SURPRISING POLYNOMIAL IDENTITIES ARISING FROM A CLASSICAL INTERPOLATION PROBLEM

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1. Discussion on interpolation of cubes

This is the story of a student with a deep curiosity for mathematics. Although, our young explorer was not a specialist in mathematics, however he always possessed a strong sense of mathematical beauty and aesthetics. His mathematical knowledge was limited by undergraduate level course, which includes the basics of matrix operations, basic calculus, and elementary linear algebra. One day, the student found himself observing the tables of finite differences, precisely finite differences of cubes. By observing the table

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of n^3.

The first question that triggered his mind was

Question 1.1. *How to reconstruct the value of n^3 from its finite differences?*

Precisely, the inquiry is to find a way to reconstruct the values of the sequence $\{0, 1, 8, 27, 64, \ldots\}$ given the values of finite differences in the table.

In its essence, the problem is so old that it can be traced back to ancient Babylonian and Greek times, several centuries BC and first centuries AD [?]. The process of finding new data points based on the range of a discrete set of known data points is called interpolation. Interpolation, as we know it today, was developed in 1674–1684 by Isaac Newton in his works referenced as foundation of classical interpolation theory [?]. For instance, Newton's series for n^3 is

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + 1\binom{n}{1} + 0\binom{n}{0}$$

because $f(x) = \sum_{k=0}^{d} \Delta^{d-k} f(0)\binom{x}{d-k}$, see [?, p. 190].

Great! But there is one thing, the student who has risen the question (1.1) had no clue about interpolation theory at all. What he decided then? Exactly, he decided to try to re-invent interpolation formula himself, fueled by the purest feeling of mystery. His mind was occupied by only a single thought: *All mathematical truths exist timelessly, we only reveal and describe them.* That mindset inspired our student to start his own mathematical journey.

By observing the table of finite differences (1) we can notice that the first order finite difference of cubes may be expressed in terms of its third order finite difference $\Delta^3(n^3) = 6$, as follows

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 \\
&\vdots \\
\Delta(n^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6n
\end{aligned}$$

By using sigma notation, we get

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k$$

However, there is a more beautiful way to prove that $\Delta(n^3) = 1 + 6 \sum_{k=0}^{n} k$. We refer to one of the finest articles in the area of polynomials and power sums, that is *Johann Faulhaber and sums of powers* written by Donald Knuth [**?**]. Indeed, this article is a great source to reach piece of mind in mathematics. We now focus on the odd power identities shown at [**?**, p. 9]

$$\begin{aligned} n^1 &= \binom{n}{1} \\ n^3 &= 6\binom{n+1}{3} + \binom{n}{1} \\ n^5 &= 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1} \end{aligned}$$

It is quite interesting that the identity in terms of triangular numbers $\binom{n+1}{2}$ and finite differences of cubes becomes more obvious

$$\Delta n^3 = (n+1)^3 - n^3 = 6\binom{n+1}{2} + \binom{n}{0}$$

It easy to see that

$$\Delta n^3 = \left[6\binom{n+2}{3} + \binom{n+1}{1}\right] - \left[6\binom{n+1}{3} + \binom{n}{1}\right] = 6\binom{n+1}{2} + \binom{n}{0}$$

because $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Moreover, the concept above allows to reach N-fold power sums $\sum^N k^{2m+1}$ or finite differences $\Delta^N k^{2m+1}$ of odd powers by simply altering binomial coefficients indexes. Quite strong and impressive.

We can observe that triangular numbers $\binom{n+1}{2}$ are equivalent to

$$\binom{n+1}{2} = \sum_{k=0}^{n} k$$

because $\binom{n+1}{m+1} = \sum_{k=0}^{n} \binom{k}{m}$. This leads to the identity in finite differences of cubes

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6 \sum_{k=0}^{n} k$$

An experienced mathematician would immediately notice a spot to apply Faulhaber's formula [?] to get the closed form of the sum $\sum_{k=0}^{n} k$

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Thus, the finite difference $\Delta(n^3)$ takes a well-known form, which matches Binomial theorem [?]

$$\Delta(n^3) = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k$$

And... that could be the end of the story, isn't it? Because all what remains is to say that

$$n^3 = \sum_{k=0}^{n-1} (k+1)^3 - k^3 = \sum_{k=0}^{n-1} \left(1 + 6 \sum_{t=0}^{k} t\right) = \sum_{k=0}^{n-1} 1 + 3k + 3k^2$$

Thus, the polynomial n^3 is interpolated successfully, and thus, our protégée's question (1.1) is answered positively. Because we have successfully found the function that matches n^3 from the values of its finite differences from the table (1).

However, not this time. Luckily enough (say), the student who has stated the question (1.1) wasn't really aware of the approaches above neither. What a lazy student! Probably, that's exactly the case when unawareness leads to a fresh sight to century-old questions, leading to unexpected results and new insights. Instead, our investigator spotted a little bit different pattern in $\Delta n^3 = 6\binom{n+1}{2} + \binom{n}{0}$.

Consider the polynomial n^3 as sum of its finite differences

$$\begin{aligned} n^3 &= [1 + 6 \cdot 0] \\ &\quad + [1 + 6 \cdot 0 + 6 \cdot 1] \\ &\quad + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\ &\quad + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

We can observe that the term 1 appears n times, the item $6 \cdot 0$ appears $n-0$ times, the item $6 \cdot 1$ appears $n-1$ times and so on. By rearranging recurring common terms

$$\begin{aligned} n^3 = n &+ [(n-0) \cdot 6 \cdot 0] \\ &+ [(n-1) \cdot 6 \cdot 1] \\ &+ [(n-2) \cdot 6 \cdot 2] + \cdots \\ &+ [(n-k) \cdot 6 \cdot k] + \cdots \\ &+ [1 \cdot 6 \cdot (n-1)] \end{aligned}$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

We can freely move the term n under the summation because there are exactly n iterations. Therefore,

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$$

By inspecting the expression $6k(n-k)+1$, we can notice that it is symmetric over k. Let be $T_1(n,k) = 6k(n-k)+1$ then

$$T_1(n,k) = T_1(n,n-k)$$

This symmetry allows us to alter summation bounds easily. Hence,

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

By arranging the values of $T_1(n,k)$ as a triangular array, we see that cube identities indeed are true

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $T_1(n,k) = 6k(n-k)+1$. See the sequence A287326 in OEIS [?].

The following recurrence holds for $T_1(n,k)$

$$T_1(n,k) = 2T_1(n-1,k) - T_1(n-2,k)$$

Which is indeed true, because

$$T_1(5,2) = 2 \cdot 25 - 13 = 37$$

Finally, our curious learner has reached the first milestone, by finding his own answer to the question (1.1) and the answer was positive. What an excitement it was! However, it wouldn't take long. Indeed, curiosity is not something that can be fulfilled completely, and thus new questions arise. Somehow, the inquirer got a strong feeling that something bigger, something even more general hides behind the identity $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$. That was quite intuitive. Fair enough that the next question was

Question 1.2. *Given that the identity $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$ holds for the polynomial n^3, can it be extended or generalized to higher-degree powers, such as n^4 or n^5, in a similar manner?*

However, this time it was not so easy for the young explorer to find identity for n^4 or n^5 by simply observing the tables of finite differences. The previous approach to express the difference of cubes Δn^3 in terms of $\Delta^3 n^3 = 6$ and then express the cubes as $n^3 =$

$\sum_k 6k(n-k)+1$ — was not successful. Moreover, it wasn't even clear what is the generic form of an identity our student was looking for, a lot of concerns came from a simple interpolation task. Thus, the question (1.2) was shared with the mathematical community. And there was an answer.

2. A system of linear equations

In 2018, Albert Tkaczyk published two papers [?, ?] presenting analogous identities for polynomials n^5, n^7 and n^9 derived in a manner similar to $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$. Tkaczyk assumed that the identity for n^5 takes the following explicit form

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C\right]$$

where A, B, C are yet-unknown coefficients. We denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ to reach the compact form of double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

By observing the equation above, the potential form of generalized odd-power identity becomes more obvious. One important note to add here, we define $0^x = 1$ for all x, see [?, p. 162]. This is because when $k = n$ and $r = 0$ the term $k^r(n-k)^r = n^0 \cdot 0^0$, thus we must define $0^x = 1$ for all x.

To evaluate the set of coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ we construct and solve a certain system of linear equations, which is built as follows

$$n^5 = \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0(n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1(n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2(n-k)^2$$

By expanding the sums $\sum_{k=1}^{n} k^r(n-k)^r$ using Faulhaber's formula [?], we get an equation

$$\mathbf{A}_{2,0} n + \mathbf{A}_{2,1} \left[\frac{1}{6}(n^3 - n)\right] + \mathbf{A}_{2,2} \left[\frac{1}{30}(n^5 - n)\right] - n^5 = 0$$

By multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0} n + 5\mathbf{A}_{2,1}(n^3 - n) + \mathbf{A}_{2,2}(n^5 - n) - 30n^5 = 0$$

By expanding the brackets and rearranging the terms

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

By combining the common terms, we obtain

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore,

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} & = 0 \\ \mathbf{A}_{2,1} & = 0 \\ \mathbf{A}_{2,2} - 30 & = 0 \end{cases}$$

By solving the system above, we evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$

$$\begin{cases} \mathbf{A}_{2,2} & = 30 \\ \mathbf{A}_{2,1} & = 0 \\ \mathbf{A}_{2,0} & = 1 \end{cases}$$

Thus, the identity for n^5

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

Again, the terms $30k^2(n-k)^2 + 1$ are symmetric over k. Let be $T_2(n,k) = 30k^2(n-k)^2 + 1$ then

$$T_2(n,k) = T_2(n, n-k)$$

By arranging the values of $T_2(n,k)$ as a triangular array, we see that the identity for n^5 is indeed true

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $T_2(n, k) = 30k^2(n-k)^2 + 1$. See the sequence A300656 in OEIS [**?**].

The following recurrence holds for $T_2(n, k)$

$$T_2(n, k) = 3T_2(n-1, k) - 3T_2(n-2, k) + T_2(n-3, k)$$

Which is indeed true because

$$T_2(6, 2) = 3 \cdot 1081 - 3 \cdot 481 + 271 = 1921$$

3. Recurrence relation

Assume that the following odd power identity holds

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r \tag{3.1}$$

Our main goal is to identify the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that identity above is true.

Although, the recurrence relation is already given at [**?**], a few key points in proof are worth to discuss additionally.

The main idea of Alekseyev's approach was to utilize a recurrence relation to evaluate the set of coefficients $\mathbf{A}_{m,r}$ starting from the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ recursively, continuing similarly up to $\mathbf{A}_{m,0}$.

We utilize Binomial theorem $(n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t$ and specific version of Faulhaber's formula [**?**]

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} = \frac{1}{p+1} \left[\sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1}$$
$$= \frac{1}{p+1} \left[\sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1}$$

The reason we use modified version of Faulhaber's formula is because we want to omit summation bounds, for simplicity. This would help us to collapse the common terms across complex sums, see also [**?**, p. 2]. Therefore, we expand the sum $\sum_{k=1}^{n} k^r (n-k)^r$ using Binomial theorem

$$\sum_{k=1}^{n} k^r (n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r}$$

By applying Faulhaber's formula above, we obtain

$$\sum_{k=1}^{n} k^r (n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \left[\left(\frac{1}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{t+r+1-j} \right) - \frac{B_{t+r+1}}{t+r+1} \right]$$

By moving the common term $\frac{(-1)^t}{t+r+1}$ out of brackets

$$\sum_{k=1}^{n} k^r (n-k)^r = \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \left[\sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t} \right]$$

By expanding the brackets

$$\sum_{k=1}^{n} k^r (n-k)^r = \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} \right]$$
$$- \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

By moving the sum in j and omitting summation bounds in t

$$\sum_{k=1}^{n} k^r (n-k)^r = \left[\sum_{j,t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[\sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

By rearranging the sums we obtain

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_j B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right] - \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right] \tag{3.2}$$

We can notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j=0 \\ \frac{(-1)^r}{j}\binom{r}{2r-j+1} & \text{if } j>0 \end{cases} \tag{3.3}$$

An elegant proof of the binomial identity (3.3) is done by Markus Scheuer in [?]. In particular, the equation (3.3) is zero for $0 < t \leq j$. To utilize the equation (3.3), we have to move $j = 0$ out of summation in (3.2) to avoid division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j=1}^{\infty} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right] - \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

Hence, we simplify the equation above by using (3.3) so that

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j=1}^{\infty} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1}\right]}_{(\star)} - \underbrace{\left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]}_{(\diamond)}$$

By introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms across two sums

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right]$$
$$- \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right]$$
$$= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by the odd-power identity (3.1), we obtain the following relation for polynomials in n

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{r=0}^{m}\sum_{\text{odd } \ell} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by $\ell = 2k+1$ we get

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{r=0}^{m}\sum_{k=0}^{\infty} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2k}\binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1}$$

By simplifying the term 2

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \sum_{r=0}^{m}\sum_{k=0}^{\infty} \mathbf{A}_{m,r} \frac{(-1)^r}{r-k}\binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1} \tag{3.4}$$

Basically, the relation (3.4) is the generating function we utilize to evaluate the values of $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$. We now fix the unused values of $\mathbf{A}_{m,r}$ so that $\mathbf{A}_{m,r} = 0$ for every $r < 0$ or $r > m$.

Taking the coefficient of n^{2m+1} in (3.4) yields

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

because $\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} = 1$.

That's may not be immediately clear why the coefficient of n^{2m+1} is $(2m+1)\binom{2m}{m}$. To extract the coefficient of n^{2m+1} from the expression (3.4), we isolate the relevant terms by setting $r = m$ in the first sum, and $k = m$ in the second sum. This gives

$$[n^{2m+1}]\left(\sum_{r=0}^{m}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1}+\sum_{r=0}^{m}\sum_{k=0}^{\infty}\mathbf{A}_{m,r}\frac{(-1)^r}{r-k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}-n^{2m+1}\right)$$
$$=\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}}+\sum_{r=0}^{m}\mathbf{A}_{m,r}\frac{(-1)^r}{r-m}\binom{r}{2m+1}B_{2r-2m}-1$$

We observe that the sum

$$\sum_{r=0}^{m}\mathbf{A}_{m,r}\frac{(-1)^r}{r-m}\binom{r}{2m+1}B_{2r-2m}$$

does not contribute to the determination of the coefficients because the binomial coefficient $\binom{r}{2m+1}$ vanishes for all $r \leq m$. Consequently, all terms in the sum are zero. Thus,

$$\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}}-1=0 \implies \mathbf{A}_{m,m}=(2m+1)\binom{2m}{m}$$

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d \leq m-1$ in (3.4) gives

$$[n^{2d+1}]\left(\sum_{r=0}^{m}\mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1}+\sum_{r=0}^{m}\sum_{k=0}^{\infty}\mathbf{A}_{m,r}\frac{(-1)^r}{r-k}\binom{r}{2k+1}B_{2r-2k}n^{2k+1}-n^{2m+1}\right)$$
$$=\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}}+\sum_{r=0}^{m}\mathbf{A}_{m,r}\frac{(-1)^r}{r-d}\binom{r}{2d+1}B_{2r-2d}.$$

For every $\frac{m}{2} \leq d$, the binomial coefficient $\binom{r}{2d+1}$ vanishes, because for all $r \leq m$ holds $r < 2d+1$. As a particular example, when $r = m$ and $d = \frac{m}{2}$, we have

$$\binom{m}{m+1}=0.$$

Therefore, the entire sum involving $\binom{r}{2d+1}$ vanishes, and we conclude

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}}=0 \implies \mathbf{A}_{m,d}=0.$$

Hence, for all integers d such that $\frac{m}{2} \leq d \leq m-1$, the coefficient $\mathbf{A}_{m,d} = 0$. In contrast, for values $d \leq \frac{m}{2} - 1$, the binomial coefficient $\binom{r}{2d+1}$ can be nonzero; for instance, if $r = m$ and $d = \frac{m}{2} - 1$, then

$$\binom{m}{m-1}\neq 0,$$

allowing the corresponding terms to contribute to the determination of $\mathbf{A}_{m,d}$.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ in (3.4), we obtain

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0.$$

Solving for $\mathbf{A}_{m,d}$ yields

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!\,d!\,m!\,(m-2d-1)!} \cdot \frac{1}{m-d}B_{2m-2d}.$$

Proceeding recursively, we can compute each coefficient $\mathbf{A}_{m,r}$ for integers r in the ranges

$$\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}, \quad \text{for } s = 1, 2, \ldots$$

by using previously computed values $\mathbf{A}_{m,d}$ for $d > r$, via the relation

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}.$$

Finally, we define the following recurrence relation for coefficients $\mathbf{A}_{m,r}$

Definition 3.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{3.5}$$

where B_t are Bernoulli numbers [?]. It is assumed that $B_1 = \frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 4. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [?, ?].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

4. Related works

We show related works in format of a directed graph, where node represents a work and edge represents a citation between two works. For example, the node 12 (current manuscript) cites the manuscripts 7 and 8. Nodes of the related work graph in BFS order are

- 12 is this manuscript
- 7 is *A study on partial dynamic equation on time scales involving derivatives of polynomials* [?]
- 8 is *106.37 An unusual identity for odd-powers* [?]
- 9 is *Another approach to get derivative of odd-power* [?]
- B is *A two-sided Faulhaber-like formula involving Bernoulli polynomials* [?]
- 10 is *Polynomial identity involving Binomial Theorem and Faulhaber's formula* [?]
- 11 is *Finding the derivative of polynomials via double limit* [?]
- 6 is *On the link between binomial theorem and discrete convolution* [?]

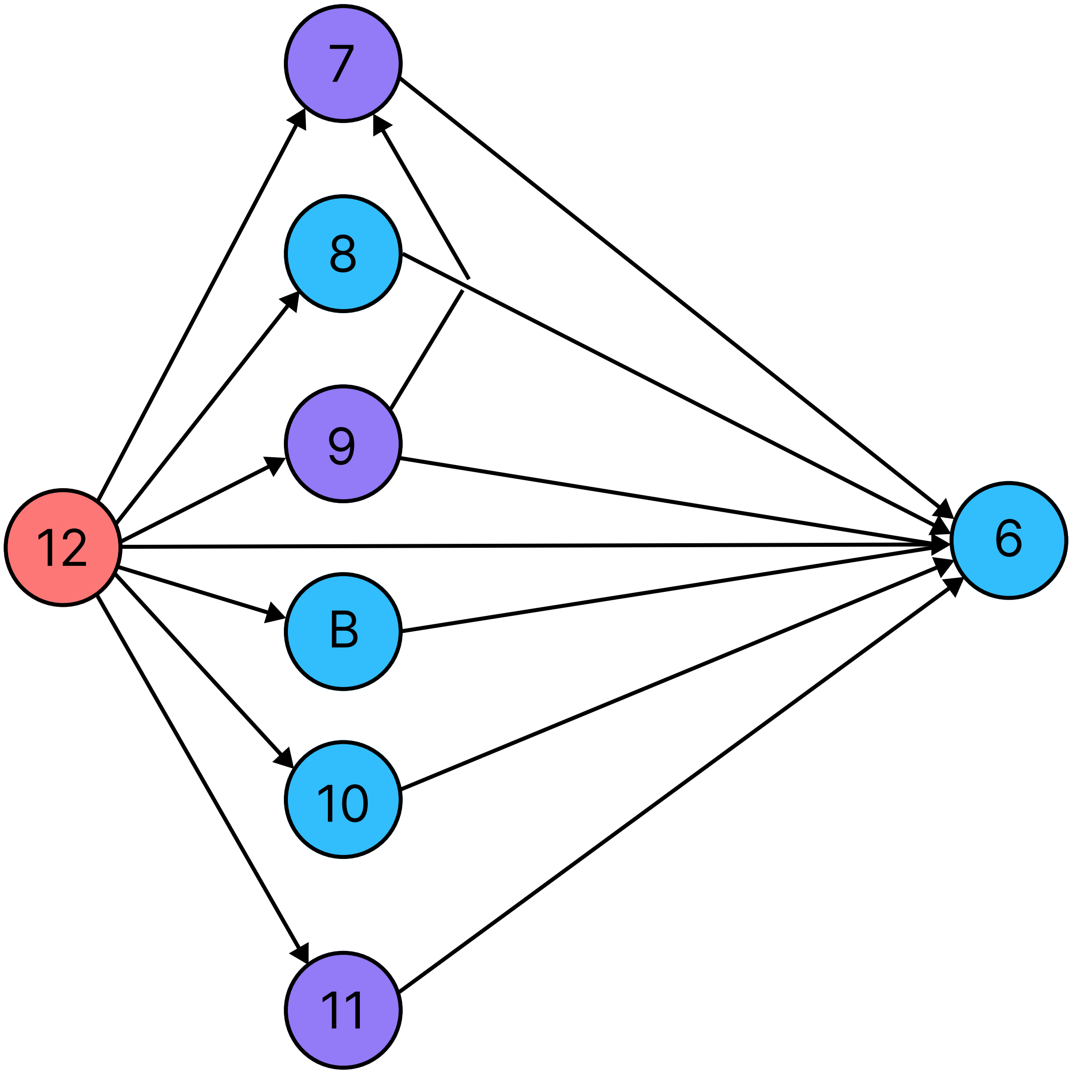

Figure 1. Related works graph.

- 12 This manuscript
- 6 In *On the link between binomial theorem and discrete convolution* [**?**]: Let $\mathbf{P}_b^m(x)$ be a $2m+1$-degree polynomial in x and $b \in \mathbb{R}$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r$$

where $\mathbf{A}_{m,r}$ are real coefficients. In this manuscript, we introduce the polynomial $\mathbf{P}_b^m(x)$ and study its properties, establishing a polynomial identity for odd-powers in terms of this polynomial. Based on mentioned polynomial identity for odd-powers, we explore the connection between the Binomial theorem and discrete convolution of odd-powers, further extending this relation to the multinomial case. All findings are verified using Mathematica programs.

- (7) In *A study on partial dynamic equation on time scales involving derivatives of polynomials* [?]: Extends the main results of (6) deriving and discussing an identity that connects the timescale derivative of odd-powered polynomial with partial derivatives of polynomial $\mathbf{P}_b^m(x)$ evaluated in particular points. For every $t \in \mathbb{T}_1$ and $(x, b) \in \Lambda^2$

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m, b, x)}{\Delta x}(m, \sigma(t), t) + \frac{\partial P(m, b, x)}{\Delta b}(m, t, t)$$

such that $\sigma(t) > t$ is forward jump operator. In addition, we discuss various derivative operators in the context of the partial cases of above equation, We show finite difference, classical derivative, $q-$derivative, $q-$power derivative on behalf of it.

- (8) In *106.37 An unusual identity for odd-powers* [?]: Explores and proves the partial case of (6) that is the polynomial identity for odd-powers

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

- (9) In *Another approach to get derivative of odd-power* [?]: Extends the results of (6) by providing a relation in terms of partial differential equations such that ordinary derivative of odd-power $2m+1$ can be reached in terms of partial derivative of the polynomial $\mathbf{P}_b^m(x)$. Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx} g_v(u)$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)'_x(u, u)$ evaluate in point (u, u) plus partial derivative $(f_v)'_z(u, u)$ evaluate in point (u, u)

$$\frac{d}{dx} g_v(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u) \tag{4.1}$$

where $f_y(x, z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r = \mathbf{P}_z^y(x)$.

- (B) In *A two-sided Faulhaber-like formula involving Bernoulli polynomials* [?]: Based on equation (3.2), the authors give a new identity involving Bernoulli polynomials and combinatorial numbers that provides, in particular, the Faulhaber-like formula for sums of the form $1^m(n-1)^m + 2^m(n-2)^m + \cdots + (n-1)^m 1^m$ for positive integers m and n.

- 10 In *Polynomial identity involving Binomial Theorem and Faulhaber's formula* [?]: proves that for every $n \geq 1,\ n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that the polynomial identity holds
$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$
which is a direct consequence of the definition of $\mathbf{P}_b^m(x)$ given in 6, reached by utilizing Binomial theorem and Faulhaber's formula.
- 11 In *Finding the derivative of polynomials via double limit* [?]: By applying the results of 6 provides another perspective of ordinary derivatives of polynomials allowing expressing them via a double limit, because
$$\lim_{h \to 0} \mathbf{P}_{x+h}^m(x) = x^{2m+1}$$
- Three sequences were contributed to the OEIS [?, ?, ?] showing the coefficients of the polynomial $\mathbf{P}_b^m(x)$ having fixed points m, b while $x \in \mathbb{R}$.
- OEIS sequences such that row sums give odd-powers [?, ?, ?].
- OEIS sequences related to the coefficients $\mathbf{A}_{m,r}$ [?, ?].

The node indexes in the related works graph are not random, persisting the same values as these works have on my personal website

kolosovpetro.github.io/math

5. Future research

- Differential equation (4.1) can also be expressed in terms of backward and central differential operators, including derivatives on time-scales so that results of [?] could be generalized further.

- Theorem (**??**) provides an opportunity to express odd-power identity in terms of multiplication of certain matrices.
- There are Taylor series and Maclaurin series versions in terms of $\mathbf{P}_b^m(x)$.
- The summation bounds of definition (**??**) can be altered so that k runs over $1 \leq k \leq b$, by symmetry.
- Prove that $\mathbf{P}_b^m(x)$ is an integer valued polynomial in (x, b).
- The definition (**??**) is closely related to discrete convolution because
$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r$$
where $\sum_{k=0}^{b-1} k^r (x-k)^r$ is the discrete convolution of x^r. It is worth to get a closer look into it so that new relations in terms of discrete convolution may be found.
- All kinds of derivatives e.g. forward, backward and central, including the derivatives on time-scales can be expressed as double limit of $\mathbf{P}_b^m(x)$ extending the results of [**?**].
- Introducing the definitions of the coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_m$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}_m$
$$\begin{bmatrix} n \\ k \end{bmatrix}_m = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$
$$\begin{Bmatrix} n \\ r \end{Bmatrix}_m = \sum_{k=0}^{n-1} \mathbf{A}_{m,r} k^r (n-k)^r$$
the novel identities can be reached, for example
$$\begin{bmatrix} 2t+1 \\ 1 \end{bmatrix}_m = \begin{bmatrix} t+2 \\ 2 \end{bmatrix}_m$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_m = \begin{bmatrix} n \\ n-k \end{bmatrix}_m$$
$$\begin{bmatrix} 2t-3r \\ r \end{bmatrix}_m = \begin{bmatrix} t \\ 2r \end{bmatrix}_m = \begin{bmatrix} 2t-3r \\ 2t-4r \end{bmatrix}_m$$
so that combinatorial sense of above is also a topic to research.
- Contribute new OEIS sequences related to $\begin{bmatrix} n \\ k \end{bmatrix}_m$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}_m$.

- An identity

$$(x-2a)^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=a+1}^{x-a} (k-a)^r (x-k-a)^r$$

  allows to provide a novel proof of power rule in terms of derivatives of polynomials.

- Following the results of https://arxiv.org/pdf/1603.02468v15.pdf, the equation (**??**) approximates the odd-power polynomial x^{2m+1} around given points x_i as it may be observed from the following plots

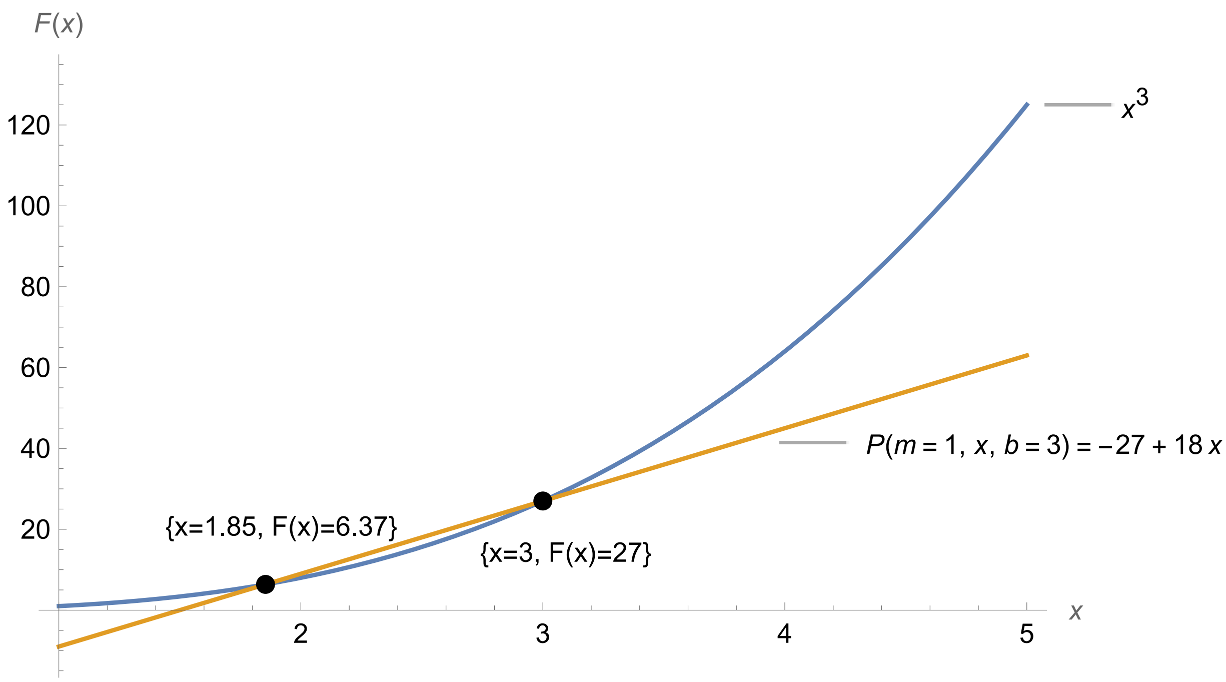

Figure 2. Approximation of x^3.

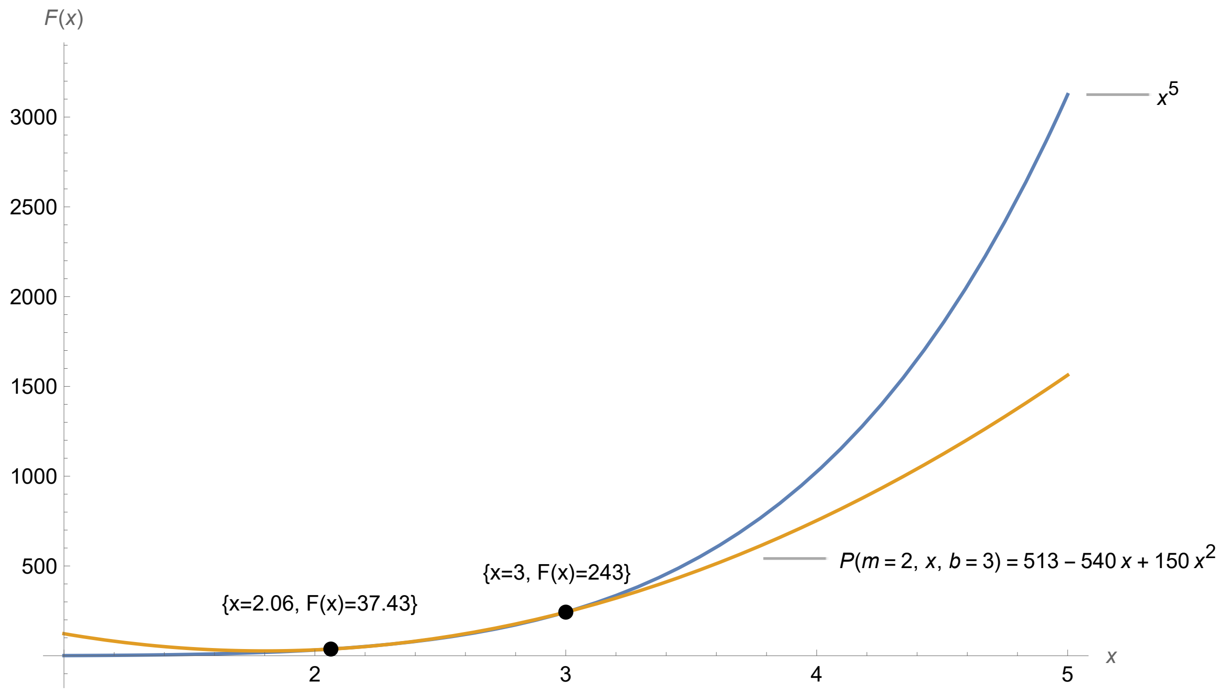

Figure 3. Approximation of x^5.

- English grammar reviews and improvements are welcome.
- Improvements and suggestions to current manuscript under open-source initiatives at https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP

6. Conclusions

In this manuscript we have successfully provided a comprehensive historical survey of the milestones and evolution of the polynomial $\mathbf{P}_b^m(x)$ as well as related works such that based onto, for instance various polynomial identities, differential equations etc. In addition, future research directions are proposed and discussed.

Acknowledgements

The author is grateful to Albert Tkaczyk for pointing out the idea of the general form of odd power identity, and for providing examples of how to build and solve systems of linear equations to evaluate $\mathbf{A}_{m,r}$. The author is grateful to Max Alekseyev (George Washington University) for providing a valuable contribution of recurrence relation for coefficients $\mathbf{A}_{m,r}$.

The author is grateful to Markus Scheuer for his help in proving the identity (3.3), the proof is elegant and beautiful. Finally, the author is grateful to OEIS editors for their valuable work during the sequences related to this manuscript were submitted.