Homework 5

Section 3.4

Problem 23

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find B so that $AB = I = BA$ as follows: first equate entries on both sides of $AB = I$. Then solve for a, b, c, and d; finally verify that $BA = I$ as well.

Solution

Compute

$$AB = \begin{bmatrix} 2a + c & 2b + d \\ 3a + 2c & 3b + 2d \end{bmatrix} = I \;\Rightarrow\; \begin{cases} 2a + c = 1, \\ 3a + 2c = 0, \\ 2b + d = 0, \\ 3b + 2d = 1. \end{cases}$$

Solve: from $2a + c = 1$ and $3a + 2c = 0$ get $a = 2,\ c = -3$; from $2b + d = 0$ and $3b + 2d = 1$ get $b = -1,\ d = 2$. Hence

$$B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

Verification of $BA = I$:

$$BA = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + (-1) \cdot 3 & 2 \cdot 1 + (-1) \cdot 2 \\ -3 \cdot 2 + 2 \cdot 3 & -3 \cdot 1 + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $AB = I = BA$.

Problem 39

Use matrix multiplication to show that if x_1 and x_2 are two solutions of the homogeneous system $Ax = 0$ and c_1 and c_2 are real numbers, then $c_1x_1 + c_2x_2$ is also a solution.

Solution

We want to prove

$$A(c_1x_1 + c_2x_2) = 0.$$

Distribute

$$c_1(Ax_1) + c_2(Ax_2) = 0.$$

Plug in assumption from problem statement

$$c_1(0) + c_2(0) = 0$$

Problem 40

(a) Use matrix multiplication to show that if x_0 is a solution of the homogeneous system $Ax = 0$ and x_1 is a solution of the nonhomogeneous system $Ax = b$, then $x_0 + x_1$ is also a solution of the nonhomogeneous system.
(b) Suppose that x_1 and x_2 are solutions of the nonhomogeneous system of part **(a)**. Show that x_1 x_2 is a solution of the homogeneous system $Ax = 0$.

Solution

(a)
We want to prove

$$A(x_0 + x_1) = 0.$$

Distribute

$$Ax_1 + Ax_2 = b.$$

Plug in assumption from problem statement

$$0 + b = b.$$

(b)
We want to prove

$$A(x_1 - x_2) = 0.$$

Distribute

$$Ax_1 - Ax_2 = 0.$$

Plug in assumption from problem statement

$$b - b = 0.$$

Section 3.5

Problem 6

First apply the Inverses of 2×2 Matrices Theorem to find $\mathbf{A^{-1}}$. Then use $\mathbf{A^{-1}}$ to solve the system $\mathbf{Ax} = \mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} 4 & 7 \\ 3 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Solution

First we verify that **A** is invertible by ensuring its determinant is not equal to 0.

$$det(\mathbf{A}) = (4)(6) - (7)(3) = 3 \neq 0$$

Now, apply the theorem.

$$\mathbf{A^{-1}} = \frac{1}{3}\begin{bmatrix} 6 & -7 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{7}{3} \\ -1 & \frac{4}{3} \end{bmatrix}$$

To evaluate $\mathbf{Ax} = \mathbf{b}$, multiply both sides by $\mathbf{A^{-1}}$.

$$\cancel{(A^{-1})Ax} = (\mathbf{A^{-1}})\mathbf{b} \Rightarrow \mathbf{x} = \begin{bmatrix} 2 & -\frac{7}{3} \\ -1 & \frac{4}{3} \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 2(10) - \frac{7}{3}(5) \\ -1(10) + \frac{4}{3}(5) \end{bmatrix} = \boxed{\begin{bmatrix} \frac{25}{3} \\ -\frac{10}{3} \end{bmatrix}}$$

Problem 19

Find the inverse A^{-1} of each given matrix A.

$$\begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 1 & 4 & 5 & 0 & 1 & 0 \\ 2 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R_3 - 2R_1}{\overset{R_2 - R_1}{}} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 \\ 0 & -3 & -5 & -2 & 0 & 1 \end{array}\right] \mathbf{swap}(R_2, R_3) \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & -3 & -5 & -2 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 0 \end{array}\right]$$

$$\underset{R_1 - 4R_2}{\overset{(-\frac{1}{3})R_2}{}} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{11}{3} & -\frac{5}{3} & 0 & \frac{4}{3} \\ 0 & 1 & \frac{5}{3} & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 2 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} \frac{1}{2}R_3 \\ R_1 + \frac{11}{3}R_3 \\ R_2 - \frac{5}{3}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{2} & \frac{11}{6} & \frac{4}{3} \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{6} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

$$\therefore A^{-1} = \boxed{\begin{bmatrix} -\frac{7}{2} & \frac{11}{6} & \frac{4}{3} \\ \frac{3}{2} & -\frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}$$

Problem 27

Find a matrix X such that $AX = B$.

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 7 \\ 2 & 2 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Solution

First finding $\mathbf{A^{-1}}$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 7 & 0 & 1 & 0 \\ 2 & 2 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 5 & 1 & -2 & 1 & 0 \\ 0 & 6 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{5R_3-6R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 5 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 2 & -6 & 5 \end{array}\right]$$

$$\xrightarrow[-R_3]{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 & -5 & 5 \\ 0 & 0 & 1 & -2 & 6 & -5 \end{array}\right] \xrightarrow[\frac{1}{5}R_3]{R_1-3R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 7 & -18 & 15 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 6 & -5 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -20 & 17 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 6 & -5 \end{array}\right]$$

$$\therefore \mathbf{A^{-1}} = \begin{bmatrix} 7 & -20 & 17 \\ 0 & -1 & 1 \\ -2 & 6 & -5 \end{bmatrix}$$

and multiplying the result by $\mathbf{AX} = \mathbf{B}$

$$\cancel{(A^{-1})A}\mathbf{X} = \begin{bmatrix} 7 & -20 & 17 \\ 0 & -1 & 1 \\ -2 & 6 & -5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 17 & -20 & 24 & -13 \\ 1 & -1 & 1 & -1 \\ -5 & 6 & -7 & 4 \end{bmatrix}$$

Problem 30

Suppose that A, B, and C are invertible matrices of the same size. Show that the product ABC is invertible and that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Solution

Let A, B, and C be invertible matrices of the same size.
To show that ABC is invertible, we must find a matrix X such that

$$(ABC)X = X(ABC) = I.$$

Consider $X = C^{-1}B^{-1}A^{-1}$. Then,

$$(ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

Similarly,

$$(C^{-1}B^{-1}A^{-1})(ABC) = C^{-1}B^{-1}(A^{-1}A)BC = C^{-1}(B^{-1}B)C = C^{-1}C = I.$$

Since both products give the identity matrix,

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

Therefore, the product ABC is invertible and its inverse is the reverse product of the individual inverses.

Problem 36

Show that $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ad - bc = 0$.

Solution

Recall the formula for an inverse of a 2×2 matrix:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The given condition, $ad - bc = 0$ would result in a division by 0. i.e. undefined.

Problem 39

Let $\mathbf{E}$ be the elementary matrix $\mathbf{E_2}$ of Example 6 and suppose that $\mathbf{A}$ is a 3×3 matrix. Show that $\mathbf{EA}$ is the result upon adding twice the first row of $\mathbf{A}$ to its third row.
Example 6:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(2)R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \mathbf{E}_2$$

Solution

The elementary matrix $\mathbf{E}_2$ corresponds to the row operation $(2)R_1 + R_3 \rightarrow R_3$, meaning that twice the first row is added to the third row.
Let

$$\mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then,

$$\mathbf{EA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Performing the matrix multiplication:

$$\mathbf{EA} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{11} + a_{31} & 2a_{12} + a_{32} & 2a_{13} + a_{33} \end{bmatrix}.$$

The first two rows are unchanged, and the third row is obtained by adding twice the first row of $\mathbf{A}$ to the original third row. Therefore, $\mathbf{EA}$ is the result of applying the row operation $(2)R_1 + R_3 \to R_3$ to $\mathbf{A}$.

Section 3.6

Problem 6

Use cofactor expansions to evaluate the determinant. Expand along the row or column that minimizes the amount of computation that is required.

$$\begin{vmatrix} 3 & 0 & 11 & -5 & 0 \\ -2 & 4 & 13 & 6 & 5 \\ 0 & 0 & 5 & 0 & 0 \\ 7 & 6 & -9 & 17 & 7 \\ 0 & 0 & 8 & 2 & 0 \end{vmatrix}$$

Solution

$$\begin{aligned} |\mathbf{A}| &= ... + 5 \begin{bmatrix} 3 & 0 & -5 & 0 \\ -2 & 4 & 6 & 5 \\ 7 & 6 & 17 & 7 \\ 0 & 0 & 2 & 0 \end{bmatrix} + ... \text{ (All other terms simplify to 0)} \\ &= 5(... - 2 \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 5 \\ 7 & 6 & 7 \end{bmatrix} + ...) \\ &= 5(-2(3 \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} + ...)) \\ &= 5(-2(3((4)(7) - (5)(6))) \\ &= \boxed{60} \end{aligned}$$