Mathematical model of quadcopter

The quadcopter structure is presented in Figure 1 including the corresponding angular velocities, torques and forces created by the four rotors (numbered from 1 to 4).

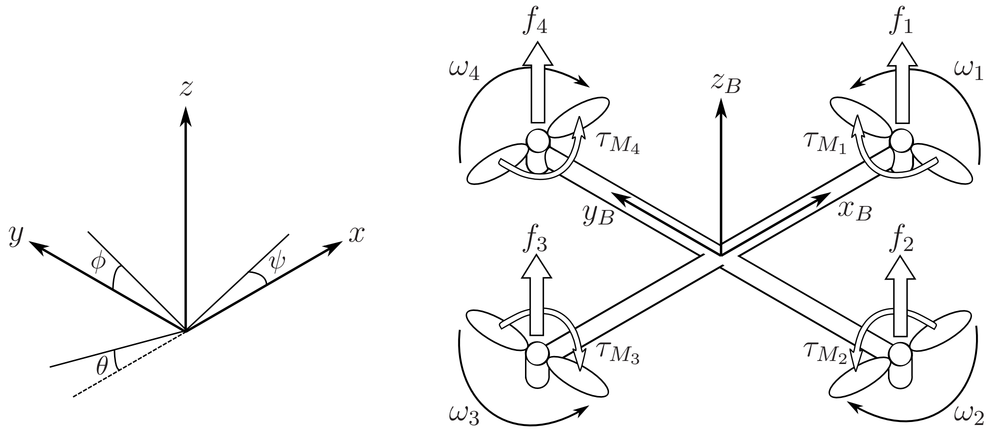

Figure 1: The inertial and body frames of a quadcopter

The absolute linear position of the quadcopter is defined in the inertial frame x,y,z-axes with $\boldsymbol{\xi}$. The attitude, i.e. the angular position, is defined in the inertial frame with three Euler angles $\boldsymbol{\eta}$. Pitch angle θ determines the rotation of the quadcopter around the y-axis. Roll angle ϕ determines the rotation around the x-axis and yaw angle ψ around the z-axis. Vector $\boldsymbol{q}$ contains the linear and angular position vectors

$$\boldsymbol{\xi} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{\eta} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix}. \tag{1}$$

The origin of the body frame is in the center of mass of the quadcopter. In the body frame, the linear velocities are determined by $\boldsymbol{V}_B$ and the angular velocities by $\boldsymbol{\nu}$

$$\boldsymbol{V}_B = \begin{bmatrix} v_{x,B} \\ v_{y,B} \\ v_{z,B} \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{2}$$

The rotation matrix from the body frame to the inertial frame is

$$\boldsymbol{R} = \begin{bmatrix} C_\psi C_\theta & C_\psi S_\theta S_\phi - S_\psi C_\phi & C_\psi S_\theta C_\phi + S_\psi S_\phi \\ S_\psi C_\theta & S_\psi S_\theta S_\phi + C_\psi C_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi \\ -S_\theta & C_\theta S_\phi & C_\theta C_\phi \end{bmatrix}, \tag{3}$$

in which $S_x = sin(x)$ and $C_x = cos(x)$. The rotation matrix $\boldsymbol{R}$ is orthogonal thus $\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$ which is the rotation matrix from the inertial frame to the body frame.

The transformation matrix for angular velocities from the inertial frame to the body frame is $\boldsymbol{W}_\eta$, and from the body frame to the inertial frame is $\boldsymbol{W}_\eta^{-1}$, as shown in [14],

$$\begin{aligned} \dot{\boldsymbol{\eta}} = \boldsymbol{W}_\eta^{-1}\,\boldsymbol{\nu}, \qquad & \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & S_\phi T_\theta & C_\phi T_\theta \\ 0 & C_\phi & -S_\phi \\ 0 & S_\phi / C_\theta & C_\phi / C_\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \\ \boldsymbol{\nu} = \boldsymbol{W}_\eta\,\dot{\boldsymbol{\eta}}, \qquad & \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\phi & C_\theta S_\phi \\ 0 & -S_\phi & C_\theta C_\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}, \end{aligned} \tag{4}$$

in which $T_x = tan(x)$. The matrix $\boldsymbol{W}_\eta$ is invertible if $\theta \neq (2k-1)\phi/2, (k \in \mathbb{Z})$.

The quadcopter is assumed to have symmetric structure with the four arms aligned with the body x- and y-axes. Thus, the inertia matrix is diagonal matrix $\boldsymbol{I}$ in which $I_{xx} = I_{yy}$

$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}. \tag{5}$$

The angular velocity of rotor i, denoted with ω_i, creates force f_i in the direction of the rotor axis. The angular velocity and acceleration of the rotor also create torque τ_{M_i} around the rotor axis

$$f_i = k\,\omega_i^2, \quad \tau_{M_i} = b\,\omega_i^2 + I_M\,\dot{\omega}_i, \tag{6}$$

in which the lift constant is k, the drag constant is b and the inertia moment of the rotor is I_M. Usually the effect of $\dot{\omega}_i$ is considered small and thus it is omitted.

The combined forces of rotors create thrust T in the direction of the body z-axis. Torque $\boldsymbol{\tau}_B$ consists of the torques τ_ϕ, τ_θ and τ_ψ in the direction of the corresponding body frame angles

$$T = \sum_{i=1}^{4} f_i = k \sum_{i=1}^{4} \omega_i^2, \quad \boldsymbol{T}^B = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix}, \tag{7}$$

$$\boldsymbol{\tau}_B = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} l\,k\,(-\omega_2^2 + \omega_4^2) \\ l\,k\,(-\omega_1^2 + \omega_3^2) \\ \sum_{i=1}^{4} \tau_{M_i} \end{bmatrix}, \tag{8}$$

in which l is the distance between the rotor and the center of mass of the quadcopter. Thus, the roll movement is acquired by decreasing the 2nd rotor velocity and increasing the 4th rotor velocity. Similarly, the pitch movement is acquired by decreasing the 1st rotor velocity and increasing the 3th rotor velocity. Yaw movement is acquired by increasing the the angular velocities of two opposite rotors and decreasing the velocities of the other two.

Newton-Euler equations

The quadcopter is assumed to be rigid body and thus Newton-Euler equations can be used to describe its dynamics. In the body frame, the force required for the acceleration of mass $m\dot{\boldsymbol{V}}_B$ and the centrifugal force $\boldsymbol{\nu} \times (m\,\boldsymbol{V}_B)$ are equal to the gravity $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{G}$ and the total thrust of the rotors $\boldsymbol{T}_B$

$$m\dot{\boldsymbol{V}}_B + \boldsymbol{\nu} \times (m\,\boldsymbol{V}_B) = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{G} + \boldsymbol{T}_B. \tag{9}$$

In the inertial frame, the centrifugal force is nullified. Thus, only the gravitational force and the magnitude and direction of the thrust are contributing in the acceleration of the quadcopter

$$m\ddot{\boldsymbol{\xi}} = \boldsymbol{G} + \boldsymbol{R}\boldsymbol{T}_B,$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{T}{m} \begin{bmatrix} C_\psi S_\theta C_\phi + S_\psi S_\phi \\ S_\psi S_\theta C_\phi - C_\psi S_\phi \\ C_\theta C_\phi \end{bmatrix}. \tag{10}$$

In the body frame, the angular acceleration of the inertia $\boldsymbol{I}\dot{\boldsymbol{\nu}}$, the centripetal forces $\boldsymbol{\nu} \times (\boldsymbol{I}\boldsymbol{\nu})$ and the gyroscopic forces $\boldsymbol{\Gamma}$ are equal to the external torque $\boldsymbol{\tau}$

$$\boldsymbol{I}\dot{\boldsymbol{\nu}} + \boldsymbol{\nu} \times (\boldsymbol{I}\boldsymbol{\nu}) + \boldsymbol{\Gamma} = \boldsymbol{\tau},$$

$$\dot{\boldsymbol{\nu}} = \boldsymbol{I}^{-1} \left(- \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx}\, p \\ I_{yy}\, q \\ I_{zz}\, r \end{bmatrix} - I_r \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega_\Gamma + \boldsymbol{\tau} \right),$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} (I_{yy} - I_{zz})\, q\, r / I_{xx} \\ (I_{zz} - I_{xx})\, p\, r / I_{yy} \\ (I_{xx} - I_{yy})\, p\, q / I_{zz} \end{bmatrix} - I_r \begin{bmatrix} q / I_{xx} \\ -p / I_{yy} \\ 0 \end{bmatrix} \omega_\Gamma + \begin{bmatrix} \tau_\phi / I_{xx} \\ \tau_\theta / I_{yy} \\ \tau_\psi / I_{zz} \end{bmatrix}, \tag{11}$$

in which $\omega_\Gamma = \omega_1 - \omega_2 + \omega_3 - \omega_4$. The angular accelerations in the inertial frame are then attracted from the body frame accelerations with the transformation matrix $\boldsymbol{W}_\eta^{-1}$ and its time derivative

$$\begin{aligned} \ddot{\boldsymbol{\eta}} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{W}_\eta^{-1}\boldsymbol{\nu}\right) = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{W}_\eta^{-1}\right)\boldsymbol{\nu} + \boldsymbol{W}_\eta^{-1}\dot{\boldsymbol{\nu}} \\ &= \begin{bmatrix} 0 & \dot{\phi} C_\phi T_\theta + \dot{\theta} S_\phi / C_\theta^2 & -\dot{\phi} S_\phi C_\theta + \dot{\theta} C_\phi / C_\theta^2 \\ 0 & -\dot{\phi} S_\phi & -\dot{\phi} C_\phi \\ 0 & \dot{\phi} C_\phi / C_\theta + \dot{\phi} S_\phi T_\theta / C_\theta & -\dot{\phi} S_\phi / C_\theta + \dot{\theta} C_\phi T_\theta / C_\theta \end{bmatrix} \boldsymbol{\nu} + \boldsymbol{W}_\eta^{-1}\dot{\boldsymbol{\nu}}. \end{aligned} \tag{12}$$

Euler-Lagrange equations

The Lagrangian $\mathcal{L}$ is the sum of the translational E_{trans} and rotational E_{rot} energies minus potential energy E_{pot}

$$\begin{aligned}\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= E_{trans} + E_{rot} - E_{pot} \\ &= (m/2)\,\dot{\boldsymbol{\xi}}^{\mathrm{T}}\dot{\boldsymbol{\xi}} + (1/2)\,\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{I}\,\boldsymbol{\nu} - mgz.\end{aligned} \tag{13}$$

As shown in [10] the Euler-Lagrange equations with external forces and torques are

$$\begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{\tau}\end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right) - \frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}. \tag{14}$$

The linear and angular components do not depend on each other thus they can be studied separately. The linear external force is the total thrust of the rotors. The linear Euler-Lagrange equations are

$$\boldsymbol{f} = \boldsymbol{R}\boldsymbol{T}_B = m\ddot{\boldsymbol{\xi}} + mg\begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{15}$$

which is equivalent with Equation (10).

The Jacobian matrix $\boldsymbol{J}(\boldsymbol{\eta})$ from $\boldsymbol{\nu}$ to $\dot{\boldsymbol{\eta}}$ is

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{\eta}) = \boldsymbol{J} &= \boldsymbol{W}_{\boldsymbol{\eta}}^{\mathrm{T}}\,\boldsymbol{I}\,\boldsymbol{W}_{\boldsymbol{\eta}}, \\ &= \begin{bmatrix} I_{xx} & 0 & -I_{xx}S_\theta \\ 0 & I_{yy}C_\phi^2 + I_{zz}S_\phi^2 & (I_{yy}-I_{zz})\,C_\phi S_\phi C_\theta \\ -I_{xx}S_\theta & (I_{yy}-I_{zz})\,C_\phi S_\phi C_\theta & I_{xx}S_\theta^2 + I_{yy}S_\phi^2 C_\theta^2 + I_{zz}C_\phi^2 C_\theta^2 \end{bmatrix}.\end{aligned} \tag{16}$$

Thus, the rotational energy E_{rot} can be expressed in the inertial frame as

$$E_{rot} = (1/2)\,\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{I}\,\boldsymbol{\nu} = (1/2)\,\ddot{\boldsymbol{\eta}}^{\mathrm{T}}\boldsymbol{J}\,\ddot{\boldsymbol{\eta}}. \tag{17}$$

The external angular force is the torques of the rotors. The angular Euler-Lagrange equations are

$$\boldsymbol{\tau} = \boldsymbol{\tau}_B = \boldsymbol{J}\,\ddot{\boldsymbol{\eta}} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{J})\,\dot{\boldsymbol{\eta}} - \frac{1}{2}\frac{\partial}{\partial\boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^{\mathrm{T}}\boldsymbol{J}\,\dot{\boldsymbol{\eta}}\right) = \boldsymbol{J}\,\ddot{\boldsymbol{\eta}} + \boldsymbol{C}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})\,\dot{\boldsymbol{\eta}}. \tag{18}$$

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in which the matrix $\boldsymbol{C}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})$ is the Coriolis term, containing the gyroscopic and centripetal terms.

The matrix $\boldsymbol{C}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})$ has the form, as shown in [9],

$$\boldsymbol{C}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix},$$

$$\begin{aligned}
C_{11} &= 0 \\
C_{12} &= (I_{yy} - I_{zz})(\dot{\theta} C_\phi S_\phi + \dot{\psi} S_\phi^2 C_\theta) + (I_{zz} - I_{yy})\dot{\psi} C_\phi^2 C_\theta - I_{xx}\dot{\psi} C_\theta \\
C_{13} &= (I_{zz} - I_{yy})\dot{\psi} C_\phi S_\phi C_\theta^2 \\
C_{21} &= (I_{zz} - I_{yy})(\dot{\theta} C_\phi S_\phi + \dot{\psi} S_\phi C_\theta) + (I_{yy} - I_{zz})\dot{\psi} C_\phi^2 C_\theta + I_{xx}\dot{\psi} C_\theta \\
C_{22} &= (I_{zz} - I_{yy})\dot{\phi} C_\phi S_\phi \\
C_{23} &= -I_{xx}\dot{\psi} S_\theta C_\theta + I_{yy}\dot{\psi} S_\phi^2 S_\theta C_\theta + I_{zz}\dot{\psi} C_\phi^2 S_\theta C_\theta \\
C_{31} &= (I_{yy} - I_{zz})\dot{\psi} C_\theta^2 S_\phi C_\phi - I_{xx}\dot{\theta} C_\theta \\
C_{32} &= (I_{zz} - I_{yy})(\dot{\theta} C_\phi S_\phi S_\theta + \dot{\phi} S_\phi^2 C_\theta) + (I_{yy} - I_{zz})\dot{\phi} C_\phi^2 C_\theta \\
&\quad + I_{xx}\dot{\psi} S_\theta C_\theta - I_{yy}\dot{\psi} S_\phi^2 S_\theta C_\theta - I_{zz}\dot{\psi} C_\phi^2 S_\theta C_\theta \\
C_{33} &= (I_{yy} - I_{zz})\dot{\phi} C_\phi S_\phi C_\theta^2 - I_{yy}\dot{\theta} S_\phi^2 C_\theta S_\theta - I_{zz}\dot{\theta} C_\phi^2 C_\theta S_\theta + I_{xx}\dot{\theta} C_\theta S_\theta.
\end{aligned} \tag{19}$$

Equation (18) leads to the differential equations for the angular accelerations which are equivalent with Equations (11) and (12)

$$\ddot{\boldsymbol{\eta}} = \boldsymbol{J}^{-1}\left(\boldsymbol{\tau}_B - \boldsymbol{C}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})\,\dot{\boldsymbol{\eta}}\right). \tag{20}$$

Aerodynamical effects

The preceding model is a simplification of complex dynamic interactions. To enforce more realistical behaviour of the quadcopter, drag force generated by the air resistance is included. This is devised to Equations (10) and (15) with the diagonal coefficient matrix associating the linear velocities to the force slowing the movement, as in [15],

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{T}{m} \begin{bmatrix} C_\psi S_\theta C_\phi + S_\psi S_\phi \\ S_\psi S_\theta C_\phi - C_\psi S_\phi \\ C_\theta C_\phi \end{bmatrix} - \frac{1}{m} \begin{bmatrix} A_x & 0 & 0 \\ 0 & A_y & 0 \\ 0 & 0 & A_z \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \tag{21}$$

in which A_x, A_y and A_z are the drag force coefficients for velocities in the corresponding directions of the inertial frame.

Several other aerodynamical effects could be included in the model. For example, dependence of thrust on angle of attack, blade flapping and airflow distruptions have been studied in [1] and [2]. The influence of aerodynamical effects are complicated and the effects are difficult to model. Also some of the effects have significant effect only in high velocities. Thus, these effects are excluded from the model and the presented simple model is used.

Simulation

Parameter values from are used in the simulations and are presented in Table 1. The values of the drag force coefficients Ax, Ay and Az are selected such as the quadcopter will slow down and stop when angles ϕ and θ are stabilised to zero values.

Table 1: Parameter values for simulation

Parameter	Value	Unit
g	9.81	m/s^2
m	0.468	kg
l	0.225	m
k	$2.980 \cdot 10^{-6}$	
b	$1.140 \cdot 10^{-7}$	
I_M	$3.357 \cdot 10^{-5}$	kg m^2

Parameter	Value	Unit
I_{xx}	$4.856 \cdot 10^{-3}$	kg m^2
I_{yy}	$4.856 \cdot 10^{-3}$	kg m^2
I_{zz}	$8.801 \cdot 10^{-3}$	kg m^2
A_x	0.25	kg/s
A_y	0.25	kg/s
A_z	0.25	kg/s