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On relationship between the parametric maximum likelihood method and the constrained ILC method

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ABSTRACT

There are two main approaches to obtain a clean map of the cosmic microwave background (CMB) from multi-frequency data of the microwave sky. One is the internal linear combination (ILC) method and the other is the parametric method. The former is a blind method, as it makes no assumption about the nature of the contamination, such as the number and frequency dependence of foreground emission components. The latter relies on knowledge of the contamination. However, recent development in the component separation method makes this boundary somewhat fuzzy. In this note, we attempt to clarify the relation between the so-called "constrained" ILC method and the parametric maximum likelihood method.

1. PARAMETRIC MAXIMUM LIKELIHOOD METHOD

Suppose that we work with maps of the microwave sky in n_ν frequency bands, and each map contains n_p pixels. For simplicity we assume that noise is Gaussian and uncorrelated over pixels or frequencies. Equivalently we can work with spherical harmonics coefficients, which are Gaussian and uncorrelated over (ℓ, m) or frequencies. We can also work in the wavelet domain.

The likelihood of a map (or spherical harmonics coefficients or wavelet coefficients) at a given pixel, (ℓ, m) or any other appropriate domain is (e.g., Eriksen et al. 2006)

$$-2\ln\mathcal{L} = (\mathbf{d} - \mathbf{a}s - \sum_i^m \mathbf{b}_i g_i)^{\mathrm{T}}\mathbf{N}^{-1}(\mathbf{d} - \mathbf{a}s - \sum_i^m \mathbf{b}_i g_i) + \ln|2\pi\mathbf{N}|\,, \tag{1}$$

where $\mathbf{d}$ is a vector of the observed map (containing n_ν elements), $\mathbf{N}$ is a n_ν-by-n_ν noise covariance matrix, $\mathbf{a}$ and $\mathbf{b}_i$ are spectral response vectors of CMB and the ith component we wish to eliminate (each containing n_ν elements), and s and g_i are values of CMB (temperature, Stokes Q, or Stokes U) and the ith component we wish to eliminate.

When $\mathbf{a}$ and $\mathbf{b}_i$ are known, the maximum likelihood solution for s is given by (e.g., Stompor et al. 2009)

$$s = \mathbf{e}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{d}\,, \tag{2}$$

where $\mathbf{A} = (\mathbf{a}\ \mathbf{b}_1 \ldots)$ and $\mathbf{e}^{\mathrm{T}} = (1\ 0 \ldots)$.

The assumption in this derivation is two folds:

- The sky contains only s and g_i, and
- We know $\mathbf{b}_i$.

We usually do not know $\mathbf{b}_i$, with the exception of a few components such as the thermal Sunyaev-Zeldovich effect. But if we do know all of $\mathbf{b}_i$ (or we estimate them from the data, often in parametrised forms), then the above result is the best solution for CMB.

In the special case in which the number of frequency channels is equal to the number of components in the sky, i.e., $n_\nu = m + 1$, $\mathbf{A}$ is invertible and we obtain

$$s = \mathbf{e}^{\mathrm{T}}\mathbf{A}\mathbf{d}\,. \tag{3}$$

2. CONSTRAINED ILC

In the ILC method, we write

$$s = \mathbf{w}^{\mathrm{T}}\mathbf{d}\,, \tag{4}$$

and determine the weights $\mathbf{w}$ by minimizing the variance of s. Let us assume that the sky contains s, g_i, and a collection of other components ϵ with the spectral response of $\mathbf{c}$, i.e., $\mathbf{d} = \mathbf{a}s + \sum_i^m \mathbf{b}_i g_i + \mathbf{c}\epsilon + \mathbf{noise}$.

In the standard ILC (Bennett et al. 2003; Tegmark et al. 2003; Eriksen et al. 2004; Hinshaw et al. 2007), we determine $\mathbf{w}$ by minimizing the variance of s. The spectral response of the foreground is "learned" from the frequency dependence of the covariance matrix estimated from data, $\mathbf{C}$. In the constrained ILC (Remazeilles et al. 2011a,b; Hurier et al. 2013), we deproject the components with known $\mathbf{b}_i$. The solution is

$$s = \mathbf{e}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{d}\,, \tag{5}$$

where $\mathbf{C}$ is the total covariance matrix estimated from data.

In the special case in which the number of frequency channels is equal to the number of components to deproject plus one, $n_\nu = m + 1$, $\mathbf{A}$ is invertible and we obtain

$$s = \mathbf{e}^{\mathrm{T}}\mathbf{A}\mathbf{d}\,, \tag{6}$$

which agrees with Eq. (3). Therefore, the maximum likelihood method and the constrained ILC method are equivalent in this special case.

3. COMPARISON BETWEEN THE MAXIMUM LIKELIHOOD AND CONSTRAINED ILC METHODS

Eqs. (2) and (5) are similar but there is a clear difference: the former uses $\mathbf{N}^{-1}$ whereas the latter uses $\mathbf{C}^{-1}$. In the limit that we ignore the other components ϵ, i.e., we know the spectral response of all components in the sky, Eq. (2) is the optimal solution. Is the constrained ILC solution given in Eq. (5) equivalent to Eq. (2) in this case? The answer is yes[1].

3.1. *Standard ILC*

The solution of the standard ILC is $s = \mathbf{a}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{d}/(\mathbf{a}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{a})$ (Tegmark et al. 2003; Eriksen et al. 2004). When the sky contains only CMB and noise, the covariance matrix is $\mathbf{C} = \mathbf{N} + \sigma^2_{\mathrm{CMB}}\mathbf{a}\mathbf{a}^{\mathrm{T}}$. We then use the Woodbury formula $(A+UBV)^{-1} = A^{-1} - A^{-1}U(B^{-1} + VA^{-1}U)^{-1}VA^{-1}$ with $A = \mathbf{N}$, $B = \sigma^2_{\mathrm{CMB}}$, $U = \mathbf{a}$ and $V = \mathbf{a}^{\mathrm{T}}$ to write

$$\mathbf{C}^{-1} = \mathbf{N}^{-1} - \frac{\mathbf{N}^{-1}\mathbf{a}\mathbf{a}^{\mathrm{T}}\mathbf{N}^{-1}}{\sigma^{-2}_{\mathrm{CMB}} + \mathbf{a}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{a}}\,. \tag{7}$$

We thus find $s = \mathbf{a}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{d}/(\mathbf{a}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{a}) = \mathbf{a}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{d}/(\mathbf{a}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{a})$.

3.2. *Constrained ILC*

When the sky contains only CMB, noise, and foreground components with the known frequency spectra, the covariance matrix is $\mathbf{C} = \mathbf{N} + \sigma^2_{\mathrm{CMB}}\mathbf{a}\mathbf{a}^{\mathrm{T}} + \sum_{i=1}^{m}\sigma^2_{\mathrm{FG},i}\mathbf{b}_i\mathbf{b}_i^{\mathrm{T}}$. We write this as

$$\mathbf{C} = \mathbf{N} + \mathbf{A}\mathbf{D}\mathbf{A}^{\mathrm{T}}\,, \tag{8}$$

where $\mathbf{D} = \mathrm{diag}(\sigma^2_{\mathrm{CMB}}, \sigma^2_{\mathrm{FG},1}, \ldots)$. Using the Woodbury formula, we find

$$\mathbf{C}^{-1} = \mathbf{N}^{-1} - \mathbf{N}^{-1}\mathbf{A}(\mathbf{D}^{-1} + \mathbf{A}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{N}^{-1}\,, \tag{9}$$

and Eq. (5) becomes

$$s = \mathbf{e}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{d} = \mathbf{e}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{d}\,. \tag{10}$$

Therefore, if we replace $\mathbf{C}^{-1}$ with $\mathbf{N}^{-1}$ in Eq. (5), we make the assumption that the sky contains only the components that we deproject. Leaving $\mathbf{C}^{-1}$ in Eq. (5), we allow for ϵ, whose spectral response is unknown and whose contribution is minimised via the minimisation of variance of s.

4. SUMMARY

The constrained ILC solution given in Eq. (5) is equivalent to the maximum likelihood solution given in Eq. (2), provided that the sky contains only CMB, noise, and foreground components with the known frequency spectra that we deproject. Leaving the total covariance matrix $\mathbf{C}$ in Eq. (5) allows for further minimisation of additional foreground components whose frequency spectra are not known.

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REFERENCES

Bennett, C. L., Hill, R. S., Hinshaw, G., et al. 2003, ApJS, 148, 97, doi: 10.1086/377252

Eriksen, H. K., Banday, A. J., Górski, K. M., & Lilje, P. B. 2004, ApJ, 612, 633, doi: 10.1086/422807

Eriksen, H. K., Dickinson, C., Lawrence, C. R., et al. 2006, ApJ, 641, 665, doi: 10.1086/500499

Hinshaw, G., Nolta, M. R., Bennett, C. L., et al. 2007, ApJS, 170, 288, doi: 10.1086/513698

Hurier, G., Macías-Pérez, J. F., & Hildebrandt, S. 2013, A&A, 558, A118, doi: 10.1051/0004-6361/201321891

Remazeilles, M., Delabrouille, J., & Cardoso, J.-F. 2011a, MNRAS, 410, 2481, doi: 10.1111/j.1365-2966.2010.17624.x

—. 2011b, MNRAS, 418, 467, doi: 10.1111/j.1365-2966.2011.19497.x

Stompor, R., Leach, S., Stivoli, F., & Baccigalupi, C. 2009, MNRAS, 392, 216, doi: 10.1111/j.1365-2966.2008.14023.x

Tegmark, M., de Oliveira-Costa, A., & Hamilton, A. J. 2003, PhRvD, 68, 123523, doi: 10.1103/PhysRevD.68.123523