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Answer 1

a) We can treat the number of binary string of length n that meet the given specification as a recurrence relation problem. We can create a binary string of length n either by concatanating a "0" to a string of length $n-1$ or by concatanating a "10" to a string of length $n-2$. Thus, the formula for the linear recurrence relation is the following:

$$a_n = a_{n-1} + a_{n-2}$$

Where $a_1 = 1$ (0), and $a_2 = 2$ (10, 00). Clearly, what we get resembles the *fibonacci sequence*.

$$1, 2, 3, 5, 8, 13, 21, 34, 55 \ldots$$

However since "$00\ldots0$" is not considered a valid string, we will subtract 1 from our final result. $a_9 = 55$ The answer is 54.

b) Notice that the order of digits should be taken into consideration, and 1s and 0s are indistinguisable. We have 3 cases of bit strings in terms of the number of 1s and 0s they consist of.

i) 8 1s and 2 0s:

$$\frac{10!}{8!2!} = 45$$

ii) 9 1s and 1 0:

$$\frac{10!}{9!1!} = 10$$

iii) 10 1s and no 0s:

$$\frac{10!}{10!} = 1$$

By **sum rule**, there are $45 + 10 + 1 = 56$ bit strings that meet the specifications.

c) The number of onto functions from a set with m elements to a set n elements is

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m$$

By *Theorem 1 from section 8.6 (textbook)*. For $m = 4$ and $n = 3$, there are $81 - 48 + 3 = 36$ onto functions.

d) Since books of the same courses are identical, and thus, indistinguisable, there are only three possible configurations:

   (a) S,S,S,D
   (b) S,S,D,D
   (c) S,D,D,D

   The answer is 3.

Answer 2

a) Let us start with comparing subsets of the sets $A = \{1, 2, 3 \dots n\}$ and $B = \{1, 2, 3 \dots (n-1)\}$ that contain no consecutive numbers, (for the rest of question we will ommit the phrase "that do not contain consective numbers" to avoid repetition. The reader should assume that all subsets that are mentioned follow the given criterion.). All subsets of B are also subsets of A. Additionally A has some subsets containing n.

   Let $C = \{1, 2, 3 \dots (n-2)\}$. All subsets of B that are not also subsets of C contain $(n-1)$. The number of those subsets that contain $(n-1)$ is $a_{n-1} - a_{n-2}$. The remaining subsets of B do not contain $(n-1)$, and there are a_{n-2} of them. Moving back to subsets of A, we can produce subsets that contain n by "adding" n to the subsets of B that do not contain $(n-1)$ (there are a_{n-2} of them). Hence, we can write a_n in terms of a_{n-1} and a_{n-2}:

$$a_n = a_{n-1} + a_{n-2}$$

   Where $a_1 = 2$ ($\varnothing$, $\{1\}$), and $a_2 = 3$ ($\varnothing$, $\{1\}$, $\{2\}$).

b) Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$ be our generating function for a_n.

$$\begin{aligned} G(x) - xG(x) - x^2G(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} a_{k-1} x^k - \sum_{k=2}^{\infty} a_{k-2} x^k \\ &= a_0 + a_1 x - a_0 x + \sum_{k=2}^{\infty} (a_k - a_{k-1} - a_{k-2}) x^k \\ &= a_0 + a_1 x - a_0 x \end{aligned}$$

   Hence,

$$G(x) = \frac{1+x}{1-x-x^2}$$

   We want to write this in the following form: $\frac{1}{1-ax}$.

$$
\begin{aligned}
G(x) &= \frac{1+x}{1-x-x^2} \\
&= -\left(\frac{A}{x+\frac{1-\sqrt{5}}{2}} + \frac{B}{x+\frac{1+\sqrt{5}}{2}}\right)
\end{aligned}
$$

$$(A+B=1 \text{ and } \tfrac{A+A\sqrt{5}}{2} + \tfrac{B-B\sqrt{5}}{2} = 1,\ A = \tfrac{1+\sqrt{5}}{2\sqrt{5}}, B = \tfrac{-1+\sqrt{5}}{2\sqrt{5}})$$

$$
\begin{aligned}
G(x) &= \frac{1+x}{1-x-x^2} \\
&= \frac{6+2\sqrt{5}}{4\sqrt{5}}\left(\frac{1}{1-\frac{\sqrt{5}+1}{2}x}\right) + \frac{6-2\sqrt{5}}{4\sqrt{5}}\left(\frac{1}{1-\frac{\sqrt{5}-1}{2}x}\right) \\
&= \frac{6+2\sqrt{5}}{4\sqrt{5}}\sum_{k=0}^{\infty}\left(\frac{\sqrt{5}+1}{2}\right)^k x^k + \frac{6-2\sqrt{5}}{4\sqrt{5}}\sum_{k=0}^{\infty}\left(\frac{\sqrt{5}-1}{2}\right)^k x^k
\end{aligned}
$$

Answer 3

a_n is in the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$
The characteristic equation of the given recurrene relation is the following:

$$r^3 - 4r^2 - r + 4 = 0$$

whose roots are $r_1 = 1$, $r_2 = (-1)$, $r_3 = 4$. To find α_1, α_2, and α_3, we need to solve the following system of equation for α_1, α_2, and α_3.

$$
\begin{array}{ccccccc}
\alpha_1 & + & \alpha_2(-1)^0 & + & \alpha_3 4^0 & = & 4 \\
\alpha_1 & + & \alpha_2(-1)^1 & + & \alpha_3 4^1 & = & 8 \\
\alpha_1 & + & \alpha_2(-1)^2 & + & \alpha_3 4^2 & = & 34
\end{array}
$$

We see that $\alpha_1 = 1$, $\alpha_2 = 1$, and $\alpha_3 = 2$. Hence,

$$a_n = 1 + (-1)^n + 2 \times 4^n$$

Answer 4

For R to be a equivalence relation, it needs to be *reflexive*, *transitive*, and *symmetric*.

1. If $(x_1, y_1)R(x_1, y_1)$ then R is reflexive. This is a trivial thing to do since $3x_1 - 2y_1 = 3x_1 - 2y_1$, obviously.

2. For R to be transitive, the following should be true: "If $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$, then $(x_1, y_1)R(x_3, y_3)$." Let $3x_1 - 2y_1 = k$. If $(x_1, y_1)R(x_2, y_2)$, then $3x_2 - 2y_2 = k$, also. Similarly, if $(x_2, y_2)R(x_3, y_3)$, then $3x_3 - 3y_3 = k$, also. Finally, since $3x_1 - 2y_1 = 3x_3 - 2y_3$, $(x_1, y_1)R(x_3, y_3)$. Thus, R is transitive.

3. For R to be symmetric the following should be true: "If $(x_1, y_1)R(x_2, y_2)$, then $(x_2, y_2)R(x_1, y_1)$." Again, this is a trivial thing to do. $3x_1 - 2y_1 = 3x_2 - 2y_2$, then $3x_2 - 2y_2 = 3x_1 - 2y_1$, and $(x_2, y_2)R(x_1, y_1)$. Thus, R is symmetric.

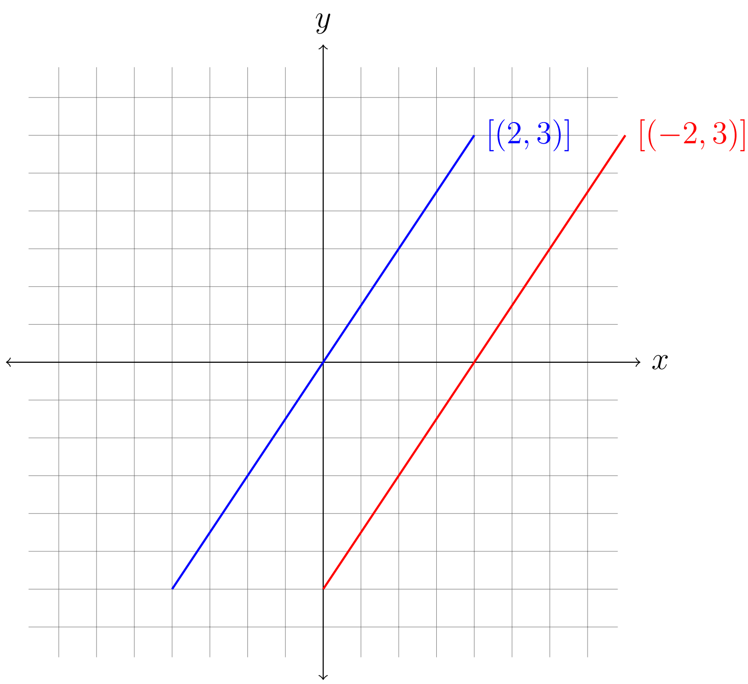

Figure 1: The graphical representation of the given equivalence classes