1 Infiniteness of delay-1 unary deterministic oritatami system

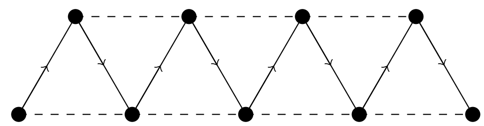


Fig. 1. zig-zag conformation

1.1 Introduction

In this section, we consider on finiteness of structures produced deterministically at delay 1. In result, cases of arity 1 and 3 can only yield finite structures which is size of $\mathcal{O}(n)$, and cases of arity 4 and more can only yield finite structures which is size of $mathcalO(n^2)$, and a case of arity 2 can yield infinite structures but they are only the zig-zag conformation shown in Fig.1.

Let Ξ be a deterministic oritatami system of delay 1 and arity 2. Assume its seed σ consists of n beads. For $i \geq 0$ let C_i be the unique elongation of σ by $w[1..i]$ that is foldable by Ξ. Hence $C_0 = \sigma$.

Let us consider the stabilization of the i-th bead a_i upon C_{i-1}. The bead cannot collaborate with any succeeding bead $w[i+1], w[i+2], \cdots$ at delay 1. There are just two ways to get stabilized at delay 1. One way is to be bound to another bead. The other way is through a *tunnel section*. A tunnel section consists of four beads that occupy four neighbors of a point (Fig.2). Accordingly, how they are stabilized can be described by a binary sequence S of b's (bound) and t's (tunnel section); priority is given to t, that is, $S[i] = t$ if the i-th bead w_i is stabilized not only by being bound but also through a tunnel section.

Assume that four of the six neighbors of a point p are occupied by beads $a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}$ while the other two are free. We call such a bead p as $inside\ of\ a\ tunnel$ and such beads p' as *entrance of a tunnel* without a case that p' is inside of a tunnel. If the beads $w[i-2]$ and $w[i-1]$ are stabilized respectively at one of the two free neighbors and at p one after another, then the next bead $w[i]$ cannot help but be stabilized at the other free neighbor. In this way, $w[i]$ can get stabilized without being bound.

We say that point p is reachable from a conformation C if there exists a directed path P' from the last point of C that does not cross the path of C. We define *binding capability* with reachable.

Definition 1 (binding capability). *Let B_i be $(\{(h,i) | {}^\forall h < i\} \cup \{(i,j) | {}^\forall j > i\}) \cap H$. Moreover, let R_i be a set of neighbors of $w[i]$ that are free and reachable from C_j where C_j is a conformation which stabilized until $w[j]$. We represent* *the number of binding capabilities of a conformation C_j as $\#bc(C_j)$. $\#bc(C_j)$ is defined by $\sum_{k=-n+1}^{j} \min\{|B_k|, |R_k|\}$.*

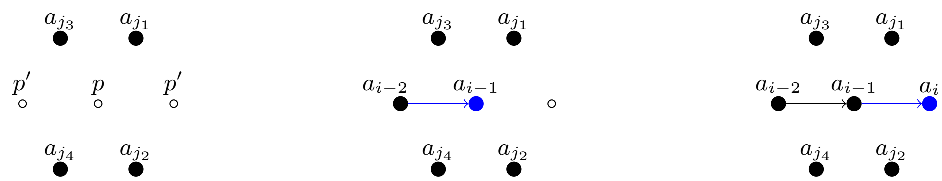


Fig. 2. Through a tunnel section

Theorem 1 (Tunnel Troll Theorem). *Let Ξ be a unary oritatami system of $\delta = 1, \alpha \geq 2$. If there are indices i and j such that $S[i..j+1] = bt^{j-i}b$, then $\#bc(C_i)$ ¿ $\#bc(C_j)$.*

Proof. Assume Ξ is deterministic. Each of beads in transcript are bound either inside of tunnel or outside (Fig. 4). If a bead is stabilized at inside of tunnel, then its successor is already decided the position either inside of a tunnel or outside. Moreover, If a bead is stabilized at outside of a tunnel, then its position is an entrance of a tunnel or otherwise.

Tunnel sections have three possible shape with considering symmetry such as straight, acute turn and obtuse turn (Fig. 3).

Let us consider each of cases of tunnel A, B, and C. Accordingly, We use lemma 1

Lemma 1. *If a bead does not have any bond, then neighbors of it must be occupied by $\alpha + 2$ beads at $\delta = 1$ and unary except first and last beads.*

Proof (lemma 1). A bead in transcript needs predecessor and successor except first and last beads. If the bead does not have ant bond, then it use hand with α neighbors. Thus, lemma 1 is clearly true.

Let us consider tunnel sections only tunnel A and B. See Fig. 4. From appendix (Entrance of Tunnel A, B and Exit of Tunnel), we add maximum number which each transitions provide binding capabilities for each edge where a is number of consuming binding capabilities when the bead is stabilized at position of *successor in outside*.

Next, we consider on tunnel C section. If $w[i]$ is stabilized by tunnel C and $S[i+1]$ is t, then $w[i+1]$ is stabilized by tunnel A or B because if $w[i+1]$ is stabilized by tunnel C, then C_{i+1} is a terminal. Hence, tunnel C section is divided cases such as Figure 5. Cases of $S[i...] = bt^l (l \geq 2)$ are already considered (Upper). According to appendix (Tunnel C), cases of $S[i..i+2] = btb$ also consume some binding capabilities (Lower).

Thus, if a bead is stabilized through a tunnel section, then it consume some binging capabilities.

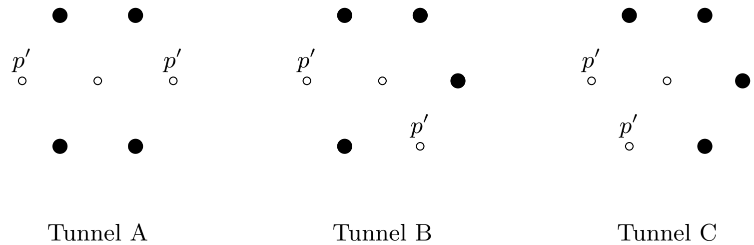


Fig. 3. All possible tunnel sections: straight, acute turn, and obtuse turn

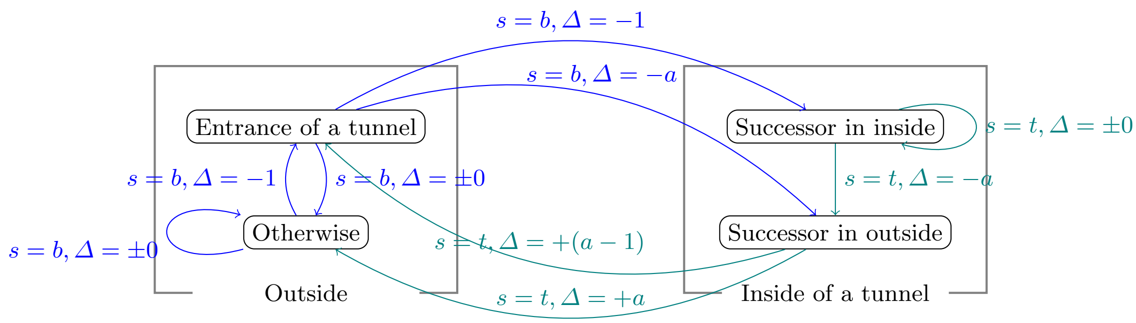


Fig. 4. Increment on Tunnel A,B

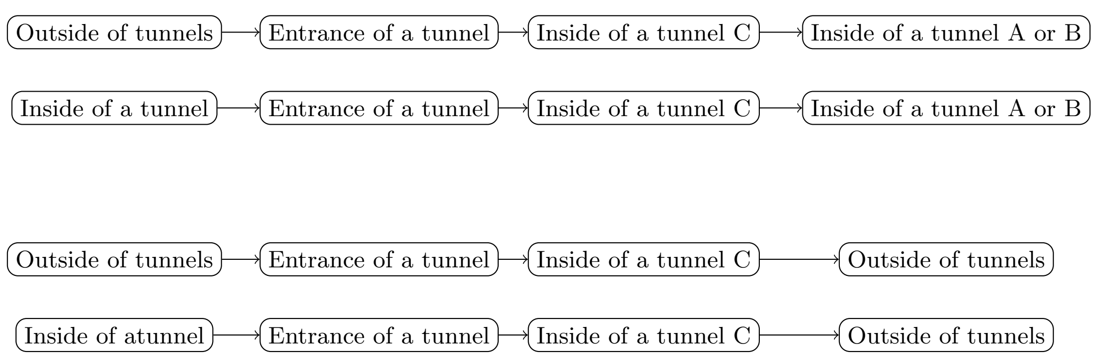


Fig. 5. Case of Tunnel C

1.2 Appendix of Tunnel Troll Theorem

Entrance of TunnelA,B Fig.6 exhibits all the three kinds of entrance of tunnel A, B. Any cases in $\delta = 1, \alpha = 2$ consume some binding capabilities into the follows.

- Case of t_0
  Let us consider points of n_3, n_4 either occupied or not. A point n_3 or n_4 is free because if both of them are occupied, p' is inside of tunnel. If n_3 is free, then p' has to be bound to a bead except n_1 due to deterministically stabilize. In this situation, at least three neighbors of n_1 are free that is n_1 leave at least one bond from lemma 1. Hence, p' must be bound to n_1. Thus, a case of t_0 consumes two binding capabilities and it does not supply any binding capabilities.
- Case of $t_{\pm 60}$
  In this case, a point n_4 or n_5 is free, too. If n_5 is free, p' has to be bound to n_1 or n_2. If n_5 is occupied, then n_4 is free. This time, n_2 has some binding capabilities so p' has to be bound to n_2.
  In this situation, p' is able to supply a binding capability. if this capability is active, p' bind a bead into n_4 or n_5. However, n_2 and n_3 are exist in back bone. According to Jordan curve theorem, any successors of p' cannot reach a point n_4 or n_5 so this capability is inactive. Thus, a case of $t_{\pm 60}$ consumes some binding capabilities.
- Case of $t_{\pm 120}$
  Binding capabilities that p' supply are inactive according to Jordan curve theorem on n_1 and n_2. Moreover, p' has to be bound to one of n_3, n_4, n_5 in order to deterministically stabilize. Thus, a case of $t_{\pm 120}$ consumes some binging capabilities.

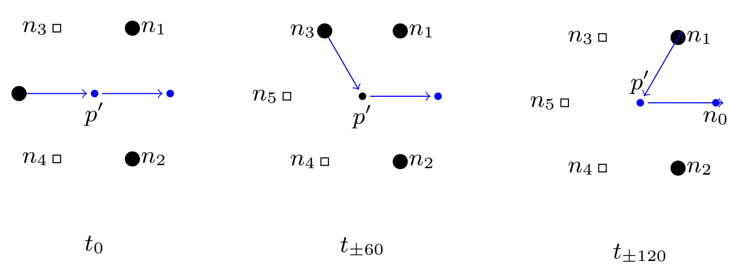


Fig. 6. Direction into a entrance

Exit of Tunnel Fig.7 exhibits all the two kinds of exit of tunnel. At least one of point n_1 or n_2 is free because if both of them are occupied, p' is inside of tunnel.

$\delta = 1, \alpha = 2$ Any cases of $\delta = 1, \alpha = 2$ supply at most a binding capabilities into follows where a is number of predecessor of p' consumes binging capabilities.

- Case of n_1 and n_2 are free
  This case can be regarded same situation as entrance. See Fig.7 (Left). Predecessor n_5 has to be bound n_4 and n_5 because each of n_3 and n_4 leave binding capabilities. Hence, at least $a = 2$. This time, $\alpha = 2$ that is this case supply at most only a binding capabilities.
- Case of c is occupied
  See Fig.7 (Right). If n_1 is occupied, then n_2 is free so that n_5 has to be bound n_4. Hence, at least $a = 1$. This case can supply two binding capabilities but p' can bind to only one of n_0 or n_2 because n_0 or n_2 will be occupied a successor of p'. Therefore, this case supply at most only $a = 1$ active binding capability.

$\delta = 1, \alpha \geq 3$ Any cases of $\delta = 1, \alpha \geq 3$ consume some binding capabilities into follows.

- Case of n_1 and n_2 are free
  In $\alpha \geq 3$, if three neighbors of a bead leave, then it can supply two binding capabilities. Therefore Predecessor n_5 has to be bound n_3 and n_4, and p', too. In this case, at least consumes four bindings and supplies at most two bindings. Thus, it consumes some binding capabilities, totally.
- Case of c is occupied
  In this case, n_4 leave at least two bindings and n_3, n_1 also leave at least one binding. Therefore n_5 has to be bound n_3 and n_4, and p' also has to be bound n_1 and n_4. In this case, at least consumes four bindings and supplies at most one binding. Thus, it consumes some binding capabilities, totally.

Point n_1 and n_2 are free   Point n_1 is occupied

Fig. 7. Exit of Tunnel

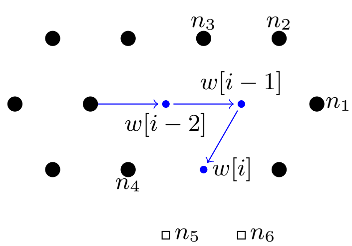
Tunnel C Assume $w[i]$ is a bead which stabilized by tunnel C. Let us consider kinds of stabilization $S[i-2..i] = tbt$ or $S[i-2..i] = bbt$ except cases of $w[i]$ is inside of tunnel A, B.

Case of $\boldsymbol{S[i-2..i] = tbt}$ Fig.8 exhibits all the two kinds of stabilization depending on structures of tunnel C.

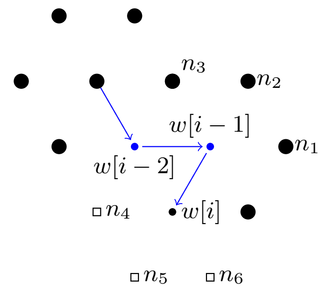
- Left of Fig.8
  In this figure, Bead n_4 has at least one binding so that $w[i-2]$ has to bound n_4. Moreover, $w[i-1]$ has to bound one of n_1, n_2, n_3 in order to stabilize deterministically. On the other hand, $w[i]$ can supply two bindings but free neighbors of $w[i]$ are two points. One of them is occupied by a successor. Therefore $w[i]$ can only bind one of n_5, n_6 that is $w[i]$ supplies at most one binding. Thus, this case consumes some binding capabilities.
- Right of Fig.8
  This cases are divided on number of capabilities that $w[i-2]$ consumes.
  - $w[i]$ does not consume any bindings
    $w[i-1]$ has to bound one of n_1, n_2, n_3 in order to stabilize deterministically. $w[i]$ has to be bound to $w[i-2]$ because $w[i-2]$ has bindings. This time, let us consider either n_4 is occupied or not. If n_4 is occupied, then $w[i-2]$ has no active bindings that is this situation consumes some binding capabilities. If n_4 is free and $w[i+1]$ is stabilized in n_4, then $w[i-2]$ has to bind $w[i+1]$. Therefore, In this case, stabilization of $w[i-2..i+1]$ consumes some bindings. If n_4 is free and $w[i+1]$ is stabilized except n_4, then this oritatami system has to use two binding capabilities in order to bind $w[i+1]$. Therefore, in this case consumes some bindings. Thus in this cases consume some binding capabilities.
  - w_i consumes one binding
    In this case, w_{i-1} has to be bound one of n_1, n_2, n_3. In addition, $w[i-2]$ and $w[i]$ are not supply any bindings. Thus, in this cases consume some binding capabilities.
  - w_i consumes two bindings
    In this case, $w[i-2]$ already consumes two binding. $w[i-1]$ has to be bound. $w[i]$ supplies two bindings. Thus, in this cases consume some binding capabilities.

Case of $\boldsymbol{S[i-2..i] = bbt}$ Let us consider number of consumed by $w[i-2]$ (Fig.9).

- $w[i-2]$ consumes one binding
  In this situation, $w[i-2]$ supplies one active binding whereas $w[i]$ consumes this binding. In addition, $w[i-1]$ has to bound to one of n_1, n_2, n_3. Thus, in this cases consume some binding capabilities.
- $w[i-2]$ consumes two bindings
  In this case, $w[i-2]$ already consumes two binding. $w[i-1]$ has to be bound. $w[i]$ supplies at most two bindings. Thus, in this cases consume some binding capabilities.



Pattern 1



Pattern 2

Fig. 8. Case of $S[i-2..i] = tbt$

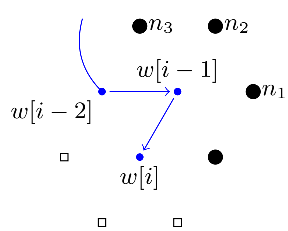


Fig. 9. Case of $S[i-2..i] = bbt$

By Tunnel Troll Theorem, any tunnel sections which represented in bbt^+ or bt^+bt^+ consume binding capabilities. If the sequence S is free from any subsequence of the form bt^+bt^+, then it can factorize as $S = u_1u_2u_3\cdots$ for some $u_1, u_2, u_3, \cdots \in \{b\} \cup bbt^+$. Assume the length of σ is n, seed supplies at most $2n$ binding capabilities. Therefore formula 1 hold.

$$\exists i \in \mathbb{N} \quad s.t. \quad u_{i-1}, u_i, u_{i+1}, u_{i+2}, \cdots \in \{b\} \tag{1}$$

Let us represent S as $S[i.i+1...] = v_iv_{i+1}v_{i+2}\cdots$ for some $v_i, v_{i+1}, v_{i+2}, \cdots \in \{a, o\}$ where if v_k is a, then v_{k+1} is bound to v_{k-1}, if v_k is o, then v_{k+1} is NOT bound to v_{k-1}.

Let us consider the case of v_k is o. See Fig.10. $w[i-1]$ consumes some binding capabilities because v_{i-1} is b. If the number of $w[i-1]$'s bindings is one binding, then $w[i+1]$ has to be bound except n_1 or n_2 so that $w[i+1]$ must consumes two bindings except the case of n_1 and n_2 are occupied and $w[i]$ consumes at least one binding. If n_1 and n_2 are occupied, then $w[i-1]$'s bindings are inactive that is $w[i-1]$ consumes two binding capabilities. Therefore, this case consumes binding capabilities. If $w[i-1]$ dose Not have any bindings, then $w[i-1]$ already consumes two bindings. In addition, $w[i]$ and $w[i+1]$ consume at least one binding. Therefore this case consumes binding capabilities. Thus, the formula 2 hold and according to the formula 1 and the formula 2, the formula 3 is hold.

$$\exists j \in \mathbb{N} \quad s.t. \quad u_j, u_{j+1}, u_{j+2}, \cdots \in \{a\} \tag{2}$$

$$|S| > \forall m \in \mathbb{N} \quad \rightarrow \quad \exists n \in \mathbb{N} \quad s.t. \quad S[n], S[n+1], \cdots \in \{a\} \tag{3}$$

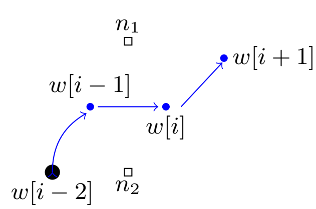


Fig. 10. Case of $S[i]$

$\alpha = 3$

Theorem 2 **($\delta = 1, \alpha = 3$).** *Let Ξ be a unary oritatami system of $\delta = 1, \alpha = 3$. It can yield only a finite structure whose size is $\mathcal{O}(n)$.*

Lemma 2. *Let p be a point whose neighbors is occupied at least two point. If $w[i]$ is not stabilized and $w[i-1]$ includes neighbors of p, then $w[i]$ is stabilized at p with at least one bond, $w[i]$ is stabilized at another point of p otherwise with at least two bond except any neighbors of p is occupied.*

Proof (proof of lemma). Assume the transcript is stabilized until $w[i-1]$. One of neighbors of p is not $w[i-1]$ where this bead regards n_1. If $w[i-1]$ include neighbors of p and $w[i]$ is stabilized at another point of p with one bond. Then, any neighbors do not have bond without $w[i-1]$. Neighbors of n_1 have to be occupied at least five according to lemma 1 and two of them include neighbors of p where each of them regards n_2, n_3. In the same way, five neighbors of n_2 and n_3 are occupied and each of one of them includes neighbors of p where they regard n_4, n_5. one of n_5's neighbors includes neighbors of p where it regards n_6. Then, any neighbors of p are occupied. That is if some neighbors of p are free, then there exists a bead which has bonds in neighbors.

Proof. Let us show that $\#bc(C_{i-1}) > \#bc(C_i)$, that is when $w[i]$ is stabilized, $w[i]$ uses at least two hands. Let us assume $w[i]$ is able to be stabilized with using one hand. Fig.11 exhibits all the three kinds of possibility of stabilized $w[i]$. Then, $w[i]$ can be also stabilized at n_3.

Case of straight

- Case of n_3 is free
  According to assumption, $w[i]$ uses only one hand. Therefore, any neighbors of n_3 are occupied according to the lemma2. n_3 and the point which is stabilized $w[i]$ are free so that n_1 has some bond by lemma1. Accordingly, this situation is non-deterministic. Thus, n_3 and n_4 have to be occupied because of symmetry.
- Otherwise
  Because of $S[i] = b$, at least one of n_1 and n_2 have to be free. Let us regard that n_1 is free. Neighbors of n_1 have to be occupied and at least two neighbors of n_{-1} have to be free for n_1 and $w[i]$. According to lemma1, n_{-1} have some hand. Therefore $w[i]$ can be also stabilized n_1 that is this situation is non-deterministic. Thus, one of n_3 and n_4 has to be free.

Therefore, this case is false.

Case of obtuse

- Case of n_3 is free
  Any neighbors of n_3 have to be occupied but the point which is stabilized $w[i]$ is free. Thus n_3 has to be occupied.
- Case of n_4 is free

- Case of n_2 is free
  Any neighbors of n_2 have to be occupied so that n_0 is occupied. Any neighbors of n_0 except n_2 have to be also occupied but the point which is stabilized $w[i]$ is free. Thus n_2 has to be occupied.

- Case of n_0 is free
  Any neighbors of n_0 have to be occupied so that n_{-1} is occupied. Any neighbors of n_{-1} except n_0 have to be also occupied but the point which is stabilized $w[i]$ is free. Thus n_0 has to be occupied.

Therefore, any situations contradict $S[i] = b$.

Case of acute

- Case of n_4 is free
  n_4 and a point which is stabilized $w[i]$ are free so that $w[i-2]$ has some hands according to lemma1. However, $w[i]$ can be also stabilized n_4 in this case. Thus, n_4 has to be occupied.
- Case of n_2 is free
  According to lemma2, n_0 has to be occupied. n_1 has to be also occupied because of lemma1. We consider this case just like case of obtuse and that n_4 is free. Then if $w[i-2]$ binds $w[i]$, any n_{-1}, n_{-2}, n_{-3} are occupied. If n_1 binds $w[i]$, this case is same. Also if n_1 and $w[i-2]$ do not have any hand, any n_{-1}, n_{-2}, n_{-3} are occupied. Therefore, $w[i+1]$ cannot be provided.
- Case of n_0 is free
  Any neighbors of n_0 have to be occupied so that n_1 is occupied. Any neighbors of n_1 except n_0 have to be also occupied but the point which is stabilized $w[i]$ is free. Thus n_0 has to be occupied.
- Case of n_1 is free
  Any neighbors of n_1 have to be occupied so that n_{-1} is occupied. Any neighbors of n_{-1} except n_1 have to be also occupied but the point which is stabilized $w[i]$ is free. Thus n_1 has to be occupied.

Therefore, any situations contradict $S[i] = b$.

Hence, assumption that w[i] is able to be stabilized with using one hand is false. Therefore, when w[i] is stabilized, w[i] uses at least two hands.

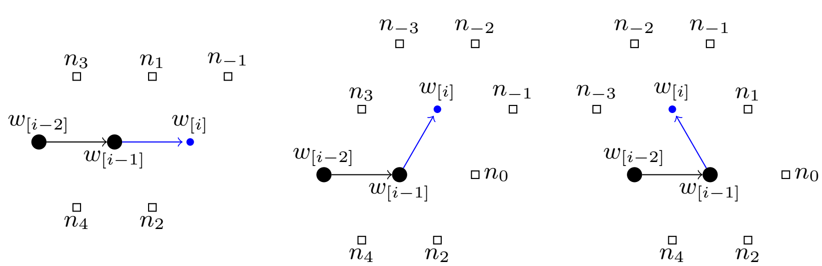


Fig. 11. All possible directions of $w[i]$: straight, obtuse, acute.

$\alpha = 4$

Theorem 3 **($\delta = 1, \alpha = 4$).** *Let Ξ be a unary oritatami system of $\delta = 1, \alpha = 4$. It can yield only a finite structure whose size is $\mathcal{O}(n^2)$.*

Lemma 3. *Any beads which are already stabilized by some bonds use at least two bonds.*

Proof (proof of lemma). Let us consider when $w[i]$ is stabilized by only one bond. See Fig.12. According to lemma1, if n_3 is free, $w[i-2]$ has some hands. Thus, n_4 has to be occupied in order to stabilize deterministically. Moreover, also n_2 has to be occupied for deterministic and also n_0, n_1. n_1 has some hands because n_3 is free. Therefore, $w[i]$ is stabilized at n_3 and it has to use at least two hands. It contradict assumption.

Proof. According to lemma3, when $w[i]$ is stabilized, it has to use at least two bonds. Let us consider when a bead $w[i]$ which is the first bead out of $\hexagon^n_{w[-n+1]}$ is stabilized. See Fig.13. any n_0, n_1, n_3, n_5 is free because if some of them is occupied, $w[i]$ is not the first bead out of $\hexagon^n_{w[-n+1]}$. At least two neighbors of $w[i]$ except predecessor have to be occupied in order to bind. In this case, a point which is able to put a bead is only n_2. Therefore, any transcript cannot be stabilized in out of $\hexagon^n_{w[-n+1]}$. Hence oritatami system can yield only a finite structure whose size is $\mathcal{O}(n^2)$ in $\delta = 1, \alpha = 4$.

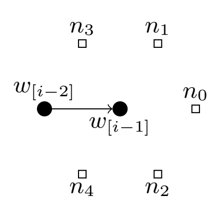


Fig. 12. $\alpha = 4$: when $w[i]$ is stabilized

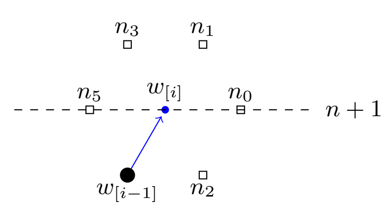


Fig. 13. the first bead out of $\hexagon^n_{w[-n+1]}$