300785 [?].

However, constructing and solving a system of linear equations for every odd-power $2m+1$ requires a huge effort

Assumption 2.1. *There must be a formula that generates a set of real coefficients* $\mathbf{A}_{m,0}, \mathbf{A}_{m,0}, \ldots \mathbf{A}_{m,m}$ *for each fixed* m *such that*

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r$$

I thought.