# 1 Infiniteness of delay-1 unary deterministic oritatami system

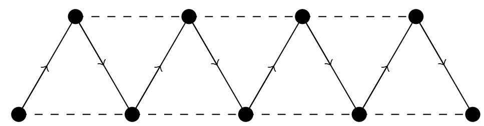

**Fig. 1.** zig-zag conformation

## 1.1 Introduction

In this section, we consider on finiteness of structures produced deterministically at delay 1. In result, cases of arity 1 and 3 can only yield finite structures which is size of $\mathcal{O}(n)$, and cases of arity 4 and more can only yield finite structures which is size of $mathcalO(n^2)$, and a case of arity 2 can yield infinite structures but they are only the zig-zag conformation shown in Fig.1.

Let $\Xi$ be a deterministic oritatami system of delay 1 and arity 2. Assume its seed $\sigma$ consists of $n$ beads. For $i \geq 0$ let $C_i$ be the unique elongation of $\sigma$ by $w[1..i]$ that is foldable by $\Xi$. Hence $C_0 = \sigma$.

Let us consider the stabilization of the $i$-th bead $a_i$ upon $C_{i-1}$. The bead cannot collaborate with any succeeding bead $w[i+1], w[i+2], \cdots$ at delay 1. There are just two ways to get stabilized at delay 1. One way is to be bound to another bead. The other way is through a *tunnel section*. A tunnel section consists of four beads that occupy four neighbors of a point (Fig.2). Accordingly, how they are stabilized can be described by a binary sequence $S$ of $b$'s (bound) and $t$'s (tunnel section); priority is given to $t$, that is, $S[i] = t$ if the $i$-th bead $w_i$ is stabilized not only by being bound but also through a tunnel section.

Assume that four of the six neighbors of a point $p$ are occupied by beads $a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}$ while the other two are free. We call such a bead $p$ as $inside\ of\ a\ tunnel$ and such beads $p'$ as *entrance of a tunnel* without a case that $p'$ is inside of a tunnel. If the beads $w[i-2]$ and $w[i-1]$ are stabilized respectively at one of the two free neighbors and at $p$ one after another, then the next bead $w[i]$ cannot help but be stabilized at the other free neighbor. In this way, $w[i]$ can get stabilized without being bound.

We say that point $p$ is reachable from a conformation $C$ if there exists a directed path $P'$ from the last point of $C$ that does not cross the path of $C$. We define *binding capability* with reachable.

**Definition 1 (binding capability).** *Let $B_i$ be $(\{(h,i) | {}^{\forall} h < i\} \cup \{(i,j) | {}^{\forall} j > i\}) \cap H$. Moreover, let $R_i$ be a set of neighbors of $w[i]$ that are free and reachable from $C_j$ where $C_j$ is a conformation which stabilized until $w[j]$. We represent* *the number of binding capabilities of a conformation $C_j$ as $\#bc(C_j)$. $\#bc(C_j)$ is defined by $\sum_{k=-n+1}^{j} \min\{|B_k|, |R_k|\}$.*

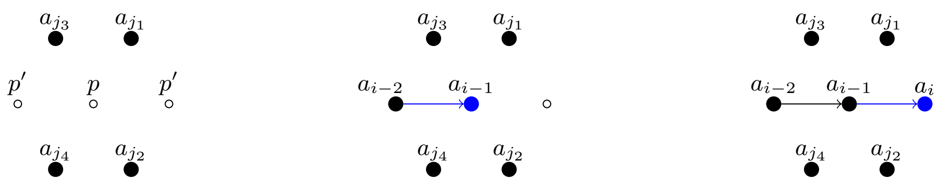


**Fig. 2.** Through a tunnel section

**Theorem 1 (Tunnel Troll Theorem).** *Let $\Xi$ be a unary oritatami system of $\delta = 1, \alpha \geq 2$. If there are indices $i$ and $j$ such that $S[i..j+1] = bt^{j-i}b$, then $\#bc(C_i)$ ¿ $\#bc(C_j)$.*

*Proof.* Assume $\Xi$ is deterministic. Each of beads in transcript are bound either inside of tunnel or outside (Fig. 4). If a bead is stabilized at inside of tunnel, then its successor is already decided the position either inside of a tunnel or outside. Moreover, If a bead is stabilized at outside of a tunnel, then its position is an entrance of a tunnel or otherwise.

Tunnel sections have three possible shape with considering symmetry such as straight, acute turn and obtuse turn (Fig. 3).

Let us consider each of cases of tunnel A, B, and C. Accordingly, We use lemma 1

**Lemma 1.** *If a bead does not have any bond, then neighbors of it must be occupied by $\alpha + 2$ beads at $\delta = 1$ and unary except first and last beads.*

*Proof (lemma 1).* A bead in transcript needs predecessor and successor except first and last beads. If the bead does not have ant bond, then it use hand with $\alpha$ neighbors. Thus, lemma 1 is clearly true.

Let us consider tunnel sections only tunnel A and B. See Fig. 4. From appendix (Entrance of Tunnel A, B and Exit of Tunnel), we add maximum number which each transitions provide binding capabilities for each edge where $a$ is number of consuming binding capabilities when the bead is stabilized at position of *successor in outside*.

Next, we consider on tunnel C section. If $w[i]$ is stabilized by tunnel C and $S[i+1]$ is $t$, then $w[i+1]$ is stabilized by tunnel A or B because if $w[i+1]$ is stabilized by tunnel C, then $C_{i+1}$ is a terminal. Hence, tunnel C section is divided cases such as Figure 5. Cases of $S[i...] = bt^l (l \geq 2)$ are already considered (Upper). According to appendix (Tunnel C), cases of $S[i..i+2] = btb$ also consume some binding capabilities (Lower).

Thus, if a bead is stabilized through a tunnel section, then it consume some binging capabilities.

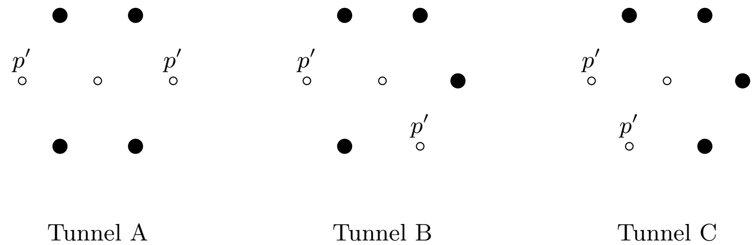


**Fig. 3.** All possible tunnel sections: straight, acute turn, and obtuse turn

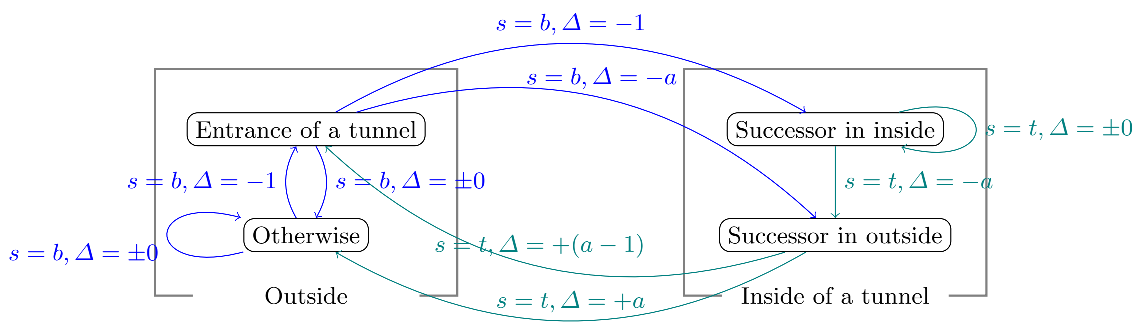


**Fig. 4.** Increment on Tunnel A,B

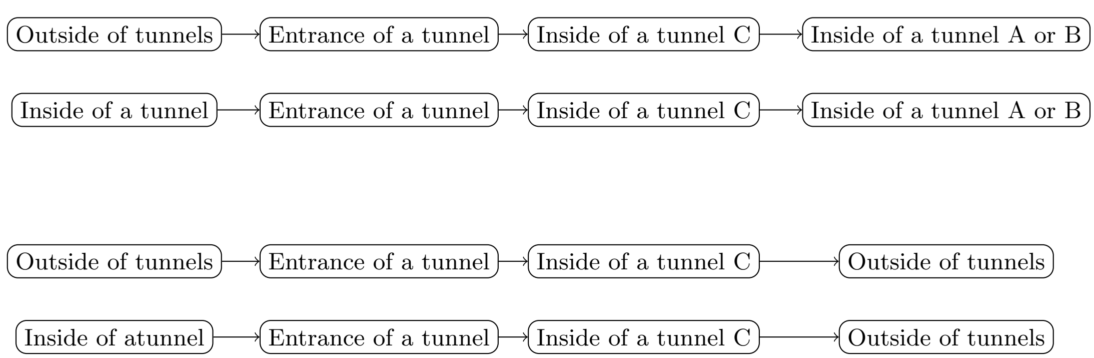


**Fig. 5.** Case of Tunnel C

### 1.2 Appendix of Tunnel Troll Theorem

**Entrance of TunnelA,B** Fig.6 exhibits all the three kinds of entrance of tunnel A, B. Any cases in $\delta = 1, \alpha = 2$ consume some binding capabilities into the follows.

- Case of $t_0$
  Let us consider points of $n_3, n_4$ either occupied or not. A point $n_3$ or $n_4$ is free because if both of them are occupied, $p'$ is inside of tunnel. If $n_3$ is free, then $p'$ has to be bound to a bead except $n_1$ due to deterministically stabilize. In this situation, at least three neighbors of $n_1$ are free that is $n_1$ leave at least one bond from lemma 1. Hence, $p'$ must be bound to $n_1$. Thus, a case of $t_0$ consumes two binding capabilities and it does not supply any binding capabilities.
- Case of $t_{\pm 60}$
  In this case, a point $n_4$ or $n_5$ is free, too. If $n_5$ is free, $p'$ has to be bound to $n_1$ or $n_2$. If $n_5$ is occupied, then $n_4$ is free. This time, $n_2$ has some binding capabilities so $p'$ has to be bound to $n_2$.
  In this situation, $p'$ is able to supply a binding capability. if this capability is active, $p'$ bind a bead into $n_4$ or $n_5$. However, $n_2$ and $n_3$ are exist in back bone. According to Jordan curve theorem, any successors of $p'$ cannot reach a point $n_4$ or $n_5$ so this capability is inactive. Thus, a case of $t_{\pm 60}$ consumes some binding capabilities.
- Case of $t_{\pm 120}$
  Binding capabilities that $p'$ supply are inactive according to Jordan curve theorem on $n_1$ and $n_2$. Moreover, $p'$ has to be bound to one of $n_3, n_4, n_5$ in order to deterministically stabilize. Thus, a case of $t_{\pm 120}$ consumes some binging capabilities.

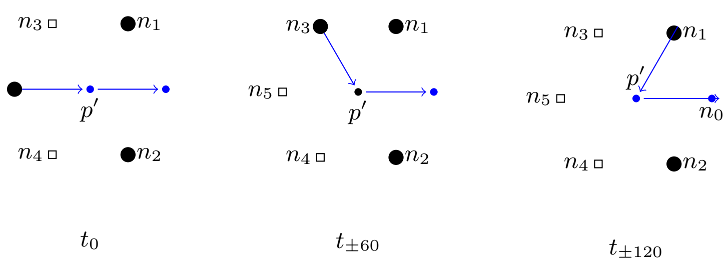


**Fig. 6.** Direction into a entrance

**Exit of Tunnel** Fig.7 exhibits all the two kinds of exit of tunnel. At least one of point $n_1$ or $n_2$ is free because if both of them are occupied, $p'$ is inside of tunnel.

$\delta = 1, \alpha = 2$ Any cases of $\delta = 1, \alpha = 2$ supply at most $a$ binding capabilities into follows where $a$ is number of predecessor of $p'$ consumes binging capabilities.

- Case of $n_1$ and $n_2$ are free
  This case can be regarded same situation as entrance. See Fig.7 (Left). Predecessor $n_5$ has to be bound $n_4$ and $n_5$ because each of $n_3$ and $n_4$ leave binding capabilities. Hence, at least $a = 2$. This time, $\alpha = 2$ that is this case supply at most only $a$ binding capabilities.
- Case of $c$ is occupied
  See Fig.7 (Right). If $n_1$ is occupied, then $n_2$ is free so that $n_5$ has to be bound $n_4$. Hence, at least $a = 1$. This case can supply two binding capabilities but $p'$ can bind to only one of $n_0$ or $n_2$ because $n_0$ or $n_2$ will be occupied a successor of $p'$. Therefore, this case supply at most only $a = 1$ active binding capability.

$\delta = 1, \alpha \geq 3$ Any cases of $\delta = 1, \alpha \geq 3$ consume some binding capabilities into follows.

- Case of $n_1$ and $n_2$ are free
  In $\alpha \geq 3$, if three neighbors of a bead leave, then it can supply two binding capabilities. Therefore Predecessor $n_5$ has to be bound $n_3$ and $n_4$, and $p'$, too. In this case, at least consumes four bindings and supplies at most two bindings. Thus, it consumes some binding capabilities, totally.
- Case of $c$ is occupied
  In this case, $n_4$ leave at least two bindings and $n_3$, $n_1$ also leave at least one binding. Therefore $n_5$ has to be bound $n_3$ and $n_4$, and $p'$ also has to be bound $n_1$ and $n_4$. In this case, at least consumes four bindings and supplies at most one binding. Thus, it consumes some binding capabilities, totally.


Point $n_1$ and $n_2$ are free

Point $n_1$ is occupied

**Fig. 7.** Exit of Tunnel

**Tunnel C** Assume $w[i]$ is a bead which stabilized by tunnel C. Let us consider kinds of stabilization $S[i-2..i] = tbt$ or $S[i-2..i] = bbt$ except cases of $w[i]$ is inside of tunnel A, B.

**Case of $S[i-2..i] = tbt$** Fig.8 exhibits all the two kinds of stabilization depending on structures of tunnel C.

- Left of Fig.8
  In this figure, Bead $n_4$ has at least one binding so that $w[i-2]$ has to bound $n_4$. Moreover, $w[i-1]$ has to bound one of $n_1, n_2, n_3$ in order to stabilize deterministically. On the other hand, $w[i]$ can supply two bindings but free neighbors of $w[i]$ are two points. One of them is occupied by a successor. Therefore $w[i]$ can only bind one of $n_5, n_6$ that is $w[i]$ supplies at most one binding. Thus, this case consumes some binding capabilities.
- Right of Fig.8
  This cases are divided on number of capabilities that $w[i-2]$ consumes.
  - $w[i]$ does not consume any bindings
    $w[i-1]$ has to bound one of $n_1, n_2, n_3$ in order to stabilize deterministically. $w[i]$ has to be bound to $w[i-2]$ because $w[i-2]$ has bindings. This time, let us consider either $n_4$ is occupied or not. If $n_4$ is occupied, then $w[i-2]$ has no active bindings that is this situation consumes some binding capabilities. If $n_4$ is free and $w[i+1]$ is stabilized in $n_4$, then $w[i-2]$ has to bind $w[i+1]$. Therefore, In this case, stabilization of $w[i-2..i+1]$ consumes some bindings. If $n_4$ is free and $w[i+1]$ is stabilized except $n_4$, then this oritatami system has to use two binding capabilities in order to bind $w[i+1]$. Therefore, in this case consumes some bindings. Thus in this cases consume some binding capabilities.
  - $w_i$ consumes one binding
    In this case, $w_{i-1}$ has to be bound one of $n_1, n_2, n_3$. In addition, $w[i-2]$ and $w[i]$ are not supply any bindings. Thus, in this cases consume some binding capabilities.
  - $w_i$ consumes two bindings
    In this case, $w[i-2]$ already consumes two binding. $w[i-1]$ has to be bound. $w[i]$ supplies two bindings. Thus, in this cases consume some binding capabilities.

**Case of $S[i-2..i] = bbt$** Let us consider number of consumed by $w[i-2]$ (Fig.9).

- $w[i-2]$ consumes one binding
  In this situation, $w[i-2]$ supplies one active binding whereas $w[i]$ consumes this binding. In addition, $w[i-1]$ has to bound to one of $n_1, n_2, n_3$. Thus, in this cases consume some binding capabilities.
- $w[i-2]$ consumes two bindings
  In this case, $w[i-2]$ already consumes two binding. $w[i-1]$ has to be bound. $w[i]$ supplies at most two bindings. Thus, in this cases consume some binding capabilities.

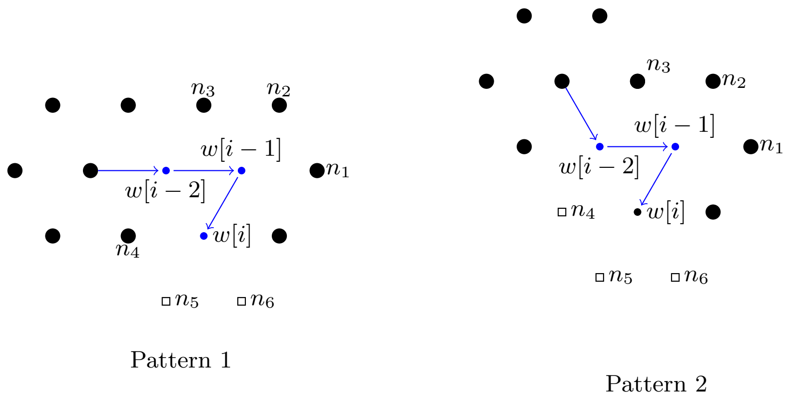


**Fig. 8.** Case of $S[i-2..i] = tbt$

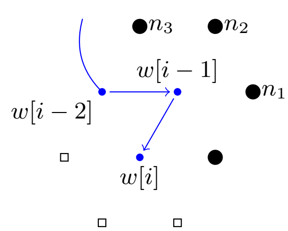


**Fig. 9.** Case of $S[i-2..i] = bbt$

**$\alpha = 2$** By Tunnel Troll Theorem, any tunnel sections which represented in $bbt^+$ or $bt^+bt^+$ consume binding capabilities. If the sequence $S$ is free from any subsequence of the form $bt^+bt^+$, then it can factorize as $S = u_1u_2u_3\cdots$ for some $u_1, u_2, u_3, \cdots \in \{b\} \cup bbt^+$. Assume the length of $\sigma$ is $n$, seed supplies at most $2n$ binding capabilities. Therefore formula 1 hold.

$$\exists i \in \mathbb{N} \quad s.t. \quad u_{i-1}, u_i, u_{i+1}, u_{i+2}, \cdots \in \{b\} \tag{1}$$

Let us represent $S$ as $S[i.i+1...] = v_iv_{i+1}v_{i+2}\cdots$ for some $v_i, v_{i+1}, v_{i+2}, \cdots \in \{a, o\}$ where if $v_k$ is $a$, then $v_{k+1}$ is bound to $v_{k-1}$, if $v_k$ is $o$, then $v_{k+1}$ is NOT bound to $v_{k-1}$.

Let us consider the case of $v_k$ is $o$. See Fig.10. $w[i-1]$ consumes some binding capabilities because $v_{i-1}$ is $b$. If the number of $w[i-1]$'s bindings is one binding, then $w[i+1]$ has to be bound except $n_1$ or $n_2$ so that $w[i+1]$ must consumes two bindings except the case of $n_1$ and $n_2$ are occupied and $w[i]$ consumes at least one binding. If $n_1$ and $n_2$ are occupied, then $w[i-1]$'s bindings are inactive that is $w[i-1]$ consumes two binding capabilities. Therefore, this case consumes binding capabilities. If $w[i-1]$ dose Not have any bindings, then $w[i-1]$ already consumes two bindings. In addition, $w[i]$ and $w[i+1]$ consume at least one binding. Therefore this case consumes binding capabilities. Thus, the formula 2 hold and according to the formula 1 and the formula 2, the formula 3 is hold.

$$\exists j \in \mathbb{N} \quad s.t. \quad u_j, u_{j+1}, u_{j+2}, \cdots \in \{a\} \tag{2}$$

$$|S| > \forall m \in \mathbb{N} \quad \rightarrow \quad \exists n \in \mathbb{N} \quad s.t. \quad S[n], S[n+1], \cdots \in \{a\} \tag{3}$$

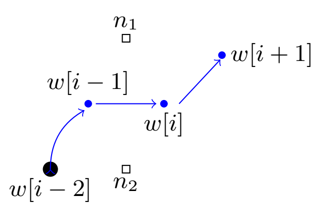


**Fig. 10.** Case of $S[i]$

**$\alpha = 3$**

**Theorem 2** **($\delta = 1, \alpha = 3$).** *Let $\Xi$ be a unary oritatami system of $\delta = 1, \alpha = 3$. It can yield only a finite structure whose size is $\mathcal{O}(n)$.*

**Lemma 2.** *Let $p$ be a point whose neighbors is occupied at least two point. If $w[i]$ is not stabilized and $w[i-1]$ includes neighbors of $p$, then $w[i]$ is stabilized at $p$ with at least one bond, $w[i]$ is stabilized at another point of $p$ otherwise with at least two bond except any neighbors of $p$ is occupied.*

*Proof (proof of lemma).* Assume the transcript is stabilized until $w[i-1]$. One of neighbors of $p$ is not $w[i-1]$ where this bead regards $n_1$. If $w[i-1]$ include neighbors of $p$ and $w[i]$ is stabilized at another point of $p$ with one bond. Then, any neighbors do not have bond without $w[i-1]$. Neighbors of $n_1$ have to be occupied at least five according to lemma 1 and two of them include neighbors of $p$ where each of them regards $n_2, n_3$. In the same way, five neighbors of $n_2$ and $n_3$ are occupied and each of one of them includes neighbors of $p$ where they regard $n_4, n_5$. one of $n_5$'s neighbors includes neighbors of $p$ where it regards $n_6$. Then, any neighbors of $p$ are occupied. That is if some neighbors of $p$ are free, then there exists a bead which has bonds in neighbors.

*Proof.* Let us show that $\#bc(C_{i-1}) > \#bc(C_i)$, that is when $w[i]$ is stabilized, $w[i]$ uses at least two hands. Let us assume $w[i]$ is able to be stabilized with using one hand. Fig.11 exhibits all the three kinds of possibility of stabilized $w[i]$. Then, $w[i]$ can be also stabilized at $n_3$.

*Case of straight*

- Case of $n_3$ is free
  According to assumption, $w[i]$ uses only one hand. Therefore, any neighbors of $n_3$ are occupied according to the lemma2. $n_3$ and the point which is stabilized $w[i]$ are free so that $n_1$ has some bond by lemma1. Accordingly, this situation is non-deterministic. Thus, $n_3$ and $n_4$ have to be occupied because of symmetry.
- Otherwise
  Because of $S[i] = b$, at least one of $n_1$ and $n_2$ have to be free. Let us regard that $n_1$ is free. Neighbors of $n_1$ have to be occupied and at least two neighbors of $n_{-1}$ have to be free for $n_1$ and $w[i]$. According to lemma1, $n_{-1}$ have some hand. Therefore $w[i]$ can be also stabilized $n_1$ that is this situation is non-deterministic. Thus, one of $n_3$ and $n_4$ has to be free.

Therefore, this case is false.

*Case of obtuse*

- Case of $n_3$ is free
  Any neighbors of $n_3$ have to be occupied but the point which is stabilized $w[i]$ is free. Thus $n_3$ has to be occupied.
- Case of $n_4$ is free
  According to lemma2, $n_2$ has to be occupied because $n_4$ is free. Also $n_0$ has to be occupied from lemma1. Thus, only one of $n_0, n_3$ leave some hands or both of them do not leave any hands because $w[i]$ use only one bond.

If $n_0$ has some hands, then $n_3$ does not have any hands so that $n_{-3}$ is occupied. Also $n_{-3}$ must not have any hands so that $n_{-2}$ is occupied and also $n_{-1}$ is occupied. Therefore any neighbors of $w[i]$ are occupied so that $w[i+1]$ cannot provide.
If $n_3$ has some hands, then $n_0$ does not have any hands so that $n_{-1}$ is occupied. In the same previous way, any $n_{-2}, n_{-3}$ are occupied. Therefore any neighbors of $w[i]$ are occupied.
If both of $n_0, n_3$ do not have any hands, then both of $n_{-1}, n_{-3}$ are occupied. If one of $n_{-1}, n_{-3}$ has some hands, the other does not have any hands so that $n_{-2}$ is occupied. If both of $n_{-1}, n_{-3}$ do not have any hands, $n_{-2}$ has to be occupied and $n_{-2}$ has some hands.Therefore any neighbors of $w[i]$ are occupied so that $w[i+1]$ cannot provide.
Thus $n_3$ has to be occupied in order to yield infinite structures.

- Case of $n_2$ is free
  Any neighbors of $n_2$ have to be occupied so that $n_0$ is occupied. Any neighbors of $n_0$ except $n_2$ have to be also occupied but the point which is stabilized $w[i]$ is free. Thus $n_2$ has to be occupied.
- Case of $n_0$ is free
  Any neighbors of $n_0$ have to be occupied so that $n_{-1}$ is occupied. Any neighbors of $n_{-1}$ except $n_0$ have to be also occupied but the point which is stabilized $w[i]$ is free. Thus $n_0$ has to be occupied.

Therefore, any situations contradict $S[i] = b$.

*Case of acute*

- Case of $n_4$ is free
  $n_4$ and a point which is stabilized $w[i]$ are free so that $w[i-2]$ has some hands according to lemma1. However, $w[i]$ can be also stabilized $n_4$ in this case. Thus, $n_4$ has to be occupied.
- Case of $n_2$ is free
  According to lemma2, $n_0$ has to be occupied. $n_1$ has to be also occupied because of lemma1. We consider this case just like case of obtuse and that $n_4$ is free. Then if $w[i-2]$ binds $w[i]$, any $n_{-1}, n_{-2}, n_{-3}$ are occupied. If $n_1$ binds $w[i]$, this case is same. Also if $n_1$ and $w[i-2]$ do not have any hand, any $n_{-1}, n_{-2}, n_{-3}$ are occupied. Therefore, $w[i+1]$ cannot be provided.
- Case of $n_0$ is free
  Any neighbors of $n_0$ have to be occupied so that $n_1$ is occupied. Any neighbors of $n_1$ except $n_0$ have to be also occupied but the point which is stabilized $w[i]$ is free. Thus $n_0$ has to be occupied.
- Case of $n_1$ is free
  Any neighbors of $n_1$ have to be occupied so that $n_{-1}$ is occupied. Any neighbors of $n_{-1}$ except $n_1$ have to be also occupied but the point which is stabilized $w[i]$ is free. Thus $n_1$ has to be occupied.

Therefore, any situations contradict $S[i] = b$.

Hence, assumption that w[i] is able to be stabilized with using one hand is false. Therefore, when w[i] is stabilized, w[i] uses at least two hands.

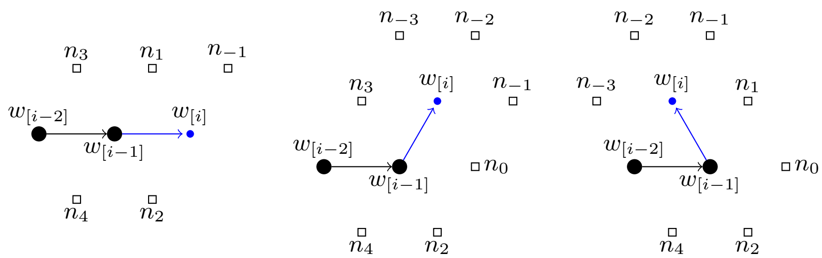


**Fig. 11.** All possible directions of $w[i]$: straight, obtuse, acute.

**$\alpha = 4$**

**Theorem 3** **($\delta = 1, \alpha = 4$).** *Let $\Xi$ be a unary oritatami system of $\delta = 1, \alpha = 4$. It can yield only a finite structure whose size is $\mathcal{O}(n^2)$.*

**Lemma 3.** *Any beads which are already stabilized by some bonds use at least two bonds.*

*Proof (proof of lemma).* Let us consider when $w[i]$ is stabilized by only one bond. See Fig.12. According to lemma1, if $n_3$ is free, $w[i-2]$ has some hands. Thus, $n_4$ has to be occupied in order to stabilize deterministically. Moreover, also $n_2$ has to be occupied for deterministic and also $n_0, n_1$. $n_1$ has some hands because $n_3$ is free. Therefore, $w[i]$ is stabilized at $n_3$ and it has to use at least two hands. It contradict assumption.

*Proof.* According to lemma3, when $w[i]$ is stabilized, it has to use at least two bonds. Let us consider when a bead $w[i]$ which is the first bead out of $\hexagon^n_{w[-n+1]}$ is stabilized. See Fig.13. any $n_0, n_1, n_3, n_5$ is free because if some of them is occupied, $w[i]$ is not the first bead out of $\hexagon^n_{w[-n+1]}$. At least two neighbors of $w[i]$ except predecessor have to be occupied in order to bind. In this case, a point which is able to put a bead is only $n_2$. Therefore, any transcript cannot be stabilized in out of $\hexagon^n_{w[-n+1]}$. Hence oritatami system can yield only a finite structure whose size is $\mathcal{O}(n^2)$ in $\delta = 1, \alpha = 4$.

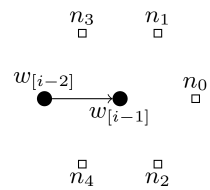


**Fig. 12.** $\alpha = 4$: when $w[i]$ is stabilized

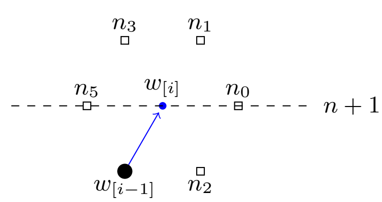


**Fig. 13.** the first bead out of $\hexagon^n_{w[-n+1]}$