Robust Optimization of Structural and Aerodynamic Designs

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Why Uncertainty Quantification? I

- Design variables and input parameters are subject to variations
  - Treat as random variables
- Optimization under uncertainty
  - Robust optimization
  - Reliability based optimization
- Benefits:
  - Determine the effects of uncertainties on the design (robust or vulnerable)
  - Obtain confidence intervals for results (range of possible outcomes)
    - 95% probability (confidence) that the target C_L is achieved
    - 1% probability of violation of constraint $\#10$
  - Identify the limitations of the design (and improve)
  - Reliability analysis for certification and quality assurance purposes

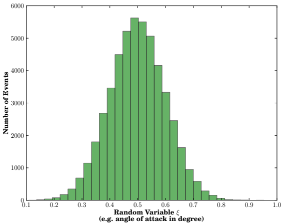
Uncertainty Types

- Aleatory / Irreducible / Type A
- Epistemic / Reducible / Type B
- Mixed

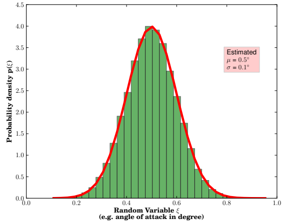
Aleatory Uncertainties I

Characteristics

- Inherent randomness or variations:
  - input parameters (Youngs modulus, shear force)
  - design variables
  - operating environment (cruise settings, temperature)
- Input probability distributions are known (sometimes assumed)
- Goal is to determine the output distribution



Available data

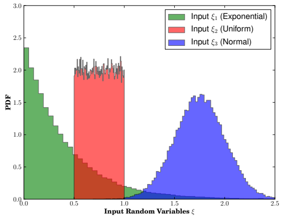


Fitted/Assumed distribution

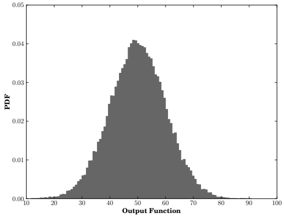
Aleatory Uncertainties II

Quantifying Aleatory Uncertainties

- Need to know the input–output relationship of uncertainties
- Use Monte Carlo Sampling (MCS)
- Need thousands of simulations
- Use **surrogate models** to approximate the simulation output (kriging, polynomial chaos)

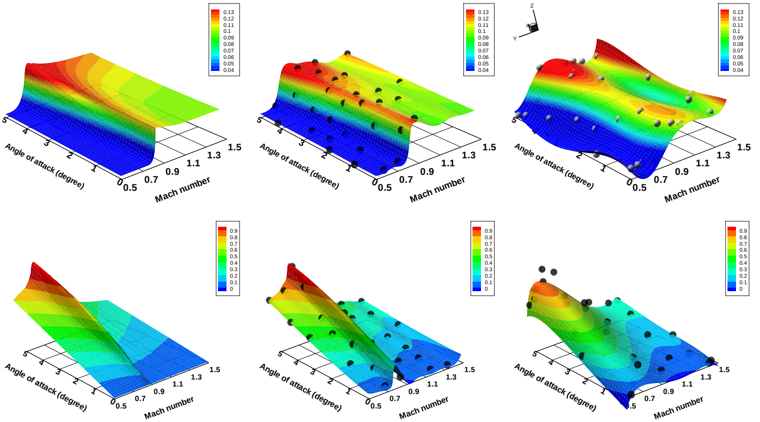


Input distributions



Output distribution

Aleatory Uncertainties III



Contours of exact database (left), kriging (middle) and PCE (right) for drag (top) and lift coefficients (bottom) with 30 training points.

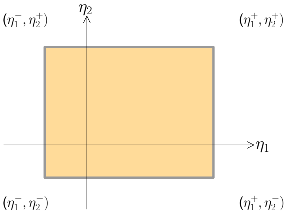
[1] K. Boopathy and M.P. Rumpfkeil, "Unified Framework for Training Point Selection and Error Estimation for Surrogate Models", AIAA Journal. 2014. 10.2514/1.J053064.

[2] K. Boopathy and M.P. Rumpfkeil, "A Multivariate Interpolation and Regression Enhanced Kriging Surrogate Model", 21st AIAA Computational Fluid Dynamics Conference, San Diego, June 2013. AIAA Paper 2013-2964.

Epistemic Uncertainties I

Characteristics

- Lack of knowledge about the appropriate value
- Only bounds can be specified $I(\boldsymbol{\eta}) = [\boldsymbol{\eta}^-, \boldsymbol{\eta}^+] = [\bar{\boldsymbol{\eta}} - \boldsymbol{\tau}, \bar{\boldsymbol{\eta}} + \boldsymbol{\tau}]$
- Goal: determine the worst and best scenarios within the interval $I(\boldsymbol{\eta})$



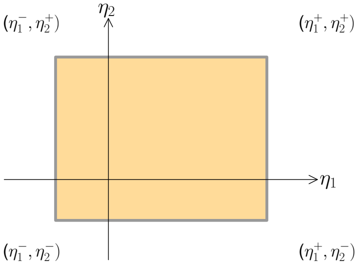
Bounds on epistemic variables

Epistemic Uncertainties II

Goal: determine the worst and best scenarios within the bounds

1. Extensive Sampling

- Need $10^3 - 10^6$ simulations
- Prohibitively expensive for bigger problems



Bounds on epistemic variables

Epistemic Uncertainties III

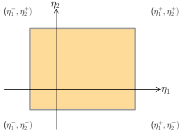
Goal: determine the worst and best scenarios within the bounds

2. Box Constrained Optimization

- Optimization problem:

$$\begin{aligned} \underset{\boldsymbol{\beta}}{\text{minimize/maximize}} \quad & f = f(\boldsymbol{\eta}), \\ \text{subject to} \quad & \boldsymbol{\beta} \in I(\boldsymbol{\eta}) = [\bar{\boldsymbol{\eta}} - \boldsymbol{\tau}, \bar{\boldsymbol{\eta}} + \boldsymbol{\tau}]. \end{aligned}$$

- L-BFGS optimizer (needs gradients)
- Attractive even for bigger problems (scales linearly)



Bounds on epistemic variables

Mixed Aleatory & Epistemic Uncertainties

Quantifying Mixed Uncertainties

- Comprise of both aleatory ξ and epistemic uncertainties η
  - Naive approach: Nested Sampling
    - Very expensive (millions of function evaluations)
    - Not computationally affordable
  - Our approach: IMCS+BCO [1,2]
    - Surrogate models for aleatory uncertainties
    - Box constrained optimization for epistemic uncertainties
    - Few hundred (or thousand) function evaluations (manageable)

1.Helton, J. C., Oberkampf, J. D. J. W. L., and Sallaberry, C. J., "Representation of Analysis Results Involving Aleatory and Epistemic Uncertainty", Tech. Rep. SAND2008-4379, Sandia National Laboratories, 2008.
2.B.A. Lockwood, M. P. Rumpfkeil, W. Yamazaki and D. J. Mavriplis., "Uncertainty Quantification in Viscous Hypersonic Flows using Gradient Information and Surrogate Modeling", 49th AIAA Aerospace Meeting and Exhibit, Orlando, Jan 2011. AIAA paper 2011-885.

Optimization Problem Formulation I

Deterministic Optimization

$$\min_{\mathbf{d}} \quad J = J(f, \mathbf{q}, \mathbf{d}),$$
$$\text{s.t.} \quad R(\mathbf{q}, \mathbf{d}) = 0,$$
$$g(f, \mathbf{q}, \mathbf{d}) \leq 0.$$

Robust Optimization

$$\min_{\boldsymbol{\xi}, \boldsymbol{\eta}} \quad \mathcal{J} = \mathcal{J}(\mu_{f*}, \sigma^2_{f*}, \mathbf{q}, \boldsymbol{\xi}, \boldsymbol{\eta}),$$
$$\mathcal{J} = w_1 \boxed{\mu_{f*}} + w_2 \boxed{\sigma^2_{f*}}$$
$$\text{s.t.} \quad R(\mathbf{q}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0,$$
$$g^r = g(\boxed{\mu_{f*}}, \mathbf{q}, \boldsymbol{\xi}, \boldsymbol{\eta}) + k \boxed{\sigma_{f*}} \leq 0.$$

$$\boxed{\mu_{f*}} \approx \frac{1}{\widetilde{N}} \sum_{k=1}^{\widetilde{N}} \widehat{f^*}(\boldsymbol{\alpha}^k)$$

$$\boxed{\sigma^2_{f*}} \approx \left(\frac{1}{\widetilde{N}} \sum_{k=1}^{\widetilde{N}} \widehat{f^*}^2(\boldsymbol{\alpha}^k) \right) - \mu^2_{f*}$$

So far ...

Deterministic Optimization

Deterministic input $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)$

Optimization Under Uncertainty

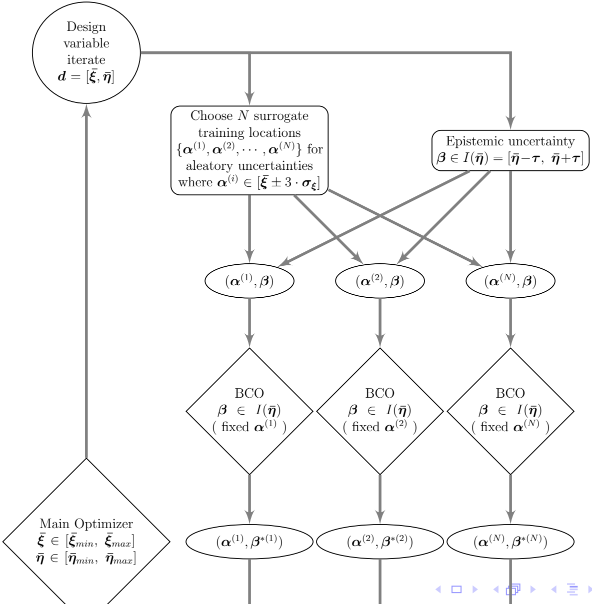
- Aleatory (known probability distributions)

  e.g. $\boldsymbol{\xi} = (\mathcal{N}(10^7, 10^3), \mathcal{U}(2^\circ, 3^\circ), \dots)$
- Epistemic (known interval)

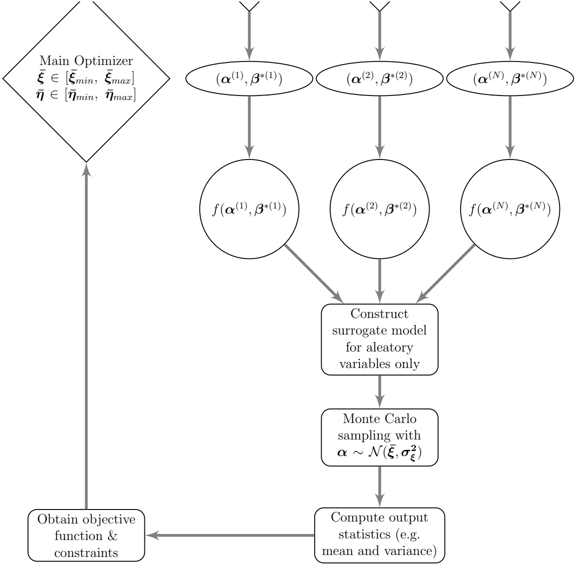
  e.g. $\boldsymbol{\eta} = ([299, 301\ K],\ [0.95, 1.05\ in], \dots)$
- Mixed

  e.g. $d = (\mathcal{N}(10^7, 10^3), \dots, [0.95, 1.05\ in], \dots)$

Mixed OUU Framework: IMCS+BCO I

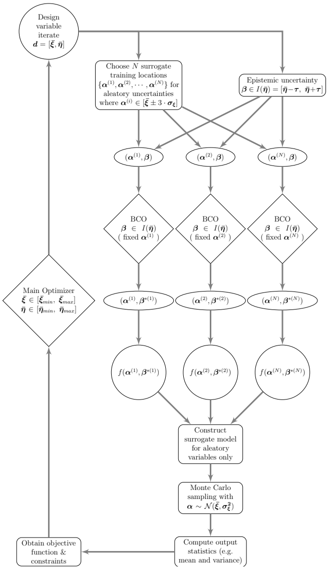


Mixed OUU Framework: IMCS+BCO II



Framework for optimization under mixed aleatory and epistemic uncertainties.

Mixed OUU Framework: IMCS+BCO III



Airfoil Optimization I

Lift constrained drag minimization

Deterministic Problem

$$\begin{aligned} \underset{\mathbf{d}}{\text{minimize}} \quad & \mathcal{J} = C_d, \\ \text{subject to} \quad & g = C_l - {C_l}^{+} \geq 0. \end{aligned}$$

- Need one flow and adjoint solution per optimizer iteration.

Robust Optimization Problem

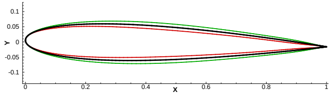
$$\begin{aligned} \underset{\boldsymbol{\xi},\boldsymbol{\eta}}{\text{minimize}} \quad & \mathcal{J} = \mu_{C_{d_{max}}} + \sigma^2_{C_{d_{max}}}, \\ \text{subject to} \quad & g = (\mu_{C_{l_{min}}} + k\sigma_{C_{l_{min}}}) - {C_l}^{+} \geq 0. \end{aligned}$$

- Need two surrogate models (C_L and C_D) per optimizer iteration
- **max**, **min** reflects the best/worst case scenario from BCO
- Target $C_L^{+} = 0.6$
- k= 0,1,2,3 (robustness parameter)

Airfoil Optimization II

Table Data for robust optimization of airfoil

Random Variable	Description	Uncertainty Type	τ_{min}	τ_{max}	Standard Deviation
$\eta_{1,2,13,14}$	Shape design variables	Epistemic	-0.00125	0.00125	-
η_{3-12}	Shape design variables	Epistemic	-0.01	0.01	-
ξ_α	Angle of attack	Aleatory	-	-	0.1°
ξ_M	Mach number	Aleatory	-	-	0.01



The NACA 0012 airfoil (in black) and airfoils resulting from perturbations of ± 0.0025 (in gray).

- Seven shape design variables at 20%, 30%, 40%, 50%, 60%, 80%, and 90% chord
- Flow variable bounds: $0^\circ \leq \alpha \leq 4^\circ$ and $0.6 \leq M \leq 0.78$
- 2D Euler solver [1]
- Henne-Hicks sine bump function

[1] Mani, K., Mavriplis, D.J, "Unsteady Discrete Adjoint Formulation for Two-Dimensional Flow Problems with Deforming Meshes", AIAA Journal, Vol. 46-6, June 2008, pp. 1351-1364

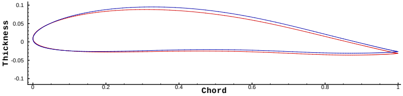
Optimization Results

Table Optimization results for airfoil

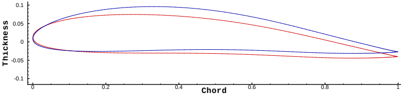
Type	k	P_k	$\mu_{c_{d_{max}}}$	$\sigma^2_{c_{d_{max}}}$	$\mu_{c_{l_{min}}}$	$\sigma_{c_{l_{min}}}$	α	M	No. of F/FG Evals. & Iterations
Initial	-	-	$4.72 \cdot 10^{-4}$	-	0.335	-	2.000°	0.650	
Deterministic	-	-	$1.17 \cdot 10^{-3}$	-	0.600	-	2.510°	0.600	49/49 – 24
Robust-KR	0	0.5000	$2.72 \cdot 10^{-3}$	$2.03 \cdot 10^{-7}$	0.600	$1.84 \cdot 10^{-2}$	2.013°	0.600	844/844-23
Robust-PC	0	0.5000	$2.62 \cdot 10^{-3}$	$5.80 \cdot 10^{-8}$	0.600	$1.82 \cdot 10^{-2}$	2.389°	0.600	675/6751-16
Robust-KR	1	0.8413	$2.93 \cdot 10^{-3}$	$3.07 \cdot 10^{-7}$	0.619	$1.86 \cdot 10^{-2}$	2.065°	0.600	434/434-13
Robust-PC	1	0.8413	$2.73 \cdot 10^{-3}$	$2.50 \cdot 10^{-7}$	0.618	$1.84 \cdot 10^{-2}$	3.058°	0.600	434/434-15
Robust-KR	2	0.9772	$3.10 \cdot 10^{-3}$	$4.46 \cdot 10^{-7}$	0.637	$1.88 \cdot 10^{-2}$	2.179°	0.600	831/831-19
Robust-PC	2	0.9772	$3.20 \cdot 10^{-3}$	$8.58 \cdot 10^{-7}$	0.637	$1.89 \cdot 10^{-2}$	2.193°	0.600	710/710-22
Robust-KR	3	0.9986	$3.28 \cdot 10^{-3}$	$6.23 \cdot 10^{-7}$	0.657	$1.90 \cdot 10^{-2}$	2.301°	0.600	650/650-21
Robust-PC	3	0.9986	$3.25 \cdot 10^{-3}$	$9.83 \cdot 10^{-7}$	0.658	$1.92 \cdot 10^{-2}$	2.352°	0.600	1145/1145-21
Robust-KR	4	0.9999	$3.56 \cdot 10^{-3}$	$9.50 \cdot 10^{-7}$	0.677	$1.93 \cdot 10^{-2}$	2.421°	0.600	620/620-15
Robust-PC	4	0.9999	$3.65 \cdot 10^{-3}$	$1.25 \cdot 10^{-6}$	0.677	$1.93 \cdot 10^{-2}$	2.427°	0.600	2104/2104-36

- Deterministic: Optimum sought at the boundary of lift-constraint hyperplane
- Robust: Optimum sought at at a distance of $k-$ standard deviations away from the lift-constraint hyperplane
- 20 times more simulation requirements compared to deterministic optimization

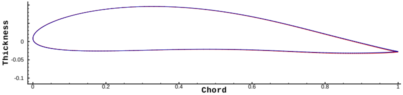
Airfoil Shapes I



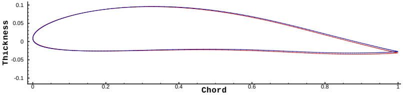
Robust Airfoils $k = 0$



Robust Airfoils $k = 1$



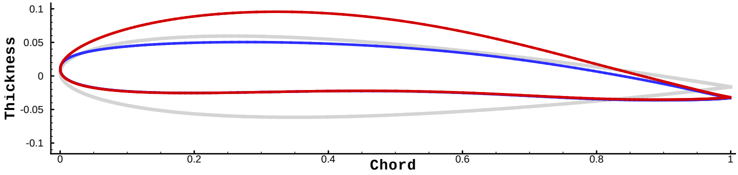
Robust Airfoils $k = 2$



Robust Airfoils $k = 3$

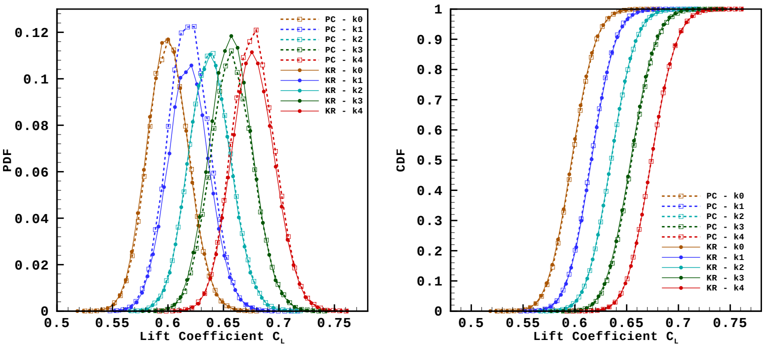
Red=Polynomial Chaos, Blue=Kriging

Airfoil Shapes II



NACA 0012, Deterministic, Robust Airfoils corresponding to $k = 4$.

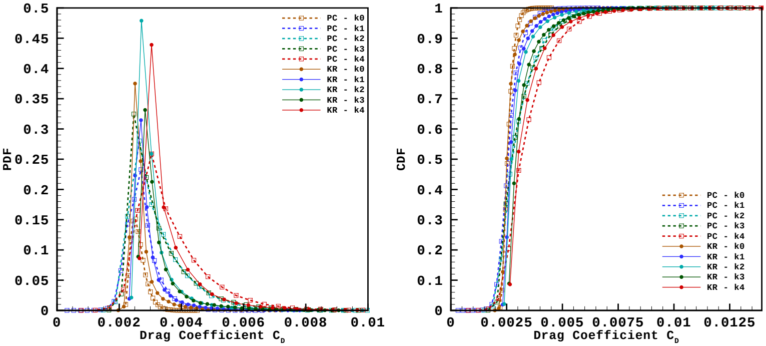
Output Distributions I



PDF and CDF lift coefficient at the optimum design.

- Shift to the right with increasing $k \rightarrow$ more lift is generated
- Only very less chances of falling short of the target $C_L^+ = 0.6$

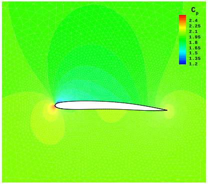
Output Distributions II



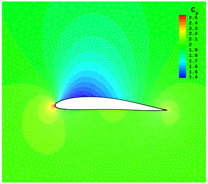
PDF and CDF drag coefficient at the optimum design.

- Shift to the right with increasing $k \rightarrow$ more drag (robustness is achieved at the expense of an increased cost function-drag penalty)

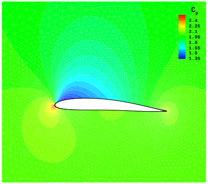
Pressure Distributions



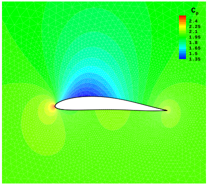
Deterministic



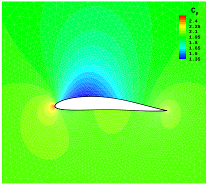
Robust $k = 0$



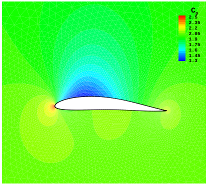
Robust $k = 1$



Robust $k = 2$



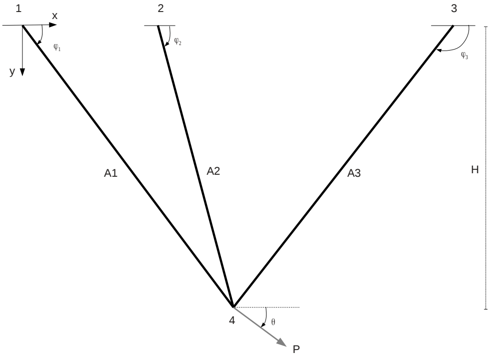
Robust $k = 3$



Robust $k = 4$

Contour plots of pressure coefficients C_p at different optimum designs using polynomial chaos.

Three Bar Truss I



A schematic of the three-bar truss structure.

- Minimum weight truss design
- 8 constraints (6 stress, 2 displacement)
- Design variables (areas A_i and orientations ϕ_i)

Three Bar Truss II

Mathematical Formulation

$$
\begin{aligned}
\underset{\mathbf{d}}{\text{minimize}} \quad & W = \frac{A_1 \gamma H}{\sin(\phi_1)} + \frac{A_2 \gamma H}{\sin(\phi_2)} + \frac{A_3 \gamma H}{\sin(\phi_3)}, \\
\text{subject to} \quad & g_1 = \frac{\sigma_1}{\sigma_{1_{max}}} - 1 \le 0, \\
& g_2 = \frac{\sigma_2}{\sigma_{2_{max}}} - 1 \le 0, \\
& g_3 = \frac{\sigma_3}{\sigma_{3_{max}}} - 1 \le 0, \\
& g_4 = -\frac{\sigma_1}{\sigma_{1_{max}}} - 1 \le 0, \\
& g_5 = -\frac{\sigma_2}{\sigma_{2_{max}}} - 1 \le 0, \\
& g_6 = -\frac{\sigma_3}{\sigma_{3_{max}}} - 1 \le 0, \\
& g_7 = \frac{Q_{4x}}{Q_{4x_{max}}} - 1 \le 0, \\
& g_8 = \frac{Q_{4y}}{Q_{4y_{max}}} - 1 \le 0.
\end{aligned}
$$

Bounds

$$
\begin{aligned}
0.25\ in^2 \le\ & A_1,\ A_2,\ A_3 \le 5.0\ in^2, \\
30^\circ \le\ & \phi_1 \le 60^\circ, \\
60^\circ \le\ & \phi_2 \le 120^\circ, \\
120^\circ \le\ & \phi_3 \le 150^\circ.
\end{aligned}
$$

Solver

- Stresses and displacements using hand-coded FEA procedure

Three Bar Truss III

Table Design data for three-bar truss.

Quantity	Description	Value	Unit
P	Load	30000	*lb*
θ	Loading angle	50	*deg*
E	Young's modulus	10^7	*psi*
γ	Weight density	0.1	lb/in^3
H	Reference length (projection on $y-$axis)	10	*in*
$\sigma 1_{max}$	Allowable axial stress on bar 1	5000	*psi*
$\sigma 2_{max}$	Allowable axial stress on bar 2	10000	*psi*
$\sigma 3_{max}$	Allowable axial stress on bar 3	5000	*psi*
$u4x_{max}$	Allowable x-displacement at 4	0.005	*in*
$u4y_{max}$	Allowable y-displacement at 4	0.005	*in*
ϵ_1	Constraint violation tolerance	10^{-3}	-
ϵ_2	Norm of design change $\|\Delta \mathbf{d}\|$	10^{-3}	-

Three Bar Truss IV

Robust Optimization Problem

$$\begin{aligned} \underset{\boldsymbol{\xi},\boldsymbol{\eta}}{\text{minimize}} \quad & \mathcal{J} = \mu_W + \vartheta_W, \\ \text{subject to} \quad & g_i^r = \mu_{g_i} + k\sigma_{g_i} \leq 0, \quad \text{for } i = 1, \ldots, 8 \end{aligned}$$

- Area design variables A_i (epistemic with $\tau_i = 0.1\ in^2$)
  - Propagated via BCO
- Orientation design variables ϕ_i (aleatory with $\sigma_i = 0.1°$)
  - Propagation via surrogate sampling
  - Kriging and PCE built with 70 training points

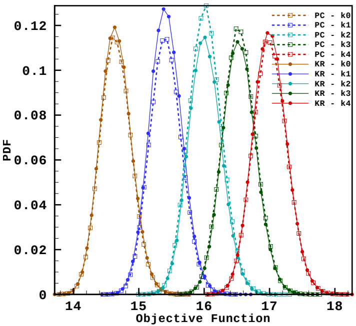
Three Bar Truss V

Table Optimization results for three-bar truss problem.

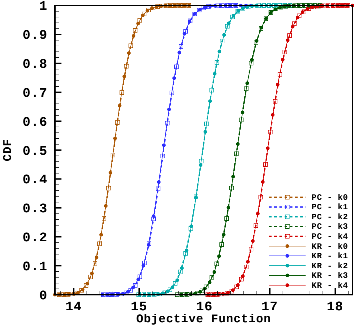
Type	k	P_k	A_1 in^2	A_2 in^2	A_3 in^2	ϕ_1 deg	ϕ_2 deg	ϕ_3 deg	μ_W lb	σ_W lb	C_v -	No. of F/FG Evals. & Iterations
Initial design	-	-	2.0	2.0	2.0	45.0	90.0	135.0	7.66	-	-	-
Det $F_s = 1.0$	-	-	5.00	1.42	2.30	37.6	60.0	150.0	14.45	-	-	108/108-12
Det $F_s = 1.3$	-	-	5.00	4.95	5.00	39.5	60.0	143.6	22.00	-	-	126/126-14
Robust-KR	0	0.5000	5.00	1.45	2.37	37.7	60.0	150.0	14.65	0.24	0.0162	17559/17559-12
Robust-PC	0	0.5000	5.00	1.45	2.37	37.7	60.0	150.0	14.65	0.24	0.0162	17615/17615-12
Robust-KR	1	0.8413	5.00	1.66	2.66	37.5	60.0	149.3	15.41	0.24	0.0159	21963/21963-14
Robust-PC	1	0.8413	5.00	1.66	2.66	37.5	60.0	149.3	15.41	0.24	0.0159	20555/20555-13
Robust-KR	2	0.9772	5.00	1.84	2.92	37.5	60.0	148.6	16.02	0.25	0.0155	23594/23594-13
Robust-PC	2	0.9772	5.00	1.84	2.92	37.5	60.0	148.6	16.02	0.25	0.0155	33555/33555-18
Robust-KR	3	0.9986	5.00	1.99	3.15	37.5	60.0	148.2	16.54	0.25	0.0153	20771/20771-12
Robust-PC	3	0.9986	5.00	1.99	3.15	37.5	60.0	148.2	16.54	0.25	0.0153	17938/17938-12
Robust-KR	4	0.9999	5.00	2.13	3.36	37.6	60.0	147.9	17.00	0.26	0.0151	31178/31178-17
Robust-PC	4	0.9999	5.00	2.13	3.36	37.6	60.0	147.9	17.00	0.26	0.0151	19500/19500-12

- A deterministic design with no F_s is 15% lighter than a robust design specified by $k = 4$.
- A deterministic design with F_s of 1.3 is 29% heavier than a robust design specified by $k = 4$.

Objective Function Distribution:



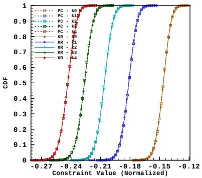
PDF



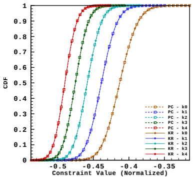
CDF

- Cost function increases with desired robustness.

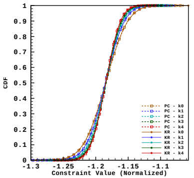
Three Bar Truss VII



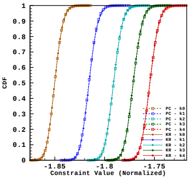
Constraint 1



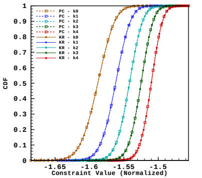
Constraint 2



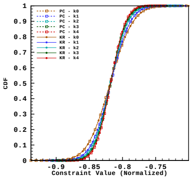
Constraint 3



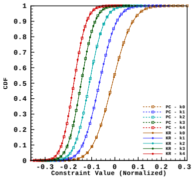
Constraint 4



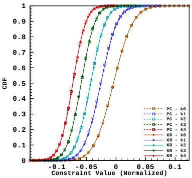
Constraint 5



Constraint 6



Constraint 7



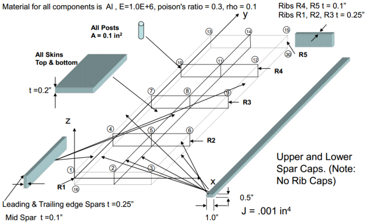
Constraint 8

Cumulative distribution function of constraints at robust and deterministic optimum designs.

- Constraints become more negative with desired robustness (prevents failure due to uncertainties).

Summary and Future Work

- Aleatory uncertainties using surrogate models
- Epistemic uncertainties using box constrained optimization
- Mixed uncertainties using IMCS+BCO
- Robust optimization under uncertainty
  - Airfoil Optimization
  - Threebar Truss Design
  - Cantilever beam Design (in paper)
- Application to wing structural optimization (Scitech 2015)



Acknowledgments

1. Graduate Student Summer Fellowship (University of Dayton)
2. Wataru Yamazaki – Kriging surrogate
3. Karthik Mani – Euler Solver

References

1. K. Boopathy and M.P. Rumpfkeil, “A Unified Framework for Training Point Selection and Error Estimation for Surrogate Models”, AIAA Journal. 10.2514/1.J053064.
2. Helton, J. C., Oberkampf, J. D. J. W. L., and Sallaberry, C. J., “Representation of Analysis Results Involving Aleatory and Epistemic Uncertainty”, Tech. Rep. SAND2008-4379, Sandia National Laboratories, 2008.
3. K. Boopathy and M.P. Rumpfkeil, “A Multivariate Interpolation and Regression Enhanced Kriging Surrogate Model”, 21st AIAA Computational Fluid Dynamics Conference, San Diego, June 2013. AIAA Paper 2013-2964.
4. B.A. Lockwood, M. P. Rumpfkeil, W. Yamazaki and D. J. Mavriplis., “Uncertainty Quantification in Viscous Hypersonic Flows using Gradient Information and Surrogate Modeling”, 49th AIAA Aerospace Meeting and Exhibit, Orlando, Jan 2011. AIAA paper 2011-885.
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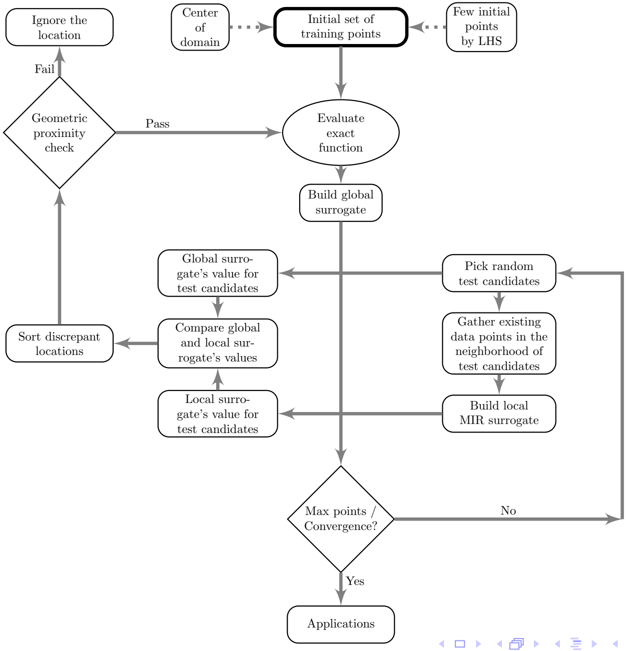
6. Keane, A. and Nair, P., “Computational Approaches for Aerospace Design”, John Wiley & Sons, 2005.
7. Yamazaki, W. and Mavripilis, D. J., “Derivative-Enhanced Variable Fidelity Surrogate Modeling for Aerodynamic Functions,” AIAA Journal , Vol. 51, No. 1, 2013, pp. 126–137.

Questions

Any Questions?

- Email: KomahanBoopathy@gmail.com
- https://github.com/komahanb/kriging.git
- https://github.com/komahanb/pchaos.git

Dynamic Training Framework



Aleatory Gradients

Aleatory gradients

$$\frac{d\mathcal{J}}{d\xi} = \frac{\partial\mathcal{J}}{\partial\mu_{f*}}\frac{d\mu_{f*}}{d\xi} + \frac{\partial\mathcal{J}}{\partial\vartheta_{f*}}\frac{d\vartheta_{f*}}{d\xi} = w_1 \boxed{\frac{d\mu_{f*}}{d\xi}} + w_2 \boxed{\frac{d\vartheta_{f*}}{d\xi}} \tag{1}$$

$$\boxed{\frac{d\mu_{f*}}{d\xi}} \approx \frac{1}{\widetilde{N}}\sum_{k=1}^{\widetilde{N}} \frac{d\widehat{f^*}(\boldsymbol{\alpha}^k)}{d\boldsymbol{\alpha}^k}\frac{d\boldsymbol{\alpha}^k}{d\xi} = \frac{1}{\widetilde{N}}\sum_{k=1}^{\widetilde{N}} \frac{d\widehat{f^*}(\boldsymbol{\alpha}^k)}{d\boldsymbol{\alpha}^k} \tag{2}$$

$$\boxed{\frac{d\vartheta_{f*}}{d\xi}} \approx \left(\frac{2}{\widetilde{N}}\sum_{k=1}^{\widetilde{N}} \widehat{f^*}(\boldsymbol{\alpha}^k)\frac{d\widehat{f^*}(\boldsymbol{\alpha}^k)}{d\boldsymbol{\alpha}^k}\right) - 2\mu_{f*}\frac{d\mu_{f*}}{d\xi} \tag{3}$$

Epistemic Gradients I

Epistemic gradients

$$\frac{d\mathcal{J}}{d\boldsymbol{\eta}} = \frac{\partial\mathcal{J}}{\partial\mu_{f*}}\frac{d\mu_{f*}}{d\boldsymbol{\eta}} + \frac{\partial\mathcal{J}}{\partial\vartheta_{f*}}\frac{d\vartheta_{f*}}{d\boldsymbol{\eta}} = w_1 \frac{d\mu_{f*}}{d\boldsymbol{\eta}} + w_2 \frac{d\vartheta_{f*}}{d\boldsymbol{\eta}} \tag{4}$$

Approximations

$$\frac{d\mu_{f*}}{d\boldsymbol{\eta}} \approx \left.\frac{df^*}{d\boldsymbol{\eta}}\right|_{(\boldsymbol{\xi}=\bar{\boldsymbol{\xi}},\boldsymbol{\eta}=\bar{\boldsymbol{\eta}})} \quad \text{and} \quad \frac{d\vartheta_{f*}}{d\boldsymbol{\eta}} \approx 0 \tag{5}$$

Cantilever Beam Design I

Problem Formulation

$$
\begin{aligned}
\underset{b,d}{\text{minimize}} \quad & A(b,d) = bd, \\
\text{subject to} \quad & g_1(b,d,\mathcal{M}) = \frac{6\mathcal{M}}{bd^2\sigma_{allow}} - 1 \le 0, \\
& g_2(b,d,\mathcal{V}) = \frac{3\mathcal{V}}{2bd\tau_{allow}} - 1 \le 0, \\
& g_3(b,d) = \frac{d}{2b} - 1 \le 0, \\
\text{bounds} \quad & 100\ mm \ \le\ b,d\ \le\ 600\ mm,
\end{aligned}
$$

Cantilever Beam Design II

Table Data and assumed uncertain parameters for cantilever beam design problem.

Random Variable	Description	Uncertainty Type	τ_{min}	τ_{max}	Mean	Standard Deviation	Unit
b	Breadth	Epistemic	-10	10	-	-	mm
d	Width	Epistemic	-10	10	-	-	mm
$\mathcal{M}$	Bending Moment	Aleatory	-	-	$40 \cdot 10^6$	40000	$N \cdot mm$
$\mathcal{V}$	Shear Force	Aleatory	-	-	$150 \cdot 10^3$	1500	N

Robust Optimization Problem

$$\begin{aligned} \underset{b,d}{\text{minimize}} \quad & A(b,d) = \mu_A + \sigma_A^2, \\ \text{subject to} \quad & g_1^r(b,d,\mathcal{M}) = \mu_{g_1} + k\sigma_{g_1} \leq 0, \\ & g_2^r(b,d,\mathcal{V}) = \mu_{g_2} + k\sigma_{g_2} \leq 0, \\ & g_3^r(b,d) = \mu_{g_3} + k\sigma_{g_3} \leq 0. \end{aligned}$$

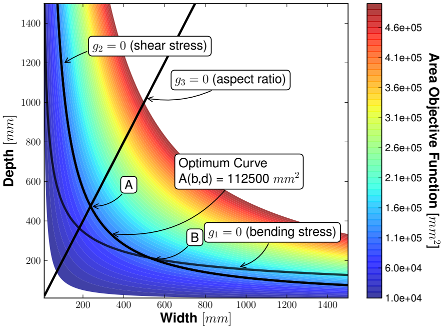
Cantilever Beam Design III

Table Optimization results for cantilever beam design problem.

Type	k	P_k	Width b mm	Depth d mm	Area A $\cdot 10^3\ mm^2$	No. of F/FG Evals. & Iterations
Initial Design	-	-	300	300	90.0	-
Det ($F_s = 1.0$)	-	-	335.5	335.4	112.5	33/33-7
Det ($F_s = 1.5$)	-	-	595.5	283.4	168.7	45/45-8
Robust-KR	0	0.5000	347.4	343.4	126.3	7046/3523-7
Robust-PC	0	0.5000	347.4	343.4	126.3	7917/7917-8
Robust-KR	1	0.8413	349.7	344.5	127.5	7146/3573-7
Robust-PC	1	0.8413	349.7	344.5	127.5	8037/8037-8
Robust-KR	2	0.9772	398.5	305.4	128.8	7686/3843-7
Robust-PC	2	0.9772	398.5	305.4	128.8	9661/9661-9
Robust-KR	3	0.9986	386.5	317.8	130.0	8694/4347-8
Robust-PC	3	0.9986	386.5	317.8	130.0	11669/11669-10
Robust-KR	4	0.9999	356.6	347.5	131.1	7286/3643-7
Robust-PC	4	0.9999	356.6	347.5	131.1	8196/8196-8

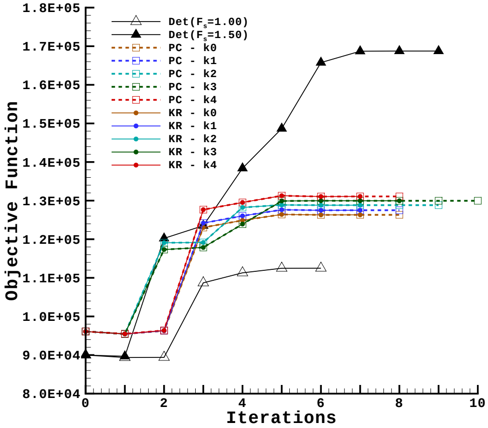
- The cross-sectional area increases by roughly 17% for a design corresponding to $k = 4$ compared to a deterministic design with no factor of safety.
- The robust design ($k = 4$) is 29% lighter than a deterministic design with a factor of safety of 1.5.

Cantilever Beam Design IV



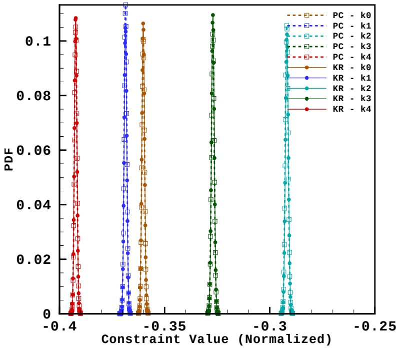
Graphical solution to the minimum area beam design problem.

Cantilever Beam Design V

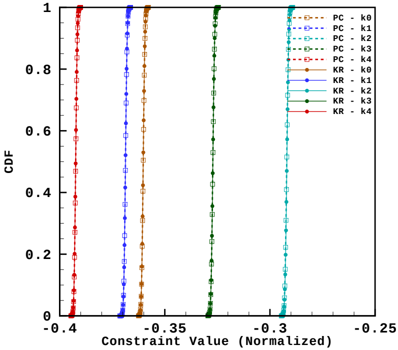


Iteration history for the beam design problem.

Cantilever Beam Design VI

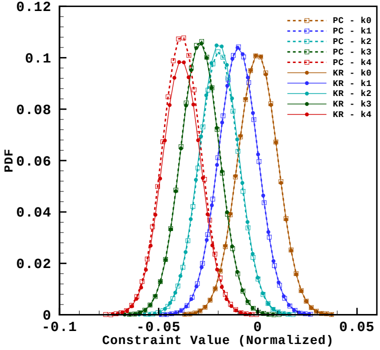


PDF (g_1)

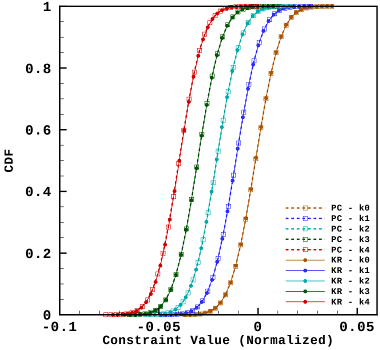


CDF (g_1)

Cantilever Beam Design VII



PDF (g_2)



CDF (g_2)

Output PDF (left) and CDF (right) of constraint g_1 and g_2.