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Andrew Comech
Alexander Komech
Mikhail Vishik
Editors

Partial Differential Equations and Functional Analysis

Mark Vishik: Life and Scientific Legacy

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Trends in Mathematics

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Editors
Andrew Comech
Department of Mathematics
Texas A&M University
College Station, TX, USA

Alexander Komech
Faculty of Mathematics
University of Vienna
Vienna, Austria

Mikhail Vishik
Department of Mathematics
The University of Texas at Austin
Austin, TX, USA

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Andrew Comech
Alexander Komech
Mikhail Vishik

Introduction

Mark Vishik is one of the leading mathematicians of the twentieth century. He made a fundamental contribution to the theory of partial differential equations, giving a new appearance to this field which constitutes the basis of the modern science. His work received a wide international recognition. In 1990 he was elected an honorary member of the American Academy of Arts and Sciences and in 1994 became a member of the National Academy of Sciences of Italy (the "Academy of the Forty").

The scientific works of Mark Vishik belong to the area of partial differential equations, functional analysis, and mathematical physics, including statistical hydrodynamics and global theory of nonlinear wave processes. Mark Vishik provided the mathematical foundation to these areas which determined their future development.

His work turned the theory of partial differential equations into one of the subareas of modern functional analysis. Undoubtedly, an important role in this was played by his connection to the Lwów school of Stephan Banach. The research of Mark Vishik is the highest point of that school, which gave the maximal realization of this school's potential.

All his life Mark Vishik carried a huge pedagogical work. He participated in the scientific formation of hundreds of undergraduate and graduate students. Many dozens of professors and habilitated doctors consider him as their scientific advisor. His pupils are now an essential part of the scientific community.

All who knew Mark Vishik were charmed by his delicacy and kindness. He was a very open person, in spite of his seeming temperance, and was ready to help all his students and colleagues.

All of this is remembered in this book by friends and pupils of Mark Vishik. The reader sees the whole epoch and the number of remarkable scientists, whose names are connected with the life of Mark Vishik: S. Banach, J. Schauder, I. N. Vekua, N. I. Muskhelishvili, L. A. Lyusternik, I. G. Petrovsky, S. L. Sobolev, I. M. Gelfand, M. G. Krein, A. N. Kolmogorov, N. I. Akhiezer, J. Leray, J.-L. Lions, L. Schwartz, L. Nirenberg, and others.

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Part I
Memoirs

Notes About My Father

Mikhail Vishik

My notes are based on the stories of my father, Mark Vishik, the way I remember them, as well as on my own memories. I cannot guarantee the absolute accuracy of these sources.

Mark loved to tell stories about his childhood in Lwów. His family lived in poverty after the death of Mark's father, Josef-Haim, when Mark was 8 or 9 years old. Mark told me that after his father's death, he recited Kaddish every day for a year at the synagogue. Clearly, he had been raised in a fairly strict Orthodox Jewish tradition. Mark was still able to sing excerpts from the Haggadah many years later (Eliyahu Hanavi, Eliyahu Hatishbi, Eliyahu Hagiladi...) I have been unable to determine his father's exact occupation. Mark had two brothers—Bernhard, the eldest, and Josef, the youngest, born after their father's death. Mark adored his younger sister Hela. It is possible that she was blonde. He remembered going on walks with her holding on to his finger.

At the Lwów Gymnasium №9 he met, and became lifelong friends with, Władysław (Wladek) Elievich Lyantse, who later went on to become a renowned mathematician, specializing in functional analysis and spectral theory. Mark's recollections about school were largely centered around food. Or, rather, breakfasts (lunches) that children's parents would pack for them. He was struck by a chocolate sandwich (perhaps with chocolate spread) that one of the other students had

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M. Vishik (✉)
Department of Mathematics, The University of Texas at Austin, Austin, TX, USA
e-mail: vishik@math.utexas.edu

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regularly brought for breakfast. After the death of Mark's father, his family only had enough money for the most basic foods. According to Mark, his mother Rivka (Rebecca, Regina) worked as a librarian after her husband's death. At one point, she has to sell her engagement ring in order to obtain enough money for the school breakfasts. The two friends, Mark and Wladek, would often go for walks around the city on Fridays, discussing the latest news from the realm of science fiction (for example, the canals of Mars), having filled their pockets beforehand with pastries that Mark's mother baked on Fridays before sunset. Wladek's family was better off financially: his father was a broker at a Warsaw curtain-selling firm.

Mark's mathematical ability manifested early. At the gymnasium, he was fortunate to have a mathematics teacher, Freilich, who nurtured his mathematical talent. During a lesson on logarithms, where the logarithm tables were used and the linear interpolation was applied on intervals between the points from the table, Mark asked, “How do we know that the logarithm is a linear function on these intervals?” The professor threw down his chalk and began looking for the student's name in the class list. He was so impressed by Mark's comment that he spoke very highly of the pupil to the rabbi at a teachers' meeting. On the next Religion lesson (there were separate religion lessons for different faiths) the rabbi showered Mark with praise, although Mark had not been very good at the rabbi's own subject. Mark did even more poorly in Latin, although many years later he was still able to recite from memory some excerpts from Caesar's Commentaries and Ovid's Metamorphoses (“Look, Niobe comes...”).

After the Red Army's “liberation” of Lwów in 1939, Mark and Wladek got the opportunity to apply to the University of Lwów. Mark was noticed early by Stefan Banach, a renowned mathematician who had founded the Lwów school of mathematics. Mark attended Banach's seminar at the University of Lwów. After the war ended, Stefan Banach was overjoyed to see his old student during his visit to Moscow (Banach survived through the German occupation in Lwów).

A turning point in Mark's life was his escape from Lwów and the death his family and every one of his Lwów relatives during the War. These events were deeply engraved in his mind, but in his conversations with me, he only discussed certain aspects of them, while much remained and continues to remain in shadow. In June 1941, as the Wehrmacht was entering the city, Mark left Lwów on foot and headed east. In his pocket was his passport, his Komsomol membership card, and his University of Lwów transcript. No personal effects were with him. This is when his hungry wanderings began, sometimes with groups of refugees. One of his fellow escapees asked for some bread at a peasant's house, and was killed in front of Mark's eyes. The chaos that reigned during the early stages of the German offensive and the Red Army's flight has been well-documented in literature. Mark recalled the bombing of the retreating Red Army units, and civilian refugees, by German aircraft. The first few days of the German occupation of Lwów were accompanied by *pogroms*—in which the local population actively participated—of the city's Jewish neighbourhoods. The fate of his family was something Mark tried to uncover after the war, when he came to Lwów in the late 1940s with his wife, Asya (Rachel) Moiseevna, my mother. They questioned the neighbours who had remained in the city during the German occupation. This attempt to reconstruct what had happened was not 100% successful. Now I know that Mark's older brother either died of hunger, or was shot in the concentration camp on Yanovskaya Street (Janowska concentration camp). It appears that his younger brother starved to death in the Lwów ghetto. This conclusion can be drawn from the cemetery records. His mother and sister were likely sent to the Bergen-Belsen concentration camp before the liquidation of the Lwów ghetto. Their fate is unknown, but it can be assumed with high probability that they perished there. Wladek's family managed to buy him a ticket for one of the last eastbound trains from Lwów. (Mark, as I recall, mentioned the word "ticket", purchased by Wladek's parents. It had been utter chaos. All of the Party and NKVD leaders had fled, city services had not been operational. "Ticket" could have meant anything: some amount of money, for example, paid for the opportunity to leave, etc.)[1]

Mark has many relatives all over the world: in the United States, Australia, Scotland. They are the descendants of those relatives who had left Poland before the war. Orthodox Jewish families, as a rule, have many children. As far as I know, Mark had no contact with the descendants of his relatives who had survived the war.

Traveling on foot, sometimes by train or with military echelons, Mark risked his life avoiding NKVD patrols. On the Arzamas–Derbent–Baku train, after another lucky coincidence when he evaded arrest by railway police, Mark decided to get off in Makhachkala: routes to Baku were guarded because it was the center of the production of oil, a strategic commodity. Mark was able to obtain a position at the Pedagogical Institute in Makhachkala. He humorously described a lesson at the Institute, where a discussion arose between Mark and the Russian language teacher. One of them used the word "kuz", while the other said "kuza". What they had both meant was the Russian word "kuzov", a boot of a lorry. Obviously, Mark had not been perfectly fluent in Russian at the time. He would go on to speak perfectly correct Russian, with a light accent. His writing was very beautiful, with wonderfully contoured letters, in calligraphic handwriting.

An important period in his life began after his move to Tbilisi, where he was a graduate student in mathematics at the Tbilisi State University (TSU). Ilia Vekua and Nikoloz Muskhelishvili nobly took an interest in his fate. Mark would always gratefully remember the help that was given to him by the Georgian mathematicians and his friends there. For instance, Muskhelishvili called the Minister of Light Industry of Georgia in order to obtain a "uniform" for Mark, having noticed that Mark's only outfit had gotten very shabby over his years of wanderings. Mark patched up the holes in his "uniform" as well as he could, but, in his own words, he had not been entirely well-versed in the art of securing threads. His friendship with Vekua lasted the rest of his life. In Tbilisi, Mark made another pair of equally close friends: Karen Ter-Martirosyan, who would later be Landau's student and then a

[1] Wladek's family described his escape differently: Wladek later told his son Edward that he left Lwów on foot after the university officials ordered the students to do so. Bella Naumovna Garshtein—Mark and Wladek's classmate—later doubted that such an order was given.

prominent physicist specializing in the high energy physics, and Alexander Hvoles, a mathematician who had just been demobilized due to a serious injury. Mark's roommates at the TSU student dormitory were the Abkhaz poet Bagrat Shinkuba (later the Chairman of the Presidium of the Supreme Soviet of the Abkhaz ASSR) and the famous Abkhaz historian Shalva Inal-Ipa.

In the spring of 1945, Muskhelishvili arranged (on his own initiative) Mark's transfer from postgraduate study at TSU to postgraduate study at the Steklov Mathematical Institute of Russian Academy of Sciences in Moscow, where Lazar Aronovich Lyusternik had agreed to be his supervisor. Lazar and Mark later collaborated on a famous series of works on equations with a small parameter at the highest derivative.

Mark's friendship with Lazar lasted all his life. Many times Lazar would invite our whole family to his dacha on the 42nd kilometre of the Moscow–Kazan railway. From these visits I remember the French seventeenth to eighteenth century chronicles, which Lazar would read in the original language. Lazar was extremely knowledgeable in Russian literature, and wrote wonderful poems. His speeches were wise and profound. Much later, I learned with horror about the case of Nikolai Luzin.[2] Many years after Lazar's death, when I asked Mark about this, he said, "Oh, they were all forced into it." Of course, Lazar and Mark were both witnesses to very terrible times. Lazar Lyusternik, in Mark's eyes, was always the highest authority, a mathematical giant, an old and wise comrade, and an example of a pure mathematical talent.

In the spring of 1945, Mark met his future wife Asya, my mother. This was, in a certain sense, the end of his wanderings. Mark would always note that, during their courtship, Asya demanded that he read the most important works of Russian literature. In particular, she required that he read all of Alexander Pushkin and all Leo Tolstoy. "A very cultured family," Mark would often say. Asya's family accepted Mark into their circle. The process of being accepted into a new family was, apparently, not completely free of conflict. In the early days, Mark's future father-in-law, Moisey Naumovich Guterman, would love to say, in Mark's presence: "He'll run off, oh, he'll run off! He'll get his permanent registration and run off!" (This was in reference to Mark's Moscow residency permit.) Mark would laughingly recall this maxim all his life, always to his wife's great displeasure. Asya did not deny the facts. But she would indignantly remark, her voice raised: "Markus, stop right now, you don't understand anything!" Moisey Naumovich also expressed doubts about Mark's origin: "But we do not know anything about his family, he's some kind of cosmopolite!" Mark's future father-in-law used the explosive word "cosmopolite" in its original sense. However, in the papers and in public discussion, this wonderful word held a completely different meaning. When bed

[2] Nikolai Luzin (1883–1950) was a Russian mathematician who worked in the descriptive set theory, mathematical analysis, and point-set topology, who formed his highly successful school of pupils (informally named "Luzitania"). He was persecuted in 1936, during the Great Purge. After the trials, Luzin was cleared of political accusations and was not arrested, yet his department in the Steklov Mathematical Institute was closed and he lost his official positions. —*ed.*

bugs appeared in the communal apartment, Mark's future mother-in-law confidently determined that "the bugs came from Mark". After some time, the initial problems were smoothed over, and Mark became a sure-footed member of the family. As for Moscow's bed bug problem, my colleagues tell me it has not been solved to this day. Bed bugs survived Gorbachev's Perestroika, the dawn of capitalism, and are alive and well today in Moscow's cheaper hotels.

Mark also entered Asya's circle of friends, which included Miklail Levin, Alexander and Lydia Kronrod, Akiva Yaglom, Isaak Yaglom, Boris Rosenfeld, and many others. The beginning of Mark's acquaintance with the great twentieth century mathematician Israel Gelfand, who would have a great influence on Mark both mathematically and personally, also occurred during this period.

After the graduate school, Mark was assigned to a position at the Moscow Power Engineering Institute (MEI). (During that period, the head of the Department of Mathematics at MEI, following the dismissal of V. Levin, was N. Lednyov.) Mark used to tell that his workload had been enormous, and was distributed as follows: a couple of lectures, a 2 hour break, a couple more lectures, another 2 hour break, and so on, that is, a maximally inconvenient lecture schedule. During the breaks, Mark tried to focus on his research. His works from that period, which received significant acclaim, were also written under the conditions of his communal apartment on Sivtsev Vrazhek Lane (close to Arbat Street), where ten families lived. All ten families shared one kitchen and one toilet.

In Mark's recollection, at MEI he gave lectures with great enthusiasm. Once, when Taylor's theorem on the approximation of a function by a polynomial came up during a lecture, Mark asked the audience to rise. Thus, in his opinion, one ought to have shown respect for this central fact of mathematical analysis. This "All rise!" was often recalled by the MEI graduates whom he would later meet. And there were thousands and thousands of future engineers who went through his lecture courses. I would not have dared to initiate a similar "All rise!" in my own lectures on mathematical analysis at the University of Texas.

During this same period, Mark also actively participated in Petrovsky's seminar at MGU on partial differential equations. Ivan Georgievich Petrovsky, the Rector of Moscow State University (MGU), wanted to bring Mark into his Chair at Mechmat MGU,[3] but was categorically discouraged from doing so. Petrovsky was told that Mark was close with N. Lednyov at MEI. It was the first, but not the last, example of academic politics in my father's life. Lednyov was a refuter of Einstein's theory of relativity, and took part (perhaps sincerely, out of ignorance—it is hard for me to judge) in some unseemly "personnel transfers" in Odessa, at the Department of Physics at MGU, and at MEI, during the period of the anti-cosmopolitan campaign. Lednyov's reputation in Petrovsky's eyes was, to put it mildly, not very high. Petrovsky nevertheless asked Gelfand, "How is it that Mark, such a capable mathematician, is friends with Lednyov?" Gelfand replied: "Who

[3] Chair of differential equations at the Department of Mechanics and Mathematics, Moscow State University —*ed.*

told you that, Ivan Georgievich?" The conversation ended there. Without any further discussion, Petrovsky called Mark and offered him a Professor's position at MGU, at his Chair of differential equations. From this moment on, Mark's academic life and teaching were both linked with MGU, one of the most important centres of mathematics in the USSR, and, at that time, in the world.

It should be noted that Petrovsky had already been a great help to Mark at an earlier time, with regard to his dissertation defense. Having been submitted to the Scientific Council of the Institute of Mathematics of the Academy of Sciences of the Soviet Union at the very start of the 1950s (in the midst of the anti-cosmopolitan campaign), the dissertation lay there untouched for half a year. Mark called the Council secretary, but was told repeatedly that the Council was overloaded. Petrovsky, having learned of this, went to see Ivan Vinogradov, the director of the Institute of Mathematics. Mark's defense took place shortly after their conversation. His opponents were Israel Gelfand, Sergei Sobolev, and Andrei Tikhonov.

Students and coauthors played an enormous role in Mark's life. The entire list is well-known. The apartment in Lefortovo, across from the Lefortovo Prison, and later, the apartment beside Leninsky Prospekt metro station, were both visited in various years by Mikhail Krasnov, Grigory Makarenko, Mikhail Agranovich, Yulii Dubinskii, Gregory Eskin, Victor Grushin, Mikhail Shubin, Andrei Fursikov, Alexander Demidov, Alexander Komech, Alexander Shnirelman, Anatoli Babin, Pavel Blekher, Sergei Kozlov, Sergei Kuksin, Vladimir Chepyzhov, and others. Mark found great importance in his interpersonal relations with people, not just in their mathematical ability. Mark made great efforts to set up the lives of his students; when possible, he asked for the assistance of Petrovsky and other colleagues. I recall many dinners at Lefortovo and at Leninsky Prospekt with Mark's students and coauthors, which invariably included discussions on mathematics, as well as any other topics, including also politics.

Mark's seminar on partial differential equations at MGU was a tremendously successful endeavour of his. I hope that Mark's students and other regular participants of the seminar will write about it in more detail. Mark had a tradition of inviting the speakers to his home (I remember multiple visits by future speakers to the apartment in Lefortovo). They would discuss an upcoming talk, at times going into amounts of detail that would have been impossible to fit into the standard academic schedule of an MGU seminar. The future speaker was surely invited to dinner. This tradition remained unchanged throughout the entire time Mark ran his seminar at MGU, except, perhaps, the last few years, when I was no longer a direct witness to it.

One of my first memories of my father is attending an entrance exam to a music school in the Nemetskaya Sloboda,[4] across from the German cemetery, not far from the apartment in Lefortovo. The exam was hosted by the headmaster, and I flunked it. He sat down at the piano and invited me to sing the song «Shiroka strana moya rodnaya» («Wide is my motherland. . .»), which I proceeded to sing with pleasure,

[4] A neighborhood in the northeast of Moscow which translates as *German Quarter —ed.*

because I knew the words well, but all on one note. Mark loved music, all of classical music. I first heard Johann Sebastian Bach when Mark played one of the records in his collection: a recording of the Passacaglia in C minor (BWV 582) performed by an orchestra conducted by Leopold Stokowski. At various times over the years, Mark would become particularly fond of Brahms, Beethoven, Mozart, Haydn, Tchaikovsky, Wagner, Bach's concerti… A lighter music he loved in later years included songs by Charles Aznavour and Joe Dassin.

With Mark, we would regularly go hiking: skiing in winter and mountain climbing or kayaking in summer. I recall with nostalgia a multi-day hike (about 10 days) through the passes of the Greater Caucasus Mountain Range: Baksan—Chegem—Tvibersky Pass—Mestia—Cheget—Baksan. The backpacks were quite heavy, and all of the (canned) food, tents, and sleeping bags had to be carried. Before we reached the Tvibersky Pass, we slept in a tent on a glacier, going out at about four in the morning, when the sun was just beginning to illuminate the mountain peaks of the Greater Caucasus Mountain Range. In Mestia, where, like in Florence, every house had towers, we drank ayran and, for some reason, berry moonshine.

I remember my brother Senya preparing for the entrance exams to MGU, along with his friends from school, Alexander Komech and Fedya Bogomolov. For this occasion, all three lived in a dacha that my parents rented in the Moscow region. All three of them had been successfully accepted to Mechmat MGU. I still find it difficult to understand Mark's role in their preparation for the mathematics examination. But it was a very joyful time, with group walks, on foot and by bicycle, around the neighbourhood, swimming in the pond, and interesting conversations at the table. Sometimes, their school friend Alexander Vilenkin, who lived nearby, would join the company.

I remember one of the dinner parties in the apartment by Leninsky Prospekt metro station. It was my mother's birthday. She was telling with inspiration stories about her teacher, who she had just run into on the street (obviously, her former teacher was elderly by that point). My mother was full of admiration for her energy and love of life. Mark was unenthusiastic about alcoholic beverages, but here was a bottle of cognac nearby, and he drank a little more than usual. His unexpected comment, utterly out of place: "Are you still alive, my dear old lady?" (a line from Sergei Esenin),—brought the house down. With not a word, Asya took the bottle of cognac and carried it away to the kitchen.

My closest friend, the early departed Volek Fishman, once pointed out: "Mark, for me, sets an example of a Christian attitude towards life." It had then sounded unexpected and counterintuitive, but now this seems to me like quite a fair assessment. Here is the reason why. Mark was invited to a St. Petersburg apartment to the Easter celebration. The sister of the apartment's famous owner, who had prepared a holiday dish, asked: "But will the heathen eat the Easter meal?" The hostess, Mark's close friend, was trilling with laughter. Mark really liked the word "heathen", and would use it often after that occasion, tastefully and with pleasure.

Quite early on, he gave up the pursuit of academic regalia (though many say he deserved it). This made him happier. He said that his potential competitors for academic titles began to treat him better. He never spoke ill of people who made his life more complicated.

Mark gratefully accepted the Humboldt Prize and laureate diploma from the hands of the President of the Federal Republic of Germany. Many of Mark's students and coauthors were invited to Berlin for a conference at the Free University of Berlin on the occasion of Mark's being awarded an honorary doctorate (honoris causa). I do not know whether he was aware of the symbolic meaning of this occasion in the context of his own life. This symbolism was mentioned by the officials of the Free University administration in their speeches. Unfortunately, I did not ask, and Mark never said anything on the subject. Roger Temam, an old French friend of Mark's, gave a talk where he recreated the route of Mark's wartime wanderings on the map: Lwów—Makhachkala—Tbilisi—Moscow.

After my departure to Chicago and Texas with my family, for a long period of time, my communication with Mark mainly took place over the phone.

Mark and Asya visited us twice in Austin, the capital of Texas. Mark was very happy to hear of his granddaughter Inna's successes in school. In different years, we met in Bloomington (Indiana), Berlin, Leipzig, and Irvine (California). He was happy to have the opportunity, which had opened up after 1991, to travel the world and collaborate with new coauthors. Mark had a particularly close collaborative relationship and friendship with Bernold Fiedler. Asya usually accompanied Mark in his trips, though not always. The meeting in Leipzig was especially wonderful, where all of us, including Mark's granddaughter Inna, visited the apartment museum of J. S. Bach, went to see Bach's tomb in the Thomaskirche, listened to Bach's famous organ, and visited numerous local restaurants, where Asya paid special attention to the desserts. Mark spoke wonderful German; his German colleagues were thrilled by his complete lack of accent. Apparently, this had to do with the fact that his family in Lwów knew Yiddish, and Mark's parents, who were born in Austro-Hungary, spoke German. Mark learned French at school and comfortably gave lectures in French language while in France. His English—at least, his mathematical English—was also quite good; he had learned it much later. My colleagues, who had invited him to give a talk at my university, showered him with compliments. It was a masterful talk, and one of the very few of his talks that I ever heard.

Mark was later very interested in my daughter Inna's study of physics. She received her Bachelor's degree, and then her PhD. in experimental physics at Stanford. He was pleased with her articles which were published in good physics journals. Inna ran marathons with good, though amateur, times (her record time was 3 hours 5 minutes). Naturally, this required daily training. Mark did not approve of such serious sports endeavours, and was persistent in advising her to drop them. He himself played tennis very well, but treated that partly as relaxation and partly as entertainment. Almost everything in his life was governed by his scientific pursuits. My daughter did not heed his advice, but nevertheless became an experimental physicist specializing in high-temperature superconductivity. Inna is currently a Physics Professor at the University of California, Davis.

I will especially mention Mark's Sunday outings. Mark had a habit of going for walks around the Moscow region every Sunday. Usually from one railway station to another, usually located on a different line. His knowledge of the Moscow region was phenomenal. Sometimes his walks were rather long, about 20 km, and if skiing in winter, then up to 25–30. His main walking partner was Iosif Abramovich Ovseevich, but in various years, they were joined by other friends: Vladimir Stein, Dodik Ivinskii, Alexander Povzner, Sava Goldin, and others. I mostly took part in the winter skiing. Lunch was the focal point of these Sunday outings. Supplies and thermoses full of tea would be taken out of backpacks. The discussions would often concern politics, and one could hear the latest political (and also everyday) anecdotes. Iosif Ovseevich, who was married to Asya's sister Vitaliya (Talya), was an inexhaustible source of anecdotes and jokes (not always wholesome). The friends were completely unanimous in their political views, with one interesting exception: Alexander Povzner, a wonderful mathematician and an interesting, original man, would always approve the latest party line and would invariably find a deep meaning in the decisions of the Brezhnev Politburo. In the winter, temperatures sometimes dropped below −30°C, but that did not stop Mark. I remember some bitterly cold skiing trips with him in the Moscow region's Balashikha. During long trips from station to station, without any obvious landmarks, it was easy to get lost. Mark was very principled about how, and from whom, to recover one's geographic location and directions to the destination. Amid the crowds of train passengers, Mark and his friends stood out by their rather tattered appearance, shabby clothes, darned and holey jackets. However, in contrast to many of the passengers, they were sober. I think Mark would have liked Venedikt (Venichka) Yerofeyev's brilliant novel (poem), «Moscow—Petushki». Unfortunately, he never read it.

In the winter of 1973, Mark took the family to the village of Mozzhinka near Moscow, to a resort that housed visitors in academic dachas donated by Stalin to the academics and corresponding members of the USSR Academy of Sciences. Mark Graev, with whom we spoke often, was on holiday with us. Our skiing trips, including ones on the ice of the Moscow river, left magical memories. I remember Mark's conversation with Dmitri Anosov, who dropped by our table during dinner. In the conversation, he called Mark "urchapist" (PDEist), in return Mark remarked: "And what, as though 'obyknovenschik' (ordinary-schik) sounds better?" Mark highly regarded Anosov's world-famous works on dynamical systems theory. It was in Mozzhinka where Mark learned the tragic news of the death of Ivan Georgievich Petrovsky.

After the death of my mother in 2009, Mark's life changed greatly. My mother was ill for a long time before her death. Doctors were unable to establish the exact nature of her illness. I remember I visited her at the hospital, not far from Losinoostrovskaya railway station. Mark was constantly with her in the hospital ward, and would spend his nights at the hospital. He was also the one who discussed her procedures with the doctors and nurses. When she was no more, he continued to work as always, in a disciplined and strictly regimented way. But one day, about 2 years after her death, he remarked, "Asennka entirely freed me from the daily routine." This phrase is ambiguous, and I will leave it here without comment. Mark was strong enough for long walks. During my visits to Moscow, he and I would take relatively long walks in the *Neskuchnyy Sad*,[5] walk down to the Moscow river, and to Park Kultury.[6]

I am reaching the final and most difficult part of my notes. Mark courageously faced the inevitable fading of his strength and ability to work in his advanced age. More than once, during our private conversations, he told me that he thought of death without horror. About 2 years before his death, he told me point-blank: "I am at the finishing line". His voice was completely calm, as if he were stating an obvious fact. He tried as well as he could to reasonably distribute his strength, and get some work done in at least the first half of every day. During this period, he listened to a lot of music, and, unfortunately, watched television—sometimes even political programmes. Here, the famous Greek poem comes to mind:

> The fatal hour comes, and the black Keres appear
> To people: in the hands of the first is the heavy lot of old age,
> The second holds death.
> So short-lived
> Is the sweet fruit of youth: the sun rises—it fades.

(Mimnerm, late seventh century BC; translated by Maria Komech)

During this period, I spoke to Mark every day on the phone. His ability to remember specific events from the distant past, and people he knew, was incredible. However, his memories were sometimes coloured by his imagination, and he would recall events in which he could not have participated. It appears to me that Mark was endowed by nature with a strong immune system, good health and high energy levels. Malaria, which he had lived through during the war, and liver disease had both had a negative impact on him. However, I think he could have lived much longer due to his exceptionally strong immune system. But doctors say that one can die of a stroke even at a young age.

[5] A historic park along the right shore of the Moskva river, close to Mark Vishik's flat —*ed.*

[6] A central park in Moscow —*ed.*

Mark Iosifovich and Asia Moiseevna: Random Reminiscences

Claire Vishik and Inna Vishik

How It All Started for Me I can say without being untruthful that my first meeting with Mark Iosifovich (MI) and Asya Moiseevna (AM) was caused by engaging (perhaps not entirely wisely) in intellectual pursuits. I was an undergraduate student at the time, and although I was normally careful with money, on this occasion, I got carried away and acquired two thick tomes for what amounted to almost all of my disposable funds for that month. My mother in St. Petersburg, who was generally unsympathetic to such excesses, was nonetheless worried that I would go hungry. And, through friends of friends, she found someone (MI and AM, in this case) traveling in the right direction and willing to deliver a package containing a bit of food and a small financial infusion to tide me over.

Late in April, AM called me to arrange a delivery of the package. I was given the address on Ordzhonikidze Street and told to be sure to arrive on time. I didn't plan to stay longer than it takes to ring the bell, open the door and receive the package. After all, MI was a professor at the best university in the country, a god to someone like me, and surely too busy to spend his valuable time chatting about nothing. I made other plans for that afternoon, but these plans were destined to go awry.

In order to avoid confusion, this short memoir is written as Claire's story, by both Inna and Claire.

C. Vishik (✉)
Austin, TX, USA

I. Vishik
Department of Physics and Astronomy, UC Davis, Davis, CA, USA

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Trends in Mathematics, https://doi.org/10.1007/978-3-031-33681-2_2

The Tea Ritual Instead of a short and impersonal package transfer, I was invited to come in and have tea. Naturally, I protested that I was not hungry (although I was ravenous), but MI and AM refused to listen, walked me to their tiny kitchen where AM served sweet tea with all kinds of chocolates, cookies, and other goodies that would lift the spirits of any student living away from his or her family. The conversation around the table was lively. Misha (my future husband as it turned out) and MI gently teased AM and asked me a lot of questions, mostly to make sure I felt comfortable and was included in the conversation. MI's level of politeness towards a random person drifting in surprised me. Most of my professors were nice, but showed no interest in us beyond their subject. But here, in this tiny space (6 square meters) everyone was relaxed and warm-hearted.

MI: Preparing His Informal Conversations in Advance On this occasion, I heard—for the first time—several of MI's maxims that he liked to repeat. One of the sentences that I heard for the first time then and that was repeated many times over the years was, "It is better to have two dinners than none," a pronouncement made to vigorously support my hesitant acceptance of the fourth cookie. I discovered later that MI had a wide selection of various prepared statements, some constructed to be able to pursue small talk while thinking about something else, and some designed to avoid in-depth discussions on a subject he preferred to avoid. The combination of striking (and mostly correct) insights and frequently used prefabricated sentences created a conversational style that was his own and that I have not encountered anywhere else. It allowed MI to have easy, friendly conversations that could be in-depth, but didn't need to be absorbing. For example, most questions about MI's childhood in Lwów met with this response: "Before my father passed away, we lived well." In general, for topics he didn't wish to discuss at length (and his time in Lwów was one of them), these carefully crafted sentences dominated the conversation.

Exceptional and Understated Kindness This first meeting with MI and AM was delightful, and, if it was a test, I must have passed it, since I was invited to dinner, an invitation I wasn't able to accept until late fall of that year. After that, I was a recipient of MI's and AM's exceptional kindness (and abundant food) on many occasions.

Feeding the younger generation was then an academic tradition, but MI's and AM's support of their young visitors went well beyond this. I have observed them extending their helping hand in many situations, going well beyond generic hospitality and well beyond providing the kind of assistance that didn't cost any effort. I was an object of their great kindness on many occasions, including very early in our acquaintance in conjunction with my "propiska," or the residency permit. I was registered in a dormitory but lived in a rented apartment. The dormitory registration made it impossible for me to do a few useful things, such as seeing a doctor. Since my parents lived in the "second capital" St. Petersburg, my frequent visits home resolved many issues, but not all. MI heard about these difficulties and offered to register me temporarily with them. Those who are familiar with the rules at the time and the huge value of any type of propiska in Moscow, can understand the enormity of this gesture. MI barely knew me, but he probably remembered his first year in Moscow, in similar circumstances, and was touched—in an actionable way—by the plight of a virtual stranger. This was not the last time MI and AM provided assistance without advertising it or expecting anything in return.

Many years later, when both my mother and father passed away, AM, who was not demonstrative and always posited that affection is deserved, not given without a reason and should not be granted to family over friends, took me aside and said, "You are an orphan now, but MI and I will be your Mom and Dad. We are adopting you." It was a sweet, loving way to express her sympathy, and the sentiment that still makes me smile when I remember this evening in Irvine in 2004.

MI's Focus on Work, and AM's Hard Work to Make His Life Easier In the Soviet Union, there was considerable interest in intellectual pursuits. In many cases, erudition was treated as a proxy for intelligence and of being part of the "elite." We didn't have many other outlets for the natural desire to learn and communicate, with travel and entertainment severely limited, and so reading, for example, natural philosophy or "sophisticated" literature, listening to and becoming experts in classical music, learning foreign languages was our way of indicating that we belonged to this "intellectual circle." Being cultured, frequently in an old-fashioned and abstract way, was definitely a virtue. But being ambitious in one's career or devoted to one's work in an overt way was not always interpreted positively, because taking this direction was frequently associated with considerable levels of conformity and opportunism required by the system that was intensely disliked by everyone we knew.

MI's complete dedication to his work (in addition to his family and community) resulted in what was, in the Soviet Union, a somewhat unusual lifestyle. His days were very carefully and consistently regimented, with little lightheartedness allowed. He woke up at dawn, and, after 30 minutes of exercising on the balcony, started his workday. Most of the time went to work and work-related activities, only interrupted by long walks over the weekend and very occasional visitors unrelated to the world of mathematics as well as 1 hour of recreational reading before going to sleep. Most of the people we knew dedicated more time to recreation, socializing outside of work, and to ensuring greater comfort for their families.

MI and AM were also an unusual couple from a different point of view. From what I could see at home and among my parents' friends, in the white collar families at the time, men and women in the Soviet Union had equal engagement in everything, from work outside of their home to house work. This equality was born out of necessity, not ideology or conviction. But in this family, AM was responsible for 100% of household management. It wasn't a life of drudgery, by any definition: AM enjoyed the interesting people who came to the house, very happy family life, and she traveled with MI on many of his work trips, especially later in life. MI was AM's life, and she welcomed it. But sometimes she would tell me that she had sacrificed her career for her sons and her husband, and she sounded wistful. I assume it was not always easy for her.

I realized much later how demanding MI's and AM's lifestyle was and how much courage it required to live like that, separating themselves from the constant hassle that even the simplest everyday chores, like buying slightly better food, required. At the beginning I thought that MI's choice (supported by AM) to live without many comforts and with almost no relaxation was a kind of penance he assigned himself for surviving while the rest of his family perished. But this lifestyle was not motivated by the enormous loss MI suffered losing all his family in the Holocaust; it was the price he was willing to pay to continue his great enjoyment of mathematics. Mathematics was truly his life.

Reading MI's and AM's choice of reading materials was very different. AM was more conventional and trended towards anything that had real or perceived educational value, regardless of the subject. I remember many discussions with AM while she was steadily moving through a book in English dedicated to different types and history of sailboats. Every time I visited, AM quizzed me on English maritime terms, and I had to plead ignorance to most questions. After 6 months, when the book was finally conquered, we both learned a lot of new words. For MI, the choice of books was random and opportunistic, ranging from Macaulay to Bashevis Singer simply because these books were readily available at home. But reading was the time of relaxation for MI and he read carefully, remembered what he read, and liked to talk about it. Unlike many conversations that included considerable generic content, every book was always met with a certain level of absorption. Conversations over tea would frequently gravitate to the book of the day. A random tome of Macaulay's History of England brought forth animated discussions on the fine points of history that were no more than footnotes to more important events. And "The Family Moscat" by Singer became a source of many dinner time conversations as well as several nicknames, including one of my own, Hadassat (from the incorrect reading of Hadassah, one of the protagonists in the book) while one of Misha's friends became Reb Meshulem. When talking about Singer's book, MI occasionally mentioned small details of his life in Poland, e.g., that his grandmother was deeply religious and wore a wig.[1] It was clear that Singer's descriptions aligned with what he remembered from his life in 1920s and 30s, but it didn't make MI more open about this period of his life.

Music MI loved music. It always elicited an emotional response from him. Classical music was always playing in the background when he worked by himself, and sometimes when he worked with others. The selection was consistent: MI had recordings for all major operas, music by Beethoven, Mozart, and Schubert was played frequently, and Brahms was one of the favorite composers, surprisingly for

[1] Orthodox Jewish women who were observant covered their hair after they were married.

me. He preferred large forms, and I frequently arrived to pick up Inna[2] to the sounds of a symphony or a sonata. MI always regretted not having had any training in music. He mentioned that his older brother Bernhard had violin lessons as a child. The lessons had to be cancelled when their father died. Despite his lack of formal education in music, MI had excellent memory for the pieces he heard and a very pleasant signing voice. He could reproduce correctly even the most challenging melodies. He could have been a very talented musician, as my brother, a concert pianist, pointed out when they first met.

By contrast, AM was not interested in music or affected by it, but she was always supportive of MI's interest in classical music.

MI had only one recording that was not classical music: the American/French chansonnier Joe Dassin. It was a present from one of his visitors, and MI played it only for one of his former graduate students. Occasionally, when I picked up Inna to the sounds of Joe Dassin instead of a classical opera, I knew who was visiting.

Religion MI's views on religion are hard to define. He grew up in a Jewish Orthodox family but longed to be secular as a young man. He certainly was not practicing any religion, in the conventional sense of this term. But MI had a very strong, although also very vague, religious feeling going beyond the desire to be an ethical person. He told us once that he always said a prayer before going to sleep. It was a way to organize his priorities at the same time that it was an appeal to a higher power. In this, MI was a complete opposite of AM. She had zero sympathy for any kind of religion, both in theory and in practice. She referred to discussions on religious topics as "figly-migly."[3] This view was highly ingrained in her life philosophy and it has not changed as she grew older.

Striking Gestures, Sometimes MI and AM spoke in an understated, matter of fact way. They were sometimes enthusiastic, but not inclined to be effusive, differently from many others in their generation. But there were situations when something unusual and memorable needed to happen. And it did. MI told us how he asked his class to rise when talking about what he considered beautiful mathematics. AM knelt near the Newton's tomb in London, to recognize his genius. And she became effusive when talking about MI's success and international recognition, usually quoting Pushkin's poetry in illustration. AM was equally effusive in her praise of Inna's work, from her little paintings at age four to her popular science articles when Inna was a student at Stanford. AM was always the most enthusiastic cheerleader for MI and Inna, even as she continued to be mostly reserved in other everyday interactions.

Order Out of Chaos MI's days were regimented and predictable. Whenever possible, breakfast, dinner, and collaborators' visits were scheduled at the same time, and the bedtime was non-negotiable. The New Year Eve was a day he hated

[2] Inna Vishik is MI's and AM's granddaughter.

[3] mumbo-jumbo.

because he had to stay up until midnight to participate in the inevitable festivities and usher in the new year. MI's schedule was somewhat defined by his commitments and external obligations, but AM's days were also well scheduled. She went to grocery stores at the same time on same days of the week, cooked dinner for several days at the beginning of the week, shopped for birthday presents about 2 weeks prior to family birthdays. They were both comfortable with this predictability. When MI went away on lengthy work trips and it was impossible for AM to accompany him, she filled her days with projects. For example, AM created, by hand, a card catalog for all the books in their extensive library when MI spent a few months in France.

This desire to optimize the chaos of daily existence revealed itself in many ways, including lunch and dinner menus. AM's everyday cooking was very simple. Ordinarily, she would prepare a large pot of vegetable soup and a big chunk of beef or a whole large chicken slowly cooked in water, to accommodate MI's dietary requirements. The food was designed to last several days. These meals seemed somewhat monotonous to me until I understood that it was the most efficient way to feed many guests, some of them unexpected. Thus, everyday cooking was a duty, although lovingly performed. But for special occasions, cooking became art. AM didn't shy away from recipes that require many hours of work. The results of these efforts were memorable: many guests who joined MI and AM for various celebrations remember her amazing gefilte fish or bean paste cooked with nuts in homemade pomegranate juice.

Family History—Not a Favorite Topic: MI While we frequently discussed books and amusing events, MI and AM almost never talked about their childhood in any detail. MI mentioned funny occurrences, such as extending his middle finger in his jacket pocket during Christian prayers at his school, on the advice of the school's rabbi, or that he envied folks who had chocolate butter sandwiches for lunch, while his mother only packed boring cheese sandwiches for him. Most of his childhood stories were about food. Occasionally, there were other tidbits: for example, MI remembered being called up to the Torah at his synagogue, which is how we learned his Hebrew name, "Mordkhe-Yitshak ben Yoseph-Chaim." He talked about saying Kaddish (memorial prayer) for his father who died when MI was nine. He sometimes talked about his only friend, Wladek Lyantse, and how they walked around the city for hours on Saturdays, with their pockets full of baked goods their mothers supplied. They talked about what they considered scientific topics, for example, if there were people on the Moon and canals on Mars. Sometimes MI talked about his high school, mostly remembering how his interest in mathematics began and was encouraged by one of the teachers. He talked of the university life frequently, but in general terms, mostly describing the joy (and overwhelming interest) in learning mathematics that took all his time, leaving only 2 hours a week for dating.

When pressed about his childhood memories, late in life, MI talked about being "a strange child," with only one friend, Wladek Lyantse. MI's father died young, when MI was not yet ten, and his mother Regina took care of the four children. The youngest child, Josef, was born after his father passed away. MI's brother Bernhard was less than 2 years older than MI, and his sister Hela was 2–3 years younger. The kids went to the Jewish schools except for MI, who switched to the Polish track. MI doesn't appear to have been close to his older brother, but he always spoke warmly about his younger sister and his mother. Little Josef was described as "cheerful." Overall, MI very rarely talked about his three siblings. According to MI, his older brother Bernhard worked as a paralegal after he graduated from the Jewish gymnasium. There seemed to be considerable sibling rivalry between the two older boys. MI's younger sister Hela was always characterized as beautiful and sweet, but the only memory about her that he shared was how he took her to the movies and how she was always holding onto his finger not to be separated in the crowd. Bernhard and Josef perished in Lwów in May and November of 1942 respectively. We don't know what happened to Hela.

MI's Yiddish name was Mordkhe-Yitzkhak, but he was Markus Wischik to his teachers at school. To his fellow students in his Polish Gymnasium and then Lyceum, he was Mietek, short for Mieczysław. A couple of photographs from the 1940s in our collection are addressed to "Mietek," so the name stuck for a few years past his departure from Lwów. A few years ago, I was thrilled to find a reference to "Markus Wischik" in a Polish newspaper in Lwów for 1936, but when I downloaded the photocopy, with a lot of effort, the notice published there was that Markus Wischik lost his school ID. A tiny peak in MI's life in Lwów before Galicia was annexed to the Soviet Union. AM always called MI Markus, the only one who used his Polish "official" name.

Many books were written about the battle of Lwów in 1939, including 2 weeks of fighting in the streets. This must have been a difficult and memorable time for the residents, but MI never talked about it.

. . .**and AM** While MI's childhood was derailed and defined by his father's early death and geopolitical events, AM's family spent the 1920s, 1930s and 40s on the brink of a different, but terrible tragedy. AM's family was wealthy before the October Revolution, and in 1920s many of her relatives became "lishentsy" (losing their voting and other rights). Then in the 1930s, many of her politically active family members who were Mensheviks or connected to the wrong communist party fractions perished in Stalin's terror. Among them were Grigory Zinoviev[4] (relative by marriage), her Uncles Abram and Boris Guterman (both prominent Mensheviks), and her beautiful cousin Eva who committed suicide when her husband Vasily Yarotsky was arrested. Probably as a result of having to be always on her guard, AM almost never talked about her childhood; I heard more about it from her cousins. When I met AM, these "dangerous" connections didn't matter anymore, but she continued to be very reserved when the topic came up. Of her childhood, she only said that she had many friends at school, that she was a popular girl, and that her father instructed her mother not to teach the girls household management or cooking skills because this would negatively affect their careers, in his opinion. AM was the

[4] https://en.wikipedia.org/wiki/Grigory_Zinoviev.

youngest of three children. Her older brother Somik perished in WWII; he enlisted as a volunteer.

AM also volunteered, and she was sent to a military hospital. The war period, difficult and tragic as it was for her family, was nonetheless a major highlight of her life. Among AM's small archive we inherited, we have an article about army nurses, with AM's comments written on the margin. These comments are full of passion; the memories of her war work were always fresh in her mind.

In a small pile of AM's papers that were sent to us, there were several secondary school report cards. AM was an excellent student, and we noticed that, very unusually, it was her father and not her mother who signed the report cards. Her father was a professor at the Academy of Forestry, a graduate of the Imperial Mechanical Institute that later was known as Bauman School of Engineering (Baumanskoe Uchilische).

Reminiscences of MI's and AM's Life Together While childhood memories were not a favorite or welcome topic, both MI and AM took pleasure talking about their joint experiences, starting from joining a large group of friends for a vacation in Optina Pustyn' during the summer of 1945. MI was proud that he was viewed as the most practical of this group of students, and thus sent to the farmers' market for provisions, in hopes that he could negotiate optimal prices. Among the favorite stories was one about registering their marriage (they were asked, "death or birth?"), one about the general admiration their son Senya's good looks elicited, and the story about their younger son Misha throwing a snowball at MI's dissertation adviser Lyusternik. Most of the stories were humorous, very well adapted to be told again and again. Later, when MI and AM started traveling internationally, they liked to talk about places they visited and people they met. AM frequently added a philosophical dimension to such conversations, for example, she was convinced that the general population was happier and kinder in countries that were neutral during WWII. WWII was a defining event in her life, one that never lost its crucial importance even decades later.

Granddaughter The predictability of MI's and AM's days evolved when our daughter Inna was born and gradually became an integral part of life for MI and AM, taking over what used to be Misha's room. Until Inna's birth, there were only boys in her generation in our families, and I was the only girl in the previous generation. Thus, having a little girl around proved to be fascinating, especially for AM. AM and Inna got on very well together, an attachment that lasted through all their interactions. Inna was so frequently sick going to daycare as a young toddler that her grandparents had to step in. AM and occasionally MI looked after Inna several times a week when she was between two and 4 years old. Their lives were very busy, but they generously offered help. When Misha and I were both sick with a very serious flu, AM took Inna to stay with them for a couple of weeks. By that time, the transition was seamless, Inna was almost a permanent resident in her grandparents' home.

As a young child, Inna was present when MI's collaborators visited, she listened to the conversations and frequently shared with us insights from what she heard. I don't think she participated in the conversations, as she did with our guests, but she learned a lot of words and subjects that she used freely to strengthen her argument in many toddler style debates. It was inevitable, then, with the company MI and AM kept, that she also became an academic, a professor of physics.

When Inna was very young, MI would bang kitchen pots together to amuse her and, hopefully, make her eat. This serious mathematician who would not tolerate tomfoolery from his own children put on a show for his toddler granddaughter to get a few morsels of food in her gullet. When Inna's sled got stolen and she was despondent, he rubbed his hands together and described how he would crush those terrible boys who stole the sled.

When we settled in the United States, Inna's interactions with grandparents were less frequent but nevertheless memorable. When Inna was 16, our family spent some time reunited in Leipzig where MI chose to work for a few months after receiving the Humboldt Prize. AM was not feeling well that summer, and MI gave us the grand tour of Leipzig, walking us through all his favorite spots. Everyone else's German was hopeless to nonexistent, but MI spoke on our behalf. Always modest, he downplayed his language skills, saying, "I speak like a child."

The last time Inna saw her grandparents was when they spent a quarter in UC Irvine in 2004. MI, AM, Misha, Inna, and I spent a nice day walking around Irvine and returned to their apartment for a meal. MI and AM prepared a banquet for us, covering the table with dozens of dishes, and encouraged us to eat by claiming all of the treats had zero calories.

As Inna got older, MI's and AM's cherished phone conversations with her developed into a familiar interchange led by MI: "Are you by your parents?" "You study physics… not math." (Spoken in a resigned tone). "You must to walk, and you must to think." (A reproach for spending too much time training for marathons when Inna became a prolific marathon runner). "You must to find your specialty."

It was all very charming in its predictability. While MI led the conversation, AM would take the phone from him towards the end of the call and implore Inna to learn Russian.

From these conversations and a few meetings that she can remember, Inna was especially touched by the devotion that MI and AM always shared. She supported him in his career, and later in life he looked after her with unfailing devotion.

Work—Above All Else MI and AM invited Misha and me to share their apartment after we married. The reason put forward by MI was that sharing housework will give us more time to be successful in our professions. It was a very kind offer that we ultimately declined, but I remember how it was worded: it was for us, and for our professional future.

The same focus on success in research was frequently communicated to Inna when she was a student. MI firmly believed that activities like sports or travel were distractions and wasted valuable time. While this humorous pressure may have been difficult for a teenager to accept, the language barrier changed that. Inna's Russian is far from perfect as was MI's English, while AM could read, but not speak

English. The need for translation and constant guessing made these conversations lighthearted and humorous.

Saving Asen'ka As MI and AM reached their late 80s, it became even clearer how important AM was in MI's life. There was still the focus on work, but, as AM's health deteriorated, the main theme became "saving Asen'ka." No effort was spared, private hospitals and doctors were engaged, some with outlandish courses of treatment. There were many successes, too. But some things can't be remedied or reversed. AM passed away in September 2009, leaving MI alone, after 60+ years of marriage. He was devastated.

Final Words One can write a book about MI and AM, their lives, their adventures, their friends, MI's research (of course), their kindness, their fascinating views, and a lot more. We hope these random reminiscences and anecdotes could shed some light on some of the aspects of MI's and AM's lives and their life together.

Meetings with Mark Iosifovich Vishik

Ekaterina Kalikinskaya

Mark Iosifovich Vishik was my husband Vladimir Chepyzhov's advisor when he was a student at Mechmat MGU. Subsequently, they worked together for almost 30 years. This is why I often talked to him while not being a mathematician myself: we would speak on the phone, see each other on joint trips, and at a few holidays. In his last years, Mark Iosifovich was left without his loyal friend and devoted wife. My husband and I decided to visit him regularly to brighten up his increasing loneliness. Many of his other acquaintances did the same. I can say without exaggeration that Mark Iosifovich liked talking to me about everyday matters; to reminisce about Asya Moiseevna, his friends and family, to tell me about his youth, the many remarkable encounters that had enriched his life. It occurred to us to write down our conversations with him and to publish them as a book. Unfortunately, there were only few such conversations to follow. Our last conversation took place in January 2010. The book never materialized, but an article did; it was published in English in [Kal14] as an interview with Mark Iosifovich.

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E. Kalikinskaya (✉)
Moscow, Russia
e-mail: chep@iitp.ru

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In this article, I attempted, through the content of our conversations, to recount his memories of the people who played important roles in his life.

1 Stefan Banach

A legendary person that Mark Iosifovich Vishik was lucky to meet in his youth. A genius scientist and university professor, with whose lectures, perhaps, Mark Iosifovich's introduction to science began. He was also a man of a very similar spirit, with similar life goals.

At the University of Lwów, which I entered in 1939, the Dean of the Mathematics Department was Stefan Banach, a genius mathematician. Apart from him, we were taught by the most outstanding professors of the Banach school: Bronisłav Knaster taught us analytical geometry, Juliusz Schauder taught theoretical mechanics, Professor Stanisłav Mazur taught differential geometry. Professor Władisław Orlicz taught algebra. All of the classes were in Polish. Only the vice Dean, Miron Zarycki, taught in Ukrainian.

I attended all of Banach's lectures, and they made me feel extraordinarily happy: I could study at such a university! After all, Banach was already considered the first classic of Polish mathematics during his lifetime...

However, showing his greatness was not in his nature. His characteristics were simplicity, friendliness, and cheerfulness; he had a great sense of humour. During his seminar, he would always participate, ask questions, could play a friendly joke on the speaker. But no one feared that: the overall atmosphere was very cheerful and comforting. Before the seminar, Banach loved to hide professor Stanislav Saks's briefcase, so that he would look for it and everyone would be a little entertained on this occasion.

Every day, he and Professor Zarycki would go to dinner, and when they returned, it was clear that they had allowed themselves to drink a little bit as well. Banach was a life-loving person: big, noisy, vociferous.

He had a demanding wife, and partially for that reason he wrote not only academic books, but also middle school textbooks and books on mechanics. But this was to everyone's benefit: more than one generation grew up on his superb textbooks. Professor Schauder taught us using Banach's book. We were very fortunate to have studied using those books. Schauder also treated his students very respectfully: before every new lecture, he always asked one of the students, whoever wanted to, to summarize the basics of the previous lecture or theorem we had studied. And only after that, he would begin his next lecture.

Abraham Plessner's seminars also made a big impression on me. He repatriated from Warsaw to Russia, and later I met him at MGU, where he became a professor. I remember he asked me to referee the paper by Hermann Weyl (it was assumed that he would be the Head at Göttingen after Hilbert). At that time, it was considered a good tone, after the completion of the work, to go give a talk on it at Göttingen to Hilbert.

Once, Banach and Schauder were called to Kiev and given the assignment to "sovietize" the University of Lwów. Upon returning, they did not do anything special in terms of ideology, but decided to strengthen the university's work by organizing student conferences, which we did not have at the time. Professor Edward Szpilrajn gathered the students and told us about Hausdorff spaces. This was how our student conferences began. Banach also attended these conferences and very merrily reacted to the students' comments on these topics.

I studied for about 2 years. And eventually my university professors saved my life, because they organized a collaboration between Lwów University and Tbilisi University. On this occasion came Professor Nikolai Muskhelishvili.[1] This collaboration was called a "socialist competition", and we were accepted as members both to the Trade Union and the Komsomol; I was very proud at that time. When the War began, we were on duty every day at the University, and we were asked to always come with our documents, record-books, Komsomol ticket. On June 28, 1941, one of the Komsomol activists came to the University and said that fascist troops were already in Lwów, yet at the other end of the city. And if we wanted to fight Hitler, we must immediately move to the East to join the Soviet army. So, without parting with families and without taking anything, straight from the University, myself and my peers went to the East. Many years after the War I found out about the fate of my relatives remaining in Lwów: all of them were taken to the Ghetto and all were killed.

During the first day we have passed 60 km, slept in the gutter. In Zhmerynka, we joined a freight train and began moving towards Kiev. We had no money, nor food. From hunger, I lost consciousness several times. Two weeks we traveled to Kiev and, finally, came to the Central Committee of Komsomol. And they told us: those who were not born in Russia have no right to be in the Army. They sent us to help with harvest to Kuban region, Timashyovskaya station, and we stayed there for 2 months. Then I went to Krasnodar on the rooftop of a freight car, then moved on and had undergone many ordeals: fainted from hunger, fell sick with malaria, traveled on the platform with combat equipment and on foot, until I got to Tbilisi, where I completed my studies at the University. I believe that I was saved from death during the War by Georgian mathematicians.

[1] Nikoloz Muskhelishvili —*ed.*

After the war, in 1945 in Moscow, I saw Banach once more. He was invited to Moscow by Academician Kolmogorov, who wanted Banach to teach at MGU. Banach was staying at the university hotel on Tverskaya, and I visited him there. He was unrecognizable: he had gotten much thinner, and became somehow flat, one-dimensional. He told me that during the years of the war, he saved himself by feeding lice. The fascist doctors in Lwów performed experiments on humans: they studied the effects of lice on the body, to save their armies from them. And someone brought Banach to those doctors to save him from death.[2] They tied a bag of lice to Banach's hand, and he was required to feed them. He was even paid something, so that he could support his family and survive himself.

But when all of these horrors were left behind, here's what Banach asked me first: what mathematical works did I read during the war? I described to him my scientific interests, and he told me: "Be like Schauder—study methods of functional analysis and differential equations."

Banach died of lung cancer that same year. He was a heavy smoker.

In the fifties, I went to give lectures at the Banach Center in the Institute of Mathematics of the Polish Academy of Sciences. I was invited by Banach's student, Professor Władysław Orlicz. He had been a professor in Poznań, and later returned to Wrocław. After the lectures, he and I discussed how one needs to help one's graduate students. At the time, there were strict requirements in Poland (and in Russia) for advisors to ensure that their graduate students would pass their candidate dissertations. Orlicz asked me how many graduate students I had. I answered, "Six." He also had six. He asked me what I did if the students did not pull their weight. I said I helped them. He confessed that he helped his students, too. We were happy that we were doing the same thing.

2 Ivan Georgievich Petrovsky

For Mark Iosifovich, this was a man who actively and heartily helped him in difficult situations at the beginning and middle of his scientific journey; a great scientist and a leader; Mark Iosifovich remembered him with gratitude and reverence. For him, he was a legend of a man, and at the same time, a person he was close to, whom he always tried to understand, and whom he would remember more than others.

When I moved to Moscow in 1945, I spent all my time in the old university building on Mokhovaya. I. G. Petrovsky was then the Dean of the Department of Mechanics and Mathematics and the Head of the Chair of differential equations. He gave lectures on partial differential equations at the university and wrote three

[2] During the German occupation of Lwów, the Polish biologist Rudolf Weigl—the inventor of the typhus vaccine and the director of the Lwów university of epidemiology, whose responsibilities included vaccine creation—received permission to hire employees at his own discretion, and used this opportunity to save many people's lives. —*ed.*

books dedicated to ordinary differential equations, partial differential equations, and integral equations. (At that time, the rector of the university was someone named Galkin, whom people at the university were displeased with.) Ivan Georgievich made an enormous impression on me from the very first glance. I knew that he was one of the greatest scientists, and read his recently published article in «Uspekhi Mat. Nauk» where he outlined the main direction of mathematical development in the near future.

Every day, I would rush, on my own two feet, from my wife's house on Sivtsev Vrazhek to the university. I did not miss a single one of I. G. Petrovsky's seminars, not a single one of his lectures on ordinary differential equations and partial differential equations! Petrovsky's students would mostly be the ones speaking at his seminars. I remember Gena Landis, who would come to the seminars in a military uniform. There were also Olga Oleinik, Anatoliy Myshkis. There was "the small seminar" as well.

Ivan Georgievich treated me very kindly, despite the fact that I did not speak Russian very well at the time: I graduated from the Polish-language gymnasium in Lwów, I knew Ukrainian and German well, but I began to study Russian in Georgia, so my speech was peculiar. But I could fluently read Russian scientific literature.

Petrovsky's seminar was held at a very high level. Petrovsky made sure that the seminar's themes dealt with general, major problems; he himself solved one of Hilbert's problems. Many people solved this problem for particular cases, for instance, Sergei Natanovich Bernstein, while Ivan Georgievich proved it for general systems.

The atmosphere at Petrovky's seminar was unusually friendly, and there were speakers every week. But on the days when, for some reason, there were not enough of them, I would always volunteer to present, and Ivan Georgievich really valued my activity at his seminar and would often take care of me in a fatherly way.

I was overjoyed to find myself at the Moscow university. First of all, I was stunned by the high level of mathematics at MGU. That was why I went to every seminar I could: Abraham Ezechiel Plessner's, Lazar Aronovich Lyusternik's, Israel Moiseevich Gelfand's... as well as to Moscow Mathematical Society meetings, to lectures for students which were read—as an extracurricular activity—by famous mathematicians: for example, Andrey Nikolaevich Kolmogorov. At the time, I studied mathematics for 12 hours a day.

Ivan Georgievich played an important role in the defence of my candidate dissertation, as well as my doctoral dissertation. When I prepared my candidate dissertation in 1947, he personally became my opponent. My other opponent was Academician Sergei Lvovich Sobolev whom I had known since 1946.

Petrovsky helped me, with great kindness, in 1951, when I completed my doctoral dissertation. I was officially attached to the Steklov Institute, and brought the fruit of my labour where I was supposed to. I had never been interested in any kinds of intrigues, or human relationships in general, and I had no idea that the whole place was entirely unsuitable for me. At the time, I absolutely could not have imagined that people "with the fifth line" in their paperwork[3] were really not liked at the Steklov Institute, and that I should not even think about doing my defense there!

The Academic Secretary took my papers, but told me dryly that the Academic Council was overloaded. He even refused to take all four copies of my thesis, and took only one of them, adding that they would call me when there is any progress. I packed the three "unnecessary" copies into my father-in-law's briefcase (I did not have my own at the time) and went home. This was in January. Of course, I never received any calls. The situation seemed hopeless. Six months went by—and silence. One day at the seminar, Petrovsky asked me how my doctoral dissertation was coming along. I told him I'm waiting for a phone call from the Steklov. And then, Ivan Georgievich, without saying a word to me, got into his, rector's, car and drove over to Academician Ivan Matveevich Vinogradov, to tell him his opinion of me and my scientific work. The Academic Secretary called me that very day! He demanded that I immediately bring over my character reference, the three copies of my thesis... The process was under way! My conversation with Petrovsky took place some time in June, and the Academic Council does not meet in the summer. But by October I was doing my defense.

Academician Vinogradov could not ignore Academician Petrovsky: he only had five storeys at the Institute, while Petrovsky, who was by that point the Rector of MGU, had a few more than that. In the future, that would turn into quite a number of storeys.

That episode was very characteristic of Ivan Georgievich. He always actively interfered in everything that seemed unfair. He acted quickly and decisively, without waiting to be asked. For example, his student Genya Landis, for whom "the fifth line" also got in the way of working at MGU, was hired at the Chair of differential equations only after Petrovsky's own initiative.

I remember another episode from my youth. When I was still a student at the university, Petrovsky's textbook on partial differential equations was just being prepared for publishing, and at our lectures, we were handed out sheets from the future book. One student found a mistake in the textbook and told Ivan Georgievich about it. Petrovsky not only thanked him, but expressed his gratitude in the text of the book itself. This is what kind of man he was! To him, truth meant more than anything else.

Ivan Georgievich did not change at all when he became the university Rector. The kind of dean he was—caring, compassionate—was the kind of rector he became. He was offered the position of Rector more than once, but he refused because he wanted to do the scientific work, and was actively working at the time. During the war, when the university was evacuated to Kazan, Petrovsky was the Dean of Mechmat and proved himself to be an excellent leader. He cared not only about abstract concepts,

[3] People of Jewish origin, as indicated in an employee's card at the Personnel Office in the fifth field, designated for the identification of *nationality* (ethnic origin), which followed after surname, first name, patronymic, and date of birth. —*ed.*

but specifically about the bread and butter for each of his colleagues. And during the war he was, again, offered the position of rector. But, again, he refused.

This is when the Party members of MGU went to a meeting with comrade Stalin and reported that the university found the previous rector to be unsatisfactory. Stalin allegedly asked:

– Is there someone there who would be both a prominent scientist and a good leader?
– There is, but he refuses.
– Tell me his name.
– Ivan Georgievich Petrovsky.

According to a legend, Stalin took a sheet of paper and wrote on it, “To appoint I. G. Petrovsky as a Rector of MGU.” And the very same day, Ivan Georgievich began receiving congratulations from his colleagues on his new position, although nothing had been officially announced yet. A long journey lay ahead of official appointment. But he could no longer refuse. Besides, Petrovsky always thought about how to be the most useful to the university. Once, Ivan Georgievich told me, “I’ve decided: if I remain the Head of the Chair, I will create dozens of major scientific works, but if I become Rector, thousands of such works will be created in the university.” That cemented his decision.

Thanks to Petrovsky, I began to teach at MGU, though not right away. After I graduated university, I was directed to the Moscow Power Engineering Institute, but in 1951, they started firing every professor “with the fifth line” (except Party members). For example, V. I. Levin, the Head of the department of higher mathematics at MEI, was fired. I was also on the list of those to be fired. But the Rector, Professor Mikhail Grigorievich Chilikin, crossed me out, since I was doing important work. N. A. Lednyov was hired in Levin’s place; he was a kind of mathematical Lysenko. At every seminar and department meeting, he raged about Petrovsky and Sobolev, and promised to throw out everyone with the fifth line. As for me, someone spread a rumour, totally untrue, of course, that I was Lednyov’s best friend. And thanks to that, I was not fired.

Lednyov still worked as part-time Professor at MGU, and Academician Andrei Nikolaevich Tikhonov decided to get rid of him in a very ingenious way: he gave him half time Professorship, but with no teaching. And when it was time to report on 2 years work results, Lednyov did not report anything. And he was fired. After that Lednyov started complaining to Rector Chilikin, who decided that Lednyov was crazy and got rid of him as the Head of the department.

In 1965, Petrovsky decided to refresh his Chair of differential equations, and began consulting Gelfand about who take on to strengthen his teaching and academic staff. Gelfand said, “It’s obvious: Vishik!” Ivan Georgievich cautiously asked, “Isn’t he Lednyov’s best friend?”—“Who told you that? It’s not true. I don’t know a more reliable man than Vishik,” answered Gelfand. That decided that. The very same day, Petrovsky called me at home and told me what I must do in order to become a Professor at his Chair. And by June, that is what I became.

We had a very good relationship until the end of his days. I visited Ivan Georgievich often at home; I was well acquainted with his wife Olga Afanasievna. He was an educated, very cultured man. And something I did not expect: he was a religious man. In one of his rooms hung a large canvas by Korin, which depicted some monks. The painting made an enormous impression on me.

Petrovsky really loved the music of Bach. I remember one graduate student saying indignantly: you come to see Ivan Georgievich, and he might listen to Bach for 2 hours! Just Bach. He considered Bach's work a treasure. I also really love Bach. You can listen to him even after Beethoven. But when Olga Arsenievna Oleinik and I would go to Paris, Ivan Georgievich always asked us to sneak him in some records of Orthodox liturgical hymns.

I had the opportunity, at the time, to travel abroad, and this was also thanks to Ivan Georgievich. In 1967, Igor Rostislavovich Shafarevich, a great mathematician, returned from an academic conference in France and said, "The French would like to invite Vishik." At the time, Grisha Eskin, Mikhail Semyonovich Agranovich and I were working on very interesting things. And the French knew me very well, because Jacques-Louis Lyons, their head, a talented organizer and a leader of the research direction, was Laurent Schwartz's student. Schwartz was the one who introduced generalized functions into the science, using Sobolev's results. They immediately used generalized functions and were far ahead of us. Our O. A. Ladyzhenskaya worked in H^1, while they worked in H^s—the whole scale of spaces to infinity.

When I went to France for the first time in 1967, it was very difficult to organize. But Petrovsky sent me in exchange for some French guest; I'm not sure whom, I never asked. Ivan Georgievich personally vouched for me. The rules about that at the time were strict.

Being young, I was proud that I joined the Komsomol. I was delighted that I became a member of the Trade Union and I never particularly thought to it; just "like everybody".

I knew a short course of the VKPB[4] history and believed in what was written. Only when I settled in Moscow, my wife Asya began to form me politically. A husband of her sister, Iosif Abramovich Ovseevich, was the Deputy Director at the Institute for information transmission problems. Of course, they understood what was happening. It generally did not interest me much, I was not in that reality. I understood nothing, neither in politics nor in life. I knew well the Polish and Ukrainian languages, but I knew Russian not very well. Being at Lwów University I learned the poem of Nekrasov "how a weeping willow cannot raise its drooping branches". My new relatives were aware of this, so they did not try very hard to "re-educate" me. Gradually, about 7 years after, my eyes opened to what was happening.

[4] An earlier name of the Communist Party of the Soviet Union —*ed.*

I remember the story with the letter in defense of a mathematician Alexander Esenin-Volpin.[5] He was confined in a psychiatric hospital because he said something anti-Soviet. The University was visited by people, by students who had signed the letters in his defense. In my memory it was the first protest against the authorities in the scientific circles in the Soviet time. But I understood absolutely nothing in politics. So I told them that I will answer tomorrow. Coming home I called Ovseevich, who was very well versed in politics. And he said: "In no case do not sign the letter; it will do nothing to protect Esenin-Volpin, our signature doesn't matter, but you could get hurt. Professors Kurosh and Lyusternik have signed this letter, and they were invited to the Party Committee of the University, where they withdrew their signatures".

The Communist Party organization worked well in that time and the signatories of the letter were considered enemies. In particular, they were not allowed to travel to scientific events abroad, but I had this chance, thanks to Ivan Georgievich.

There was a woman in the university administration who did nothing but arrange these business trips abroad, following these rules. For instance, each year, one could not spend more than a month abroad. If you spent 2 weeks in Bulgaria, you could then spend 2 weeks in Sweden. There was a time when I was invited by Lars Gårding for a month, but I had already "spent" 2 weeks, and the lady told me: "I would love to, Mark Iosifovich, but I cannot: you are only entitled to a month in a year."

Ivan Georgievich led the Chair and the seminar until the end of his days. Sometimes he would be very late to the seminar, and we would sometimes wait for an hour. There was nothing to be done: he had to solve enormous administrational problems. But he always came. By the end of the talk, he would usually ask the presenter to which extent the imposed restrictions were essential. He respected only deep theoretical works. On the other hand, he really welcomed applied research; for example, in mathematical physics. I remember he very much liked a talk by Professor Sergei Konstantinovich Godunov from Novosibirsk, who went on to become an academician.

When Petrovsky was the Rector, one could reach him in his office at any time. He had no office hours: he was always available to all the people and problems at the university.

Ivan Georgievich, I believe, was also a great leader. He was interested in one thing only: whether a person could be useful to the university. Neither one's nationality, nor one's Party membership, nor one's origin played any role in that; only the academic weight. His friends were Samarii Aleksandrovich Galpern and Mikhail Aleksandrovich Kreines. At the time, there was a campaign against Jews in science, and the Department of Physics had not been accepting professors or students "with the fifth line" for a long time. But Ivan Georgievich insisted that Lev Davidovich Landau and Evgeny Mikhailovich Lifshitz became a part of the Physics

[5] A poet and mathematician, a dissident, a political prisoner, and a leader of the human rights movement, who was repressed by the authorities on numerous occasions and incarcerated in a psychiatric hospital. In 1972 he was allowed to emigrate to the USA. —*ed.*

Department. Although they themselves did not really want to be distracted from their scientific work. But on Ivan Georgievich's initiative, this great deed was done. Academician I. S. Shklovsky once calculated that during his time as the Rector, Ivan Georgievich must have done at least ten thousand good deeds.

All of these worries shortened his life by a few years. Because he had a heart condition, but he never spared himself, and did everything on his own. For example, he fought to ensure that in order to gain admission to the university, one would need to take not an oral and a written examination, but two written ones. It was possible to fail anyone at an oral examination. Ivan Georgievich would constantly go to the Central Committee of the Party about this, asking that the examination procedure be altered. They always answered, "You, Ivan Georgievich, are a prominent scientist and a leader, but the question of how many examinations are necessary for university admission is not your problem." During one of these meetings, he became ill: he had a heart attack right in the lobby of the Central Committee building. He fell to the floor. They began to lift him up: the employees tried to put him on a chair, "It's wrong to lie here, it's not allowed." If they had called the ambulance immediately, without lifting him, perhaps it may have been possible to save him. But because of the delay, they lost precious time.

I do not remember Petrovsky ever celebrating his birthdays. But I do remember his funeral very well. The great violinist Leonid Kogan played «Elegie» by the French composer Jules Massenet. I stood right behind his violin. Mikhail Andreevich Suslov, of the Party's Central Committee, came to attend. I left the room briefly to see what was going on, when suddenly they began to crowd everyone and say, "Go on through, go on through." That was how they insured Suslov's arrival, so that no unwanted people were present when he arrived. They had their own rules.

3 Jacques-Louis Lions and Jean Leray

Mark Iosifovich was kind and respectful to his wonderful French colleagues, but he liked to sometimes make good-natured jokes about them. Many of them were younger than him and, in his opinion, often walked along beaten paths that he had forged. Mark Iosifovich, by nature of his extraordinary mind, was a pioneer, and often paved the way in science, and his ideas formed the basis of other scientists' fundamental works. He spoke about this with a kind smile, without any resentment. He was impressed by a great deal of the French mathematicians' work, and his personal relationships with them were very warm.

In 1967, I found myself in France for the first time, at the Paris University "Paris VI". Jacques-Louis Lions, a professor at that university, wrote five or six books based on my collaboration with S. L. Sobolev, and on my own works. Academician Sobolev worked on general problems for hyperbolic and elliptic equations. One day, I figured out how to solve these problems and wrote to him. And he came and visited me at home. Back then, I still lived with my wife's parents on Sivtsev Vrazhek, in one room with two children. And so this academician arrives on his black "Volga", with a chauffeur, and sits down with me at one table to solve problems together. From this episode it becomes clear what a true, great scientist Sobolev was. He was ecstatic to work on pure science; he was bored with working at the Kurchatov Institute, where he was the Head of the department,[6] and was indifferent to formalities.

We published two brief reports on this subject, while Jacques-Louis Lions and Enrico Magenes worked on it a great deal afterwards and wrote three volumes on the subject. Sobolev suggested that we write a larger paper on the subject, but I could not, because at the time, I was working with Lyusternik on boundary layers, and also thought that our articles already contained everything.

I was wonderfully received in France. French mathematicians were very interested in what we did and what we thought. There was a young man there, Alain Haraux, who took an interest in my research and wrote a very good work based on it. He and I became friends.

Lions was pretty young back then. He was a very educated man. The French have one remarkable trait: they know a great deal and are superbly erudite. With us, it is common to come up with everything on our own, and the erudition is not important. But with them, they usually combine erudition with their own talent. And that takes root in French history: in the nineteenth century, France had many great mathematicians: Laplace, Lagrange, Cauchy, Galois. French mathematicians are deeply educated people, and they are very interested in new ideas.

Lions came from the city of Grasse, of which his father was the mayor. When the fascists came, the first thing they did was arrest his father so that he would not organize a resistance, and he spent the entire war in prison. And Lions himself, though he was at the time a 15-year-old boy, joined the Maquis—partisans—and fought the fascists in the mountains as a partisan.

He treated me very well, and invited me over. While his wife prepared everything to receive us, she called us and asked to delay for a couple of hours, because she does not have enough time to finish everything properly. So Lions came to my office and said, "So in the meantime, tell me about interpolation." I went to the board and began to explain.

Finally, we went to his home, and some other well-known French professors came. They said, "Mark Iosifovich, we are so grateful to you! This is the first time we've been to Lions's place, thanks to you. If it wasn't for you, we never would have gotten to be here."

Lions was once presented a wine cellar in Grasse. Such a cellar contains 40-year-old wine. But once you remove a bottle of wine from it, you must replace it with one of the best wines of the nearest year. This is a French law: a wine cellar must live on for future generations. And Lions would sometimes bring dusty old bottles from Grasse, wrapped in beautiful paper, and would treat connoisseurs to it.

[6] S. L. Sobolev was I. V. Kurchatov's deputy in the development of the Soviet atomic bomb project.

Unfortunately, I am not a gourmet and do not understand anything about food or wine. And I am unfit for such degustations because I've had an enlarged liver since the wartime years: in Dagestan, I was ill with malaria, and was treated with Quinine, which can cause one's hearing and liver to deteriorate. So I cannot drink fine wine. Once, after my lecture in Collège de France, I was invited to a famous restaurant by the Seine, where the owners made their own wine from the vines on the riverbank. They wanted to surprise me with this famous brand of wine. They ceremoniously poured wine in everyone's glasses. But I diluted it with water, right in front of the stunned faces of the French. Accompanied by cries of, "What are you doing? You'll have frogs breeding in your stomach!" But, unfortunately, I could not drink that wine because of my diseased liver.

When I visited the Collège de France, the Head of the department was Academician Jean Leray. He worked with Schauder, my first teacher and first consultant. When I arrived in France, Leray immediately invited me to a restaurant. He very much loved Schauder, and he knew how Schauder died: he was mauled to death by dogs in the ghetto. At the restaurant, Jean Leray and I spoke on mathematical topics. He was interested in what I was working on, and treated me to a lavish dinner. I was not used to eating that much. After the wonderful dinner, when they also brought us ice cream and an entire basin of chocolate, I was already unable to touch anything.

Leray was interested in the problem of the three-dimensional Navier–Stokes equation. He was saying that it was impossible to prove the smoothness of the solution to the system for large Reynolds numbers. In 2000, that problem was announced as one of the millennium problems.

I spoke to Leray a few times afterwards. He really liked when I would visit and tell him about my ideas: he always welcomed me with great enthusiasm.

In France, I realized for the first time that a new idea is an extremely valuable thing. They did not think so in Moscow at the time. I gave talks at Petrovsky's, Tikhonov's, Lyusternik's, and Plessner's seminars, and spoke about the things I was working on. I did not think about whether or not they were published, submitted to a publisher, not to mention that the work could be lying at the publisher's for a year, waiting in line to be published. In France, however, this was considered the most indecent question to ask: "Tell me what you're working on right now." This is a secret, it is indecent to ask about it. Asking about previous, published work—by all means.

And so in France I understood the value of new ideas. Generating ideas is in my nature, and did not like to read a lot. I consider myself a poorly educated mathematician: although I went to many seminars, I never liked to read articles. I think something up on my own and only then I'll check what others have done on the subject. Not the other way around, as is customary. People were often surprised by this ability of mine.

— Mark Iosifovich, how do You like to rest when tired from doing mathematics?

After seven in the evening I cannot do science, so I listen to broadcasts from the Conservatory. It turns out that every night, without leaving one's home, one can visit most concerts. Beethoven, Schubert, Bach,—this is amazing. Even more impressive, when you watch concerts on the «Kultura» channel, you see the orchestra conductor who addresses the bassoon, trumpet, other musical instruments. This is my main form of recreation now. Earlier, we were still walks in any weather in the *Neskuchny Sad*, but now, with age, these walks have disappeared.

4 Asya Moiseevna Guterman

The significance of this incredible woman in the great scientist's life cannot be overstated: she was a wife, and a friend, and every so slightly a mother (Mark Iosifovich's mother perished in the ghetto in 1941), and a guardian, and a catalyst. Mark Iosifovich was proud of the fact that his wife during the war had been an x-ray technician, and considered her authority on medical and other matters to be indisputable. I always spoke to Asya Moiseevna with pleasure: she was a deeply intelligent, tactful, and sensitive woman. She and her closest relatives, primarily, of course Iosif Abramovich Ovseevich, created the atmosphere where Mark Iosifovich could live in peace. After Asya Moiseevna's passing, he, as a "mere mortal", became more vulnerable to injuries and illnesses; his memory began to weaken with the loss of their shared memories. Although good care and attention were provided to him, Mark Iosifovich would occasionally tell me, with surprise, that "now I have to do the housework, now I know where the cups and the kettle are, but with my little Asya around, I didn't know these things." He missed, of course, her tireless and selfless care for him.

Asya and I lived together serenely and happily for 63 years. We were united in everything from the very moment we met in university on Victory Day 1945 to her passing in 2009. Sometimes I would joke that my little Asya married me and thus saved me from starvation and death. After the terrible life I led during the war, I joined a beautiful family, I gained a home. And Asya took upon herself everything to do with the day-to-day. I never knew where anything was kept, what needed to be taken from where. I was freed from all of that in order to occupy myself with science. Asya in our everyday life was a generalissimo: she solved every problem except the scientific ones. And of course it is simply impossible for me to imagine how I will make it without her...

Reference

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My Scientific Advisor Mark Iosifovich Vishik

Sybille Handrock-Meyer

In 1963 I came from the German Democratic Republic (GDR) to Moscow in order to apply to the Department of Mechanics and Mathematics of Lomonosov Moscow State University. Mechmat had three divisions: mathematics, computational mathematics, and mechanics. I was at mathematics, and when we needed to finalize our choice of the Chair at the end of second year, I chose the Chair of differential equations.

In our third year, we had lectures on partial differential equations. These lectures were read by Mark Vishik, and I was enthusiastic of the clarity of his explanation. He was a brilliant lecturer who could express complex mathematical facts simply and clearly. That is why I began to participate in the seminar on differential equations under Mark Vishik's guidance, and attended several of his special courses. My classmates, Andrei Fursikov and Sasha Demidov, and my friends, Sasha Komech and Ljuda Aramanovich, also attended the seminar under his guidance.

In our fifth year, we were meant to write our thesis. During that time, one of the main directions of development in the theory of differential equations were pseudodifferential operators, and Mark Iosifovich suggested, as my thesis subject, «Degenerate hyperbolic convolution equations». The theory of symbols, which was brand new at the time, made it possible to investigate such problems in a very elegant way.

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S. Handrock-Meyer (✉)
Chemnitz, Germany
e-mail: handrock@mathematik.tu-chemnitz.de

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After successfully defending my thesis in 1968, I returned to the GDR and began to work at the Technische Hochschule Karl-Marx-Stadt (now the Chemnitz University of Technology). At the time, a group of young mathematicians there was studying singular integral equations under the guidance of Professor Siegfried Prößdorf. I was a member of that group, and in 1973 I defended my dissertation, “Einige Beiträge zur Theorie singulärer Integralgleichungen nicht normalen Typs” («Some contributions to the theory of singular integral equations of non-normal type»).

Although I was in the GDR, I stayed in touch with MGU and Professor Vishik. In September 1973, I came again to Moscow for a year-long internship, and again started to attend Vishik’s seminar. Mark Iosifovich drew my attention to his work with L. A. Lyusternik on the asymptotic of solutions to differential equations with a singular perturbation with a small parameter at the leading derivative (the method of small parameter, or Vishik–Lyusternik’s method). He suggested that I do a detailed investigation of the case when, besides a small parameter at the highest order derivatives, there are also degeneracies of the coefficients at the highest order derivatives in independent variables of the differential equation. One had to find the conditions when the asymptotics still exists.

In the 1970s, Mark Iosifovich visited the GDR, and I took the opportunity to invite him for a few days to the Technische Hochschule Karl-Marx-Stadt (now Technische Universität Chemnitz). We discussed mathematical problems, and I showed him some of the sights in the Karl-Marx-Stadt region—for example, the Augustusburg Castle. I remember him being very interested in it. In 1981, I defended my doctoral dissertation, “Ein Beitrag zur Theorie singulär gestörter Gleichungen” («A contribution to the theory of singularly perturbed equations»).

From 1994 to 1997, I worked at the Weierstrass Institute for Applied Analysis and Stochastics (WIAS) in Berlin. This institute inherited the Mathematics Department from the Academy of Sciences of the GDR. Mark Iosifovich was at the time a member of the Scientific Committee of the Weierstrass Institute. The Berlin universities invited him, and I had the opportunity to meet with him and his wife Asya Moiseevna. Asya Moiseevna was a very pleasant, very clever woman, who was interested in everything, and I respected her very much. We had long and interesting conversations.

My last meeting with Mark Iosifovich Vishik took place at the Berlin Symposium in December 2001. On December 20, he received an honorary doctorate (Doctor Honoris Causa) from the Free University of Berlin. Those present and myself were enthusiastic of his talk «The sources of my work»,[1] which he gave in English. He could give lectures in multiple languages: for example, in French and in German.

[1] See the transcript of M. Vishik’s lecture on p. 121. —*ed.*

I feel a deep sense of respect and gratitude towards my scientific advisor, Professor Mark Iosifovich Vishik. I remember him as a wonderful person. May he rest in peace.

M. I. Vishik in My Life

Andrei Fursikov

In 1965, when I was a third-year student, Mark Iosifovich Vishik became my scientific advisor. At the time, the theory of linear partial differential equations, which he was then engaged in, was going through a period of turbulent development. I, along with his other students, enthusiastically began to study its foundations. Specifically, we studied the recently emerged, absolutely amazing theory of generalized functions and a number of other topics. I remember with pleasure the fascination with which we—students—studied these things.

Mark Iosifovich did more than lead the special seminar for students: his famous all-Moscow scientific seminar was already taking place. It is there that I was able to see many famous mathematicians for the first time. At the time, one of the main directions of development in the theory of partial differential equations was the theory of pseudodifferential operators. Many scientists around the world, including the participants of M. I. Vishik's scientific seminar, actively studied it, sharing at that seminar their results. I attempted to master this theory by listening to their talks and reading the relevant articles. However, I was only able to truly, deeply understand it due to the fact that during my fourth year I attended M. I. Vishik's remarkable special course on that topic. After all, Mark Iosifovich had a deep understanding of the theory of pseudodifferential operators, having been one of its main creators. In addition to that, he was a brilliant lecturer who was able to convey the most nontrivial facts of this theory in an exceptionally simple and clear way.

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A. Fursikov (✉)
Moscow, Russia

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I was very lucky to have attended those lectures. What happened is that during my fourth year Mark Iosifovich suggested that I begin to study degenerate elliptic operators. After some time, having read a remarkable work by V. A. Kondratiev on parabolic equations, I was able to understand that the theory of pseudodifferential operators may be effectively used to study one type of degenerate elliptic operators. That idea turned out to be a new one, and I used it in my master thesis and candidate dissertation.

I consider myself very fortunate to have had my long scientific collaboration with Mark Iosifovich (from 1973 to 1979) in my life. Naturally, during that period, I was able to get to know him much better. His scientific work was certainly his life's purpose, and everything else was subordinate to it. His daily routine was strict: he would wake up at 7 in the morning, then exercise, have breakfast, and from 9 in the morning to 7 or 8 at night he would work, with a break for dinner. Afterwards, dinner and a walk before bed. On Sundays, there would be 20–30 kilometre walks in the Moscow region. And his wife, Asya Moiseevna, supported Mark Iosifovich in everything, devoting her life to creating the necessary conditions for his scientific work. She was truly his guardian angel.

During the years of our collaboration, we usually met at his apartment once or twice a week, and would spend the entire day doing mathematics. And even during the other days, every spare minute would go towards our common work. The work gave me great satisfaction and joy, and I now look back on those years as very happy ones.

We began our collaboration with solving a problem on the construction of the first integrals to quasilinear parabolic equations (posed by Mark Iosifovich), which are solutions to the corresponding linear partial differential equation with infinite number of independent variables. So we were constructing these first integrals as solutions to such equations, analytically depending on the initial data. Rather soon it became clear that with the help of functional-analytic expansions with respect to the initial data one can also build solutions to quasilinear parabolic equations [95, 96, 97, 98, 99, 100, 101, 103, 104, 105, 106].[1] Expanding the class of parabolic equations, we transferred this theory of the first integrals and the method of constructing solutions via functional-analytic expansions onto the case of the Navier–Stokes system which describes the flow of viscous incompressible fluid [102, 107, 108, 109, 110]. This way we came from an abstract theory of equations with infinite number of variables to a rather interesting and rich in content area of mathematical physics.

[1] Numbered references are given with respect to M. I. Vishik's bibliography at the end of the book; alphabetical references are located the end of the chapter. —*ed.*

I should note that at that time, besides enormous amounts of literature on statistical hydromechanics, written at the physical level of rigour (see [MY71, MY75] and references cited therein), there were already published rigorous works of C. Foiaş [Foi72, Foi73]. We started to develop this area, hoping to use here our theory of the first integrals and functional-analytic expansions. These plans played off completely; moreover, besides the direct transfer of these facts, for the first time there was built a mathematically rigorous theory of moments of statistical solutions in the case of small Reynolds numbers. The moments were defined, the infinite hierarchy of equations for moments was derived, its unique solvability was proved, and also the closure problem of this hierarchy was formulated and proved [132].

Much more significant in hydrodynamics is the case of large Reynolds numbers. For the case of arbitrary Reynolds numbers we developed new methods which allowed to justify the existence of spatially homogeneous statistical solutions [111, 112, 114, 116, 123, 124, 125, 126, 127, 130, 132]. Spatially homogeneous statistical solutions—or, more exactly, the corresponding moment functions—appear in many important applications of statistical hydromechanics. They are defined on the whole space $\mathbb{R}^d$, $d = 2, 3$, and have infinite energy. This fact made the methods of proofs of existence of statistical solutions available at the time inapplicable to the spatially-homogeneous case since they were all based on the energy inequality. Yet we managed to build a certain analogue of the energy inequality and with its aid to prove the existence theorem for spatially homogeneous statistical solution.

After the construction of spatially homogeneous statistical solutions there arose a natural question on the proof of existence of solutions to the Navier–Stokes system with infinite energies. The set of these solutions is the support of the probability measure corresponding to the statistical solution. It turned out that the statistical solution, introduced in [Foi72, Foi73], is not suitable to the proofs of such theorems. This is why the notion of the space-time statistical solution was introduced [123, 124, 125, 126, 127], and with its aid a new theorem on the existence of solutions to the Navier–Stokes system with infinite energy, defined on the whole space $\mathbb{R}^d$, $d = 2, 3$, was obtained. The notion of the space-time statistical solution also turned out to be useful in other cases, and was repeatedly used by us and by other authors (see, for example, one of the recent articles [BFMT19]). Our main results in statistical hydromechanics were presented in our monograph «Mathematical Problems of Statistical Hydromechanics», which was translated into German and English [132, 173, 186]. I should also note that at the time, M. I. Vishik and A. I. Komech were studying stochastic Navier–Stokes systems with white noise [113, 115, 122, 125, 126, 127, 128, 131, 132, 133].

I undoubtedly gained a lot as a result of our many years of collaborative work. I sensed this very clearly when, after it was over, I began to successfully work in a completely different area of mathematics.

Mark Iosifovich and I maintained a close contact until the end of his days. Most of all, naturally, we would meet at his seminar, but not only there. I have already noted that on Sundays he would regularly go for walks in the Moscow region. At a certain point (during our collaboration) he invited me along, and for several years I took part in those outings quite often. The last trip took place after a long break: he and I hiked the Moscow region in the winter, without skis, and walked more than 10 kilometres. This was in 2002–2003, that is, when Mark Iosifovich was over 80. That was how remarkably in shape he was at the time!

Mark Iosifovich was a wise man. He and I would often discuss various situations in my life as they occurred, and he would give me his advice. I did not always follow it, but he was the one who practically always would prove to be correct.

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Un Grand Mathématicien, le Professeur Vishik

Gérard Tronel

Gerard Tronel (1934–2017) was a student of Jacques-Louis Lions and always maintained close business relationships with him. Tronel was strongly interested in the theory of hysteresis, and even learned Russian in order to read the works of M. A. Krasnoselsky in the original. He came to the Soviet Union back in the 1970s to talk personally with Mark Aleksandrovich! This was the first breakthrough of the iron curtain for Western mathematicians!

Gerard did a lot to popularize mathematics and modernize the school system in France, and also to organize written graduation examinations with anonymous grading in Paris.

Gerard was M. I. Vishik's close friend and talked a lot with him both in Paris and in Moscow. He was a very kind and devoted friend of all mathematicians from the Soviet Union. He met them all in Paris as his closest friends, even if he saw them for the first time. He met them at the airport, placed them at his home for the first days, organized for them free university housing in Paris, and took care of them as of his own children.[1]

Gerard was a very cultured, delicate, and decent person. Friendship with him was a blessing.

Alexander Komech

Les nombreux élèves du professeur Vishik parleront mieux que moi de son oeuvre mathématique, je voudrais apporter un témoignage simple de l'influence qu'il a eu sur quelques mathématiciens français, notamment sur ceux de l'école de Jacques-Louis Lions, pour cela je ferai appel à des souvenirs personnels.

[1] See http://www.ljll.math.upmc.fr/Hommage-Gerard-Tronel-12janv2018.

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G. Tronel
(1934–2017), Paris, France

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Dans les années 1950, en France, malgré les difficultés des communications, le nom du Professeur Vishik était déjà connu, notamment parmi les mathématiciens et les mécaniciens qui utilisaient les résultats qu'il avait obtenus dans l'étude des équations aux dérivées partielles dépendant de petits paramètres ; en fait ce type d'équations est très fréquent dans des modèles que l'on rencontre en mécanique,—couches limites, plaques et coques minces—mais pour ce qui concerne les mathématiques, des outils utilisés dans les études asymptotiques des solutions de certaines équations aux dérivées partielles, les théories de l'homogénéisation, sont déjà largement construits dans les articles publiés par le Professeur Vishik jusqu'aux années 1950 ; à propos de ses premiers travaux, il faut souligner une curiosité : le résultat connu sous le nom de théorème de Lax–Milgram devrait être un théorème de Vishik puisque qu'il figure dans un article antérieur à celui dans lequel on rencontre ce théorème pour la première fois dans les travaux de Lax et Milgram.

Mais c'est dans les années 1960 que j'ai vu pour la première fois le Professeur Vishik, il avait été invité à faire une conférence au séminaire Lions–Schwartz à l'Institut Henri Poincaré, il était l'un des premiers mathématiciens soviétiques à venir à Paris depuis le début de la guerre froide. L'amphithéâtre Hermite était complet. L'auditoire avait été immédiatement séduit par ce conférencier qui s'exprimait dans un français remarquable, si mes souvenirs sont exactes sa conférence portait ses derniers résultats sur le thème évoqué plus haut : les équations aux dérivées partielles dépendant de paramètres. Les auditeurs ont été enchantés par la clarté, la rigueur, la précision des énoncés des résultats ; il faut rappeler qu'à cette époque l'école de Bourbaki régnait sur les mathématiques françaises et que des sujets à l'interface entre des mathématiques pures et des mathématiques appliquées n'étaient pas spécialement appréciés par les grands maîtres français de l'époque qui bien entendu n'ont pas assisté à la conférence, mais ils ont eu tort car ils auraient apprécié une prouesse remarquable celle d'un mathématicien russe exposant à la « Bourbaki » un sujet relevant des mathématiques qu'ils considéraient comme non pures.

Ma deuxième rencontre avec le Professeur Vishik remonte au Congrès International de Mathématiciens à Nice en 1970. Ce congrès était tout à fait exceptionnel, il était présidé par un grand mathématicien français, Jean Leray, qui avait gardé des contacts étroits avec les mathématiciens soviétiques malgré les obstacles de la guerre froide. Cette année 1970 a été pour les mathématiciens soviétiques l'année du dégel puisque une importante délégation avait été autorisée à assister au congrès, ce qui ne s'était pas produit pour les congrès précédents puisque les autorités politiques soviétiques avaient refusé les autorisations de sorties des mathématiciens. Je n'ai pu que saluer le Professeur Vishik car, pendant le congrès, il a été très sollicité par ses collègues venus de tous les continents. Je me souviens d'avoir assisté à de nombreux exposés auxquels assistait le Professeur Vishik, il était très assidu et il prenait des notes. C'est à partir de ce congrès que les citoyens soviétiques, les mathématiciens notamment, ont pu répondre plus facilement aux invitations de leurs collègues étrangers.

A partir des années 1970, j'ai effectué plusieurs missions en Union soviétique, les villes ouvertes aux étrangers étaient limitées à Moscou, Léningrad et Novosibirsk, je me suis trouvé très souvent à Moscou pour travailler avec des équipes dirigées l'une par le Professeur Oleinik sur le thème de l'homogénéisation et l'autre par le Professeur Krasnoselskii sur le thème de l'hystérésis, ces thèmes étaient assez loin des préoccupations mathématiques du Professeur Vishik mais à chaque séjour à Moscou j'avais plaisir à la rencontrer pour parler de mathématiques, celles qui se faisaient à Paris et celles qui se développaient à Moscou ; c'est toujours avec plaisir que j'assistais à son séminaire au MGU, ce séminaire était pour moi plus accessible qu'un autre séminaire, celui du Professeur Gelfand. Même si les sujets traités au cours de ce séminaire Vishik ne correspondaient pas à mes préoccupations, j'arrivais toujours à glaner quelques idées, mais ce que j'appréciai surtout était l'atmosphère de ce séminaire qui se déroulait dans une certaine décontraction favorable aux discussions, à l'opposé de l'atmosphère toujours tendue des séminaires parisiens. Le Professeur Vishik n'intervenait pas constamment comme avait l'habitude de le faire certains de ces collègues, ses interventions se limitaient à des remarques brèves lorsqu'il estimait que le conférencier s'était trompé ou que l'exposé manquait de clarté au point de le rendre incompréhensible. Ce n'est qu'à la fin de la conférence que le Professeur Vishik prenait la parole pour dégager les grandes idées qui devaient être retenues, pour faire des observations de fond et de forme, bien entendu il rectifiait des erreurs, mais il insistait sur la présentation des résultats fondamentaux : il fallait que les théorèmes soient clairement formulés, les hypothèses devaient être introduites dans l'ordre dans lequel elles seraient utilisées dans les démonstrations, les conclusions devaient être aussi présentées dans l'ordre où elle seraient obtenues dans les démonstrations ; si les démonstrations étaient trop longues il recommandait d'introduire des lemmes intermédiaires qui pouvaient être utiles dans d'autres circonstances. Ce qui frappait était les qualités du grand mathématicien : clarté, rigueur précision, concision, mais ses interventions étaient toujours faites sur un ton amical, sans agressivité, il acceptait qu'un mathématicien, surtout jeune, commette des erreurs, mais je ne l'ai jamais entendu critiquer sévèrement un conférencier sous prétexte que son exposé n'était pas satisfaisant. J'y reviendrai plus loin mais ces remarques me permettent d'écrire que le Professeur Vishik était un grand mathématicien.

Sur la période 1980–2010, le Professeur Vishik a effectué de nombreux séjours à Paris et dans d'autres pays, notamment en Italie, pendant son séjour en Italie il a publié un petit ouvrage merveilleux qui reflète ses conceptions sur « Asymptotic behaviour of solutions evolutionary equations »; on trouvera dans ce livre toutes les idées fondamentales sur ce qui allait devenir un des thèmes principaux de ses recherches ultérieures : la théorie des attracteurs. Le Professeur Vishik invité au Collège de France par Jacques-Louis Lions avait donné dans les années 1980 un cours remarquable que l'on peut retrouver partiellement dans son livre cité ci-dessus ; bien que le sujet ait été assez éloigné de mes préoccupations mathématiques du moment, j'avais été attiré par le fait qu'il traiterait des équations de Navier–Stokes, un sujet qui pose encore aujourd'hui de sérieux problèmes à la communauté mathématique internationale ; à partir du problème des attracteurs dans les problèmes de Navier–Stokes, le cours développait tous les outils utiles à la théorie des attracteurs dans un cadre non linéaire abstrait. Il n'y a pas eu à ma connaissance de notes sur ses cours, mais je me souviens bien de la présentation : premières leçons faisait l'inventaire des problèmes qu'il souhaitait aborder, puis dans les leçons suivantes il avait donné les grandes lignes des modèles qu'il traiterait, dans les exposés suivants il a introduit les méthodes qu'il utiliserait, puis dans les dernières leçons il a donné des idées essentielles mises en oeuvre dans quelques démonstrations de points délicats et enfin son cours se terminait une liste de problèmes ouverts ; ce que j'ai retenu en assistant à ces cours confortait mes opinions sur les immenses qualités du mathématicien : clarté précision, rigueur, économie de langage, aucun mot de trop, un souci contant de vouloir être compris traduisant de très grandes qualités pédagogiques. Le Professeur Vishik répondait toujours avec gentillesse aux questions, pour lui il n'y avait jamais de questions stupides. Ce cours a laissé dans la mémoire des auditeurs d'excellents souvenirs.

Je pourrais continuer à égrener des souvenirs laissés par mes rencontres avec le Professeur Vishik, mais je voudrais terminer par quelques remarques sur sa vie et qu'il m'a raconté, il parlait assez peu de lui même, par modestie sans doute ; mais c'est au cours d'invitations chez lui à Moscou, en présence de Madame Vishik, une grande dame très cultivée et qui savait recevoir les invités de son mari, que le Professeur Vishik m'a dit que pendant la dernière guerre, celle de 1940–1945, les nazis avaient déporté et assassiné toute sa famille et qu'il voulait prendre les armes pour les venger, mais un mathématicien russe de renom l'en a dissuadé, car ses dons pour les mathématiques lui promettaient une très grande carrière, il est quelques fois bon quand on est jeune d'écouter les conseils ! L'autre souvenir qu'il a évoqué est lié au sort dramatique du mathématicien Schauder très connu : à Lvov où se trouvait exilé dans un hôpital le Professeur Banach, ce dernier apprend que Schauder allait être arrêté par les nazis, inquiété par cette nouvelle, Banach veut le sauver et pour ce faire écrit à son collègue allemand, Bieberbach, mathématicien très influent dans les sphères du régime nazi, le résultat ne s'est pas fait attendre, quelques jours après l'envoi de cette lettre, Schauder a été arrêté et envoyé dans un camp d'extermination d'où il ne reviendra pas ; voici une bien triste histoire qui montre que les mathématiques ne peuvent pas être déconnectés de la société, aussi bonne soit-elle avec Banach et aussi cruelle soit-elle avec Bieberbach. C'est aussi un exemple qui montre que si les mathématiques peuvent être neutres, les mathématiciens ne le sont pas.

Professeur Vishik, les mathématiciens qui vous ont rencontré, ceux qui ont utilisé vos résultats ne vous oublierons pas ainsi que tous ceux qui les utiliseront, vos qualités laisseront de vous le souvenir d'un grand mathématicien et d'un grand humaniste.

A Great Mathematician, Professor Vishik

Gérard Tronel

Many of Professor Vishik's students will speak better than I of his mathematical work. I would like to present a simple testimony of the influence he had on some French mathematicians, notably on those belonging to the school of Jacques-Louis Lions; for that I will call for my personal memories.

In the 1950s, in France, in spite of communication difficulties, the name of Professor Vishik was already known, in particular, among mathematicians and mechanics who used the results he had obtained in the study of partial differential equations depending on small parameters. As a matter of fact, this type of equations is very common in models which one encounters in mechanics,—boundary layers, thin plates and thin shells—but as far as mathematics, the tools used in the studies of asymptotics of solutions of certain partial differential equations, the theories of homogenization, had already largely been constructed in the articles published by Professor Vishik by the 1950s. Regarding his first works, one curiosity should be emphasized: the result known as the Lax–Milgram theorem should be Vishik's theorem since it appeared in an article prior to the one in which one encounters this theorem for the first time in the works of Lax and Milgram.

But it was not until the 1960s that I saw Professor Vishik for the first time. He had been invited to give a talk at the Lions–Schwartz seminar at the Henri Poincaré Institute. He was one of the first Soviet mathematicians to come to Paris since the beginning of the Cold War. The Hermite Amphitheater was full. The audience was immediately seduced by this speaker who in remarkable French was giving a talk, if I remember correctly, on his latest results on the aforementioned topic: partial differential equations depending on parameters. The listeners were enchanted by the clarity, the rigour, the precision of his statements of the results. It is worth remembering that at that time the school of Bourbaki reigned on French mathematics and that subjects at the interface between pure mathematics and applied mathematics were not especially appreciated by the great French masters of the time, who of course did not attend the talk, but they were wrong, since they would have appreciated a Russian mathematician's remarkable feat—presenting *à la Bourbaki* the mathematics that they considered not pure.

My second meeting with Professor Vishik goes back to the International Congress of Mathematicians in Nice in 1970. This congress was really exceptional; it was chaired by a great French mathematician, Jean Leray, who had kept close contacts with Soviet mathematicians despite the obstacles of the Cold War. Year 1970 was for Soviet mathematicians the year of the thaw, as a large delegation had been authorized to attend the Congress. This had not happened for the previous

Translated from the original by Andrew Comech and Maria Komech.

congresses because the Soviet authorities had refused to authorize foreign travel for mathematicians. There was nothing more but to welcome Professor Vishik since during the congress he was in high demand among his colleagues who came from every continent. I remember attending many of the talks that were also attended by Professor Vishik; he was very diligent and was taking notes. It is after this congress that Soviet citizens, mathematicians in particular, were able to respond more easily to invitations from their foreign colleagues.

From the 1970s onwards, I made several trips to the Soviet Union. Since the cities open to foreigners were limited to Moscow, Leningrad and Novosibirsk, I very often came to Moscow to collaborate with Professor Oleinik's group on the subject of homogenization, and with another group, Professor Krasnoselskii's, on the theme of hysteresis. Although these themes were quite far from the mathematical interests of Professor Vishik, during each of my stays in Moscow I had a pleasure of meeting him to talk about mathematics: the one that was being done in Paris and the one that was developing in Moscow. It was always with pleasure that I attended his seminar at the MGU, the seminar that was more accessible to me than another seminar, that of Professor Gelfand. Even if the subjects treated during this Vishik seminar did not match to my preoccupations, I always managed to glean some ideas. But what I appreciated most of all was the atmosphere of this seminar, which took place in a certain relaxed setting, fruitful for discussions, unlike the ever tense atmosphere of Parisian seminars. Professor Vishik did not intervene constantly, like some of his colleagues used to do; his interventions were limited to brief remarks when he felt that the speaker had made a mistake or that the presentation lacked clarity to the point of becoming incomprehensible. It was not until the end of the talk that Professor Vishik took the floor to highlight the great ideas that needed to be retained, to make observations of form and substance. Of course he corrected errors, but he insisted on the presentation of the fundamental results: the theorems had to be clearly formulated, the hypotheses were to be introduced in the order in which they would be used in the proofs, the conclusions also had to be presented in the order they would be obtained in the proof; if the proofs were too long, he recommended introducing intermediate lemmas that could be useful in other circumstances. What was striking were the qualities of a great mathematician: clarity, rigor, precision, conciseness. Meanwhile his interventions were always made in a friendly tone, without aggression; he accepted that a mathematician, especially a young one, made mistakes, but I never heard him severely criticize a speaker under the pretext that the presentation was unsatisfactory. I will return to this later, yet these remarks allow me to write that Professor Vishik was a great mathematician.

From 1980 to 2010, Professor Vishik made numerous visits to Paris and to other countries, especially to Italy. During his stay in Italy he published a small wonderful book that reflected his concepts of «Asymptotic behavior of solutions to evolutionary equations»; in this book, one will find all the fundamental ideas on what was to become one of the principal themes of Vishik's later research: the theory of attractors. In the 1980s, Professor Vishik, having been invited by Jacques-Louis Lions to the College de France, gave a remarkable course that can be partially found in the book mentioned above; although the subject was quite far from my preoccupations at that time, I was attracted by the fact that the course would deal with the Navier–Stokes equations, a subject which still poses serious problems for the international mathematical community. Starting with the problem of attractors in the problems of Navier–Stokes, the course developed all the tools useful to the theory of attractors in an abstract nonlinear framework. To my knowledge, there were no lecture notes for this course, but I remember the presentation well: the first lectures creating the inventory of the problems he wished to tackle, in the following lectures he gave the outline of the models he would treat, in the subsequent lectures he introduced the methods he would use; then in the last lectures he gave key ideas implemented in some of the proofs of delicate points, and finally his course ended with a list of open problems. What I took away from attending this course enforced my opinion on the immense qualities of this mathematician: clarity and precision, rigor, economy of language, no excess words, a constant desire to be understood reflected very high pedagogical qualities. Professor Vishik always kindly answered questions; to him, there were no stupid questions. This course left its listeners with remarkable memories.

I could continue recounting the memories left by my meetings with Professor Vishik, but I would like to end with a few remarks about his life and about what he told me. He spoke rather little of himself,—out of modesty, no doubt. It was during the times I was invited to his home in Moscow, in the presence of Mrs. Vishik, a very cultured great lady who knew how to receive her husband's guests, that Professor Vishik told me that during the last war, that of 1940–1945, the Nazis had deported and murdered his entire family. He had wanted to take up arms to avenge them, but a renowned Russian mathematician dissuaded him from this, since his talent for mathematics promised a very great career; it is sometimes good for someone who is young to listen to advice! The other story that he mentioned is about the dramatic fate of a very well-known mathematician, Schauder. In Lvov, where Professor Banach was in a hospital, he learned that Schauder was to be arrested by the Nazis. Alarmed by this news, Banach wanted to save Schauder, and wrote to his German colleague, Bieberbach, a mathematician who was very influential in Nazi circles. The result did not take long: a few days after Banach's letter was sent, Schauder was arrested and sent to an extermination camp, from which he would never return. This is a very sad story which shows that mathematics cannot be disconnected from society, which would be so good with Banach and so harsh with Bieberbach. It is also an example which shows that while mathematics can be neutral, mathematicians cannot.

Professor Vishik, the mathematicians who have met you, those who have used your results, as well as all those who will use them in the future, will not forget you, your qualities will leave a memory of you as a great mathematician and a great humanist.

Recollections of a Former Mechmat Student

Ljudmila Meister

In 1964 I entered the Department of Mechanics and Mathematics of Moscow State University, the famous "Mechmat MGU". The early sixties can be considered Mechmat's prime. Lectures were read by such remarkable mathematicians as A. G. Kurosh (algebra), M. A. Kreines (mathematical analysis), V. I. Arnold (differential equations), V. M. Tikhomirov (functional analysis), M. I. Vishik (his Special Topics course on differential equations) and many, many others.

Mechmat at the time had three divisions: mathematics, computational mathematics ("computers" for short), and mechanics. They were all located from the twelfth to sixteenth floors of the Main Building of MGU. Later, in 1970, the "computers" became the Department of Computational Mathematics and Cybernetics, and moved to a separate building, while the mathematicians and mechanics remained in the Main Building. It is interesting that at Moscow State University, the Department is called the Department of Mechanics and Mathematics (Mechmat), and at Leningrad (now St. Petersburg) State University, it is Matmech. I selected the Mathematics Division (the choice of the division needed to be specified when we submitted our application materials).

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L. Meister (✉)
Darmstadt, Germany

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The first year flew by quickly and somehow chaotically. Apparently this was due to the exhaustion and anxiety caused by my school finals and MGU entrance examinations, an abundance of new impressions, and other factors: for, as the popular song goes, "you're only eighteen once."

On the fourteenth floor of Mechmat was a notice board mottled with printed and handwritten announcements about various special courses and special seminars. I did not know where to look: all the tempting titles and the famous names of the professors... It was tempting to be everywhere.

In the beginning of my second year I saw, among the others, an announcement on the board about the commencement of a seminar on differential equations, under the guidance of M. I. Vishik. Having gone to a few different seminars by that point, I realized that I liked this one in particular, and began regularly attending the lectures. I was drawn to Mark Iosifovich's teaching manner: calm and respectful. I was also drawn to the fact that chapters and problems from the recent book by Laurent Schwartz, «Mathematics for the physical sciences», were discussed at the seminar, and physics always interested me. Mark Iosifovich's seminar was mostly attended by senior undergraduates and graduate students—there were few younger students in attendance—but the themes and discussions were organized in such a way that even we (I) were able to understand something.

One day, Mark Iosifovich organized something of a test: every participant was given five problems on the topic of "generalized functions", and Mark Iosifovich walked between the rows of desks to see how we were doing. By the time he reached the desk where I was sitting with another girl, my neighbour and I had each solved one problem. Mark Iosifovich looked over it and said, with such clear joy, "Correct! You've solved everything correctly." It was evident that he was genuinely happy for us, and pleased that his students had understood even a little bit of the rather challenging material.

Most often at the seminar we would discuss problems related to the topic of Mark Iosifovich's scientific interest. He had, at the time, published several consecutive papers on the theory of boundary value problems for elliptic and parabolic equations, pseudodifferential operators, and problems with discontinuous boundary conditions. I remember one time, a student named Ada and I were given an assignment "for two": to analyze one of the proofs and present it at the seminar. Ada began the presentation at the seminar. Somewhere in the middle of the long proof, she stopped and fell silent. Then I approached the board and continued the presentation. It is interesting that Mark Iosifovich was the one to remind me of this story, when he visited my husband and I in Germany many years later. It showed that he was very attentive to the doings of even the youngest participants of the seminar. Even what one might think were tiny, insignificant details related to the seminar's participants were retained in his memory.

Of the second year students who were going to Mark Iosifovich's seminar, I especially became close with Sasha Shnirelman and Sasha Komech. The three of us would go to the seminar together, and after our second year, when we needed to choose a specialization, we chose the Chair of differential equations. The head of the Chair was the MGU rector, Ivan Georgievich Petrovsky, and Mark Iosifovich read the special course and led the special seminar. He became our scientific advisor. I should note that at the time, certain areas of mathematics were in fashion—algebraic and differential topology, group theory, and there was an infatuation with the axiomatization à la Bourbaki.

In the summer of 1966 in Moscow, the Fifteenth International Congress of Mathematicians took place. Students worked as volunteers and helped the foreigners orient themselves in MGU. The foreigners' questions were sometimes very funny. For example, "Where can one buy what the Russians eat every day,—black caviar?" Or—and this sounds like a joke—"Is it true that the letters "M" and "Zh" on the bathroom doors stand for "Madame" and "Gentleman"?". Many years later, I spoke to a German mathematician who took part in that congress. He told me about the dinner that was hosted for the participants, and added, with a dreamy smile, that he has never eaten so well in his life, and that he remembers that dinner to this day.

There were many talks at the Congress about topology and the set theory, fashionable at the time, and this sparked the students' interest even further to these areas of mathematics, so the choice of a Chair as "old-fashioned" as differential equations raised bewildered questions from some of my friends. However, M. I. Vishik's seminar was so interesting that I had no desire to transition to fashionable topology.

With every year, the discussions at M. I. Vishik's seminar became clearer and clearer to me, and his special course made the terms "generalized functions", "pseudodifferential operator", and "left and right regularizers" became part of our vernacular. We even had jokes like "The Soviet right regularizer is the rightest in the world."

In 1967, M. I. Vishik for the first time went abroad, to France. He was gone for an entire month, during which people took turns replacing him. His absence could be strongly felt, and it was very dull without him. When he finally returned, so many people came to the seminar that there was not enough room for them all, and additional chairs needed to be brought in. Everyone asked in chorus for him to tell us about France. He, with apparent pleasure, agreed, and spent a whole hour telling us about his impressions of France and his meetings with colleagues. I especially remember what he said about his meeting with Jacques-Louis Lions: "Imagine, Lions said to me, 'You Soviet mathematicians can't even imagine how fortunate you are. You can peacefully think about your problem as long as you need to: a year, 2 years, 3 years... while we are simply mandated to publish something every year.' " This comment was very unexpected in the sixties, which is why I still remember it. Among the countless questions about France, there was one about French fashion. To that, Mark Iosifovich replied shortly: "Everyone walks around in the winter without their hats on! But it's warm there, so please, don't follow that fashion."

Sometimes at the seminar Mark Iosifovich would tell us how he writes papers: "When you are solving a problem, write your drafts down neatly. Number them and lay them out in front of you. When the problem is solved, you will have a nearly finished article before you." That advice was useful more than once in my work.

Lectures, seminars, more lectures, more seminars, mandatory ones for the course, mandatory ones on our choice of a Chair, and special courses and special seminars of the students' own choosing: our study time was full to the brim. Nevertheless, we did occasionally have some free moments. I remember the three of us standing on the fourteenth floor, by the window, and heatedly discussing the connection between Hungary and the Khazar empire. Grigory Isaakovich Barenblatt was walking by (he taught us the course on continuum mechanics, and we took part in a computational mathematics workshop under his guidance). He approached us and asked what our heated discussion was about. We told him that it was about Hungarians and Khazars. He laughed, then sighed heavily and said, "Ah! I wish I had your problems."

In our fifth year, we were required to write a thesis, and its defense was a type of examination. In September 1968, Mark Iosifovich suggested to me the topic «Elliptic boundary value problems with discontinuous boundary conditions». I had a rough idea of what needed to be done: by the transformation of the space to remove the critical boundary point of discontinuity to infinity, so that the domain changes into an infinite "cylinder", in whose section the problem will have a solution, depending on a section parameter. One needed to obtain this solution, and then by the inverse transformation to "return" to the original space. It seemed to me that I should figure out the details independently, and not bother my advisor with them. As well, I wanted to understand whether any real processes were described by this model, or whether it was a purely abstract construction. I toiled for 3 months at my desk without raising my head. Finally, the thesis was almost ready. Thanks to the course on continuum mechanics, I was even able to find out that my problem with discontinuous boundary conditions describes the processes that arise from the separation of airflow from the trailing edge of an aircraft wing. In early December 1968, I went to see Mark Iosifovich with a thick stack of papers covered with writing. When he saw me, he exclaimed in amazement, "Hello, Ljuda! Where did you disappear to? I'd started to worry!" I was ashamed that he had been worrying about me, and answered with embarrassment that I had been working. I handed him my stack of papers. We went into a half-empty auditorium (there were no empty ones at Mechmat!), sat down, and Mark Iosifovich began to leaf through the sheets of paper, covered in my neat handwriting, occasionally asking questions and listening to my comments. Gradually his face cleared more and more, because the thesis was really almost ready. When I saw that everything was all right, I even started telling him about the "trailing edge of an aircraft wing". Mark Iosifovich listened attentively, but told me, "Ljuda, are you a mathematician or what? Let's not deviate from the topic."

In June 1969, our thesis defenses and the state examinations (mathematics and philosophy) were behind us, and we went to a restaurant to celebrate the end of student life. After graduating MGU, I was invited to the Moscow Institute of Thermal Technology, where I had to work on all kinds of problems, and I would often remember that question, asking myself: "Am I a mathematician or what?"

Now let's take a long break (work, children) and skip straight to 1993. In the autumn of that year I received DAAD[1] Scholarship and was invited for 2 months

[1] Deutscher Akademischer Austauschdienst (German Academic Exchange Service) —*ed.*

to the Technical University of Darmstadt in Germany. There, I met Professor Erhard Meister, who headed the Chair of mathematical methods in physics. A few months later, we were married, and I have lived in Germany ever since. When my husband learned that my academic advisor at MGU had been M. I. Vishik, he was absolutely delighted. He told me that in the sixties he was a mathematics professor at the University of West Berlin, and organized a group there to study Mark Iosifovich's work. My husband even tried to learn Russian in order to be able to read his papers immediately, without waiting for a translation. "We must invite Professor Vishik to our university," said he decisively. In May 1997, Mark Iosifovich and Asya Moiseevna came to Darmstadt and stayed for a few days at our house. The men discussed mathematical and philosophical issues, and I played hostess and spoke to Asya Moiseevna on "everyday topics". When the conversation touched on the Mechmat, Mark Iosifovich remembered many small details and very warmly spoke of all of his students. On May 14, 1997, he gave a talk at the Technical University of Darmstadt: «Trajectory attractors for nonautonomous dynamical systems». After the talk, Mark Iosifovich, Asya Moiseevna, my husband and myself, and some other department members went to a restaurant, where we could continue our conversations in an informal atmosphere. Asya Moiseevna quietly asked me, "Ljuda, tell me, did the listeners like the lecture?" It was clear that she cared deeply about her husband and took his work to heart. A day later, Mark Iosifovich and Asya Moiseevna departed. Those few days are memorable to me because they are yet another confirmation of what a tactful and delicate man Mark Iosifovich was, the warmth with which he spoke of his colleagues and remembered all his students, past and present.

In these brief notes, of course, it is impossible to convey in its entirety the deep respect that I have towards my doctoral advisor, Professor Mark Iosifovich Vishik. Time passes. But the warmth and gratitude one feels in one's soul after having contact with a wonderful person remain forever.

A Word About M. I. Vishik

Alexander Demidov

Mark Iosifovich Vishik was a great man. I could just limit myself to listing the names of the mathematicians who spoke at his renowned MGU seminar. Among them were the academicians I. M. Gelfand, V. E. Zakharov, A. M. Ilyin, S. V. Konyagin, V. P. Maslov, S. P. Novikov, and A. T. Fomenko. The foreigners included Laurent Schwartz, Lars Hörmander, Jacques-Louis Lions, Louis Nirenberg, and many other exceptional mathematicians.

S. P. Novikov sent his congratulations on M. I. Vishik's ninetieth birthday: "Dear Mark Iosifovich, it is with great pleasure that my wife Elya and I congratulate you, an outstanding mathematician and one of our teachers, with such a wonderful anniversary. We are all happy that you are celebrating your birthday being full of so much energy and activity."

Mark Iosifovich's teaching career began at the Moscow Power Engineering Institute. There, he very successfully taught an enormous number of various math courses. And it was also there that he organized and led his scientific seminar, which would become the center of gravity for Moscow's mathematicians who worked in the areas of differential equations and functional analysis. This elevated MEI's prestige to an extent that can be demonstrated by the following fact: in those not-so-simple times, in the mid-1950s, the MEI administration provided Mark Iosifovich with a two-room apartment.

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A. Demidov (✉)
Tarusa, Russia

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On October 19, 1971, about thirty of his friends and colleagues met in that very apartment to celebrate his fiftieth birthday. And before that, in the same apartment, Mark Iosifovich spent about 2 hours of his birthday at his writing desk with me, discussing the text of my candidate dissertation (he was my advisor). That writing desk now stands in the faculty room of the Department of General Problems of Control in the "first humanitarian" building (where the Department of Computational Mathematics and Cybernetics is located).

In 1976, a very representative conference on singular perturbations was held in Lyon. It would be enough to list these names: Jacques-Louis Lions, Luc Tartar, Roger Temam, Bernard Malgrange, Milton Van Dyke, Wiktor Eckhaus, Robert O'Malley Jr., Évariste Sanchez-Palencia. The organizers were very sorry that Mark Iosifovich was unable to attend. But they asked me to give, in their words, "academician Vishik" an art book and some other souvenirs.

When Magenes welcomed Vishik to Pavia, he brought him to his luxurious villa and said, "I built this house on my royalties from the book «Non-Homogeneous Boundary Problems», where Lions and I developed the ideas that you and Sobolev expressed in your short 1965 article." The impulse that led to the writing of that article was something that Mark Iosifovich expressed in these words: "I realized that the standard boundary condition in the Neumann problem is not physical. Physically motivated is not specifying the normal derivative at the boundary points, but rather specifying of the flux through the boundary, that is, the functional on smooth functions; in other words, it can even be a distribution".

The day before Mark Iosifovich's funeral, I called my classmate, Natasha Sevastyanova. She was unable to be there to say goodbye, as she was taking her examinations at the Moscow Aviation Institute. Here, I would like to relay, though not as vividly as she did, her words about Vishik. She said it was not her place to speak about Vishik's greatness as a mathematician, but that she could speak about his exclusively respectful attitude to his students as to colleagues, which awed her then, and which she now abided by herself. Here are but two examples from her recollections. Vishik is leading his seminar on generalized functions. One of his students arrives late. Vishik greets him with a joyful exclamation, "Oh, Petya is here," as though a great scientist has arrived. Another story: at Vishik's seminar in Auditorium 13–06, Misha Shubin gives a talk on Hörmander's work. Vishik senses that the participants of the seminar (and there are some outstanding mathematicians among them) are far from understanding everything. Then Vishik says, "Misha, allow me, I will try to explain what I understood from your talk." And Vishik not only clearly expressed the essence of Hörmander's work, but also showed the beauty of his result. And Misha thanked Vishik, saying that he himself now understood Hörmander's work better.

Mark Iosifovich had an enviable sense of humour, and knew how to easily and delicately diffuse conflict situations. Here are but two examples. In the late sixties, Mark Iosifovich taught a specialized course on elliptic pseudodifferential problems in huge Auditorium 01. The auditorium, including the balcony, was full of listeners. And so, someone in the balcony said something impudent, which could have led to a scandal. But Mark Iosifovich instantly, softly (one could even say gently) reduced everything to a joke and continued the lecture.

And here is a story told by Georgiy Georgievich Magaril-Ilyaev. He witnessed this scene at a bank office. Mark Iosifovich was standing in line behind another man. Suddenly, a woman appears and says with irritation that she had been standing behind that man. Mark Iosifovich, knowing that the lady's words were just a fantasy, answered, "Oh, yes, of course, get behind him." The woman, not expecting such delicacy, froze in amazement and once again began to repeat, rather vigorously, that she really had been in line behind that man. Mark Iosifovich just kept playing along.

And a few other instances I would like to mention.

In an interview that Mark Iosifovich gave Katya Kalikinskaya (Volodya Chepyzhov's wife), this is how he answered her question about Asya Moiseevna's role in his life: "Asya and I lived together serenely and happily for 63 years. We were united in everything from the very moment we met in university on Victory Day 1945 to her passing in 2009. Sometimes I would joke that my little Asya married me and thus saved me from starvation and death. After the terrible life I led during the war, I joined a wonderful family, I found a home. And Asya took upon herself everything to do with the day-to-day. I never knew where we kept tea cups, where the tea was, where was what. I was freed from all of that in order to occupy myself with science. Asya in our everyday life was a generalissimo: she solved every problem except the scientific ones. And of course it was simply impossible for me to imagine how I will make it without her..."

I should add that in the last years of her life, Asya Moiseevna was very ill. And Mark Iosifovich took extraordinary care of her. Once, when I brought them to the veterans' hospital where Asya Moiseevna was going to be treated, one of the people in the waiting room told me, "They're like newlyweds."

After Asya Moiseevna's passing, Mark Iosifovich deteriorated greatly. Two years before his death, he had a severe stroke. He was hospitalized at the Academic Hospital in Yasenevo. Vishik's condition was extremely complicated. The doctors prepared us for the very worst. But a miracle occurred. Probably as a result of his constant mental work, the function of the affected part of his brain was taken on by the other part of it. And after that, Mark Iosifovich continued to lead his scientific seminar at MGU until its very last meeting during the 2012 spring semester, thus concluding 50 years of its running. And 2 months later, literally a minute before his death, to the question, "What hurts, Mark Iosifovich?" he answered, for the first time, "Everything hurts."

To end my remarks on M. I. Vishik, I cannot fail to mention the deepest respect and awe that Mark Iosifovich felt all his life towards those dear to his heart: I. G. Petrovsky, L. A. Lyusternik, I. N. Vekua, N.İ. Muskhelishvili, S. L. Sobolev, who had provided him with their support in the most trying moments of his difficult life. The "iconostasis" of these great teachers and colleagues occupied a place of honour in his office.

Mark Iosifovich Vishik

Alexander Shnirelman

1 Two Lessons from Mark Iosifovich

One day, Vladimir Arnold asked me to write a review of an article submitted to «Uspekhi Mat. Nauk». The paper had to do with the classification of stationary two-dimensional flow of an ideal incompressible fluid. In my eyes, the work was uninteresting and vapid, full of nothing but trivialities. That is what I wrote in my review.

When I saw Arnold a few days later, he looked perplexed and a little lost. He even somewhat timidly asked me: “Sasha, why were you so harsh on him? Perhaps you wouldn’t mind giving it another look?”

This was very unlike Arnold; I vaguely sensed that I’d done something wrong, that I’d broken some rules of the game. And that is when I decided to go ask for Mark Iosifovich’s advice.

Mark Iosifovich’s reaction was unexpectedly passionate, as though my words had struck him to the core. I had never seen him so excited. He straightened up and solemnly stated: “Sasha! Forget everything, think only of whether your review will benefit Mathematics! Don’t think of anything else! Let nothing distract you from

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A. Shnirelman (✉)
Department of Mathematics and Statistics, Concordia University, Montreal, QC, Canada
e-mail: alexander.shnirelman@concordia.ca

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the most important thing! Of course, it would sound very striking that Shnirelman rejected an article favoured by Arnold. But that is nothing compared to what is most important! Think only of Mathematics!"

I thought about it, then thought about it some more... and wrote a positive review. The article was published.

I could draw two lessons from this little story.

1. In difficult cases, when it is unclear which decision to make, it can be useful to look at the situation from a higher perspective. As I remember, Mark Iosifovich did so time and time again.

2. Of course the article had been clumsy, and it would have been fine not to publish it. But the very topic of stationary flows eventually turned out to be much more interesting than it had originally seemed, and it revealed some quite unexpected connections. In fact, a fairly large part of my own work is dedicated to them. It seems as though Arnold had already suspected that this trail could lead us very far. Mark Iosifovich also somehow understood this from my story. From here we could draw the second lesson: not to act like an overconfident schoolboy, particularly with people like Mark Iosifovich and Arnold.

2 Of Classics and Romantics

Mark Iosifovich and I once had a discussion about classics and romantics in science. Suddenly Mark asked me: "Sasha, what do you think, are you a classic or a romantic?" I found myself at a loss and mumbled something like, "Both would be nice." But as for Mark Iosifovich himself, he was undoubtedly a classic. He liked to repeat: "One needs to clean up after oneself," meaning that if you came up with something good, it needed to be properly written down and published, so as not to leave everyone bewildered as to what had been done, and what had not. Mark Iosifovich also knew when was the right moment to change the research topic. He remembered some renowned mathematician lamenting that Vishik had stopped working in one area and switched to another: "Mark Iosifovich, you have such enormous potential here, why are you dropping it?" On that occasion, Mark Iosifovich quoted Hilbert, who had said about some area of mathematics that it was "abgegrasene Wiese" ("an overgrazed meadow").

Mark Iosifovich himself always abided by these rules. He left behind several grandiose creations: namely, his works on the theory of boundary value problems as the extension theory for differential operators, his works on quasilinear equations, on asymptotic methods, on convolution equations, on statistical hydrodynamics, on the attractor theory for partial differential equations... His works overall are a magnificent example of how mathematics should actually be done.

I witnessed, to a certain extent, Mark Iosifovich's work on boundary value problems for the convolution equations. At the 1966 International Mathematical Congress in Moscow, Mark Iosifovich gave a 30-minute talk on this theory in Lecture Hall 02 (on the blackboard, with chalk, no slides). At the time, I barely understood anything; and no wonder. Afterwards, Mark Iosifovich taught a brilliant one-semester course on elliptic boundary value problems, adapted for students, in which he included elements of his theory. And after that, I spent several years absorbing this theory, dedicated two papers and my dissertation to it, and finally began to understand some of it. This theory represents a colossal expansion and generalization of the established theory of elliptic and parabolic equations that existed at the time. This theory included pseudodifferential equations and systems, as well as, for example, difference equations. For a long time, it was unclear to me why such distant generalization was necessary, where it came from and where it would lead. It seemed superfluous for elliptic differential equations since the results of similar completeness are obtained through other, more economical methods. Mark Iosifovich himself would avoid these questions (which is also characteristic of classics), leaving it up to me to find the answers. The idea that maximal generalization was a positive thing in general was not good enough for me. It took me a long time to comprehend the motives that moved Mark Iosifovich to such a grandiose undertaking.

The first publications on the theory of boundary value problems for convolution equations were four short and very concentrated notes that were published in 1964 in the proceedings of the USSR Academy of Sciences. These notes outlined almost all of the results of the future theory: of course, without any explanations or proofs. At the time, untangling this cryptogram was difficult for me. But in hindsight, having studied the great and detailed works that have appeared in the following few years, everything began to fall into place for me.

Of course, the very theory of boundary value problems for elliptic differential equations can develop on its own, without going into the theory of convolution equations (in other words, pseudodifferential equations). But the need in this generalization arises in the consideration of less classical problems and, above all, the Zaremba problem. Let me remind that in the "classical" theory the elliptic boundary value problem consists of an elliptic differential equation $Au = f$ in a region G with a smooth boundary Γ of codimension 1, and the boundary condition of the form $Bu|_\Gamma = g$ (for the higher order equations there could be several such conditions). Then, under certain algebraic conditions, the problem is "almost solvable" (that is, of Fredholm type) in certain functional spaces. This was all well-known by the early 60s.

However, even before World War I, the Polish mathematician Stanisław Zaremba stated the following problem: assume that the boundary Γ is split into two regions, Γ_1 and Γ_2, which have a mutual boundary Δ of codimension 2, and assume that in each of them there is its own boundary condition: $B_1 u = g_1$ on Γ_1 and $B_2 u = g_2$ on Γ_2. For example, there is the Dirichlet boundary condition specified on Γ_1, with the Neumann boundary condition on Γ_2 (this was the original Zaremba problem). One had to formulate the conditions so that a given problem would be Fredholm; when doing so, one had to determine the functional spaces where this property would be satisfied.

It turned out that after reduction to the boundary this problem leads to the system of coupled equations, that is, pseudodifferential equations with a discontinuity along the manifold Δ. We arrive at the problem which contains all the difficulties: the equations are nonlocal, the index is not constant, ... One could not overcome all the difficulties at once, one has to separate the problem into parts and to solve them in succession. The first step is the convolution equations and systems in a region with smooth boundary under the assumption that the "smoothness condition" is satisfied; this is the closest to the classical theory. The next step is the equations in a region with a varying index at the boundary, which requires that one introduces weighted spaces of varying smoothness and operators of varying order (hardly anybody would study such things without a particular need). And only then it becomes possible to approach the Zaremba problem itself. As a result, a whole new theory appears, with the description of exotic, yet natural for this problem, functional spaces, with the determination of the necessary number of boundary conditions or supplementary potentials, or degrees of freedom (what Boris Yurievich Sternin so aptly called "co-conditions") at the boundary Γ and at the separating surface Δ, with the accurate description of their regularity and so on.

This colossal work took several years. I remember Mark Iosifovich and myself working in his office at home, while Asya Moiseevna was proofreading yet another article in the other room. From time to time she would come into the office with a list of corrections and ask, "Markus, what should be in this place?" They referred to the articles from that series as "red", "blue" and so on, in accordance with the colours of the magazine covers.

If I were to try to recover the genesis of these works, I would suppose that they had originated in the study of Zaremba's problem by Mark Iosifovich, and that the theory was being developed as a preparation of tools necessary for the solution of that problem. Or, maybe, everything was different (after all, with classics everything is not like with romantics). Regardless, as a whole, it was a grandiose example of how to do mathematics.

Teacher and Friend

Alexander Komech

My first encounter with Mark Iosifovich (MI) occurred in 1964, when I was studying at Moscow school №444 with his son, Senya, and would sometimes visit their home. After we graduated, the Vishiks invited us, along with Fedya Bogomolov, to their dacha in Kratovo to prepare for the Mechmat entrance exams. The extremely intensive training sessions continued throughout most of June for the three of us (Senya, Fedya, and I) under the guidance of a highly experienced teacher who lived nearby. MI was present at every session and paid close attention to our answers and mistakes. As a result, we all successfully completed our examinations and were accepted, though there had been absolutely no guarantee of that happening.

Our second meeting occurred in September 1965, when MI began leading a seminar for second year students along with Olga Arsenievna Oleinik and Victor Vasilievich Grushin. Victor Vasilievich masterfully taught us the theory of generalized functions and gave us a lot of wonderful problems to solve. MI always very attentively and strictly, but also exceedingly tactfully and benevolently watched the students' reaction and their activeness in solving the problems. This combination of strictness with tact and goodwill was always very characteristic of MI's interactions with undergraduate and graduate students. Many famous mathematicians came

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A. Komech (✉)
Faculty of Mathematics, University of Vienna, Vienna, Austria

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through this seminar, its first line-up and the following generations: Anatoli Babin, Pavel Blekher, Andrei Gabrielov, Alexander Demidov, Andrei Fursikov, Alexander Shnirelman, and many others. This seminar was essentially the first to fully introduce Schwartz's theory of distributions into Moscow's mathematics.

We were all deeply impressed by the special courses MI taught at MGU during that time: the theory of distributions, pseudodifferential operators, elliptic boundary value problems... These lectures were rather fundamentally publicized, which was clearly organized by Ivan Georgievich Petrovsky, the Rector of MGU: they took place during the day in the giant sunny Auditorium 01 of the Main Building of MGU, which was almost entirely full of undergraduates, graduate students, and professors! The enthusiasm of the lector and of the listeners mutually strengthened each other.

MI read his lectures very clearly and artistically, loudly and with immaculate diction. He acted very bold and confident, and only occasionally after the lectures one could see how much energy this cost him. He considered it his duty to bring the maximum benefit to his listeners. He was also very punctual.

He wrote words and formulas with chalk on the board in very large and beautifully written letters, like in cursive. He would repeat some explanations several times, varying his arguments. He brilliantly applied methods of complex analysis, for which he had most likely developed a taste at Muskhelishvili's and Vekua's Tbilisi school. He thoroughly explained the application of functional analysis methods developed by Sergei Sobolev as well as Schauder's topological methods.

By the way, Juliusz Schauder, according to MI, was his first scientific advisor in Lwów. At the time, MI studied daily after dinner at the university library. Schauder noticed that MI was reading mathematics books in German, approached him and started a conversation about mathematics. After that, they would schedule regular meetings there every week and have long discussions.

Schauder had a rank of *Docent*, but until 1939 he held a position of a schoolteacher at Lwów gymnasium № 2 (where he was not paid for being a Docent) and an assistant at the Lwów University. He perished in the Lwów ghetto in 1943.

Mark would often recall his studies at the Lwów gymnasium, where his interest to Mathematics was awaken, although his main passion at that time was the football. A strong influence on MI, it seems, was caused by the atmosphere that prevailed in the gymnasium. For example, there were public disputes on the foundations of religion between teachers of Orthodox Christianity, Catholicism, and Judaism, held in the atmosphere of mutual respect.

By the way, I once asked MI whether it is better to call him Mark or Marko. He answered that either would be fine. He was called Mark from birth, but when he needed to get a Soviet passport, he was listed as Marko. He went to the Chief Officer of the militsiya department to explain that his name was Mark, but the Officer waved him away: "Go, go,—this way is better!"[1]

At the Lwów university, MI entered the famous Lwów school of Mathematics: Stefan Banach, Hugo Steinhaus, Juliusz Schauder, Władysław Orlicz, Stanisław Mazur. . . Banach had a great sense of humor. He loved to tease his colleagues and would tell that every day one needs to take two hundred grams of Port wine since without it it is impossible to do science!

MI was extremely tactful with other people. He would teach us that one should always pay special respect to women. He would recall how he came to Germany to visit a famous professor. The professor headed to greet him, but MI first approached and bowed to his secretary, and only then turned to her boss.

Overall, MI's tactfulness was extraordinary. He spoke to his colleagues and to his students in the exact same manner! I was very surprised by the amount of attention he gave to some of my fantasies on the topic of Maxwell's equations, when I was just a second year student. I asked him a rather non-standard question, and he started very seriously questioning me where I got that idea from, and gave me some advice. I often remember that when I work with younger colleagues.

MI's friends played a large role in his life. He spoke enthusiastically about the wisdom of Iosif Abramovich Ovseevich, his loyalty to the Institute (IPPI, Institute for Information Transmission Problems), and his military merits. He deeply respected Victor Borisovich Lidskii as a brilliant mathematician and a former front line soldier who would bring in "tongues"[2] on his own shoulders from the reconnaissance. MI and Lazar Aronovich Lyusternik had the most tender friendship—Lyusternik was his doctoral advisor and coauthor of his famous works on equations with a small parameter, which MI was very proud of. He often warmly remembered Sergei Lvovich Sobolev, Mstislav Vsevolodovich Keldysh, Mark Grigorievich Krein, and his collaboration and contacts with them. MI was devoted to the mathematical genius of Israel Moiseevich Gelfand, although he did not always approve of the style in which Gelfand communicated with his colleagues. He attended Gelfand's seminar every week with no exception for 33 years. He would say: "For many years we were unable to travel the world, and it was difficult for us to obtain scientific information—but we had Gelfand!"

MI had an extraordinary sense of humour: he loved a good joke. He recalled Lyusternik would ask him, "Mark, when do you get up in the morning?"—"At seven o'clock, Lazar Aronovich!"—"Does this time exist at all?" Once they were playing charades, and MI asked, "What kind of animal does this person resemble?" Someone said, "An opossum." MI asked what an opossum looked like, and others replied that it was such a black pig. Then MI cried, "So it's you, Lazar Aronovich?"

MI always rose early. For an hour to an hour and a half he would walk and exercise in the *Neskuchny Sad* (when he lived in his new apartment). At 8 o'clock

[1] *Marko* is a Ukrainian name, while *Mark* could suggest a Jewish origin; this must have been the officer's motivation.

[2] Russian military slang for enemy soldiers captured for interrogation —*ed.*

he would have breakfast, and at 9 o'clock he would sit down at his work desk. One time I came to visit him in the morning to discuss some business, and noticed a stack of clean paper on the absolutely empty writing desk—about fifty sheets. I asked him, "MI, what are you planning to write?"—"I don't know yet!" He did not allow his research schedule to be disrupted for any reason. Concerts or plays—as the rarest exceptions—two or three times a year.

His health, after having survived malaria in 1942 in Makhachkala, was not the greatest. He once consulted about this with Isaak Grigorievich Barenblatt, a famous endocrinologist who worked at the Kremlin hospital since 1930. Barenblatt wrote him an entire heap of prescriptions, and when MI came to the pharmacy on (Old) Arbat, the pharmacists asked him in a sympathetic chorus, "It must be Professor Barenblatt who prescribed this all?" As a result, he was given a pile of medications. MI left the pharmacy; the winter sun was blinding, and water loudly dripped from the icicles. MI shoved this entire pile of medications into a nearby garbage bin and decided to lead a healthy way of life.

MI told the following story about Barenblatt: during the "Doctors' plot" (1952),[3] a nurse came running to him at the Kremlin hospital: "There's a patient in your ward yelling that her *syrniki*[4] were poisoned and that her stomach hurts!" Isaak Grigorievich ran to the ward, quickly swallowed the remaining *syrniki* and explained that they were a bit bitter: "They put too much soda!" So the "evidence" was eaten, and Isaak Grigorievich stayed free! Unfortunately, he was not able to evade the labour camp during the more "vegetarian" times in 1958 for his disrespectful remarks about Khruschev—which his closest friends relayed, as appropriate, to the authorities... (This story is described in detail by his son, Grigory Isaakovich Barenblatt, a wonderful mathematician, in his memoirs dedicated to Andrei Sakharov [Bar96].)

From that point on, MI cultivated an active lifestyle. Every morning he would exercise intensively, and EVERY Sunday he would go on a 20–25 kilometre trip in the Moscow region, in the company of his close friends. Always with Ovseevich. He was also very fond of tennis and would delightedly tell stories about his victories over younger opponents.

After the defense of his candidate dissertation in 1947, MI began to teach at MEI. He approached teaching very responsibly, and became renowned at MEI as an extraordinary lector. However, when he was taking his *politzachyot*,[5] he suffered a fiasco: he got a difficult question about "scissors",[6] of which he did not have the slightest idea. He was very bad at this "grammar", and, in desperation, started reciting what he remembered: "The Proletariat takes power into its own hands..."

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[3] In 1951, a group of predominantly Jewish doctors from Moscow were accused of a conspiracy to assassinate Soviet leaders. This was the part of the antisemitic "anti-cosmopolite" campaign during 1947–1953. —*ed.*

[4] curd cheese pancakes —*ed.*

[5] An examination in political sciences —*ed.*

[6] This refers to the "Scissors Crisis", an event in the Soviet history in 1923 during the New Economic Policy (NEP), when there was a widening gap (*price scissors*) between industrial and agricultural prices. —*ed.*

The docent of Marxism–Leninism who was giving the examination gladly cried, “Ah, you’re a complete idiot!”—and immediately gave him a passing grade.

MI did not accept the notion of precise distinction between mathematical disciplines. He would always say that mathematics is indivisible. His own mathematical interests were very varied, although not all of those areas were ones he worked in: set theory and number theory, topology and complex analysis, functional analysis and spectral theory, measure theory and theory of elasticity…

MI was constantly and tirelessly looking for topics for the research. He considered the correct choice of a topic to be crucial for the successful development of one’s mathematical methods. He questioned everyone all the time, and was in constant doubt about what to work on. He would repeat that Hilbert considered it necessary to change the topic of one’s work every 4 years (which is exactly what Hilbert had done: algebra and algebraic geometry, integral equations, relativity, logic…) MI also started with the Weyl projections and selfadjoint extensions of the Laplace operator, then the works on small parameter with Lazar Lyusternik (which received great number of citations), well-posedness of the Cauchy problem (joint work with Sergei Sobolev, the development of which was the foundation for the three volumes by Lions and Magenes), nonlinear strongly elliptic equations, boundary value problems for pseudodifferential operators (with Gregory Eskin), analytical methods for partial differential equations (with Andrei Fursikov), equations with infinitely many variables (with Pavel Blekher and Andrei Marchenko), then statistical hydrodynamics (with Andrei Fursikov and myself), attractors of autonomous and nonautonomous equations (with Anatoli Babin and Vladimir Chepyzhov).

When choosing research topics, MI very attentively heeded Israel Gelfand’s advice. He was glad when Gelfand told him, “You are doing a hero’s work,” in relation to study of equations with an infinite number of variables. MI also deeply admired Mark Krein.

MI had been acquainted with Ivan Petrovsky since 1946, and actively participated in his seminar when enrolled in the graduate school at the Steklov Institute. Their acquaintance developed into a great friendship after 1965, when MI began to work at the Chair of differential equations at Mechmat MGU, which Petrovsky was in charge of (he had also been the Rector of MGU since 1951). They trusted each other, treated each other with the greatest respect and interest, both as people and as scientists.

Petrovsky was a legendary man, devoted to science and people. He was a member of the Presidium of the Supreme Soviet of the USSR and the Chairman of the Clemency Commission. MI recalled that Petrovsky would sometimes complain that he had not slept the night deciding someone’s fate… Sometimes he’d say, heavily, pointing upwards, “I sent another two UP THERE this morning…”

As a member of the Presidium of the Supreme Soviet, Petrovsky would receive people in his office in the main building of MGU on Mokhovaya Street. After his death, the office was turned into a memorial library to which his family donated all of his books. It is possible that the office exists to this day… All of the graduate students and workers of our Chair participated in the transportation of many, many books from Petrovsky's dacha in Abramtsevo: an entire truckload!

Petrovsky's friend, Iosif Shklovsky, a famous astrophysicist, wrote about him, "The fate of MGU Rector and Academician Ivan Georgievich Petrovsky was deeply tragic. It is the oldest story in the book—a good man in a difficult position in difficult times! One should realize how hard his life must have been. I witnessed many dozens of good deeds done by this wonderful man. Being reasonably familiar with statistics, I can state with full accountability that the number of good deeds done by him during his time as the Rector must be about 10^4. How many other people do we have with such a legacy?"[7] MI recalled that some people once complained to Petrovsky that they were unable to obtain a meeting with a high-ranking bureaucrat to discuss an important decision. So Petrovsky came to see this bureaucrat himself with those people—and was also refused a meeting! After this, Petrovsky went into the bureaucrat's office and said that he himself will receive visitors in this office—he had that right as a Member of the Supreme Soviet!

Petrovsky was very scrupulous in his approach of his duty to help others. One time, two Mechmat professors wanted to apply to a good housing co-op. In order to do this, they needed very strong recommendation letters to the *Mossovet* (the City Council). One of the professors asked Kosygin, the Prime Minister during that time, to write a letter, and the other one asked Petrovsky. Well, Petrovsky's request was granted by the Mossovet, and the Premier's was denied! Petrovsky had written at the end of his letter that he requested to be notified of the decision, while the Premier requested nothing of the sort...

Petrovsky was deeply upset by the effects of the *partkom*[8]-promoted anti-semitic abuse at the entrance examinations, but he could not take on the entire system on his own. A close acquaintance of his, who lived at the dacha next door in Abramtsevo, recalled that one time Petrovsky had his closest friends—Mechmat Professors Samarii Aleksandrovich Galpern and Mikhail Aleksandrovich Kreines—over for a visit. This was in early July, right when entrance examinations were taking place. Kreines came in from the garden and said that there was a young man standing by the gate. In an hour—he was standing there. In 3 hours—he was still standing there! So Petrovsky asked, "Call him over here." It turned out, of course, that the young man had just been failed at his Mechmat oral entrance examination; he'd received "unsatisfactory". The three professors asked him how the examination had gone—everything is clear... Then the three of them put together an examination commission and re-examined him. They gave him "excellent" and made up a report. All three of them signed it. What happened then—I do not know...

Everything that had to do with MGU's successful operation was an absolute priority for Petrovsky. He hired Lev Davidovich Landau and Evgeny Mikhailovich Lifshitz to work at the Department of Physics, overcoming the stubborn resistance of

[7] Iosif Shklovsky, «Five Billion Vodka Bottles to the Moon: Tales of a Soviet Scientist». Novosti, Moscow, 1991.

[8] Local committee of the Communist Party —*ed.*

the anti-Semites there. In 1964, he organized a new Chair of Chemical Mechanics at Mechmat, led by Veniamin Grigorievich Levich. There had probably been no other such Chair anywhere in the world at the time. A unique collective of gifted scientists assembled there. Major theoretical physicists in the field of statistical physics and the theory of superconductivity, such as Anatoly Ivanovich Larkin and Igor Ekhielevich Dzyaloshinskii, would go on to collaborate with that Chair. They subsequently became frequent participants in the Petrovsky seminar on mathematical physics, and both read brilliant Special Topics courses at Mechmat. In 1972 the Chair was dissolved since Levich filed a request to go to Israel. Petrovsky was unable to prevent the closure, but he kept the ENTIRE faculty of the Chair, transferring them to his Chair of differential equations. Thus, all of us at that Chair found ourselves in such brilliant company while Petrovsky was alive. Later, of course, many of those physicists were forced out. After that Larkin and Dzyaloshinskii for many years “decorated” the most prestigious universities in the world.

Petrovsky’s approachability was legendary. He would often stroll with someone or other in the vestibule of the Main Building of MGU, by the elevator halls. During those strolls, anyone could approach him to speak on any topic. One time, I personally (in my FIRST YEAR) went to the Rector’s office on the ninth floor with some silly request to sign a letter to the Moscow International Film Festival—I really wanted to attend… His secretary looked at me sympathetically, as though I were insane, and told me the Rector would see me in 10 minutes. In precisely 10 minutes, the Rector came out and sat with me on a couch. He read the letter and said he would sign it and mail it to the proper address, although he will not be requesting a ticket for himself. He amiably said goodbye…

Another time, 7 years later, he invited me to meet him when he was to hire me to his Chair of differential equations. I was very fortunate that he had come to the Chair meeting when I was presenting my candidacy dissertation. The next day, Petrovsky called MI and asked him whether he’d recommend hiring me to work at the Chair. MI said that yes, he would. When I came to see the Rector in his little office (ten square metres), he was talking to someone on the phone. Without taking the phone off his ear, he stretched out his other hand to me, literally waltzed me around his desk and sat me into an armchair. Then he said two or three words about the future process, and that was the end of it. Six months later, he was gone…

At the memorial service for Petrovsky at MGU, I stood last in line to say goodbye. Zeldovich stood in front of me. Beside us, Rostropovich played Bach on the cello.

After Petrovsky’s death, MI, along with Olga Oleinik and Sergei Novikov, ran the Moscow seminar on mathematical physics named after Petrovsky. MI never missed a single meeting in almost 20 years, even when all of the other leaders were absent. Renowned physicists and mathematicians would speak at the seminar, sharing the very latest theoretical and experimental achievements. In particular, Igor Dzyaloshinskii spoke about liquid crystals, and brought a small 3 cm circle of liquid crystal film. We pressed it with our fingers, and it would change colour. In 1975 it was completely incomprehensible why anyone would need this, but now, of course, we all know these kinds of screens from TVs, monitors, smartphones, etc.

Petrovsky put in a great deal of effort in order to send MI in the 1970s to read lectures at Collège de France, at Jacques-Louis Lions's invitation. There, MI was awarded the Collège de France Medal, which he would show to his close friends (although without particular enthusiasm). MI found in Jacques-Louis Lions a friend and mathematical follower (in particular, the three famous volumes by Lions and Magenes develop the results of MI and Sergei Sobolev's work on the correctness of the Cauchy problem, as discussed above). MI came to Paris a few other times, and read lectures at the Institut Henri Poincaré. I met MI for the last time in Paris in 1995. He spoke very warmly of his French colleagues and friends: Jean Leray, Laurent Schwartz, Gerard Tronel, Alain Bensoussan, Alain Haraux, Haim Brezis, Richard Temam, and others. Those contacts and, in particular, the dissertations of Michel Viot and Étienne Pardoux were of an important influence for MI's later research on statistical hydrodynamics; the results of Haraux strongly affected the research of MI on attractors of dissipative equations. In turn, Haraux was telling that he was greatly influenced by the lectures of MI in College de France and research contacts with him. Haraux, alongside with Nirenberg and Temam, came to Moscow in 2012 to give talks at the conference in honor of Mark Vishik on the occasion of his 90th birthday which was held at IPPI.

By the way, Jacques-Louis Lions visited Moscow often because of the French launches at the Baikonur Cosmodrome, which he always attended, being the President of the French Space Agency during 1984–1992. During many of these visits, he would drop by MI's seminar at MGU and speak to us at the Chair of differential equations. In 1976, he brought with him a large group of his young students (Temam and others, about ten people) to the first Petrovsky conference. They all attended MI's seminar.

Of MI's international contacts, it is also important to note his visits to Augsburg to see Jochen Brüning, to Stuttgart to see Wolfgang Wendland, and to Berlin, where he worked a great deal with young mathematicians (Bernold Fiedler, Arnd Scheel, and others). Additionally, he visited the United States on several occasions, where he met Tosio Kato, George Sell, and others.

MI knew several languages: he freely spoke French, and was perfectly fluent in German (according to all of his German colleagues). His Russian was flawless. He freely spoke Polish and Ukrainian. He loved Ukrainian songs and would sing them. He said that Ukrainian was his first language, learned from his Ukrainian wetnurse. It was she who taught him Ukrainian songs.

Music played a very large role in MI's life. He loved to listen to it in the evenings after his mathematical work. His favourite composers were Mozart, Brahms, Beethoven... He especially admired Beethoven's Ninth Symphony: I would come visit him at home, and he would be singing Schiller's text, "Alle Menschen werden Brüder!" These words delighted him.

MI considered his seminar at Mechmat to be perhaps his greatest contribution to science. The famous seminar took place on Monday evenings in Auditorium 13–06. Many exceptional mathematicians grew up at that seminar. Many dozens, if not hundreds. Participants and lecturers would come from other cities and republics (Victor Ivrii from Magnitogorsk, Arnold Dikansky from Baku, and many others). MI would necessarily speak to every lecturer before the seminar, and would usually invite them to his home.

He would also often invite students and graduate students to his home, where they would be able to do their scientific work in a quiet atmosphere. MI was always very happy about the successes and talents of his students and took care of them. He would help them in any way he could—both in mathematics and financially (often *pro bono*).

The requirements to talks at his seminar were very strict: everything needed to be precisely and clearly formulated. Presented proofs could only be short and instructive. It was important to explain the origin of the lecture's topic, and to give an overall picture of the state of that area. One's results needed to be shown later, preferably at the very end… One needed to speak VERY loudly and to write EVERYTHING on the board! MI would often interrupt speakers and ask them to repeat themselves, he would put a hand to his ear and say, "I can't hear you! Speak louder!" or "Volume!" The seminar undoubtedly played an important role in the development of the theory of partial differential equations. The Vishik school was formed there, one of the world's leading scientific schools at Mechmat at that time, alongside the Kolmogorov school of probability theory, the Gelfand school of functional analysis and representation theory, the Shabat school of complex analysis, and others.

Many wonderful mathematicians came to MI's seminar and gave talks. Some of the ongoing participants who took part in it for many years: Mikhail Agranovich, Anatoli Babin, Lev Bagirov, Boris Vainberg, Tatiana Ventsel, Nikita Vvedenskaya, Leonid Volevich, Aizik Volpert, Andrei Gabrielov, Victor Grushin, Alexander Demidov, Yulii Dubinskii, Sergei Zelik, Arlen Ilyin, Sergei Kozlov, Elena Kopylova, Stanislav Kruzhkov, Sergei Kuksin, Anatoliy Myshkis, Victor Palamodov, Boris Paneah, Leonid Pokrovskii, Andrei Piatnitski, Elena Sitnikova, Mikhail Fedoryuk, Boris Fedosov, Andrei Fursikov, Vladimir Chepyzhov, Alexander Shnirelman, Mikhail Shubin, Gregory Eskin, and many others. Lectures were also read many times by Vasilii Zhikov, Vladimir Zakharov, Victor Ivrii, Yulii Ilyashenko, Anatolii Kostyuchenko, Mark Krasnoselsky, Olga Ladyzhenskaya, Victor Lidskii, Jacques-Louis Lyons, Vladimir Mazya, Viktor Maslov, Louis Nirenberg, Alexander Povzner, Yakov Roitberg, Pavel Sobolevskii, Zinovii Sheftel, Victor Yudovich…

As a rule, the talks at the MI seminar dealt with problems of great current interest. Often the newest, just obtained results—and sometimes even uncompleted studies—were presented. For example, I remember a remarkable talk by Yu. S. Ilyashenko in 1980, who together with A. N. Chetaev for the first time obtained the estimate ν^{-4} for the Hausdorff dimension of the attractor of Galerkin approximations of Navier–Stokes equations. Moreover, this estimate was uniform with respect to the dimension of the Galerkin approximation. These results were a revelation to all the participants of the seminar, and a great event in science.[9] This method is based, roughly speaking, on a generalization to the infinite-dimensional case of the Liouville formula for the change of the phase volume: the exponent contains a trace of linearized dynamics, which becomes negative for large dimensions since the eigenvalues tend to negative infinity. It remained to refine these results to estimates for the attractor of the infinite-dimensional Navier–Stokes system. For the first time this was carried out independently and practically simultaneously by O.A. Ladyzhenskaya, Yu. S. Ilyashenko with A. N. Chetaev, as well as MI with A. V. Babin. Since then, the method of Yu. S. Ilyashenko–A. N. Chetaev has been widely accepted and is applied in a great number of articles to more and more dissipative equations (the Schrödinger equation with dissipation, the Ginzburg–Landau equation, and so on).

I had a great luck of being taught by MI since my second year at the university and in the graduate school, and then to collaborate with him for 9 years and work together at the same Chair for more than 20 years. Our friendship lasted until his final days. As an advisor, MI was very demanding and at the same time very attentive, proper and indulgent. Scheduled meetings took place at the Chair every week at a fixed day and time.

MI would always say that one needs to read a great deal of papers, and demanded it of his students. I used to find that requirement excessive, almost unachievable! But much later I realized that there is no other way, and this turned out to be extremely important for my academic work.

In my second year, to begin with, I was told to read Israel Gohberg and Mark Krein's wonderful 1958 paper in «Uspekhi Mat. Nauk», followed by Joseph Kohn and Louis Nirenberg's paper on pseudodifferential operators. I managed to read those ones, but then, the level of the assigned papers rose sharply, and that's when I ran into trouble... My relaxedness—the one of a mathematics school graduate to whom the first 2 years were at the knee's height—took its toll. I practically failed my diploma work in my fourth year, and Oleinik reprimanded me strongly at my defense for my inadequate work. MI seemed calmer.

The summer after our fourth year, MI invited his students to his dacha in Kratovo, and offered us a selection of a few difficult problems. I chose boundary value problems in domains with edges—as MI commented, due to my interest in architectural landmarks.

Then was the fifth year with the diploma work on boundary value problems in domains with conic points, which was based on Vladimir Kondratiev's method. My nightmare with the edge problem began in the graduate school, when MI would ask me almost daily, "How is it going?"—and I would answer 3 years in a row, "Nothing's working out..." MI told me I should abandon that topic, because the "big guys" were working on it, and repeatedly suggested other topics, but nothing worked out of them either. The idea of a characteristic Riemann surface came to me rather quickly, but that's as far as it went. One day, having looked over my attempts,

[9] See also Chap. 11 written by A. V. Babin. —*ed.*

Sasha Shnirelman said that he'd seen something similar in Vadim Malyshev's book on random walks, which he had just reviewed. I began to thoroughly read through this brilliant book (of course, back then, I could not understand any of it) and gradually, through a process of elimination, found the page that I felt would be the key to solving my difficulties. But it was not clear what this key was. The book lay open on that page, on my desk, for a month or two, and then everything suddenly became clear, and the work was completed in 3 months, with the application of Malyshev's method of automorphic functions on the Riemann surface.

MI scheduled me right away to speak at his seminar, and invited my future opponent Mikhail Fedoryuk. My other opponent was Vadim Malyshev, and the "external institution" was the Leningrad State University, where I went to give a talk at the seminar led by two wonderful mathematicians, Solomon Mikhlin and Vladimir Mazya. Many years later, the connection was discovered between this set of problems and the works on the wedge diffraction by Arnold Sommerfeld, Sergei Sobolev, Joseph Keller, and many others. This area is still being developed intensively. Only MI's patience and his trust in me allowed me to work without interference for 4 years and obtain a new result. The problem he suggested turned out to be extremely fruitful.

MI raised many students in a similar way—dozens, if not hundreds. He very readily talked to all of them, shared ideas and problems, and helped everyone in any way he could. Many specialists consider themselves to be his pupils. People would come from the entire country to give talks at his seminar. As Petrovsky said, that seminar did more for the development of the theory of partial differential equations in the USSR than entire academic institutes.

After MI moved to his new apartment by Leninsky Prospekt metro station, he would walk to the 19:00 seminar. He would pace very quickly: I could barely keep up with him. He would arrive early and interact with everyone before the seminar began. He spoke to everyone, including the greener students, with delicacy and respect. He readily discussed non-mathematical topics and news, including sports news. He said that in his youth he was an avid football player.

MI would often recall his youth and the story of his flight from Lwów from the fascists [Dem08]. All the students were gathered in the university hall and told that the Germans were approaching the city, and all of the Komsomol members must leave immediately. So he left with everyone else, without going home. He would never see his mother or any of his relatives again.

For the first few days, they ran for almost 24 hours a day. People would fall over in exhaustion. MI's friend offered to let him stay at his aunt's house in the village and wait there for a few months until the situation normalizes, but MI did not take the offer. His friend was killed as soon as the Germans arrived. MI said that his life was saved by the Sergeant Ivanov, who was a part of their group. As soon as someone would stop or fall, he would run up and aim his gun: "Get up, or I shoot!" Then the hunger began. There was no food, and they had to beg the peasants for it. Once, they had to stand outside a gate for several hours until a loaf of bread was given. He fainted several times from hunger.

Hunger would follow MI throughout the war. He said that in Tbilisi, in the morning, he would eat the bread he got from his ration cards, and then study mathematics until noon, "and then the hunger would start…" His friends helped him as much as they could. Ilia Vekua and Nikoloz Muskhelishvili treated him like a son, and one of his coworkers gave him her cafeteria pass. He was often fed at Karen Ter-Martirosyan's house, who was his friend and fellow student, and who went on to become a wonderful physicist. MI retained the warmest feelings for all these people, and his friendship with them continued all their lives.

MI's home was very hospitable, and Asya Moiseevna dedicated her entire life to creating conditions for his work with collaborators. She also edited all of MI's writing, and strongly reprimanded everyone for grammatical inaccuracies… The devotion and love between these two people was extraordinary.

MI's fame, and that of his scientific contributions, grew with time. In the 1950s, the Lax–Milgram method became widely known. This method was first introduced in MI's works, according to Jean Leray (this was also noted in Leray and Lions' collaboration [LL65]). MI's works with Lazar Lyusternik on small parameters caused an extraordinary stir in many areas of mathematics and its applications. The number of times these works were quoted was unprecedented. The same goes for his works with Gregory Eskin on boundary value problems for pseudodifferential operators.

Following MI's lectures at Collège de France, he became widely known around the world. He became a member of the American Academy of Arts and Sciences (in 1992), as well as the Accademia dei XL (the National Academy of the Sciences in Italy) in 1996. I came to Rome from Milan in March 1996 to attend his acceptance ceremony to the Accademia dei XL. The presentation was made by Luigi Amerio, an Academy member, with Gaetano Fichera presiding.

Apart from this, MI did a great deal of selfless work on reviewing articles for mathematical journals. I also had to do some work, at his request, on improving some publications.

Nonetheless, the recognition in Russia of MI's scientific merit came quite late, and was totally disproportionate, despite his brilliant early work on extensions of the Laplace operator, on the small parameter, and on strongly elliptic nonlinear equations. This was primarily due to the government's hardened anti-Semitic stance, which had its peak during the start of MI's academic career (he defended his doctoral dissertation in 1951). However, Russia is dealing with the consequences of this "government" politics to this day.

The entire mathematical *beau monde* of the USSR was well-aware of the outstanding value of Mark Vishik's works. However, it is well-known that this alone does not guarantee acceptance into the Academy: much more important is being a part of various scientific syndicates. During one of MI's Sunday walks near Yasenevo, he ran into Academician Sergei Nikolsky, who very amiably waved his hand and shouted happily from afar, "Yes, yes, I know, we do not give you what you deserve!"

To MI, efficiency was an important requirement for scientific research. For example, any criteria had to be easily verifiable. The presence of examples was critical. If there were only abstract conditions and no simple, concrete examples—that would not do.

Looking back at MI's scientific legacy, I am stunned by how many discoveries he was able to make in mathematics—and such remarkable ones! He was extensively erudite and possessed a great scientific sensitivity to new problems. To me, MI will forever remain a model of scientific courage and nobility, delicacy, and human charm.

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M. I. Vishik

Anatoli Babin

In these notes I share my recollections of Mark Vishik that have remained in my memory. I tried to be precise, but a great deal of time has passed. When it comes to my work with Mark Vishik on the theory of attractors, I remember the brightest moments that are burned into my memory. I almost never mention the works of other authors that worked in the area of attractors of partial differential equations; many references to, and notes on, the history of this problem, as well as exact mathematical formulas, may be found in our book [BV92] and in my two review articles [Bab03, Bab06].

I heard of Mark Vishik from my friend Borya Stilman in 1970, in our second (or at the beginning of our third) year, when we needed to think about our specializations. I only knew that I liked things that were similar to physics, and was trying to decide between differential equations and probability theory. Borya was much better at these things, and thanks to him, I went to Vishik. At the time, there was a seminar led by Mark Vishik for students and graduate students. It was held in auditorium 13–06 before the big seminar. At this seminar, we would study, as I recall, the theory of generalized functions. These lessons were mainly led by Misha Shubin and Sasha Komech, while Mark Iosifovich would give comments. I remember he once said: “If you’re going to the Chair of differential equations, you have to know how to differentiate”. I recall another seminar where Olga Arsenievna Oleinik was present. I solved some problem, but made a mistake, and Olga Arsenievna rather harshly pointed it out. Mark Iosifovich stood up for me. Sometimes I would stay for the big seminar, but did not understand almost any

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A. Babin (✉)
University of California in Irvine, Irvine, CA, USA

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of it. During the third year I attended a mandatory courses on partial differential equations, which was taught wonderfully by M. Vishik.

I recall that the main direction of the seminar's work at the time was the theory of pseudodifferential operators, and relevant problems. Mark Iosifovich suggested several articles for us to study. But afterwards, he told me, Borya, and Fima Shifrin to "do something nonlinear", and entrusted Sasha Shnirelman to help us with that. It is since then that I study nonlinear equations. I chose the theme of my candidate dissertation as a result of two influences: Sasha Shnirelman's ideas on the importance of quasilinear structure and numerous talks at M. Vishik's seminar, where the proofs of finiteness of the dimensions of kernel and cokernel for different linear elliptic problems and the index theorem were discussed. In around 1972, during my fifth year or at the start of my graduate studies, I was able to pull through the methodology of linear elliptic theory through quasilinear difficulties, and obtained "Finite dimensionality of the kernel and cokernel of quasilinear elliptic equations". Then, for the first time, Mark Iosifovich invited me to his home, and I spent several hours sitting beside him at the desk and relaying the details. When I finished, he asked: "Are you absolutely certain that everything is correct?" I think I said that it was difficult to be absolutely certain of anything in general. Mark Iosifovich laughed: "Oh, I understand, so mathematics is a probabilistic science." After that, I presented my work at two or three sessions of the small seminar in a row.

After my dissertation defense in 1976, I became interested in the linearization of nonlinear operators with the aim of inverting them, and then interested in the iterations of linear differential operators with analytic coefficients and their connection with the theory of weighted approximations by polynomials. In approximately 1981, Mark Iosifovich invited me to speak about my academic work. He especially perked up when I said that I have been working on not only linear topics, but that I am also still very much interested in Navier–Stokes equations and nonlinear problems in general. He suggested that we collaborate on the study of behaviour of solutions to nonlinear parabolic equations for large times and to begin with the simplest half-linear equation. Mark Iosifovich assumed that the special gradient structure of the spatial part may be helpful. And indeed, on the very first day we noticed that from the monotonic energy decay one obtains asymptotic convergence of each solution to stationary ones. Mark Iosifovich asked, whether one can clarify the character of this convergence. And here my experience with linearizations was of help, so I drew at each stationary point the stable incoming and unstable outgoing manifold and the motion of a typical solution along a hyperbola, approaching an unstable manifold. It became clear, that the outgoing unstable manifolds play an important role. We got to thinking what it is that they describe since each solution converges to a stationary point. And during the discussion process we understood that if one takes a bounded set of initial data, then the image of this set under the shift along trajectories is attracted to the union of unstable manifolds (such an object we then started calling a regular attractor).

Soon I noticed, that the construction of a regular attractor, based on the attraction of the image of a bounded set, coincides with the construction of the invariant omega-limit set of the Navier–Stokes equation from the '1979 work by Foiaş and Temam, which they had sent Mark Iosifovich and which he gave me at one point to familiarize myself with. I told Mark Iosifovich that we should give a general definition, which would cover both situations, and offered a definition which would include the invariance, boundedness, and attraction of bounded sets; such a definition guaranteed the uniqueness of the object. He gave it some thought, then asked, "Are you sure we should get into this?" I answered that it is a natural approach. Mark Iosifovich hesitated, then enthusiastically slammed his hand down on the table and said that if so, we must get down to business. After consulting with V. I. Arnold and Y. S. Ilyashenko, he suggested the name "maximal attractor", to highlight the difference between that and a minimal attractor, which in a regular case consists of stable stationary points. Besides, the maximal attractor is the maximal bounded invariant set. Unlike the definition of the omega-limit set, which depends on the initial set and does not include the concept of attraction, our definition includes the attraction as the principal property, which gave the ground to use the term "attractor". Thus we arrived at the concept of a maximal (global) attractor of a partial differential equation.

In the aforementioned article by Foiaş and Temam, they proved the finiteness of the dimension of omega-limit set. Mark Iosifovich suggested that we find the explicit estimates of the Hausdorff dimension of the attractor in terms of physical parameters. In this work of great help were the contacts of Mark Iosifovich with Yu. S. Ilyashenko, who pointed out to us the geometric approaches used in the theory of dynamical systems. Thinking over the structure of attractors, I figured out that the unstable manifold of a stationary point is in the attractor, even if the attractor has a very complicated structure. Since the unstable manifold has the same dimension as a linear unstable subspace corresponding to the linearization of the equation at a stationary solution, there is a possibility to estimate the dimension of the attractor from below. I asked Mark Iosifovich, whether he remembers the works on hydrodynamical instability; they could help us to estimate from below the dimension of the attractor of the Navier–Stokes. Mark Iosifovich said, that he recalled the talk of V. I. Yudovich, and that the reprint should be at his office at the Institute for problems in mechanics, where he worked part time. Mark Iosifovich brought the reprint and the references to the work of L. D. Meshalkin and Ya. G. Sinai. Now we could estimate the dimensions both from above and from below. One more research direction opened up, when Mark Iosifovich suggested that we look at the hyperbolic equation with friction. There is also the global Lyapunov functional, and the attractor has a regular structure. Improvement of the results on hyperbolic equations was possible with the aid of estimate techniques of Alain Haraux, whom Mark Iosifovich met during his trip to France and whom he referred to very warmly. Later we were told (unfortunately, I do not remember by whom) that this technique is also present in the works of N. F. Morozov.

Thus began our long-term collaboration. I would come to Leninsky Prospekt once or twice a week, bringing the texts that I had written on the basis of our previous meeting. We would discuss what I had written, and I would give the text to Mark Iosifovich. At the following meeting he would bring out what he had written, based on my notes, and tell me about it, then give it to me for its next iteration. After several iterations we used mainly glue and scissors. When an article was completed, Mark Iosifovich would give it to the typist, and the formulas would be written in by Asya Moiseevna. She would lament my terrible handwriting. Sometimes we would come up with something new during our discussions, and Mark Iosifovich would say, “Well, it seems that we have said enough. Write, Tolya!”

All those years, I felt like a part of Mark Iosifovich’s family. There was an atmosphere of mutual respect and understanding. I warmly remember our dinners with Mark Iosifovich and Asya Moiseevna, when we would discuss the latest news. During our work sessions, in order to rest a little bit, Mark Iosifovich would sometimes share memories with me, and, characteristically, would never speak ill of anyone. Sometimes he would lament how difficult it was to work with O. A. Oleinik, but would always add, “Mind you, Tolya, that she thinks highly of you.”

Mark Iosifovich always spoke very warmly of L. A. Lyusternik, but I remember him saying once, “When the problem was solved, Lazar Aronovich, for some reason, was not interested in turning that into a completed paper.” Mark Iosifovich did not accept that anything was trivial and always aimed for perfection. Once we were considering a problem that had many particular cases. Mark Iosifovich suggested that we consider one more particular subcase. To this I responded, with the arrogance of youth, “Well, that’s just pettiness!” In his reply, Mark Iosifovich only said, “Pettiness!” and then repeated once again, “Pettiness!” With his inherent delicacy, he said no more, but his intonation was very eloquent.

Working on the theory of attractors, I did not stop my work on constructive representation of solutions to partial differential equations and discovered very beautiful connections between the smoothness of solutions to degenerating equations and S. N Bernstein’s theory of weighted approximations. In 1985 I wrote my doctoral dissertation based on these works. Mark Iosifovich immediately took a very active part in the fate of my dissertation. He told me that I should not submit it to the Mechmat Scientific Council, because Andrei Fursikov is submitting his there: some of the council members might not like that two of Vishik’s students are doing their defences at the same time, and it could harm us both. He said that I should submit it to LOMI[1] in Leningrad, and he would ask O. A. Ladyzhenskaya to supervise me. And Mark Iosifovich immediately went over to the phone to call her. They spoke for a long time. I could sense that they had a very friendly relationship, and finally Olga Aleksandrovna agreed to hear me out. Thus began my regular trips to Leningrad. I must say that although Olga Aleksandrovna was his “old comrade”, not everything was cloudless. Their friendship lasted many decades, but conflicts did sometimes occur. Mark Iosifovich recalled how often they would speak, and he would share his thoughts with Olga Aleksandrovna. He also recalled that one day, during a walk, he asked Olga Aleksandrovna why she had not refer to him in her work on hyperbolic equations. She replied, “I was angry at you then.” Perhaps because of these old scores, Mark Iosifovich protested when I offered overly complimentary references

[1] The St. Petersburg Department of Steklov Institute of Mathematics —*ed.*

to her 1972 work. The issue was that when our first results on attractors were published in 1982, our attention was directed towards Ladyzhenskaya's 1972 article in «Scientific Seminars Notes of LOMI», which studied the omega-limit set of two-dimensional Navier–Stokes system; unfortunately, I cannot recall who pointed out to us that paper. Perhaps it had been Olga Aleksandrovna herself. She invited us to a LOMI conference in 1982 and offered to publish our paper in «Notes». Her 1972 article is remarkable: there was the construction of the invariant set, which coincided with the attractor, and the solutions on this set were studied. Later, Olga Aleksandrovna criticized us very harshly in a Russian Mathematical Surveys paper, presumably due to the lack of attention we paid to her work. Nevertheless, these priority arguments did not in any way reflect on her approach to my dissertation, and my defence was completed successfully in 1985. Mark Iosifovich was very worried that Olga Aleksandrovna was hurt; he said that it was painful for him to read her criticism not because of the criticism itself, but because he could feel her heartache.

The times began to change. Gorbachev was opening up the country. I saw him in Leningrad in 1985 in the square of Moscow Railway Station: he went to mingle with the civilians for the first time, which clearly frightened his guards, who leapt after him once he had already crossed half of the square. In 1989 George Sell organized in Minneapolis the school of dynamical systems; Mark Iosifovich and I participated in this school. We shared an apartment for a month. Unfortunately, Mark Iosifovich was not feeling very well at the time—he had problems with his liver—but he actively participated in the work of the school, and soaked up impressions of this unfamiliar American life with his inherent interest and enthusiasm. The years of our collaborative work were coming to an end. New opportunities on the one hand, and the sense of tectonic shifts on the other, drove me across borders and oceans, while Mark Iosifovich continued stubbornly to toil in Moscow. The last time we spent a meaningful amount of time together was when he came to Irvine for a month. Asya Moiseevna was already very weak. I remember we were walking along the shoreline of the Pacific Ocean when she said, "I never understood why anyone needs a lot of money. Now I know: to buy a house on the shoreline and watch the ocean." Mark Iosifovich kept working tirelessly. Once he said, "Mathematics won't let one down."

To me, Mark Iosifovich is an exemplary scientist of a very high caliber, and a tough act to follow. I learned a great deal from him. I will provide just one example. When I first began to work with him, to me the equations of mathematical physics were just particular examples of concepts of functional analysis. But gradually the picture turned over, and the equation became the foundation, on which one can build function-theoretic constructions.

Mark Iosifovich Vishik was not only my teacher: he was one of the people closest to me, and I will always remember him.

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In Mark Vishik's Own Words

Andrew Comech

Vishik *is a word I've known since birth: "Markiosich" (Mark Iosifovich) was both my parents' advisor. Not too soon after that, when already a schoolboy or even a student, I started occasionally meeting Mark Iosifovich in person.*

– *Andrew, what problems do you currently work on?*
– *In an abstract Hamiltonian system...*
– *(He listened attentively, but seemed to have lost enthusiasm). I see, I see. And... and what other problems are you currently working on?*
– *The one-dimensional model, such and such equation, under such and such assumptions and conditions of the nonlinearity...*
– *(I think that he did not even finish listening) Andrew, I think this problem is a very, very interesting and important problem. A very good problem. I think you're working on a very interesting problem!*

– *Andrew, who have you met this year? That's important, this is very important!*

After one very abstract seminar, where it was clear that the speaker was craving for an opportunity to teach: "definition 1, definition 2, definition 3, theorem 1, corollary 1, corollary 4..."

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A. Comech (✉)
Department of Mathematics, Texas A&M University, College Station, TX, USA
e-mail: comech@tamu.edu

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— We've just listened to a very interesting talk. Very interesting! Such a beautiful theory, I liked it a lot! Remarkable talk! Let's wish the lecturer to apply his beautiful theory to some interesting problems.

– *Mark Iosifovich, what did your father do for living?*
– *I am not quite sure, but he was able to do everything. We did everything when I was young, you cannot even imagine. To earn money. My mother really loved butter: not cheese, but specifically butter, and when I earned some money, I would buy butter for her.*

– *Mark Iosifovich, and what language did you speak as a child?*
– *Polish. We spoke Polish, and I also learned Ukrainian: there were Ukrainians living near Lwów, and it was cheaper to buy things at the market if you spoke to them in Ukrainian. I did not learn Russian: Russia was very far. That was later, when you liberated us...*
– *Mark Iosifovich, who did we liberate you from?*
– *From Poland!*
– *But it appears that you are Polish?*
– *Well, Andrew, that was a long time ago...*

Below are the stories Mark Iosifovich told about his life to his friends and colleagues, in Mark Iosifovich's own words; see the collection [Dem08].

Lwów

When I was still just a boy, something did not feel right in my head: something was "flying around in my brain". My mother took me to the doctor. And he said this would get well with age—it's just that my brain was developing faster than my skull. And, indeed, it got well. At school I studied well, and even distinguished myself in some ways. This was a regular Polish regional school. I could say so many little things about it. But this story took place at the lyceum, in our final year. We were studying logarithmic interpolation, when a certain value of the logarithm is absent in the logarithmic table and one needs to find this value in an interval. And the teacher said that one needs to break the interval into ten equal parts and so on. And I raised my hand and asked, "How do we know that the logarithm is linear?" This story made a big impression at the lyceum teachers' meeting. And the school director said a wonderful complimentary phrase about me. But I will not speak about this for long... I had other successes like that as well, though no one studied mathematics with me at home.

I studied at the ninth gymnasium, and then at Lyceum №5.[1] Lyceums were specialized. I was at a physics and mathematics lyceum, but there were also biology lyceums, humanities lyceums, and so on. At the time, there were no university entrance examinations. We simply wrote an application, and based on the information that we had graduated the lyceum—of course, the lyceum did give us a document stating that we had graduated—anyway, based on that information, we were accepted. And so I studied in the University at the Department of Mathematics until the war began, until the first days of war.

As I recall, our enrollment at the University of Lwów occurred in December. In early January we were already attending lectures. I went to university every day, listened to lectures for the first half of the day, and then ran home in the afternoon, where my mother would feed me dinner. Afterwards I would go to the library, and spend most of my time there. Even on weekends I would go there if the library was open.

Juliusz Schauder taught us mechanics, using Banach's book, in our second year. He had a wonderful teaching method: at the beginning of every lecture, he would ask someone from the audience to repeat the basic theorems from the previous lecture. This would take about ten minutes. Usually, the most capable young people would answer.

Schauder and I would also meet in the library. He would go there often to do his work and read journals. It was a wonderful old library of the University of Lwów. Moreover, seeing that in the library there were such young people who were zealously studying, reading different books, even in German, he once approached me and invited to talk to him sometime. Since then, he became my first consultant, let's say, my mathematical consultant. He was a very good man, he told me that there was such a thing as analytical mathematics, geometric mathematics, and so on. He was a great scientist, and had been compared to Banach himself. However, prior to the liberation of Poland, he had merely been a professor at gymnasium №2 (teachers at the gymnasiums were then called *professors*). Of course, he did speak at Banach's seminars. He is known both for his independent work and for his collaborations with Jean Leray. So Juliusz Schauder played an important role in my life. One could say, he gave me direction, he was my consultant. We met often, and for sure spoke once a week at the very least.

A little about Stanisław Saks. He was a Professor at Warsaw University. But then he basically fled from Warsaw to Lwów: apparently, he knew Banach very well, and this is what brought him to the University of Lwów. Saks participated in Banach's seminar. I also attended it. And although I was still very young and did

[1] Apparently, Mark Vishik and his friend Wladek studied at the IX boys' gymnasium named after Jan Kochanowski, which was located on Khotinskaya Street, numbers 4 and 6. Both Mark Vishik and his friend Wladek Lyantse are mentioned in the list of graduates of the gymnasium (renamed into school №59 after 1959 and located at the same address). After that, it appears that Mark Vishik was accepted to a lyceum which was specialized in physics, mathematics, and humanities. The lyceum was at boys' gymnasium №5 (Gimnazjum V Męskie im. Hetmana Stanisława Żółkiewskiego), located then at Samuel Kuszewicz Street, 5.

not understand a lot of what was being discussed, I barely missed a single seminar, because I found everything so interesting. I was delighted by everything that took place there. Of course, they were mostly occupied with set theory mathematics. The concept of the category of sets, which is still being studied, also played an important role there. So, Stanisłav Saks took part in this seminar. Banach was a very cheerful person and loved to play jokes on Saks, knowing of his tendency to be slightly absentminded. For example, he could hide his briefcase before the seminar began. Saks would begin to worry: “Where is my briefcase? Where is my briefcase?” And when he found it somewhere nearby, he would be very glad. And Banach would laugh merrily.

All in all, Stefan Banach was a brilliant mathematician. And it is totally unclear how in a city like Lwów could have appeared such a prominent school of functional analysis. “The Banach School” even organized in Lwów the publication of the journal «Studia Mathematica». About ten issues of this magazine were released. Stefan Banach was also the Dean of the Department. And I have a gradebook that was signed by him, and I still keep it like a relic. I attended all of his seminars. Although he himself did not give any lectures. And the vice Dean was Miron Onufrievich Zarycki. He taught a course on mathematical analysis. And that was the only course that was taught in Ukrainian; all of the other subjects were taught in Polish. I remember Banach and Schauder went to Kiev for new instructions. This was in 1940, I think, or in 1941. There, they were told that they have one big flaw: no student scientific conferences. And when they returned to Lwów, they immediately offered the students to make their own contribution to science and give their talks at a conference.

Edward Szpilrajn took on the work of preparing our presentations at an urgently organized student scientific conference. He was a remarkable person; he had also come to Lwów from Warsaw. And so, Szpilrajn gathered us (the ones who were always sitting in the library and studying), told us about the set theory, about Hausdorff’s book, about his spaces, and gave us some problems that could be solved quickly. They had to do for the most part with various Hausdorff spaces. And so, we sat in the library all of Saturday and Sunday, from morning till night, and thought through our “discoveries” in the set-theoretic topology.[2] And then we spoke at the student scientific conference, organized shortly thereafter. Banach was there. He listened with pleasure to our talks, sometimes giving us, I would say, somewhat humorous remarks, which were all very encouraging, and it was very pleasant that a great scientist such as Banach would say something about your work. I myself was one of the students who spoke at this conference.

Soon, Nikolai Ivanovich Muskhelishvili came to see Banach in Lwów, to arrange a *socialist competition* in the area of mathematics between the Tbilisi and Lwów Universities. I, as a member of the trade union, was also taking part in the

[2] Mark Vishik and Władysław Lyantse’s result is mentioned in «The Scottish Book» in problem № 192; see p. 133. —*ed.*

arrangement of this competition, and was very proud of that. Later, I joined the Komsomol, and was very proud of that as well.

Later Banach and I met in Moscow in 1945. That was our last meeting. He was invited to Moscow by Andrei Nikolaevich Kolmogorov, who had wanted to give him a position in Moscow and make him a member of the Academy of Sciences of the USSR. In Moscow, Banach stayed at the Academy of Sciences Hotel on Gorky Street (now Tverskaya Street). I called him and said that I'd love to get his advice on what to work on next. I came to the hotel. But when Banach came down to see me, I gasped: he had been a heavy man, but the man I saw now was, one could say, "a one-dimensional person", he'd gotten very, very thin. But he was still as benevolent as ever. First of all, Banach asked me to tell him how I escaped from Lwów. I told him about leaving Lwów on foot in the first days of the war, because the Germans were approaching, and those who wanted to fight fascism were told they must leave the city. I recounted my entire odyssey. Then, Banach asked me what I'd like to discuss with him. I answered that I do not know what to work on next. He asked me what I read. I said that, of course, I read his book on functional analysis. And, having found myself in Tbilisi, I read a great deal about differential equations, because the Tbilisi school of mathematics, led by Muskhelishvili, Vekua, and Kupradze, was specifically working on differential equations. I told him about having participated in the scientific seminar in Tbilisi. In particular, I had given a talk on Keldysh's article in «Uspekhi Mat. Nauk» on regular points of the boundary. I had also studied Wiener's work, and had given a talk on that as well. I had also given several survey talks... I also read Aleksandrov and Hopf's book in German—a very good, fat book which I studied whole days. I also read other books in Tbilisi; for example, Privalov's book on complex variables. Having heard me out, Banach said that it's very good that I'd studied all of these areas, and that I'd finally made it to Moscow. "Moscow has some wonderful specialists in functional analysis"—he named Gelfand—"and in differential equations: Ivan Georgievich Petrovsky, Sergei Lvovich Sobolev, Sergei Natanovich Bernstein." He said that I was very capable, and therefore he recommended that I should, "like Schauder," start working on problems related to both of those fields. With that, we parted. Afterwards, I learned that Banach, by the time of our meeting, already had lung cancer: he had been a heavy smoker. He did not live long after our meeting, and died in Lwów later that same year (1945).

Now, about Steinhaus. When I was still just a gymnasium student, I sometimes attended his lectures at the University of Lwów, where he spoke about interesting mathematical problems, which later were published in his book «One Hundred Problems in Elementary Mathematics». I later met him a few times at conferences. I also know that he was the one who "discovered" Banach.[3] Banach studied at the Lwów Polytechnical Institute, where he was a remarkable and very capable student.

[3] In 1916, at the Kraków *Planty* park, Hugo Steinhaus overheard a conversation of two young men about the Lebesgue integral. His interest piqued, he joined their conversation; the young men were Stefan Banach and Otto Nikodym.

Steinhaus took note of this and hauled him from the Polytechnical Institute to the University of Lwów. This was where Banach began to reflect on “his” functional analysis, the theory of linear operations. He wrote a book on the subject, which I think came out in 1923...[4] At least, it was being actively worked on at the time.

So, Banach surrounded himself with Schauder, and Mazur, and Orlicz, and other remarkable mathematicians. By the way, Mazur taught us a course on differential geometry at the University of Lwów—and taught it wonderfully. His exams were also quite unusual. He would assign you a problem and would sit nearby, drawing something to himself, sketching, solving. That is, you mind your work, and he does his. Then he would look over what you’d done, and if it satisfied him, he would simply say, “You are solving this correctly,” and would give you a “five”. I very much liked how hardworking Mazur was. Later, he became a Polish academician. And Orlicz gave us lectures on algebra.

Knaster taught us analytic geometry in Lwów. He was a comrade of Pavel Sergeevich Aleksandrov. After the war, he exchanged letters with Pavel Sergeevich in Russian. And when I had my candidate examination in Moscow with Pavel Sergeevich, he told me that Knaster remembered me and spoke warmly of me.

Here’s the story. Knaster once scheduled a test based on his lectures. Besides the usual problems, there were problems with one and with two stars. There was some difficult problem that I solved, re-proving along the way a theorem of a famous Spanish mathematician. And all my solution I wrote first in Ukrainian, and then, to fit into the allotted time, in Polish. And Knaster remembered that ever since. But we never met after the war. I did meet with Orlicz when I was invited to Warsaw. It was several times, at the Banach Center. And it was there that I once met Orlicz. One time he even came from Poznań to Warsaw specifically to meet with me. I remember the director[5] of that institute—a student of Ilia Nestorovich Vekua—hosted a wonderful reception with tea and cakes. We sat and talked about the past. By the way, Orlicz asked me, a little shyly, whether I had any graduate students. I answered that I had six of them. As it turned out, he also had six. Orlicz said that in Poland, the following situation has come up: every advisor was now responsible for ensuring that their graduate students defended their dissertations on time. Basically, their situation had become exactly the same as ours. He asked me how I cope with it. “I help!” I said. “You know, Marko Iosifovich, I help too!” You see, he also had to help. So I only met Orlicz in Warsaw. He was a very good man. I once met with Zarycki after the war, as well. I tried to meet with him again another time, having come to Lwów with Asya, my wife, but it did not work out: when my friend Wladek Lyantse and I came to Zarycki’s home to visit him, he was no longer in any state to meet us. This was around 1961, when he was already very ill.

[4] The first edition of «Teorja operacyj. Operacje linjowe» by Stefan Banach was published in Warsaw in 1931.

[5] Bogdan Bojarski.

The Departure from Lwów

From the first days of the war, we, as members of the Komsomol, were required to stand on guard at the university: in case there was a bombing, we were to remove the incendiary bombs. And, besides, we were strictly ordered to carry on us both the passport and the military identity card; that is, all of our documents. And so, on June 28, indeed when I was at the university, someone promptly found us and said that those who wished to fight against the fascists needed to urgently leave the city and go east, because the Germans were approaching.[6] And we, along with one of my mathematician friends (his last name was Tepper but I cannot remember his fist name) decided to go fight. We left off, without even going home, with the others who knew the way: we needed to reach the chaussée leading east.

On the first day, we walked with our retreating troops and equipment. Sometimes, though, we were allowed to sit on the gun carriages, although, generally speaking, this was forbidden, and so we mostly had to walk on foot. We walked for many kilometres, maybe fifty, maybe sixty. In the evening—and this was June, when the days were the longest—we collapsed to sleep. Right in some gutter. And were asleep instantly. When we arrived in Ternopil, my friend suddenly said that his aunt lived somewhere here. And that he would go and hide at hers. And then he would go back to Lwów: because all this, he believed, would probably end soon. I instead objected that this war would not end soon. Which was why we needed to keep moving with the others and then go fight. We parted, and I left with the others without him. I know nothing of Tepper's fate. We were fortunate: along the road, we came across a truck with bread. Apparently, it was nowhere to take to. And we were simply offered to take a loaf of bread each. I took one, too. Then I joined a group of Ternopil students, who were also headed east. We walked to Zhmerynka, took a freight train there and traveled to Kiev for two weeks. There was nothing to eat, I lost consciousness twice from hunger, and was fed something or other. But, finally, we reached the Kiev Komsomol Committee. Kiev was not occupied at the time. So we went there. There I met some students from the literature departments of the University of Lwów. We were told that the army will not enlist us, but that people were needed to harvest grain in the Kuban region. We were given a business trip assignment and some means. And we sailed down the Dnieper River to Dniprodzerzhynsk,[7] then to the *stanitsa*[8] of Timashyovsk. There, near Timashyovsk, I harvested grain for two months. When the harvest was over, I told these "men of letters" that I cannot sit around with them any longer and that I needed to move on to finally get to the army. Having said farewells, I went alone to the train station and left for Krasnodar. In Krasnodar, I faced my next torment. I was not accepted to study at the Pedagogical Institute. And so, to survive, I took on any work I could, but continued to study

[6] The German troops entered Lwów the night of June 30, 1941.

[7] The city of Kamianske.

[8] Village or a town (originally a stop-over).

mathematics whenever possible: I read Aleksandrov's book on the function theory; repeated the theorems I had already learned. Once, I even tried to become a loader, but the loader did not work out of me: when they loaded me up with a giant bag of onions, I, along with the bag, simply fell straight. That day I earned, I recall, only one ruble sixty.

And it so happened that right around then they started recruiting young people to the flight school. I then decided to join them. To the flight school, located outside of Krasnodar, we would go at night, for three weeks. During the day, we were hosted by the Kuban state farms. They received us very well, fed us delicious food, we even got a little bit of rest. I still remember their Kuban hospitality. Finally, an order arrived: we were all to return to Krasnodar. And when I returned there, I was finally accepted to the Pedagogical Institute. But the fascists were approaching Krasnodar. And I realized that I cannot stay in Krasnodar and that I have to go farther. There was this guy named Kratko (he was a community activist from the University of Lwów). When he learned that I'm still there, he said that I must leave immediately, because I cannot stay in Krasnodar any longer. He even offered me a ticket to Yerevan, which I bought from him. I do not know how he got that ticket, but, apparently, he bought it.

I boarded the train that followed the railroad along the Caspian Sea. This was a forbidden area, because the oil and gas for our army were transported along that route. They were very strict in checking everyone, looking at everyone's documents. By some miracle I bypassed this check. One of my fellow commuters, a lawyer, told me that I was unlikely to be so lucky in the future, and then I would get into great troubles. And when we were passing Makhachkala, he advised me to get off the train. I got off and stayed in Makhachkala. There I was accepted to the Makhachkala Pedagogical Institute, which I graduated in one year.

In Makhachkala, by the way, I was even a commander of the women's platoon, helping the military commissariat. There I kept writing applications to join the army. But they kept rejecting me and would only give me some work instead. In particular, they assigned me to deliver army summons to the local population. There were times when I would spend hours looking for some shack on the outskirts of town, but I would carry out the assignments. I began to realize, though no one told me this, that I was not accepted into the army because I had lived in Poland for eighteen years.

In the summer, we were sent to do agricultural work. This was a fruit *sovkhoz*;[9] unfortunately, there was an outbreak of malaria there. There was a hundred percent rate of infection. I also became very ill with malaria. And when I returned to Makhachkala, I was very sick and severely emaciated. I was put in the hospital, where I spent some time. I was visited by the Prokofievs, Elena Vasilievna and her

[9] State-owned farm in the USSR.

husband, docents from the Pedagogical Institute.[10] I was like a live skeleton—so thin from the malaria and malnutrition.

And when I left the hospital, a rumour was just spreading that the Germans were already in the Mozdok region. This was not far at all,[11] and I needed to urgently leave Makhachkala. I took as a fellow traveler my friend, who had also decided to leave. We understood that we could not go by train. But we saw some kind of military echelon with soldiers going south for rest. There was also military equipment, and a small airplane standing on a special platform. This was where we decided to hide: under the airplane. I could not get up there myself, but my friend raised me and carried me there. We travelled with the echelon to Baladjary station near Baku. After that, there was a turn towards Tbilisi, and we got off the train. In Baladjary we found another echelon, also a military one, also going for rest. I got into a conversation with a lieutenant from the echelon, or a senior lieutenant, I cannot remember now, who also, luckily, turned out to be a mathematician. I told him that in Makhachkala, having studied Hausdorff's book, I had even completed some work on ordered sets and sent a letter about it to Ilia Nestorovich Vekua, to Tbilisi. And Vekua had responded that «The Bulletin of the Academy of Sciences of Georgia» came out regularly, and that I should write an article on my work and send it over to the editors of that journal. And if the article were found worthy, they would publish it. And that's why I was eager to get to Tbilisi, especially since, even prior to the war, a socialist competition had been arranged between the Lwów and Tbilisi Universities. The lieutenant took pity on us, led us to the carriage with the military supplies, ammunition, and so on, sat us there, closed the carriage, and said that we'd be let out a hundred kilometres before the train reached Tbilisi. He explained that after that we could reach the city on our own, by a commuter train. That's how I got to Tbilisi.

Tbilisi

In Tbilisi, I went straight to the Rector of the university. I showed him my record book with Banach's signature. I told him that Nikolai Ivanovich Muskhelishvili, the President of the Academy of Sciences of Georgia, and the Dean of our university, Stefan Banach, had arranged—before the war—a socialist competition between the Tbilisi and Lwów Universities in the area of mathematics. Many people had been

[10] The history of the Dagestan State University, the former Pedagogical Institute mentions V. N. Prokofiev, a mathematician, who worked at the institute in the pre-war years (see dgu.ru/sveden/2329). At the same time, it seems likely that the patronymic was confused there, and that the right person is Vladimir Mikhailovich Prokofiev, who in post-war years worked in Moscow, at the Chair of Higher Mathematics at the Gubkin Institute of Oil and Gas (see kvm.gubkin.ru/istoria_kaf.html) and kept in touch with M. I. Vishik.

[11] Mozdok, a town in North Ossetia located approximately 270 kilometres from Makhachlaka, was occupied by German troops on August 23, 1942.

present at this event, including myself, and we had all been very proud of it. I also said that I'd come as a representative of the University of Lwów and was ready to continue this socialist competition at the University of Tbilisi. You can imagine the reaction by Georgians!

The Dean of the Department at that time was Ilia Nestorovich Vekua, and his approval was required in order to enroll me to the university. But he was sick. I was brought to his home, after I said that I had been in correspondence with him, and that he had even suggested that I publish my article in «The Bulletin of the Academy of Sciences of Georgia». Vekua lay ill. His wife had just prepared him *satsivi*, a very delicious chicken dish with the nut sauce. She offered me some, I then had a little enough food. And so, the ill Vekua, having spoken to me, wrote about me some incredibly kind words in Georgian so that I would be accepted to the University of Tbilisi. But the trouble was also that there was no longer a dormitory at the university: all of its rooms were taken for hospitals. And the dormitory beds had all been moved to the barracks in the university courtyard. Finally, a solution was found: they removed all of the beds, except one, from a small barrack, and settled me there, in the university courtyard. The climate was warm, so it was possible to live like that. Except, when it rained, I had to move the bed, because the ceiling dripped: the barracks were not designated for dwelling. That's how I became a fourth year student at the University of Tbilisi. Vekua, as I already mentioned, was the Dean of the department. I was taught by V. D. Kupradze and other good professors. This is where I met Karen Ter-Martirosyan, who became my closest lifelong friend. He was the same year as myself. And so that's how I began to live in that barrack.

I had to mend my clothing on my own. When my trousers wore out completely, I, having requested a needle and thread, did a very complicated combination with the thread so that the holes would not be visible. But securing the threads turned out to be my weak side. And one day, as I was talking to Nikolai Ivanovich Muskhelishvili, they came apart. And then Nikolai Ivanovich very tactfully offered to help me find some regular outfit. And he wrote a letter about that to the Ministry of Light Industry of Georgia on behalf of the President of the Academy of Sciences of Georgia, and Vekua went there with it. And they dressed me up a little.

I also had a problem with food. But one of the students from my year was the wife of a docent of the University of Tbilisi. And he had a pass to dine at the cafeteria, which was located down on Plekhanov Street. And so, this student took pity on me, that I'm so unestablished, and gave me his pass. From that point on, I began to eat, at least once a day, something broth-like, and some kind of main course... There was very little bread. I received 400 g of bread at 7 in the morning, and by 7:15 I no longer had it. Because I had a very good appetite, and there was nothing else to eat.

In the end, I graduated from the University of Tbilisi, and even received a government scholarship (then called a "Stalin scholarship") throughout my studies. After graduation, I was accepted as a graduate student to the mathematical institute of the Georgian Academy of Sciences, and I became Vekua's graduate student. And, besides, I began working as an assistant at the University of Tbilisi.

I began teaching, and even had Muskhelishvili's son as one of my students. I taught his group differential equations. As soon as I finished university, Nikolai

Ivanovich Muskhelishvili made sure that I had some kind of housing, having worked that out with the vice president who supervised Abkhaz graduate students. Uphill, near the Tbilisi funicular, they had a dormitory with several rooms. I was placed in one of these rooms, along with one more person. And so I lived in a dormitory for Abkhaz students. There was no electric light. One of my Abkhaz acquaintances was Bagrat Shinkuba, who went on to become the Chairman of the Presidium of the Supreme Soviet of the Abkhaz ASSR. He later wrote his dissertation on the grammar of the Abkhaz language. Another person who was there was Shalva Inal-Ipa, who wrote a book on Abkhaz history. They became my best friends. In the evenings, I would sit in their room and listen to their interesting stories. In the mornings, I would go to the Academy of Sciences for the entire day. I worked on mathematics there and read books. There, I found Aleksandrov and Hopf's book on topology in German, and read the entire thing. I never studied topology again in all my life. That book was enough for me, I had a very good memory. Apart from that, I read Stefan Banach's book on functional analysis and a few books on complex variables by Privalov. These were such special books. I also took part in seminars which were organized both at the Academy of Sciences and at the University of Tbilisi; they were led by Ilia Nestorovich Vekua. I would quite often speak there myself... And that is how my life went on in Tbilisi. Honestly, I did not have a bad time there, and I made many friends.

I had my own space in a four-desk room. This room was located across from the room of Arnold Walfisz. Walfisz had his own office: he was the head of the Division of number theory. He had taught several Georgians number theory, there had been no such specialty before him. Here is how he ended up there: he had studied in Warsaw before the revolution, and thanks to that he had a right to repatriate. For some time Walfisz lived in Germany, where he was Edmund Landau's student. There he became a specialist in number theory. He also married there a German lady, and came back to Warsaw with her. In Warsaw he worked somewhere. But it was difficult for him: he fell seriously ill with asthma. The doctors advised him to go somewhere warm. And that's when Walfisz remembered that he had the right to repatriate to the USSR. And that the USSR had southern cities—Tbilisi, for example—beyond the Caucasian Ridge, where the climate was warm, gentle, appropriate. He somehow arranged with Nikolai Ivanovich Muskhelishvili his move to Tbilisi, offering to create a Division of number theory at the Mathematics Institute of the Georgian Academy of Sciences, and to work there. And that is how Walfisz ended up in Tbilisi. Walfisz was a very calm, very mature, very conscientious man. And he forbade anyone from having any kind of political discussions with him. Everyone knew this. And everyone understood that he, let's say, "did not know many things". I had been to his house. There they spoke German. I understood German: I had studied it back in Lwów. And learned it fully. Particularly because I had "swiped" some of my classmate's German books: our lyceum had classes in French and German. I went through these books in two or three months, and learned many poems by heart. When I came to Germany later on, it turned out that I knew, for example, "Die Lorelei" by heart, but they did not—no matter how many different professors I asked. I learned by heart a lot of Goethe. And Schiller's poem „An die

Freude“—the one that Beethoven used in his Ninth Symphony. So German turned out to be very useful to me. As I have already said, I studied Aleksandrov and Hopf’s book in German.

The Walfisz family had two daughters. The eldest had a somewhat unfortunate fate: she married poorly. And the younger daughter had been born in Tbilisi. She studied there and became an employee of the Mathematics Institute of the Georgian Academy of Sciences when I left in 1945. I met her later at the International Congress of Mathematicians in Berlin. It turned out that she had repatriated to East Germany and made a life for herself there. I met with her a few times in Germany. I would often come to Chemnitz: when I was already working at the Moscow University, I had a graduate student who was from Chemnitz (at the time, it was called Karl-Marx-Stadt), and she would invite me to visit her there. Then I would go to Berlin, where I was invited to the Mathematical Institute of the Academy of Sciences.

Overall, Walfisz played a huge role in my life, particularly in the fact that I ended up in Moscow. It was the end of 1944, and 1945 was about to begin. Famous mathematicians from Moscow began visiting Tbilisi—Andrei Nikolaevich Tikhonov and others. And I began to ponder the question: what do I do next? I was not planning to stay in Tbilisi after all. And I began to think that it is time for me to go back to Lwów, to the wonderful Banach school that I was simply in love with—to Banach, Mazur, Orlicz, and the others. But Walfisz gradually, very carefully started persuading me that, possibly, there was no longer any school there. That only in Moscow would I find all of the mathematics specialties that interested me, and the most prominent mathematicians. And that in general one cannot compare Moscow’s scope to Lwów’s, despite of Banach being there. Then Felix Ruvimovich Gantmacher arrived from Moscow, and also began to persuade me not to return to Lwów, but to go to Moscow, which was home to the leading university in the world. Gantmacher said I would be much better off there, especially since the Georgian and Moscow mathematicians have a close connection through Nikolai Ivanovich Muskhelishvili. And in the end, I changed my wish about returning to Lwów, and set my heart on moving to Moscow.

Moscow

In 1945, my best friend Karen Ter-Martirosyan met me in the street and said that officials started to allow people business trips to Moscow. It was early in the year, in January. I immediately went to the place of my advisor, Ilia Nestorovich Vekua, and asked him to arrange my business trip to Moscow for finishing my postgraduate studies. Vekua took it without much delight: during this conversation, we were playing backgammon, and I was worried that he would break the board when throwing down the dice. But Muskhelishvili understood me: he saw how I worked and understood that my place was not in Tbilisi. And he persuaded Vekua to let me go to Moscow. Soon after that, Muskhelishvili went to Moscow and found me a wonderful advisor at the Steklov Mathematical Institute: Lazar Aronovich Lyusternik. Nikolai Ivanovich knew that Lazar Aronovich was a tremendously talented person. Furthermore, Lazar Aronovich refused no one and took under his wing people from different republics. He was a very kind man in that regard. Having learned that I come from Lwów, that I know Banach, that I studied in Tbilisi, he immediately agreed to take me as a graduate student.

Lazar Aronovich Lyusternik and Anisim Fedorovich Bermant, vice editor of the journal «Matematicheskii Sbornik», visited the University of Lwów together in 1939 to persuade the Lwów mathematicians to submit their articles to this journal. We even had on that occasion a problem with the board: Lazar Aronovich, while explaining something, wrote very energetically on the board; it fell and injured his leg, but not very badly. I remember, then the board was set on two chairs, and Lazar Aronovich continued further with his talk, which I, unfortunately, could not understand back then. Everyone was sitting there: Banach, Schauder, and all of us students. Bermant definitely spoke Russian, but most people somehow understood him. And Lyusternik could also speak Polish. But I cannot remember which language he used. I only remember that I could not understand his talk: he spoke about his work with Shnirelman. Although when Pavel Sergeevich Aleksandrov visited us, I understood him a little, because he gave his lecture in German. He knew German very well. And we all knew German, because Austria was close by and we all had some connections to it. And when I visited Germany later on, the German professors said that Pavel Sergeevich taught them how to properly speak German.

And so I was granted a business trip to Moscow. I knew the Minister of Education of Georgia, V. D. Kupradze, since I had attended his lectures; he was a Professor at the University of Tbilisi. And so I approached him to help hasten my trip, because it was war time, and business trips had to be approved by the government. He helped me. So I bought a ticket to Moscow. A student I knew from the University of Lwów—her name was Bella[12]—gave me her Moscow relatives' address on Novo-Basmannaya Street. Having arrived at the Kursky railway station in Moscow, I walked directly to their place. And I lived at theirs for some time. Then I rented a little corner from some woman: there I slept right on the floor, on a mattress. This is how I lived until I met my dear Asya. (Asya Moiseevna adds: "We met at Moscow University right on Victory Day. Julik Schrader[13] introduced us."). In short, Asya Moiseevna agreed to be my wife. We began living in her room near Arbat, on Sivtsev Vrazhek. There I got a little desk where I could finally work in peace.

[12] Bella Naumovna Garshtein.

[13] Julius Anatolyevich Schrader.

Dissertations

In 1947, I defended my candidate dissertation[14] at the Steklov Institute of Mathematics. My opponents were Ivan Georgievich Petrovsky and Sergei Lvovich Sobolev: two academicians who already knew me, because I was very active in my work. By that time, as it was already said, I was married to Asya Moiseevna Guterman. I wrote my dissertation on a topic that came to my mind when I attended Abraham Ezechiel Plessner's seminars. He asked me to give a talk on Hermann Weyl's work on the method of orthogonal projections for solutions to the Dirichlet problem. Weyl's article had been published in the «Duke Mathematical Journal», in the 7th volume, in 1940. Back then, I did not know English, and could only understand the formulas in the article. But through the formulas I was able to understand a little bit of what Weyl had done. And it also occurred to me that this could be applied to general elliptic selfadjoint positive-definite equations. And so, I spoke about Hermann Weyl's work at the seminar, while myself at home started developing the method of orthogonal projections for general selfadjoint equations of elliptic type. This was what would become my dissertation topic. At the time, I was well known to Sergei Lvovich Sobolev and Ivan Georgievich Petrovsky: I would attend their seminars and would endlessly give there talks on topics I was working on. I had met Sergei Lvovich back in 1946, while attending his special course on embedding theorems. And for my thesis all of this came in handy. He gave me as a gift his famous book «Some Applications of Functional Analysis in Mathematical Physics», I studied it thoroughly and used his methods in my candidate dissertation.

My doctoral dissertation[15] consisted of two halves. The first half had to do with the solution to the Dirichlet problem for strongly elliptic systems of partial differential equations, which are still being worked on today: for example, Mikhail Semyonovich Agranovich works on this topic. These systems of equations have a divergence form of order $2n$, positive-definite symmetric part, and a skew-symmetric part. Such systems of equations I eventually called "strongly elliptic". Lazar Aronovich came up with this name, and that was his important contribution to my doctoral dissertation. I asked his advice on what to call such systems, and he was the one who suggested using this term. The second half of my dissertation was conceived at Israel Moiseevich Gelfand's seminar, which he led for three people in 1946. The participants were Olga Arsenievna Oleinik, Olga Aleksandrovna Ladyzhenskaya, and myself. Ladyzhenskaya was graduating from the Department of Mechanics and Mathematics at MGU, and was a student of Ivan Georgievich Petrovsky. Then she moved to Leningrad, where she got married. She and I quickly became friends. She was very well brought up and friendly. And she was a very good mathematician. She would visit us often at home, in our room, she would stay with us in the summer, and ask what new things I had come up with. We would discuss

[14] *Candidate of Science* is the Russian equivalent of the Philosophy Doctor degree.

[15] *Doctor of Science* is the Russian equivalent of the Habilitation.

that, sometimes she would use my ideas (it was a pleasure to me), and sometimes shared her own. We became good friends for life. Olga Arsenievna and I at first also got along well. But then she began to, let's say, get a little jealous that the French school followed me, and not her. Lions and his school made extensive use of my monotonic differential equations, as well as my and L. A. Lyusternik's work in «Uspekhi Mat. Nauk» on small parameters: Lions went on to write an approximately 700-page book on this topic...

So, at my doctoral dissertation, my opponents were Sergei Lvovich Sobolev, Israel Moiseevich Gelfand, and Andrey Nikolaevich Tikhonov. In 1950, I wrote my dissertation by hand at my dacha. Then I was shaping it up, and by the end of the year it was ready. In early January the following year, carrying a bag with four copies of my doctoral dissertation, I went, naturally, to the Steklov Mathematical Institute—where else would I have gone in 1951? I came to the Academic Secretary... He said that the Academic Council is very busy and that I can leave one copy of the dissertation with him. And when the Council has some free time, he will give me a call. I left him my phone number and returned home with nothing. Saying nothing to anyone, I continued to actively participate in Ivan Georgievich's seminar and in other seminars. But one day in June 1951, Ivan Georgievich asked me how, in fact, things were going with my doctoral dissertation. I replied that for six months, since January, it had been lying at the Steklov Mathematical Institute. He said nothing to me, got in his car and drove over to Ivan Matveevich Vinogradov. As a result, on the same day, that same Academic Secretary called me and told me to urgently prepare all of the paperwork for my defense: my *personal characteristic*,[16] my autobiography, and so on. I'd have problems with my personal characteristic since the time when I was getting appointed a docent at MEI, where I had started teaching after my candidate dissertation defense. And I myself went to the secretary of the local Party Committee at MEI, V. A. Kirillin, for help. And he helped me. He treated me very well: after all, he saw how active I was at the institute. I tried to do the same thing at MEI that I. M. Gelfand did at the University: seminars, dissertations, and so on. I read my lectures enthusiastically. And once at a lecture, when covering the Newton–Leibniz formula, I addressed the audience in a burst of enthusiasm: "I would ask everyone to rise!" And the entire audience rose. Later students would often remind me of that incident.

Andrei Nikolaevich Tikhonov was very much worried because of my dissertation since it was filled to the brim with functional analysis and higher order differential equations. He even visited me at home so that I could help him figure it out a little bit. But his speech at my defense was very positive. I was told later how he had been appointed as an opponent at my defense: as Ivan Matveevich put it, it was "to nitpick". But Andrei Nikolaevich did not nitpick at all. On the contrary, he treated me very favorably. Because he knew me pretty well since I had given a number of talks at Sobolev's, Petrovsky's, and Tikhonov's seminars. So in 1951 I defended my doctoral dissertation. And in 1952, when Asya and I went on vacation, a letter came

[16] The character reference.

from the Higher Attestation Commission that my case had been approved. I did not even know that my work would reach them.

Shortly afterwards, Mstislav Vsevolodovich Keldysh became interested in my dissertation. The thing is that Keldysh constructed a theory of spectral decomposition for ordinary nonselfadjoint differential operators. And at the same time the second part of my dissertation was about a general form of boundary value problems for elliptic differential operators. And he wanted to construct the spectral decomposition for general boundary value problems for elliptic differential operators. Because of this, I even visited him at home a few times, where I would tell him in detail about my dissertation and in general about boundary value problems. Even prior to my dissertation defense, Mstislav Vsevolodovich had invited me to his Institute. He said that they once had been discussing how much time to allot for people to have time to finish their doctoral dissertation. He asked, “Marko Iosifovich, tell me, how much time were you given for your doctoral dissertation?” And I answered, “How much time was I given? Practically none! I continued to teach. And only on the day of the defense did I ask for someone to replace me at one lecture, so that I would have enough time to get to the Scientific Council in Steklov Mathematical Institute. So I was only given two hours off!” Mstislav Vsevolodovich was always good to me. Later, as you know, he became the leading astronautics theorist behind the preparation of Gagarin’s flight.

MEI and MGU

At Moscow Power Engineering Institute there was Naum Ilyich Akhiezer, a famous mathematician, considered in our country the second leading analyst after Sergei Natanovich Bernstein. Akhiezer’s results were used by Sergei Petrovich Novikov for his soliton theory. Naum Ilyich was a great mathematician. So, Akhiezer worked at MEI, right under Viktor Iosifovich Levin, and somehow he knew about me. And it was he who offered that I apply to their Chair. He only said that I would need to have someone well-known write me a recommendation letter. I approached Sergei Lvovich Sobolev, who would present my articles to «Doklady» of Academy of Sciences. He wrote a splendid review about me. And I went with this recommendation to M. G. Chilikin at the Rector’s office. Prior to this, Victor Iosifovich Levin, having learned my history in Lwów and Tbilisi, also wrote his letter to the Rectorate. (Levin learned about me from Naum Ilyich Akhiezer.) Chilikin liked me. And most of all he liked Sobolev’s recommendation: he had written some very strong words about me. After all, in my works I had applied many of Sergei Lvovich’s ideas and results. And I was hired to MEI. In Lwów, teaching at a technical university was considered the highest of honours. And I thought that this level should be maintained. I was immediately assigned to lectures. First I was assigned to teach analytic geometry. But I was not yet familiar with methods of teaching. However, there was a man at the Institute—Yulii Isaevich Grosberg, a docent and a war veteran. And he would tell me on the phone each time how to give an upcoming lecture. I had books, I understood everything, but did not know how to present it to the students in a comprehensible way. But gradually I became a good lecturer. Later, I already taught analysis and in parallel organized a seminar on differential equations for young instructors. This seminar very quickly became known to many, including Ivan Georgievich Petrovsky. And he began to send his finishing graduate students' dissertations (from his Chair of differential equations at Moscow State University) to MEI for review. For example, I reviewed Stanislav Nikolaevich Kruzhkov: he had a very strong candidate and doctoral dissertation. He was a strong mathematician. Apart from that, I attended all of Gelfand's and Petrovsky's seminars. From Sivtsev Vrazhek I would walk to the university by foot. So in some time everyone at MEI saw how actively I worked.

Lednyov became Head of the Chair in 1948–1949. And Levin was fired. He was a wonderful man and led the Chair very well. But with his history he could no longer be Head of the Chair: Levin had studied in England. In 1952 Lednyov gave me his article so that I could read and review it: he wanted to send it to be published in Odessa. I read it and realized that it was utter nonsense. I would not write the referee report, and he became very angry with me. He had already been a little bit out of his mind: at the doctorate dissertation defences, he opposed Sobolev and Petrovsky, saying that they were interfering with science. So, when Lednyov realized that I would not write the report, he started saying that our institute is a mess, because at the different departments—of which MEI had nine—different professors taught special courses. But it should be that one person teaches them all! And he ordered me to teach all of the special courses! I ended up having seventeen hours of lectures a week. Sometimes I would read six hours of lectures a day, from nine to three. At one point I even lost my voice. I calculated that I walked about forty kilometres a day, because I read my lectures very energetically and tended to pace back and forth. So Lednyov started treating me negatively. At the same time he was trying to create difficulties for Chilikin, saying that he had overly high standards. In the end, Lednyov went to the secretary of the District Party Committee and made a complaint on Chilikin. But the secretary knew Chilikin well. And so they decided to remove Lednyov. And soon they succeeded in doing so, and A. F. Leontiev came in his place. A very good mathematician and a wonderful person. It was during his time there that they made me a Professor. Asya and I would go to his birthday parties, and he and his wife would come to us. In short, our families were close.

At that time, at my courses, there could be a difference in age of ten years between listeners. I was assigned to read a single special course following the MGU Physics Department program—all of the subjects in one special course. I had an enormous number of lecture hours. I taught probability theory, calculus of variations—everything that is taught at the Physics Department.

From my first years of work at MEI, I immediately began to take graduate students. My first graduate students were Mikhail Leontievich Krasnov and Grigory Ivanovich Makarenko.

I am very happy with the way my life turned out. First of all, I am happy that I found myself in Moscow, where there were such scientific giants as Andrey Nikolaevich Kolmogorov, Ivan Georgievich Petrovsky, Lazar Aronovich Lyusternik, Pavel

Sergeevich Aleksandrov, Israel Moiseevich Gelfand, and others. I absorbed what was around me at the university, I basically breathed it. And the activity of our department was extraordinary: in the evenings one could not find a free auditorium to hold a seminar since they all were occupied.

In 1961 Israel Moiseevich Gelfand said to me: “Mark Iosifovich, it’s time for you to teach courses on differential equations at the university.” And I, still being a Professor at MEI, began to lead my special seminar at the Department of Mechanics and Mathematics at MGU. Even earlier, in 1956, I had taught a special course there, which was attended, in particular, by V. M. Alekseev, M. M. Lavrentiev, and N. S. Bakhvalov. Many people would attend. Auditorium 16–24 would be full. And I would enthusiastically teach my special course. Then all three of them had been taking the examination. And my special seminar, which, as I mentioned, I had begun to lead at our department in 1961, was soon being attended by G. E. Shilov’s graduate students. And Georgiy Evgenievich started trying his best that I would just transfer to work at MGU. In general, my special seminar quickly became very large. It was attended by many mathematicians. Even Olga Arsenievna attended it sometimes. And Gelfand’s wife would come to all of the seminar’s meetings. It would also be attended by Tatiana Dmitrievna Ventsel and Yuri Vladimirovich Egorov. And then Ivan Georgievich Petrovsky decided that the time had come to strengthen the faculty at his Chair of differential equations. This was in 1965. He began to consult a variety of people on how to do this, who to invite. In particular, he asked Israel Moiseevich Gelfand about it. He pointed out me. Ivan Georgievich said that it would be impossible since I was Lednyov’s friend. “Where did you get this from,” asked Israel Moiseevich. The answer was, “Olga Arsenievna told me.” To which Israel Moiseevich replied, “Ivan Georgievich, firstly, you cannot ask Olga Arsenievna, of all people, about Mark Iosifovich. And secondly, I don’t know any person more reliable than Vishik.” Ivan Georgievich Petrovsky called me that very evening and asked me to come to the Rectorate in relation to my transfer to MGU. And not long before that, in the spring of 1965, there had been a conference of all the technical university rectors. There, in particular, it had been said that for such a number of technical universities in the country, their scientific contribution was weak. The rectors attributed this to a lack of the scientific personnel. Then, Ivan Georgievich said: “What do you mean, weak contribution? For example, at MEI, there’s Mark Iosifovich Vishik, who alone accomplishes more than the entire department at the Steklov Mathematical Institute.” After that, I became something of a birthday boy at MEI. And it was this birthday boy that Ivan Georgievich invited to transfer to the university. Having transferred to MGU, I would still for some time continue teaching special courses at MEI. But most of my research and teaching load was already at MGU. My first student at MGU was Misha Shubin, who had transferred to me from Victor Pavlovich Palamodov in his third year of studies. After that, Andrei Fursikov and Sasha Demidov came to me. Later, among my students there were Sasha Komech, Sasha Shnirelman, and others...

Autumn 2011, just prior to Vishik's Monday seminar. When MI slowly, with difficulty, although, as always, with his wonderful smile, walked into Auditorium 13–06, I blurted out:

— *Mark Iosifovich, I am so glad to see you!*
— *Andrew, I am also very, very glad to see you!*

A remarkably lucid man...

Reference

[Dem08] V. Demidovich, Interview with M.I. Vishik, in *Мехматяне вспоминают (Mechmat Faculty Memoirs)* (MGU, Moscow, 2008), pp. 69–92

Remembering Wladek Lyantse

Mark Vishik

I met Wladek Lyantse at Gymnasium № 9 in the city of Lwów, where we studied for six years. (This was the equivalent of years 5 to 10 at a Russian school.)

Wladek was one of the most intelligent students in our class, both in terms of physics and mathematics and in terms of humanities disciplines. I found one of his discussions with our Polish teacher, Zosya, particularly memorable. There were worker strikes in Lwów at the time. Perhaps because of that, Zosya invited us to write an essay on the subject of what we would say to the workers to persuade them to end their strike. Wladek immediately said that he would not want to speak to the workers in such a fashion. This made our teacher, who was married to the deputy of the military commander of Lwów, quite angry. From that point on, she was negative towards Wladek, although he spoke Polish better than anyone else.

Wladek's parents would buy him popular books about different fields of natural science. Wladek surprised us with his knowledge of general problems related to the structure of the Universe, modern physics, and so on. I remember one time at a

We are grateful to Eduard Vladyslavovych Liantse (Edward Lyantse) for providing us with this text by Mark Iosifovich Vishik of his memories about Wladislaw Elievich Lyantse (1920–2007). Originally published in Russian in: *Mark Iosifovich Vishik* (editors: A. Comech, A.I. Komech, A.I. Nazarov). MCCME, Moscow, 2021. Translated by Maria Komech and Julia Ustyugova.

M. Vishik
(1921–2012), Moscow, Russia

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mathematics lesson, taught by Professor Freilich (previously the docent of Kraków University[1]), Wladek asked him why the book he was teaching from contained an equation where the sum of squares was equal to zero. It turned out that the book contained the Laplace equation. The professor explained to Wladek that it was the sum of second derivatives of the function $u(x)$.

Wladek and I became the closest of friends. In our free time we would walk around Lwów and discuss various issues. It was from Wladek that I first heard the word *science*. Under the influence of the books he had read, he told me about great scientists. I would often visit him at home, where we would continue our discussions, play chess, listen to music on the radio. Sometimes I would stay for dinner.

After graduating the gymnasium, Wladek and I were both accepted to physics and mathematics lyceums, which were unfortunately at two different gymnasiums. After graduating the lyceums in 1939, we both applied to the Mathematics Department at Lwów State University. Wladek and I attended lectures by professors who belonged to the famous Banach School.

When the war began, Wladek evacuated from Lwów by train, heading east, whereas I walked east with some of the other university students. We all wanted to fight the fascists, but none of us was admitted to the army. Wladek's parents and his entire family perished in the Holocaust.

After the war, I began to work in Moscow, and Wladek worked in Lwów. Wladek completed his candidate dissertation under the guidance of Professor Yaroslav Lopatinskii. Wladek wrote his doctoral dissertation on the spectral theory of differential operators, for which he consulted Professor Mark Naimark, whom he visited in Moscow. In Moscow, Wladek would stay with me, and we resumed our friendly outings and conversations.

In the eighties, we vacationed a few times in *Karpaty*[2] with our wives, Olga Alekseevna and Asya Moiseevna. Thanks to Wladek, I became familiar with the outskirts of Lwów. I should note that Olga Alekseevna, Wladek's wife, is a person of strong and gentle character. Asya and I have always been close friends of hers.

Unfortunately, Wladek and I never collaborated on any scientific work since we both worked in different areas of mathematics.

Sensing that he was near the end of his life, Wladek called me, but said nothing of his illness.

I am very fortunate that fate brought me together with a mathematician and a human being as exceptional as Wladek Lyantse.

[1] This is most likely a reference to Arnold Freilich, a graduate of the "Lwów Polytechnic" (Doctor, 1909), who was at one time the director of the Jewish gymnasium on Zygmuntowska Street, 17; a member of the «Leopolis Humanitarian Society» of the city of Lwów. —*ed.*

[2] The Carpathian Mountains —*ed.*

Part II
Science

Symposium in Honor of Professor Mark Vishik. Berlin, 2001

Free University of Berlin, December 17–20, 2001

LIST OF TALKS:

Andrei Afendikov: *Pulse solutions to the Navier–Stokes problem in cylindrical domains*

Mikhail Agranovich: *Spectral problems for second order strongly elliptic systems in smooth and nonsmooth domains*

Vladimir Chepyzhov: *Epsilon-entropy and dimension of attractors of evolution equations*

Klaus Ecker: *Longterm asymptotic behaviour of solutions for some nonlinear geometric equations*

Messoud Efendiev: *On the compactness of the stable set for rate independent processes*

Harald Engel: *Control of pattern formation in light-sensitive Belousov–Zhabotinskii media*

Andrei Fursikov: *Stabilization of solutions to the Navier–Stokes equations by feedback control defined on the boundary*

In this chapter we present some of the materials of the Mathematical Symposium in Honor of Professor Mark Vishik which took place on December 17–20, 2001 in Berlin. See dynamics.mi.fu-berlin.de/vishik-symposium. Reproduced from: *Mark Iosifovich Vishik* (editors: A. Comech, A.I. Komech, A.I. Nazarov). MCCME, Moscow, 2021.

A. Comech et al. (eds.), *Partial Differential Equations and Functional Analysis*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-33681-2_14

Herbert Gajewski: *On parabolic equations with nonlocal drift term*
Sergei Kuksin: *Spectral properties of solutions for nonlinear PDEs with small viscosity (analytical and numerical results)*
Alexander Mikhailov: *Controlling chemical turbulence by global delayed feedback*
Alain Miranville: *Recent results in the theory of exponential attractors*
Louis Nirenberg: *Estimates for elliptic systems from composite materials*
Giovanni Prouse: *Mathematical models associated to the motion of a rod*
Carlos Rocha: *Attractors for singular reaction-diffusion problems*
Arnd Scheel: *The impossible period-doubling of a spiral wave*
Eckehard Schöll: *Spatio-temporal pattern formation in globally coupled reaction-diffusion systems*
Alexander Shnirelman: *Inverse cascade solutions of the Euler equations*
Mikhail Shubin: *Criteria for discreteness of spectra for Schrödinger operators*
Jürgen Sprekels: *On nonlocal models for non-isothermal phase transitions*
Mikhail Vishik: *On the initial value problem for the Euler equations of an ideal incompressible fluid*
Leonid Volevich: *The Vishik–Lyusternik method in the theory of small singular perturbations of general elliptic problems*
Wolfgang Wendland: *On the degree of quasiruled Fredholm maps and nonlinear Riemann–Hilbert problems*

LIST OF CONTRIBUTORS: Gerhard Braun, Bernold Fiedler, Konrad Gröger, Mikhail Shubin, Roger Temam, Mark Vishik, Eberhard Zeidler

1 Louis Nirenberg

Courant Institute of Mathematical Sciences (New York)

A LETTER

Mark Vishik is a truly original and superb mathematician. He has made many fundamental contributions in the theory of partial differential equations (PDE) together with deep applications in fluid dynamics. He has been a world leader in these subjects for over a half century. All of us in the field have been strongly influenced by his work. He has published 250 papers and several books, and I can comment on only a small part of his work.

Vishik's early papers were on elliptic PDE: he introduced the concept of strong ellipticity, for which he obtained fundamental results. He has important work on such equations involving a small parameter. He also has seminal work on initial value problems under various boundary conditions. With L. A. Lyusternik he wrote a series of papers on a variety of problems including nonlinear problems. His papers on nonlinear elliptic and parabolic problems have had great influence. His papers with G. I. Eskin and with M. S. Agranovich are excellent. His work truly covers all aspects of PDE, linear and nonlinear, and also pseudo-differential operators; and all that is just early work. I could go on and on but let me turn to more recent things.

In the 70s Vishik published a series of important papers on differential equations involving infinitely many independent variables. Since then most of his work has been devoted to nonlinear problems—especially elliptic and parabolic ones. He treated an enormous variety of deep problems—truly astonishing. Especially striking are his papers connected with fluid dynamics and I would like to call special attention to the many papers on asymptotic behavior of solutions, in particular, attractors for PDE. These, some with A. V. Babin, are absolutely fundamental in the study of attractors; they are world famous. Their 1992 book is a principal reference. They obtained very strong estimates for the dimensions of the set of attractors for Navier–Stokes equations in fluid dynamics and for reaction-diffusion equations. Especially remarkable for the former is the estimate, from below, of the dimension. Their work on how attractors depend on various physical parameters is basic. They also studied long time behavior of solutions for nonlinear hyperbolic equations which have a Lyapunov function.

Mark Vishik continues to do absolutely first class mathematics. He is truly a world leader in PDE as well as applications. Any university honors itself in awarding him an honorary doctorate.

A personal remark

It is an enormous pleasure for me to participate in the wonderful honorary-doctorate celebration for Mark Vishik's 80th birthday. As we all know, Mark is a world master in the theory of attractors. However, for us, the main attractor is Mark himself.

I first learnt of Mark's work from his theory of strong elliptic systems, which opened new doors in partial differential equations (PDE); he is a giant in the field of PDE, but I won't talk about his research.

I have lovely memories of our first meeting in 1963: it was at a conference in Novosibirsk. We became friends immediately. I felt completely at home with him. Afterwards, every time I visited Moscow, Mark and Asya have welcomed me into their home. I am always touched by their warmth and hospitality, and feel as though I am part of their family.

Many years ago, at a party, a young woman claimed to be able to read palms. She read mine and told me that when I became old I would be rich. I laughed, of course, but now I think she was right: I am rich, not in money, but in friends. The same is true of Mark and Asya.

Mark, may you continue to thrive, do beautiful mathematics, and listen to music.

2 Roger Temam

Indiana University (Bloomington)

The name of Mark Vishik is one of the very first names that I heard when I started research in mathematics in 1964. One year before, in 1963, Mark published an article that had a very deep influence on the theory of nonlinear partial differential equations all along the 1960s, although this paper is nearly forgotten by now, and probably very few know about it. Later on I will describe in detail this part of his career that I witnessed during the preparation of my thesis.

Mark Vishik has had a very interesting and rewarding life, but also a very difficult one. He went through many difficult periods but his talent and his kindness attracted respect and sympathy for him, and, for each of the trial times in his life a good fairy came who saved his career and sometimes his life.

Mark Vishik was born on October 19, 1921 in Lvov. A first stroke of destiny hit him at the age of eight when his father passed away. His mother raised him with three other children with loving care and self-denying commitment. Mark retains warm reminiscences of his childhood despite the material difficulties.

From the gymnasium Mark remembers one of his mathematics teachers, Professor Freilich, who followed unconventional teaching patterns requiring the students to find the proofs of theorems and lemmas by themselves. This teaching method may not be appropriate for everyone, but it certainly provides a very stimulating education for a future researcher.

After gymnasium, Mark entered the physics and mathematics department of the Lvov University which was then home to the great pre-war mathematics Polish School: Banach, Schauder, Mazur, Orlicz, Steinhaus and others were teaching there. However we are in 1939 and Mark will not stay long at this University. Nevertheless he spends much time in lectures, seminars and in the library, and he stays long enough to decide that he will devote his life to mathematics.

Mark is the only one of his family who survived the Holocaust. During the first days of the German invasion of Poland[1] in World War II, Mark with some students left Lvov. He undertook an impressive trip, alone and hungry, escaping bombing many times. By foot he went to Vinnitsa, 300 kilometres away, then hidden in freight trains he traveled to Kiev, then Krasnodar, then Makhachkala some fifteen hundred kilometres away. In Makhachkala he entered the Makhachkala Pedagogical Institute from which he soon graduated. Several times he volunteered for the army but was turned down. Instead, after graduation, he was sent with a group of students to the Valley of Sun for harvesting and there he contracted malaria, reoccurrences of which still plague him to this day. He was released from the hospital very weak, but the front line was approaching again and, once more, he had to fly for his life; he was carried on a train which took him to Tbilisi.

This article appeared in: Discrete and Continuous Dynamical Systems. 2004. V. 10, № 1&2, i–vi. Reprinted with author's permission. —*ed.*

[1] Invasion of the USSR in June 1941 —*ed.*

A positive turn in his fate occurs in Tbilisi where Professors I. N. Vekua and N. I. Muskhelishvili got to know him and became very supportive of him. He was immediately accepted to the University of Tbilisi, granted State Scholarship and given housing. Mark retains a feeling of deep gratitude to the Georgian mathematicians who in fact saved his life.

After graduating in 1943, Mark became graduate student with Professor Vekua at the Mathematics Institute of the Georgian Academy of Sciences. In 1945, Muskhelishvili sends Mark to the Steklov Mathematics Institute in Moscow to continue his graduate studies. In Moscow at the age of 24, he was immediately exposed to the Moscow mathematicians who would deeply influence his research, in particular Sobolev, Petrovsky, Gelfand, Kolmogorov and others. His thesis adviser in Moscow was Lazar Aronovich Lyusternik. Soon after his arrival in 1947, Mark defended his Kandidat thesis. Then in 1951, after a very short period of four years, he defended his Doctorat Thesis, equivalent of the French or German habilitations, this at the very young age of 30.

From 1947 to 1965, Mark was successively Assistant, Assistant Professor, and then Full Professor in the Department of Higher Mathematics of the Moscow Power Engineering Institute. Then in 1965, Petrovsky invited him to join the Chair of differential equations at the Department of Mechanics Mathematics and Mechanics of Moscow State University where he worked till 1993; during that period he also held a research position at the Institute for Problems in Mechanics of the USSR Academy of Sciences.

Since 1993 he holds a position of principal researcher at the Institute for information transmission problems of the Russian Academy of Sciences and he is holding the half time position as Professor of Moscow State University. Now that traveling has become easier, Mark travels more frequently. We see him much more often in the US, in France, and in Germany where he obtained the Humboldt Award Professorship from 1997 to 2001 which made him, since then, a regular visitor of the Free University of Berlin and of the University of Stuttgart.

The impact of Mark on mathematics has been diverse, prolonged and very extensive. He has written more than 250 articles and several books. There is no way one could properly describe his work in details in a short article; I will just highlight some aspects of his work, in particular those most familiar to me.

However a technical description of his work would not be sufficient to describe his devotion to science and his deep influence. Beside his own research, Mark had and still has many students, many of whom became themselves well established mathematicians, many still around him in Moscow. For many years Mark Vishik has conducted and he still conducts a research seminar during which he proposed to his students and collaborators many open problems. His students tell of the working days with him at his home, during which they worked on these problems. His students recall also that, during the working days, Mark's wife, Asya Moiseevna, entertained them for delicious and enlightening dinners. Asya has been another of the good fairies who have looked after Mark, a very supportive life-long companion, who sailed with him through quiet and through rough waters. Together they had two sons who became mathematicians, Simeon from Temple University and Michael from the University of Texas at Austin who works in areas of PDEs at the same time close and very distinct from his father's.

On the occasion of Mark's eightieth birthday, we wish to Mark and Asya and their family many more years of happiness; and to Mark, beside good health, we wish him more students and many more years of fruitful work.

SOME ASPECTS OF MARK'S WORK

As indicated before, I will just highlight some aspects of Mark's very broad work. A more detailed description of the work of Professor Vishik can be found in the following references:

- Mathematics in the USSR during the forty years 1917–1957, *Gostekhizdat*, Moscow, 1959, v. II, 138–139.
- Mathematics in the USSR, 1958–1967. *Nauka*, Moscow, 1969, v. II, 247–249.
- Mark Iosifovich Vishik (on his sixtieth birthday), by M.S. Agranovich, I.M. Gelfand, Yu.A. Dubinskii, O.A. Oleinik, S.L. Sobolev, M.A. Shubin, Uspekhi Mat. Nauk, **37:4** (1982), 213–220, Russian Math. Surveys, **37:4** (1982), 174–184.
- Mark Iosifovich Vishik (on his seventy-fifth birthday), Uspekhi Mat. Nauk, **52:4** (1997), 225–232, Russian Math. Surveys, **52:4** (1977), 853–877.

1. Work in linear partial differential equations: the 1950s S. Sobolev introduced the spaces which bear his name shortly before World War II, and L. Schwartz discovered the theory of distributions shortly after World War II. It was clear that the theory of partial differential equations previously based on the utilization of Hölder spaces and spaces of continuously differentiable functions would have to be fully revisited and further developed using these new powerful tools.

Very young and very early, Mark was very lucky to be exposed to these new developments at the highest level, around Sobolev, Petrovsky and Gelfand. He fully benefited from his teachers and he imbedded himself in the modern theory of PDEs. Parallel developments occurred in a number of places; in particular in France and in Italy around L. Schwartz, J.-L. Lions and the Italian school (E. Magenes and G. Stampacchia and others), in Sweden around L. Gårding and L. Hörmander, in the US around L. Nirenberg, P. Lax, and others.

Particularly noticeable, among the important contributions of Mark, are the following:

- His Kandidat dissertation in 1947, «On the method of orthogonal projections for linear self-adjoint equations»;
- In 1950, a paper «On general boundary-value problems for elliptic equations», for which he obtained a Prize of the Moscow Mathematical Society;
- In 1951, his doctoral dissertation «On systems of elliptic differential equations and on general boundary-value problems».

During this period, he gave a general definition of strongly elliptic operators, described the general form of homogeneous boundary conditions—not necessarily local—for a second order elliptic differential operator for well-posedness (solving a problem set by Gelfand); he worked also on nonhomogeneous boundary value problems thus inspiring Lions and Magenes who started then the work which led to their three-volume book; he also started to work on linear time dependent problems.

Another work done during the period of 1957–1960 is the work with his thesis advisor Lazar Aronovich Lyusternik. Mark had many ideas and he did not need much help from his advisor for his theses. However, they eventually collaborated, and they became friends, a friendship which lasted until the end of Lyusternik's life.

2. Work on singular perturbations: the 1960s (1) M.I. Vishik, L.A. Lyusternik, Regular degeneration and boundary layer for linear differential equations with small parameter, Amer. Math. Soc. Transl. **20** (1962), 239–364.

This long article [ВЛ57] was and still is a reference article on singular perturbations for elliptic and parabolic problems. The other general work on this subject in the context of the modern theory of PDEs is a subsequent volume by J.-L. Lions which appeared in the Springer-Verlag Lecture Notes in Mathematics Series.

Considering the solution of an elliptic boundary value problem with a small parameter

$$L_\varepsilon u_\varepsilon = f,$$
$$L_\varepsilon = L_0 + \varepsilon L_1,$$

where $0 < \varepsilon \ll 1$ and where L_1 is of higher order than L_0, $L_1 \gg L_0$, the purpose is to study the behavior of u_ε as $\varepsilon \to 0$. A wealth of asymptotic expansion results were derived in this article. Typically

$$u_\varepsilon = w_0 + \sum_{i=1}^{m} \varepsilon^i w_i + \sum_{r=0}^{m} \varepsilon^r v_r + \varepsilon^{m+1} y_m,$$

where the w_i are the "interior" limits, the v_r are the boundary layer correctors (singular in the Sobolev spaces), and $\varepsilon^{m+1} y_m$ is the remainder.

3. Nonlinear elliptic equations monotone in their highest arguments: the 1960s (2) In the article [Виш63], M. I. Vishik started by himself the theory of monotone operators broadly studied during the 1960s and later on. The prototype problem is now known as the nonlinear Laplacian:

$$\begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u}{\partial x_i}\Big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

More generally, in the article quoted above, I. M. Vishik considered equations of the form

$$\begin{cases} -\sum_{|k|=m} D^k a_k(a, u, \ldots, D^m u) + \text{l.o.t.} = 0 \quad \text{in } \Omega, \\ \text{boundary conditions on } \partial\Omega. \end{cases} \tag{2}$$

(l.o.t.: lower order terms).

In (1), the associated nonlinear abstract operator A satisfies the monotony property

$$\langle Au - Av,\ u - v\rangle \geq 0, \quad \forall u,\ v.$$

More generally, in (2), Prof. Vishik considered operators A which are only monotone in their dominant part. This article of Mark is very technical, a tour de force. J.-L. Lions got aware of it, and detecting very early the potential novelty, he studied it and lectured about it in Italy and elsewhere in 1963 and 1964.

Parallel to this an elegant argument of monotony is proposed by G. I. Minty to study monotone integral equations [Min62]. Later on, it was found that such an argument was already used by R. I. Kachurovskii in 1960 and 1966 [Kac68, Kac66] and by M. M. Vainberg and R. I. Kachurovskii in 1959 [VK59].

F. Browder noticed that Minty's argument can be applied to equations like (1) and many others and he developed his results in [Bro63] (followed by many other articles in 1965, 1967, 1969—the latest is a review article).

Finally, J. Leray and J.-L. Lions in their only joint paper [LL65] fully recovered the results of M. I. Vishik, using the method of Minty and Browder.

Following these papers, a wealth of papers on monotone operators have appeared in the 1960s and subsequently. Let us also mention the related developments on

- the theory of nonlinear semigroups, and monotone and pseudo-monotone operators (H. Brezis, A. Pazy, and many others),
- the theory of variational inequalities (G. Fichera, J.-L. Lions and G. Stampacchia, and many others).

It is very likely that without the paper of Mark Vishik, and its dissemination by the lectures of J.-L. Lions in Italy and elsewhere, the development of this chapter of mathematics in the 1960s would have been very different.

4. Statistical theory of fluid mechanics: the 1970s In the 1970s, Prof. Vishik continues to work on nonlinear partial differential equations and he begins to work on the Navier–Stokes equations, and on the statistical theory of turbulence. This new direction of research now reflects the influence of Andrey Nikolaevich Kolmogorov. It was undertaken in collaboration with A. V. Fursikov, and it led to the reference book [ВФ80]. The topics studied in this series of articles and in this book include statistical solutions of the stochastically forced Navier–Stokes equations; Reynolds number functional analytic expansion of the solutions, connections with the problem of moments.

Other work done by M. Vishik in the 1970s includes work on degenerate elliptic problems and on pseudo-differential operators (work with Blekher in particular).

Myself in the 1970s, I worked in related but different areas: on the stochastically forced monotone equations [BT72], Navier–Stokes equations [BT73], and on statistical solutions of the Navier–Stokes equations [FT80, FT83]. These common areas of interest generated close interactions between Mark and me in the 1970s, which would amplify in the 1980s and later.

5. Attractors—Dynamical Systems: the 1980s In the 1980s, Mark undertook work on attractors and dynamical systems mostly in collaboration with Anatoli Babin. Ciprian Foiaş and myself worked on the same subject at the same time and this led to many fruitful and friendly interactions.

Prof. Vishik's work done in collaboration with Anatoli Babin led to a series of articles and then to the reference book: [БВ89] (in Russian) and [BV92] (in English). A landmark in this work is his article with A. V. Babin giving a lower bound on the dimensions of an attractor, [БВ83] (in Russian) and [BV83] (in English). In this article, studying the space periodic incompressible flow in an elongated rectangle of sides L and $L\varepsilon$, $\varepsilon > 0$ small, they obtained a lower bound of the dimension of the global attractor $\mathcal{A}$, bounding it by the dimension of the unstable manifold of a natural and simple stationary solution. They obtained a lower bound of the form

$$\frac{C}{\varepsilon} \leq \dim \mathcal{A}. \tag{3}$$

This lower bound is to be compared to the following upper bounds successively established

$$\dim \mathcal{A} \leq \frac{C'}{\varepsilon^4} \quad \text{(Babin and Vishik)},$$

$$\dim \mathcal{A} \leq \frac{C'}{\varepsilon^2} \quad \text{(Foiaş and Temam)}.$$

Finally, M. Ziane, using very involved arguments of flows in thin domains, obtained the following upper bound which matches (3) and shows that both bounds are relevant and optimal in some sense:

$$\dim \mathcal{A} \leq \frac{C'}{\varepsilon} \quad \text{(Ziane)}. \tag{4}$$

6. Recent and current work: the 1990s and 2000s Most recent work of Mark in the 1990s and in the 2000s includes an extensive work on nonautonomous dynamical systems with V. V. Chepyzhov, which led to a very recent book [CV02]. I would like also to mention a work on averaging in dynamical systems with Bernold

Fiedler, Quantitative homogenization of analytic semigroups and reaction diffusion equations with Diophantine spatial frequencies [FV01].

Of course other articles are on the way!

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3 Mark Vishik

Institute for Information Transmission Problems (*Moscow*)

THE SOURCES OF MY WORK

I am profoundly grateful for being awarded an Honorary Doctorate of the Freie Universität Berlin. I would like to speak about the sources of my mathematical work.

In 1945 academicians Muskhelishvili and Vekua helped me come to Moscow and become a graduate student of the Steklov Mathematical Institute of the Academy of Sciences. Lazar Aronovich Lyusternik was my advisor in Moscow. I was a participant of his seminar. Under his supervision I was free to do what I wanted. At Plessner's seminar, I gave a talk about the famous paper of Herman Weyl about the method of orthogonal projections. Unfortunately, my knowledge of English at that time was almost zero. Therefore I understood only the mathematical formulas and a couple of words in this paper. But the preparation of this talk was very useful for me. I found a generalization of Weyl's method for a general self-adjoint elliptic differential equation of order $2m$. Then I wrote my Ph. D. (candidate) dissertation based on this work. Lyusternik helped me to write the introduction. I. G. Petrovsky and S. L. Sobolev were the referees of my Ph. D. dissertation. I received my Ph. D. at the Steklov Institute.

For 33 years (from 1945 till 1978) I was a permanent participant of the well-known Gelfand seminar. Gelfand considered all areas of mathematics as connected with each other. All new ideas, new directions, new outstanding mathematical works were reported at this seminar. Gelfand invited the best specialists in the corresponding area and they gave talks at the seminar. But Gelfand often interrupted their presentations, asked questions, discussed, and posed some problems. He asked the lecturer to repeat some parts of the talk and to explain certain details once again. The same happened when one of the permanent participants of the seminar gave a talk on his own new work. A discussion between Gelfand and the speaker about the related problems was the main part of the talk. Gelfand always wanted to verify that everything was reasonable in the work; he posed problems connected with the talk. The seminar often became a creative discussion that was very useful for the participants. Gelfand often asked the participants about their opinion about the talk. Gelfand's seminar has set very high standards for the participants and the seminar extended their mathematical horizon very much. I. M. Gelfand taught the participants right and deep understanding of mathematics.

In 1946 Gelfand organized a small seminar for three participants: O. A. Ladyzhenskaya, O. A. Oleinik, and myself. As a result, Ladyzhenskaya started working on the problem of describing the domain of the Laplacian with homogeneous Dirichlet boundary conditions in $L_2(\Omega)$. She proved that the domain is $\mathcal{D}(A) = H^2 \cap H_0^1$. Oleinik worked together with I. G. Petrovsky at that time. I have started working on the problem of describing general boundary conditions for the second-order elliptic differential equation, i. e., when the corresponding operator with these boundary conditions is Fredholm or even an invertible operator.

From the functional analysis point of view, this problem is connected with the problem of the extension of operators in a Hilbert space. For example, you have the Laplacian operator defined on the minimal domain $\mathcal{D}_0(\Delta)$ of functions, which vanish with their first derivatives on the boundary. The maximal Laplacian is defined on all the functions from $H^2(\Omega)$ without boundary conditions. We look for all extensions of the minimal Laplacian operator which are invertible in $L_2(\Omega)$ or are Fredholm operators. I proved a general abstract theorem, where necessary and sufficient conditions for the existence of an invertible or a Fredholm extension were given. I found all such extensions for the minimal second-order elliptic differential operators and described their domains by homogeneous boundary conditions. This work was included in my habilitation doctoral thesis. For general differential operators of arbitrary order with constant coefficients, Lars Hörmander proved that the above-mentioned necessary and sufficient conditions for existence of an invertible extension of the corresponding minimal operator are fulfilled.

S. L. Sobolev had a great mathematical influence on me. In 1946 at Moscow State University he gave a course on the methods of functional analysis in partial differential equations. This course contained embedding theorems, their applications to the Cauchy problem for hyperbolic equations, their applications to boundary-value problems for polyharmonic equations and other problems. For the Cauchy problem for hyperbolic equations, he first constructed their solutions in H^s spaces. Then, by means of embedding theorems, he found conditions when the solution belongs to C^2. From here he obtained minimal conditions on the initial data when there exists a classical C^2-solution. I became an enthusiastic follower of Sobolev's ideas.

Some years later I studied the boundary-value problem for so-called strongly elliptic systems. The leading part of these systems has the form of a sum of a self-adjoint positive symmetric operator and a skew-symmetric operator. I have constructed a weak solution of the Dirichlet problem for strongly-elliptic systems. Louis Nirenberg proved that this weak solution is smooth up to the boundary of the domain provided that this boundary is sufficiently smooth.

In 1956 we wrote with S. L. Sobolev an article in Doklady of Academy of Sciences about the solutions of non-homogeneous boundary-value problems of elliptic equations belonging to some distributional class. So, for example, we considered the Neumann problem for the Laplace equation with a given measure on the boundary. The measure of a set on the boundary is equal to the integral of the normal derivative of the sought solution over this set on the boundary. The solution of this generalized Neumann problem was found by using the duality between this problem and the Dirichlet problem with corresponding smooth boundary value conditions. This paper has had far-going generalizations. Jaques Lions and Enrico Magenes wrote a book of three volumes entitled «Non-homogeneous Boundary Value Problems and Applications» [LM68a, LM68b, LM70], where, in particular, a theory of general non-homogeneous problems in various classes of distributions was developed in great detail.

In the forties I. G. Petrovsky organized a seminar on partial differential equations for graduate students and for students of the 4th and 5th years. He also had a larger seminar at the Chair of differential equations. I often gave talks at these seminars. Usually after the talk Petrovsky asked the speakers to formulate the main result once more. He asked whether all the supposed conditions were really necessary. He very much enjoyed applied mathematical talks. Petrovsky had a deep influence on me as a great mathematician and a great personality. In 1965 he invited me to be a Professor at his Chair of differential equations at Moscow State University.

In 1956 we began to work with Lazar Aronovich Lyusternik on the asymptotic behaviour of solutions of equations having a small parameter in the higher order terms. For example we studied the 4th order equations with left-hand side $\varepsilon\Delta^2 - \Delta$. As $\varepsilon \to 0+$ we have a plate equation degenerating into a membrane equation. The fourth order elliptic operator requires two boundary conditions, for example, the Dirichlet condition and the normal derivative on the boundary. The limiting membrane equation requires only the Dirichlet boundary condition. Therefore we lose one boundary condition for $\varepsilon = 0$. We proved that the solution of the 4th order equation can be asymptotically represented as a sum of a solution of the limiting Dirichlet membrane boundary-value problem and a boundary layer function. The boundary layer decreases exponentially along the normals to the boundary and has the form $\varepsilon e^{-\lambda n/\varepsilon}\psi(s)$, where n is the distance to the boundary along the normal vector. The boundary layer function satisfies a certain ordinary differential equation. The main idea of the construction was based on the hypothesis that the boundary layer function changes along the normal direction much faster than in any direction tangent to the boundary. To obtain an asymptotic decomposition of the solution for the equation with small parameter in the higher order terms we constructed two iteration processes. The first one corresponds to the behaviour of the solution inside the domain and the second one corresponds to its behaviour near the boundary. On each iteration step we tried to improve our approximation of the given boundary conditions.

We also studied asymptotic expansions of the eigenvalues and eigenfunctions of elliptic equations with small parameter in the highest derivatives.

For evolution equations with rapidly oscillating boundary values, we obtained a skin-effect first studied by Riemann. We also studied many other singularly perturbed problems.

The collaboration with such a great mathematician as Lyusternik has had a deep influence on me. For five years we have usually been working two days a week, from morning till night. We wrote 25 articles and 3 surveys in the journal Uspekhi (Surveys). During our collaboration I learnt the style of working of a great mathematician, the broadness of his interests. We worked with enthusiasm. Sometimes Lyusternik called me at 2 o'clock in the morning, when he found some useful remarks to our work. The methods developed with Lyusternik were used by many scientists in applied mathematics, mechanics, and physics.

The breaks in our working sessions were very interesting. Lyusternik was a cultured man. He knew many verses of Pushkin, Baratynskii, Tyutchev, Pasternak and other poets by heart. Lyusternik was also a poet himself. He wrote humoristic verses about some professors of mathematics. He was a person with a great sense of humor. Five years of collaboration with such a person have had a deep influence on me. After our collaboration with Lyusternik we and our families became close friends for all future time.

In the beginning of the sixties he proposed to write a book with me on boundary layer problems. Unfortunately, I could not do that because at that time I worked entirely on monotone elliptic operators. In the fifties I had tried to construct a solution of the Dirichlet problem for the nonlinear analogue of the Laplacian: $\sum \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^p = h$ (where p is an odd number for simplicity). In the beginning of the sixties I noticed that Galerkin approximations for this equation had at least one solution in a sufficiently large ball. This fact follows from a simple topological argument. By *a priori* estimates for the Galerkin approximations and the compactness of the approximating solutions I obtained the solution for the nonlinear analogue of the Laplacian. Then I constructed solutions for the general nonlinear strongly elliptic systems. Later these systems were called monotone elliptic. I also constructed solutions for general parabolic systems of equations with strongly elliptic right hand side. Many mathematicians worked in this area: F. Browder, H. Brezis, H. Gajewski, J. Leray, J.-L. Lions, Yu. A. Dubinskii, I. V. Skrypnik and many other mathematicians.

In the beginning of the fifties I worked at the Moscow Energy University. I helped to organize the Department of Mathematics and a scientific seminar on differential equations at this university. Twelve participants of this seminar received a Ph. D. degree and Dubinskii obtained a habilitation.

In the beginning of the sixties I. M. Gelfand proposed that I organize a seminar on differential equations and their applications at Moscow State University. Many young talented mathematicians were participants of this seminar. They worked on various problems in partial differential equations, on spectral theory, on the Fredholm index of elliptic operators and on many other problems. My students and graduate students, some students of Olga Oleinik and Georgiy Shilov, and many other mathematicians from different Moscow institutes participated in this seminar; there were about 50 participants. The seminar was very useful for me and, I believe, also for the participants. We reviewed and discussed new interesting articles of such outstanding mathematicians as de Giorgi, L. Hörmander, L. Nirenberg, P. Lax, J. Moser, J.-L. Lions, R. Temam, H. Brezis, L. Amerio, E. Magenes, and many other mathematicians. The participants gave talks about their new results. The seminar is now working permanently for over 40 years. I wrote many joint papers with participants of the seminar.

In the sixties we wrote some papers with M. S. Agranovich about general boundary value problems for parabolic equations of arbitrary order. First it was necessary to study the corresponding general elliptic operator with a large parameter in a sector. This part was very close to the earlier papers of Agmon and Nirenberg. I worked together with Gregory Eskin for some years. We studied problems for elliptic pseudodifferential equations in a bounded domain. Then G. Eskin wrote a book about these and other problems. The collaboration with Agranovich and Eskin was very useful for me. I learnt a lot from them.

In the beginning of the seventies we organized a small seminar to study statistical solutions of evolution equations. Eberhard Hopf was the first to give a statistical approach for the study of the Cauchy problem for the Navier–Stokes systems, in his famous paper. His approach was developed by C. Foiaş, A. Bensoussan, R. Temam, A. V. Fursikov, and myself and other mathematicians.

Let me briefly describe the statistical approach to the Cauchy problem for the 2D Navier–Stokes system.

Let a measure $\mu_0(du)$ be given at $t = 0$ on a function space, say, $L_2(\Omega)$. Here $\mu_0(du)$ is the probability of the event that the initial point $u(x)$ is in the domain du in $L_2(\Omega)$. The statistical solution corresponding to $\mu_0(du)$ is the family of measures $\mu(t, du)$, $t > 0$, which is equal to the evolution of $\mu_0(du)$ along the trajectories. The characteristic functional $\chi(t, v)$ of this measure $\mu(t, du)$ is equal to the Fourier transform of the measure $\mu(t, du)$. The characteristic functional $\chi(t, v)$ satisfies the famous Hopf evolution equation with variational derivatives in v. Andrei Fursikov and I proved that if the initial measure $\mu_0(du)$ is supported in a sufficiently small ball, then the Cauchy problem for the Hopf equation possesses a solution $\chi(t, v)$, globally in time, which is functionally analytic with respect to v. From this theorem we deduced that the Friedman–Keller infinite chain of equations for the moments of the statistical solution $\mu(t, du)$ possesses a global solution with respect to t. In the general case, when the initial measure $\mu_0(du)$ has an arbitrary support, using another method we proved with Fursikov the existence theorem of the Cauchy problem for the Friedman–Keller infinite chain of equations corresponding to the 3D Navier–Stokes equations. Fursikov and I thus constructed a statistical solution for the Cauchy problem for the 3D Navier–Stokes system.

The ideas of Andrei Nikolaevich Kolmogorov on homogeneous turbulence had a deep influence on our later work on statistical solutions. Fursikov and I proved the existence of a homogeneous statistical solution for the Navier–Stokes system which corresponds to a given initial homogeneous measure $\mu_0(du)$. The homogeneity of the measure is understood with respect to the space variables x. Our work stimulated Kolmogorov to give his last talk at the Moscow Mathematical Society on the turbulence problems which are to be solved. Some of these problems were solved in our book with Fursikov «Mathematical Problems of Statistical Hydromechanics» [BΦ80]. In the Appendix to this book written by Alexander Komech and myself we considered stochastic problems for the Navier–Stokes system and solved some of Kolmogorov's problems. But many of them still remain open.

From the beginning of the eighties, Anatoli Babin and I began to study the global attractors of differential equations and systems of mathematical physics. The theory of attractors for ordinary differential equations and some functional equations had been developed to some extent. The attractors for partial differential equations were systematically studied from the eighties on. Important contributions to this area were made by R. Temam, C. Foiaş, P. Constantin, J. Hale, G. Sell, A. Haraux, and others. Naturally, our works with Anatoli Babin were influenced by the investigations of these and other mathematicians. We studied and constructed the global attractors for the 2D Navier–Stokes system, the dissipative wave equation, some classes of reaction-diffusion systems and some other equations. When the corresponding evolution equation possesses a global Lyapunov functional we investigated the structure of the attractor and introduced the notion of a regular attractor. Under some additional conditions we proved the exponential attraction property of the global attractor. We also studied the Hausdorff dimension of the attractors. For some examples of evolution equations we gave a lower bound on the dimension of the global attractor. Our work in this area was summarized in our book with Babin, «Attractors of Evolution Equations» [БВ89].

Since 1993 I am working as a leading scientific researcher at the Institute of Information Transmission Problems, Russian Academy of Sciences. There I have nice conditions for scientific work. In particular, V. Chepyzhov and I have recently written the monograph «Attractors for Equations of Mathematical Physics» [CV02], where we exposed our works of the last ten years. I am still a Professor at Moscow State University.

With Vladimir Chepyzhov we study the global attractors for nonautonomous evolution equations with some coefficients and forcing terms depending on t. We constructed and studied global and trajectory attractors for the 3D and 2D Navier–Stokes systems with forcing term depending on t, for some classes of nonautonomous reaction-diffusion systems, for the nonautonomous Ginzburg–Landau equation, and for the nonautonomous dissipative wave equation. The Hausdorff and fractal dimension of the global attractor for nonautonomous equations is often infinite. Chepyzhov and I studied the Kolmogorov ε-entropy of global attractors for these equations, which is always finite for any $\varepsilon > 0$. We found an upper estimate for the Kolmogorov ε-entropy of the global attractor for some classes of nonautonomous equations and systems.

Recently Chepyzhov and I constructed the global attractor for the 3D Navier–Stokes system which has properties analogous to the global attractor for the 2D Navier–Stokes system. We also studied perturbation problems for global attractors, including perturbations by terms which oscillate rapidly in x and t.

Summarizing, I can say the following. My great teachers from the Moscow State University are one of the main sources of my mathematical work. With some of them I collaborated and published joint articles. I can recall the words by Isaac Newton, "In our work we stand on the shoulders of giants".

My seminar and its participants have a great influence on me. With many of the participants I collaborated personally. This seminar is an important source of my mathematical work. I am very satisfied that I had many talented graduate students. Many of them have grown into outstanding mathematicians. I learnt very much from them. They are also one of the sources of my work.

I interacted with many mathematicians of other countries. I interacted with many outstanding mathematician from France, USA, Germany, Italy, Sweden and other countries. The interaction with these mathematicians is an important source of my mathematical work.

Recently as Alexander von Humboldt Awardee I had a possibility to work in Germany. I collaborated with mathematicians of the Freie Universität Berlin, mostly with Bernold Fiedler and his colleagues. With Fiedler we wrote several joint papers. I also collaborated with mathematicians of the University of Stuttgart, mostly with Wolfgang Wendland and his mathematical group. I was in Leipzig many times, as a Visiting Professor of the University of Leipzig. Recently I visited the Max-Planck-Institut für Naturwissenschaften in Leipzig, where I collaborated with many young talented mathematicians.

Many thanks to all these people who stimulated my scientific work.

All my conscious life I am fascinated with mathematics, I like to study it. Almost all the time I think about mathematical problems, about what would be useful to do for the development of mathematics. In my life nothing else attracts me as much as this area. Such attractiveness of mathematics for me is undoubtedly one of the important source of my mathematical works.

At home my wife Asya organizes nice conditions for my mathematical work. Many thanks to her.

I am grateful to the speakers and the participants of the Symposium on Partial Differential Equations in my honor. I am grateful to Bernold Fiedler who organized the symposium.

I am profoundly grateful for the Honorary Doctorate of the Freie Universität Berlin.

Many thanks to the participants of this meeting.

Thank you very much.

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International Conference «Partial Differential Equations and Applications» in Honour of Mark Vishik on the Occasion of His 90th Birthday. Moscow, 2012

Institute for Information Transmission Problems, Moscow, June 4–7, 2012

LIST OF TALKS:

M. Agranovich: *Remarks on strongly elliptic systems in Lipschitz domains*
S. Anulova: *The parabolic Harnack inequality for non-degenerate diffusions with jumps*
A. Babin: *Relativistic point dynamics and Einstein's formula as a property of localized solutions of a nonlinear Klein–Gordon equation*
C. Bardos: *On the propagation of Monokinetic Measures with Rough Momentum Profile*
S. Bezrodnykh, V. Vlasov: *An asymptotic of a certain Riemann–Hilbert problem under singular deformation of a domain*
N. Chalkina: *Inertial manifolds for strongly damped wave equations*
V. Chepyzhov: *Trajectory attractors for equations of mathematical physics*
A. Demidov: *Vishik–Lyusternik's method and the inverse problem for plasma equilibrium in a tokamak*

Here we present the list of talks at the International conference in Honour of Mark Vishik on the occasion of his 90th birthday which took place during June 4–7, 2012 in Moscow. See http://www.komech.github.io/events/vishik.

A. Comech et al. (eds.), *Partial Differential Equations and Functional Analysis*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-33681-2_15

S. Dobrokhotov: *Pseudodifferential operator, adiabatic approximation and averaging of linear operators*
S. Dostoglou: *Statistical Hydrodynamics and Reynolds averaging*
A. Dynin: *Pseudovariational operators and Yang–Mills Millennium problem*
G. Eskin: *Acoustic and optical black holes*
M. Freidlin: *Perturbation theory for systems with multiple stationary regimes*
L. Friedlander: *Generic properties of eigenvalues of a family of operators*
A. Fursikov: *Normal parabolic equation corresponding to 3D Navier–Stokes system*
A. Grigoryan: *Negative eigenvalues of two-dimensional Schrödinger operators*
N. Gusev: *Incompressible limit of the linearized Navier–Stokes equations*
A. Haraux: *On a compactness problem*
Yu. Ilyashenko: *Bony and thick attractors*
A. Ilyin: *Sharp two-term Sobolev inequality and applications to the Lieb–Thirring estimates*
V. Imaykin: *Symplectic projection methods of deriving longtime asymptotics for nonlinear PDEs*
V. Ipatova: *On the uniform attractors of finite-difference schemes*
A. Kapustyan: *Structure and regularity of the global attractor of reaction-diffusion equation with non-smooth nonlinear term*
A.A. Komech: *Weak attractor for the Klein–Gordon equation with a nonlinear oscillator in discrete space-time*
A.I. Komech: *On global attractors of nonlinear hyperbolic PDEs*
E. Kopylova: *Dispersive estimates for magnetic Klein–Gordon equation*
V. Kozyakin: *Disprove of the commonly recognized belief that the foreign exchange currency market is self-stabilizing*
A. Krasnosel'skii: *The structure of the solution sets for generic operator equations*
S. Kuksin: *Around the Cauchy–Kowalevski theorem*
J.-P. Lohéac: *Critical manifold in the space of contours in Stokes–Leibenson problem for Hele–Shaw flow*
A. Lyapin: *The trajectory attractor of the nonlinear hyperbolic equation, contain a small parameter by the second derivative with respect to time*
V. Malyshev: *New phenomena in large systems of ODE and classical models of DC*
A. Miranville: *A Cahn–Hilliard model with dynamic boundary conditions*
S. Molchanov: *The structure of the population inside the propagating front (the qualitative analysis of FKPP equation)*
N. Nadirashvili: *Geometry of stream lines of ideal fluid*
L. Nirenberg: *On singular solutions of fully nonlinear elliptic equations*
A. Ovseevich: *Structure of the minimum-time damping of a physical pendulum*
V. Palamodov: *A uniform reconstruction formula in integral geometry*
B. Paneah: *On the general theory of multi-dimensional linear functional operators with applications in Analysis*
V. Pata: *A uniform Grönwall-type lemma with parameter and applications to nonlinear wave equations*
A. Piatnitski: *Ground state asymptotics for a singularly perturbed second order elliptic operator with oscillating coefficients*

S. Pokhozhaev: *Critical nonlinearities in partial differential equations*
O. Pyrkova: *Longitudinal correlation functions and the intermittency*
E. Radkevich: *On global solutions to the Cauchy problem for discrete kinetic equations*
N. Ratanov: *Branching random motions, nonlinear hyperbolic systems and traveling waves*
I. Rudakov: *Periodic solutions of some quasilinear evolutionary equations*
Yu. Sachkov: *Neurogeometry of vision and sub-Riemannian geometry*
V. Sakbaev: *On the blow up phenomena in differential equations and dynamical systems*
G. Sell: *On numerical methods and the study of the dynamics inside the attractor*
A. Shafarevich: *Asymptotic solutions of the Navier–Stokes equations and scenario of turbulence development*
A. Shirikyan: *Control and mixing for 2D Navier–Stokes equations with space-time localised force*
A. Shnirelman: *On the 2-point problem for the Lagrange–Euler equation*
E. Sitnikova: *Eigenfunction of the Laplace operator in a tetrahedron*
A. Skubachevskii: *Classical solutions of the Vlasov–Poisson equations in a half-space*
V. Solonnikov: *On free boundary problems of magnetohydrodynamics*
T. Suslina: *Homogenization of the elliptic Dirichlet problem: operator error estimates*
N. Tarkhanov: *Algebra of boundary value problems with small parameter*
R. Temam: *Pattern formation: The oscillon equation*
E. Titi: *Global well-posedness of an inviscid three-dimensional pseudo-Hasegawa–Mima Model*
B. Vainberg: *Weyl asymptotics for interior transmission eigenvalues*
N. Vvedenskaya, Y. Suhov: *Example of equations with nonlinearity of type* $\min[u, v]$
W. Wendland: *On the Gauss problem with Riesz potential*
F. Yashima: *Equation of coagulation process of falling drops*
V. Zakharov: *Is free surface deep water hydrodynamics an integrable system?*
S. Zelik: *Infinite energy solutions for damped Navier–Stokes equations in* $\mathbb{R}^2$

«The Scottish Book», Problem 192

Andrew Comech

The first academic student conference of the Ivan Franko Lwów State University took place during 17–23 April 1941. The physics and mathematics workshop was chaired by S. Banach, with E. Szpilrajn being his assistant. The talk given by M. Vishik, a second-year student, titled «On Cartesian product of weakly separable spaces» (21 April 1941) was awarded the fourth place. The speaker received a monetary prize of 100 rubles [Mau91].

The topic of M. Vishik's talk had to do with Problem № 192, the penultimate problem in the famous «Scottish Book» [MP17]. «Księga Szkocka» was a thick notebook bought by Lucja, Stefan Banach's wife, in 1935. The notebook was kept in the Scottish Cafe in Lwów, and mathematicians from the Banach circle would use it to write down interesting problems. Problem № 192, provided below, mentions Mark Vishik and his friend Wladek Lyantse. E. Szpilrajn later referred to the result by Vishik and Lyantse in his '1942 article published in [SM45].

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A. Comech (✉)
Department of Mathematics, Texas A&M University, College Station, TX, USA
e-mail: comech@tamu.edu

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Trends in Mathematics, https://doi.org/10.1007/978-3-031-33681-2_16

192. Definitions. 1. A topological space T has property (S) (of Suslin) if every family of disjoint sets, open in T, is at most countable. 2. A space T has property (K) (of Knaster) if every noncountable family of sets, open in T, contains a noncountable subfamily of sets which have elements common to each other.

Remarks. 1. One sees at once that the condition (K) implies (S) and, in the domain of metric spaces, each is equivalent to separability. 2. B. Knaster proved in April 1941 that, in the domain of continuous, ordered sets, property (K) is equivalent to separability. The problem of Suslin is therefore equivalent to the question whether, for ordered continuous sets, property (S) implies property (K).

Problem of B. Knaster and E. Szpilrajn. Does there exist a topological space (in the sense of Hausdorff, or, in a weaker sense, e.g., spaces of Kolmogoroff) with property (S) and not satisfying property (K)?

Remark 3. According to remark 2, a negative answer would give a solution of the problem of Suslin.

Problem of E. Szpilrajn. Is property (S) an invariant of the operation of Cartesian product of two factors?

Remarks. 4. One can show that if this is so, then this property is also an invariant of the Cartesian product of any number of (even noncountably many) factors. 5. E. Szpilrajn proved in May 1941 that property (K) is an invariant of the Cartesian product for any number of factors and B. Lance and M. Wiszik verified that if one space possesses property (S), and another space has property (K), then their Cartesian product also has property (S).

Lwów, may 1941.

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192. Definicje. 1. Przestrzeń topologiczna T ma własność (S) (Suslina), gdy każda rodzina zbiorów rozłącznych, otwartych w T, jest najwyżej przeliczalna. 2. Przestrzeń T ma własność (K) (Knastera), gdy każda rodzina nieprzeliczalna zbiorów otwartych w T zawiera podrodzinę nieprzeliczalną zbiorów mających punkty wspólne każdy z każdym.

Uwagi. 1. Widać odrazu, że warunek (K) pociąga za sobą (S) i, że, w zakresie przestrzeni metrycznych, każdy z nich jest równoważny ośrodkowości. 2. B. Knaster udowodnił w kwietniu 1941, że, w zakresie zbiorów uporządkowanych ciągłych, własność (K) jest równoważna ośrodkowości. Zagadnienie Suslina jest więc równoważne pytaniu, czy dla zbiorów uporządkowanych ciągłych własność (S) pociąga za sobą własność (K).

Zagadnienie B. Knastera i E. Szpilrajna. Czy istnieje przestrzeń topologiczna (w sensie Hausdorffa, ew. choćby w sensie słabszym, np. przestrzeń Kołmogorowa) o własności (S), a nie mająca własności (K)?

Uwaga. 3. W myśl uwagi 2 odpowiedź negatywna dałaby rozwiązanie zagadnienia Suslina.

Zagadnienie E. Szpilrajna. Czy własność (S) jest niezmiennikiem mnożenia kartezjańskiego dwóch czynników?

Uwagi. 4. Można wykazać, że jeżeli tak, to jest ona również niezmiennikiem mnożenia kartezjańskiego ilukolwiek (nawet nieprzeliczalnie wielu) czynników. 5. E. Szpilrajn udowodnił w maju 1941, że własność (K) jest niezmiennikiem mnożenia kartezjańskiego ilukolwiek czynników, a B. Lance i M. Wiszik stwierdzili, że jeżeli jedna przestrzeń ma własność (S), a druga – własność (K), to ich iloczyn kartezjański ma własność (S).

Lwów, maj 1941.

Problem 192 from the «Scottish Book». May 1941

General Elliptic Boundary Value Problems in Bounded Domains

Mark Malamud

The work of Mark Vishik [14],[1] which has now become classical, consists of two parts. Its first part has an abstract character and is devoted to extensions of a dual pair of operators.

1 Extensions of Dual Pairs

We recall necessary concepts.

Definition 1.1 *1.* Two closed densely defined linear operators A and $A^\top$ in a Hilbert space $\mathfrak{H}$ form a *dual pair* if

$$(Af, g) = (f, A^\top g), \qquad f \in \operatorname{dom} A, \quad g \in \operatorname{dom}(A^\top). \tag{1.1}$$

2. The operator $\widetilde{A}$ (not necessarily closed) is called *proper extension of a dual pair* $\{A, A^\top\}$ and denoted $\widetilde{A} \in \operatorname{Ext}\{A, A^\top\}$ if $A \subsetneqq \widetilde{A} \subsetneqq \left(A^\top\right)^*$.

[1] Numerical citations refer to Mark Vishik's bibliography in the end of the book. —*ed.*

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M. Malamud (✉)
Moscow, Russia

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Definition 1.2 ([14]) The extension $\widetilde{A} \in \mathrm{Ext}\{A, A^\top\}$ is called *solvable* if $0 \in \rho(\widetilde{A})$, that is, if the operator $\widetilde{A}$ has a bounded inverse in $\mathfrak{H}$, and *completely solvable* if it is solvable and the operator $(\widetilde{A})^{-1}$ is compact in $\mathfrak{H}$.

Theorem 1.3 ([14]) *Let the operators A and $A^\top$ have bounded inverses. Then*

1) *there is a solvable extension $A_0 \in \mathrm{Ext}\{A, A^\top\}$;*
2) *if, moreover, the operators*

$$A^{-1}\colon\ \mathrm{ran}(A) \to \mathfrak{H} \quad \text{and} \quad (A^\top)^{-1}\colon\ \mathrm{ran}(A^\top) \to \mathfrak{H}$$

are compact, then the extension A_0 can be chosen completely solvable.

M. Vishik described all solvable and completely solvable extensions $\widetilde{A} \in \mathrm{Ext}\{A, A^\top\}$, basing on the following formula for the direct decomposition of the domain $\mathrm{dom}(A^\top)^*$:

$$\mathrm{dom}(A^\top)^* = \mathrm{dom}\, A \dotplus A_0^{-1}(\ker A^*) \dotplus \ker(A^\top)^* = \mathrm{dom}\, A_0 \dotplus \ker(A^\top)^*, \tag{1.2}$$

and an analogous formula for $\mathrm{dom}(A^*)$. The formula (1.2) was a substitute for Neumann's first formula and was new even in the case of a symmetric operator $A = A^\top$.

Let us present a result by M. Vishik, assuming for simplicity that

$$\ker \widetilde{A} = \{0\}.$$

Theorem 1.4 ([1])

1. *Let the conditions of Theorem 1.3 be satisfied, and assume that $A_0 \in (\mathrm{Ext}\{A, A^\top\})$ is a fixed solvable extension of a dual pair $\{A, A^\top\}$. Each proper extension*

$$\widetilde{A} \in (\mathrm{Ext}\{A, A^\top\})$$

is a restriction of the operator $(A^\top)^*$ onto the lineal (a subspace which is not necessarily closed)

$$\mathrm{dom}(\widetilde{A}) = \mathrm{dom}\, A \dotplus (A_0^{-1} + C)\,\mathrm{dom}\, C \dotplus U_C^\perp, \quad A_0 \in (\mathrm{Ext}\{A, A^\top\}), \tag{1.3}$$

where $U := \ker A^*$, $V := \ker (A^\top)^*$, and C is a linear operator

$$C\colon\ \mathrm{dom}\, C \subset V \to U, \qquad U_C := \overline{\mathrm{dom}\, C^*}, \qquad U_C^\perp = U \ominus \mathrm{dom}\, C^*.$$

Moreover, the adjoint operator $\tilde{A}^*$ is a restriction of A^* to the domain

$$\mathrm{dom}\,(\tilde{A}^*) = \mathrm{dom}\, A^\top \dotplus (A_0^{-1*} + C^*)\,\mathrm{dom}\, C^* \dotplus V_C^\perp, \qquad A_0 \in (\mathrm{Ext}\{A, A^\top\}),$$

$$V_C^\perp = V \ominus \mathrm{dom}\, C = V \ominus V_C, \qquad V_C := \overline{\mathrm{dom}\, C}.$$

2. *Under the above assumptions the extension* $\widetilde{A}$ *is solvable* (*completely solvable*) *if and only if* $U_C^{\perp} = 0$, $V_C = V$, *and the operator* C *is bounded* (*compact*). *In the latter case it is assumed that* A^{-1}, $(A^{\top})^{-1}$, *and* A_0^{-1} *are compact.*

2 Elliptic Operators

Let Ω be a bounded region in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, let $H^s(\Omega) = W^{s,2}(\Omega)$ be the Sobolev space with the norm $\|\cdot\|_s$, $s \in \mathbb{R}$, and let $\mathcal{A}$ be a symmetric differential operator of the second order in $L^2(\Omega)$,

$$\mathcal{A} := -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k} + q(x),$$

$$a_{jk} = \bar{a}_{jk} = a_{kj} \in C^1(\overline{\Omega}), \quad q = \bar{q} \in C(\overline{\Omega}). \tag{2.1}$$

We assume that the matrix $\big(a_{jk}(x)\big)$ is positive-definite for $x \in \overline{\Omega}$. Since the coefficients a_{jk} are real, the operator $\mathcal{A}$ is properly elliptic.

We denote by $A_{\min}$ and $A_{\max}$ the minimal and maximal operators associated to $\mathcal{A}$ in $L^2(\Omega)$. Here $A := A_{\min}$ is defined as the closure of the operator $\mathcal{A} \upharpoonright C_0^\infty(\Omega)$ (here and below $\upharpoonright$ denotes the restriction), while $A_{\max}$ is generated by the operator $\mathcal{A}$ on the domain

$$\operatorname{dom} A_{\max} = \{u \in L^2(\Omega) \colon \mathcal{A}u \in L^2(\Omega)\}, \tag{2.2}$$

where $\mathcal{A}u$ is understood in the sense of distributions. Definition (2.2) is equivalent to the equality $A_{\max} = (A_{\min})^*$. It is known that $H^2(\Omega)$ is dense in the lineal $\operatorname{dom} A_{\max}$ equipped with the graph norm, that is, $A_{\max}$ is the closure of the operator $\mathcal{A} \upharpoonright C^\infty(\overline{\Omega})$.

For the minimal operator $A_{\min}$ one has a two-sided *a priori* estimate

$$C_1\|u\|_2 \le \|A_{\min}u\|_0 + \|u\|_0 \le C_2\|u\|_2, \quad u \in H_0^2(\Omega), \tag{2.3}$$

which is equivalent (both algebraically and topologically) to the relation

$$\operatorname{dom}(A_{\min}) = H_0^2(\Omega).$$

The estimate like (2.3) for the maximal operator $A_{\max}$ is not possible: in the domain $\operatorname{dom}(A_{\max})$ there are only weaker inclusions available:

$$H^2(\Omega) \subset \operatorname{dom}(A_{\max}) \subset H^2_{\mathrm{loc}}(\Omega), \tag{2.4}$$

with the second inclusion corresponding to the theorem on the inner regularity (we point out that $\operatorname{dom}(A_{\max}) \neq H^2(\Omega)$). So, for the fundamental solution to the Laplace operator, $f_{x_0}(x) := |x - x_0|^{-1} \in L^2_{\text{loc}}(\mathbb{R}^3)$, corresponding to the point $x_0 \in \partial\Omega$, we have $f_{x_0}(\cdot) \in \operatorname{dom} \Delta_{\max} \setminus H^2(\Omega)$. Moreover, $\Delta f_{x_0} = 0$ in Ω. Further, let $\frac{\partial}{\partial \nu}$ be the conormal derivative:

$$\frac{\partial}{\partial \nu} = \sum_{j,k=1}^{n} a_{jk}(x) \cos(n, x_j) \frac{\partial}{\partial x_k}. \tag{2.5}$$

In the case of the Laplace operator we have $a_{jk}(x) = \delta_{jk}$ and $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial n}$, where n is the exterior normal to the boundary $\partial\Omega$. In these notations the classical Green formula takes the form

$$(A_{\max} f, g) - (f, A_{\max} g) = \int_{\partial\Omega} \Big(\frac{\partial f}{\partial \nu} \overline{g} - f \overline{\frac{\partial g}{\partial \nu}} \Big)\, d\sigma, \quad f, g \in C^2(\overline{\Omega}). \tag{2.6}$$

Further, we set

$$\begin{aligned} G_0 u &:= \gamma_0 u := u \upharpoonright \partial\Omega, \\ G_1 u &:= \gamma_0 \Big(\frac{\partial u}{\partial \nu}\Big) = \Big(\frac{\partial u}{\partial \nu}\Big) \upharpoonright \partial\Omega, \quad u \in \operatorname{dom}(A_{\max}), \end{aligned} \tag{2.7}$$

and note that according to the results of Lions–Magenes, the trace mappings

$$G_0,\ G_1 \colon C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$$

are extended onto $\operatorname{dom}(A_{\max})$:

$$G_0 \colon \operatorname{dom}(A_{\max}) \to H^{-1/2}(\partial\Omega), \quad G_1 \colon \operatorname{dom}(A_{\max}) \to H^{-3/2}(\partial\Omega), \tag{2.8}$$

with both mappings being surjective (see [LM72]). At the same time, these mappings do not allow to extend the Green formula (2.6) onto $\operatorname{dom}(A_{\max})$. We now present the construction by M. Vishik [14] which enables such an extension.

Let zero be a point of a regular type of the operator A (i.e., there exists a positive constant $c > 0$ such that $\|Af\| \geq c\|f\|$ for all $f \in \operatorname{dom}(A)$), $A_0 \in \operatorname{Ext}\{A, A\} =: \operatorname{Ext}_A$, and let $0 \in \rho(A_0)$. According to the relation (1.2), there is a direct decomposition $\operatorname{dom} A^* = \operatorname{dom} A_0 \dot{+} \ker A^*$.

Let $A_0 := A_{\gamma_0} := A^* \upharpoonright \ker \gamma_0$ be the Dirichlet realization. According to the regularity theorem,

$$\operatorname{dom} A_0 = H^{2,0}(\Omega) := H^2(\Omega) \cap H^1_0(\Omega) = \{f \in H^2(\Omega) \colon \gamma_0 f = 0\} \\ \text{and} \quad A_0 = A_0^*. \tag{2.9}$$

In this case, under the condition $0 \in \rho(A_0)$, the decomposition (1.2) takes the form

$$\operatorname{dom} A_{\max} = \operatorname{dom} A^* = \operatorname{dom} A_0 \dotplus \ker A^* = H^{2,0}(\Omega) \dotplus \ker A^*. \tag{2.10}$$

We point out that the condition $0 \in \rho(A_0)$ is automatically satisfied if the operator A is positive-definite: $(Af, f) \geq \varepsilon\|f\| \geq 0$, $f \in \operatorname{dom} A$, since in this case $A_0 = A_{\gamma_0} = A_F (\in \operatorname{Ext}_A)$ is its Friedrichs extension.

From the inclusion $G_1\big(H^{2,0}(\Omega)\big) \subset H^{1/2}(\partial\Omega)$ one can see that, according to the decomposition (2.10) with $A_0 = A_{\gamma_0}$, functions from $\operatorname{dom} A_{\max}$, having the inner regularity (see the formula (2.4)), lose the regularity at the boundary only together with functions from the kernel $\ker A^* (\subset H^2_{\mathrm{loc}}(\Omega))$.

Since the mapping γ_0 in the formula (2.4) is surjective and $0 \in \rho(A_0)$, due to the relation (2.10) for each $\varphi \in H^{-1/2}(\partial\Omega)$ there is a unique solution to the problem

$$A^* u = 0, \quad \gamma_0 u = \varphi \in H^{-1/2}(\partial\Omega). \tag{2.11}$$

Using the solution $u (\in \ker A^*)$ we introduce the Dirichlet-to-Neumann map, setting

$$\Lambda\varphi := G_1 u = \gamma_0(\partial u/\partial\nu) \in H^{-3/2}(\partial\Omega). \tag{2.12}$$

Now, following [Gru68], the main result of M. Vishik about the regularization of the Green formula can be formulated as follows.

Theorem 2.1 *Let* $\widetilde{\Gamma}_0 := G_0$ *and* $\widetilde{\Gamma}_1 := G_1 - \Lambda\gamma_0$. *Then*

1) *the mappings*

$$\widetilde{\Gamma}_1\colon \operatorname{dom}(A_{\max}) \to H^{1/2}(\partial\Omega), \qquad \widetilde{\Gamma}_0\colon \operatorname{dom}(A_{\max}) \to H^{-1/2}(\partial\Omega) \tag{2.13}$$

are surjective, and the regularized Green formula holds:

$$(A_{\max}u, v)_{L^2(\Omega)} - (u, A_{\max}v)_{L^2(\Omega)} = (\widetilde{\Gamma}_1 u, \widetilde{\Gamma}_0 v)_{1/2,-1/2} - (\widetilde{\Gamma}_0 u, \widetilde{\Gamma}_1 v)_{-1/2,1/2}; \tag{2.14}$$

2) *the mapping*

$$\begin{pmatrix}\widetilde{\Gamma}_1\\ \widetilde{\Gamma}_0\end{pmatrix} : \operatorname{dom}(A_{\max}) \to H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) \tag{2.15}$$

is surjective.

Here $(\cdot,\cdot)_{s,-s}$ stand for the pairing between $H^s(\partial\Omega)$ and $H^{-s}(\partial\Omega)$. Unlike the mapping (2.15), the mapping

$$G = \{G_0, G_1\}\colon \operatorname{dom}(A_{\max}) \to H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega) \tag{2.16}$$

is not surjective.

Each operator $K\colon H^{-1/2}(\partial\Omega) \to H^{-3/2}(\partial\Omega)$ defines a realization

$$\begin{aligned} \widehat{A}_K &:= A_{\max} \upharpoonright \operatorname{dom}(\widehat{A}_K), \\ \operatorname{dom}(\widehat{A}_K) &:= \{f \in \operatorname{dom}(A_{\max}) : G_1 f = K G_0 f\}. \end{aligned} \tag{2.17}$$

These extensions are described by the following simple lemma.

Lemma 2.2 *Let $\widetilde{A} \in Ext_A$. The following conditions are equivalent*:

1) *the extensions $\widetilde{A}$ and $A_0 = A_{\gamma_0}$ are disjoint, that is,* $\operatorname{dom}\widetilde{A} \cap \operatorname{dom} A_0 = \operatorname{dom} A$;
2) *$\widetilde{A}$ admits a realization of the form* (2.17)*, that is, $\widetilde{A} = \widehat{A}_K$*;
3) *$\widetilde{A}$ admits a special realization $\widetilde{A} = A_{K_{\mathrm{reg}}}$ of the form* (2.17), *where the graph* $\operatorname{graph}(K_{\mathrm{reg}})$ *of the operator K_{reg} is defined by the equality*

$$\operatorname{graph}(K_{\mathrm{reg}}) = G \operatorname{dom}(\widetilde{A}). \tag{2.18}$$

We point out that the mapping $K \to \widehat{A}_K$ is not bijective: from the equality $\widehat{A}_{K_1} = \widehat{A}_{K_2}$ it does not follow that $K_1 = K_2$. This is due to the fact that the mapping (2.16) is not surjective. In the case $K \neq K_{\mathrm{reg}}$, instead of equality (2.18), there is only the inclusion $G\operatorname{dom}(\widehat{A}_K) \subseteq \operatorname{graph}(K)$. At the same time, K_{reg} is uniquely determined by K:

$$K_{\mathrm{reg}} := K \upharpoonright \operatorname{dom} K_{\mathrm{reg}}, \qquad \operatorname{dom} K_{\mathrm{reg}} := \{h \in \operatorname{dom} K : (K - \Lambda)h \in H^{1/2}(\partial\Omega)\}. \tag{2.19}$$

Let, for example, $\mathbb{O} : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ be the zero operator. Then

$$\widehat{A}_{\mathbb{O}} = A_{G_1} := A^* \upharpoonright \ker G_1$$

and

$$\operatorname{dom}(\mathbb{O}_{\mathrm{reg}}) := \{f \in H^{-1/2}(\partial\Omega) : -\Lambda f \in H^{1/2}(\partial\Omega)\} = H^{3/2}(\partial\Omega).$$

Therefore $A_{G_1} = \widehat{A}_{\mathbb{O}_{\mathrm{reg}}}$ and $\operatorname{dom}(\widehat{A}_{G_1}) = \{f \in H^2(\Omega) : G_1 f = 0\}$.

Theorem 2.3 *Let $\widetilde{A} = \widehat{A}_K$ be an extension of the form* (2.17) *and let $K = K_{\text{reg}}$. There are the following equivalences:*

1) *the operator $\widehat{A}_K$ is closed $\Leftrightarrow$ the operator $K - \Lambda$ is closed*;
2) *the range* $\operatorname{ran} \widehat{A}_K$ *is closed $\Leftrightarrow$ the range* $\operatorname{ran}(K - \Lambda)$ *is closed*;
3) $\dim \ker \widehat{A}_K = \dim \ker(K - \Lambda) \Leftrightarrow \dim \operatorname{coker} \widehat{A}_K = \dim \operatorname{coker}(K - \Lambda)$;
4) *The operator $\widehat{A}_K$ is Fredholm $\Leftrightarrow$ the operator $K - \Lambda$ is Fredholm*;
5) *The extension $\widehat{A}_K$ is solvable* $\Leftrightarrow \ker(K - \Lambda) = \{0\}$*, and the operator*

$$(K - \Lambda)^{-1} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

is bounded;

6) *the extension $\widehat{A}_K$ is completely solvable* $\Leftrightarrow \ker(K - \Lambda) = \{0\}$*, and the operator*

$$(K - \Lambda)^{-1} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

is compact.

Theorem 2.3 clearly demonstrates one of M. Vishik's achievements: in the terminology of the operator $K - \Lambda$ (the regularization of the operator K) one can characterize all the above properties of the realization $\widehat{A}_K$, while in terms of the original operator K—not a single one of them, not even the closedness.

Attempts to apply the theory of extensions to the study of boundary value problems for the Laplace operator were undertaken by many American mathematicians immediately after the article of J. Calkin in 1939.[2] Yet only M. Vishik managed to do that, in his article [14] that appeared ahead of its time and became particularly well-cited during the last 25–30 years.

In today's language, Theorem 2.1 means that M. Vishik was the first to construct the boundary triple (for the terminology, see [Mal10]) for the elliptic operator of the form (2.1). Vishik's construction is particularly remarkable since the theory of traces for the elliptic operators (and Sobolev spaces) was constructed by J.-L. Lions and E. Magenes (see [LM72]) only 7–10 years later. Not having the explicit description of traces (2.8), M. Vishik proves (in today's language) the following inclusions:

$$L^p(\partial\Omega) \subset H^{-1/2}(\partial\Omega) \quad \text{for } p \geq \frac{2(n-1)}{n},$$

$$H^{1/2}(\partial\Omega) \subset L^q(\partial\Omega) \qquad \text{for } q < \frac{2(n-1)}{n-2},$$

where $n = \dim \Omega \geq 2$.

Not having an opportunity to cite several hundred papers developing or using the article [14], we only mention two works, [Gru68] and [Mal10], which are the closest

[2] J. W. Calkin, «Abstract symmetric boundary conditions», *Transactions of the American Mathematical Society* **45** (1939), 369–442. *—ed.*

in their contents to [14] (see also the references therein). After the article [AS80] was published, the results of M. Vishik's article of 1952, as well as the results of M. Birman and M. Krein, became widely available to the international audience and got known as "Birman–Krein–Vishik theory" and treated as classical.

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On the Vishik–Lyusternik Method

Alexander Demidov

There are hundreds of publications referring to three Vishik–Lyusternik's articles on asymptotics [35, 45, 49][1] and to the famous article of Vishik [10] on strongly elliptic systems (which allowed, in particular, to justify asymptotics). Most of these publications (including works [Dem75, BD11, Dem18]) are largely based on the basic, ingeniously simple ideas of the Vishik–Lyusternik method, which allows one to solve a wide range of complex problems and therefore has received such extensive application.

For the first time in the problems of singular perturbation with a small parameter $\varepsilon > 0$ in partial differential equations, a stunning simplicity method was proposed for constructing an asymptotics uniform up to the boundary of the region, where the solution u_ε of the unperturbed problem changes significantly. The "highlight" of the method lies in the fact that the solution u_0 for $\varepsilon = 0$ of the unperturbed problem (the construction of which in such problems is a relatively simple procedure), to this solution u_0 is added an amendment in the form of a solution of the simple problem on the half-line for a ordinary differential equation with constant coefficients regarding the so-called fast variable. It is in the direction of this fast variable that the u_ε solution of the original problem changes significantly. The Vishik–Lyusternik method made it easy to identify this feature. And therein lies the power of the method.

[1] Numerical citations refer to Mark Vishik's bibliography in the end of the book. —*ed.*

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A. Demidov (✉)
Tarusa, Russia

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To present the main two ideas of the Vishik–Lyusternik method, we consider the following rather simple Dirichlet problem in a disc $D = \{|x| < R\}$ of radius $R > 0$:

$$A_\varepsilon u_\varepsilon = 1 \quad \text{in} \ \ D, \quad u_\varepsilon\big|_{\partial D} = 0. \tag{1}$$

Here

$$A_\varepsilon = \varepsilon^2 \Delta - 1,$$

ε is a small positive parameter and Δ is the Laplace operator.

It is clear that a bounded solution u_ε to a simpler problem of the same kind

$$\varepsilon^2 \frac{d^2 u_\varepsilon}{dx^2} - u_\varepsilon = 1 \quad x > 0, \quad u_\varepsilon\big|_{x=0} = 0$$

differs from the limit solution $u_0 = -1$ (as $\varepsilon \to 0$) by some function $t \mapsto g(t) = e^{-t}$ that exponentially rapidly decreases in the variable (called the fast variable) $t = \frac{x}{\varepsilon}$. This simple observation led M. I. Vishik and L. A. Lyusternik to the *first idea* of their method for constructing asymptotic expansions of solutions to rather general linear singularly perturbed elliptic problems. In accordance with this idea, an asymptotic expansion should be looked for in the form of a sum of two polynomial decompositions with respect to the parameter ε:

$$u_\varepsilon(x) \simeq [f_0(x) + \varepsilon f_1(x) + \ldots] + [g_0(t) + \varepsilon g_1(t) + \ldots],$$

where $[g_0(t) + \varepsilon g_1(t) + \ldots]$ rapidly decreases near the boundary of the domain, i. e., it is the so-called boundary layer function. Here $t = \nu(x)/\varepsilon$, where $\nu(x)$ is the distance from a point x to the boundary of the domain. More exactly, the expression $\chi(x)[g_0(t) + \varepsilon g_1(t) + \ldots]$ should be considered (cf., for example, [Dem75]), where χ is a smooth cut-off function which is equal to 1 in a neighborhood of the boundary and to 0 outside of a larger neighborhood. However, we omit technical details for the sake of simplicity.

Returning to the problem (1) and following the first idea of the Vishik–Lyusternik method, we need to apply the operator A to $[f_0(x) + \varepsilon f_1(x) + \ldots]$ and $[g_0(t) + \varepsilon g_1(t) + \ldots]$. Therefore, it is necessary to represent the operator A in the corresponding variables (i. e., x and t) for obtaining two decompositions with respect to ε. In our particular case, these decompositions have the form

$$A_\varepsilon = -1 + \varepsilon^2 \Delta \quad \text{and} \quad A_\varepsilon = (\partial_{tt} - 1) - \frac{\varepsilon}{R - \varepsilon t} \partial_t = B_0 + \varepsilon B_1 + \varepsilon^2 \widetilde{B}, \tag{2}$$

where

$$B_0 = \partial_{tt} - 1\,, \quad B_1 = -\frac{1}{R} \partial_t, \quad \widetilde{B} = -\frac{t}{R^2\Bigl(1 - \dfrac{\varepsilon t}{R}\Bigr)} \partial_t. \tag{3}$$

The second idea of the Vishik–Lyusternik method concerns justification of asymptotics, i. e., estimating the remainder term

$$z_\varepsilon(x) = u_\varepsilon(x) - [f_0(x) + \varepsilon f_1(x) + \ldots] - [g_0(t) + \varepsilon g_1(t) + \ldots].$$

For this purpose, one should use *a priori* estimates for solutions to elliptic problems. Such estimates for rather general strongly elliptic problems were first obtained by Vishik [10]. In the case of the problem (1), we have

$$\|z_\varepsilon\|_\mu \le C\, \|A_\varepsilon z_\varepsilon\|_{\mu-2},$$

where $\|\cdot\|_s$ denotes the norm in the Sobolev space H^s.

We have

$$\begin{aligned} A_\varepsilon z_\varepsilon = A_\varepsilon\Big[u_\varepsilon - (f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots) - (g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \ldots)\Big] &\overset{(2)}{=} \\ \overset{(2)}{=} 1 - (-1 + \varepsilon^2\Delta)(f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots) & \\ - (B_0 + \varepsilon B_1 + \varepsilon^2 \widetilde{B})(g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \ldots) = & \\ = [1 + f_0] + \varepsilon[f_1] + \varepsilon^2[f_2 - \Delta f_0] + \ldots & \\ + [B_0 g_0] + \varepsilon[B_0 g_1 + B_1 g_0] + \varepsilon^2[B_0 g_2 + B_1 g_1] + \ldots & \end{aligned}$$

To make the norm $\|z_\varepsilon\|_\mu$ of the remainder term small, one should require the coefficients of lower powers of ε to vanish. The fact that the coefficients closed in the brackets $[\,\cdot\,]$ vanish means that

$$f_0 = -1, \quad f_1 = 0, \quad f_2 = \Delta f_0, \ldots \tag{4}$$

and the boundary layer functions g_j are solutions to the ordinary differential equations

$$B_0 g_0 = 0 \overset{(3)}{\Leftrightarrow} (\partial_{tt} - 1) g_0 = 0,$$

$$B_0 g_1 = -B_1 g_0 \overset{(3)}{\Leftrightarrow} (\partial_{tt} - 1) g_0 = -\partial_t g_0, \ldots$$

The function g_j satisfies the boundary condition $g_j(0) = f_j(R)$ which follows from the condition $u_\varepsilon(x)\big|_{|x|=R} \overset{(1)}{=} 0$, i.e.

$$\big(f_0(x) + \varepsilon f_1(x) + \ldots\big)\big|_{|x|=R} + \big(g_0(t) + \varepsilon g_1(t) + \ldots\big)\big|_{t=0} = 0.$$

As a result, we find:

$$u_\varepsilon(x) = -1 + \Big[1 - \varepsilon\frac{t}{2} + \varepsilon^2\frac{t(1+t)}{6}\Big]e^{-t} + z_\varepsilon(x), \tag{5}$$

where $t \overset{\text{def}}{=} \frac{R-|x|}{\varepsilon}$, and

$$\|z_\varepsilon\|_\mu \leq C\|A_\varepsilon z_\varepsilon\|_{\mu-2} \leq C\varepsilon^2\|\widetilde{B}_2(g_0+g_1+g_2)\|_{\mu-2} \leq C\varepsilon^{4+1/2-\mu}.$$

We note that $H^\mu \subset C^{0,\lambda}(\bar{D})$ for every $\lambda < \delta$ if $\mu = 1+\delta$, $\delta > 0$.

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Mark Vishik's Work on Quasilinear Equations

Alexander Shnirelman

In early 1960s Mark Vishik wrote a series of articles on quasilinear elliptic and parabolic equations and systems. He studied strongly elliptic equations and systems of arbitrary order of the form

$$L(u) + V(u) = \sum_{|\alpha|\le m,\ |\beta|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\beta u) + V(x, D^\gamma u) = h, \qquad (1)$$

where $u = u(x) = (u_1, \dots, u_k)$ is an unknown vector-valued function, $A_\alpha(x, \zeta_\gamma)$ and $h(x)$ are given vector-valued functions, and $V(x, \zeta_\gamma)$ stands for lower order (in a certain sense) terms. The system is to be solved in a bounded region $M \subset \mathbb{R}^n$ with smooth boundary with the boundary conditions of the first boundary value problem $D^\omega u|_{\partial M} = 0$ ($|\omega| \le m-1$). The coefficients A_α are to satisfy a number of conditions, first of all the ellipticity condition of the form

$$\sum_{|\alpha|=m,\ |\beta|\le m} \frac{\partial A_\alpha(x, \xi_\beta)}{\partial \xi_\beta} (\zeta_\alpha, \zeta_\beta) \ge \varphi(\xi)|\zeta|^2, \qquad (2)$$

where $\varphi(\xi)$ is a positive function of all ξ_β, $|\beta| \le m$. Besides, there are conditions on the growth of coefficients $A_\alpha(x, \xi_\beta)$ as $|\xi_\beta| \to \infty$. There is a separate treatment for the cases of polynomial and superpolynomial growth. One searches for the solutions in the functional space X which depends on the behavior of functions A_α: if the

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A. Shnirelman (✉)
Department of Mathematics and Statistics, Concordia University, Montreal, QC, Canada
e-mail: alexander.shnirelman@concordia.ca

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functions A_α grow polynomially in ξ_β, this is the Sobolev space; if the growth is superexponential, this is the Sobolev–Orlicz space.

These articles contain exhaustive results on the existence and uniqueness of solutions to the problems in question. Namely, for the systems in purely divergent form (without lower order terms) it is proved that the operator L defines a homeomorphism of the solution space X to its adjoint X', that is, the problem turns out to have a unique solution; in a general case the operator L maps the space X onto the entire adjoint space X', that is, the problem has a solution for all right-hand sides h, although there could be no uniqueness.

The proof of these results is based on a certain version of the Galerkin method. The well-known Galerkin method for the solution of equations of the form $F(u) = h$ is based on choosing a certain complete set of functions $v_j \in X$, and one constructs an approximate solution in the form $u_k = \sum_{j=1}^{k} C_{kj} v_j$. Here the coefficients C_{kj} are determined from the nonlinear system

$$[F(u_k), v_j] = [h, v_j] \quad (j = 1, \dots, k). \tag{3}$$

In typical cases the existence of the solution to this system is proved from the topological considerations. The next step consists in a proof of the *a priori* estimate on the norm of the solution u_k which does not depend on k. Then the considerations based on the weak compactness yield a subsequence of approximate solutions u_{k_i}, weakly convergent to a certain limit u. Finally, one proves that the equation allows taking the limit for this weakly convergent subsequence, and, consequently, u is the required solution.

At the same time, in problems treated in M. Vishik's articles the standard Galerkin method does not provide the required uniform regularity of approximate solutions u_k; roughly speaking, one lacks one derivative. To circumvent this difficulty, M. Vishik used the following improvement of the Galerkin method.

Let v_j be some sequence of smooth functions in M; its choice will be specified below. Let us look for an approximate solution in the form $u_k = \sum_{j=1}^{k} C_{kj} v_j$. The coefficients C_{kj} are determined from the following system of k nonlinear equations:

$$[B^+(L(u_k) + V(u_k)), v_j] = [B^+ h, v_j], \tag{4}$$

where

$$B^+ w(x) = M w(x) - \Delta\psi(x) w(x); \tag{5}$$

here M is a sufficiently large positive constant, and $\psi(x)$ is a positive function in M which satisfies the condition $D^\omega \psi|_{\partial M} = 0$ for $|\omega| \le 2m - 1$. This is equivalent to the equations

$$[L(u_k) + V(u_k), B v_j] = [h, B v_j] \quad (j = 1, \dots, k), \tag{6}$$

where

$$Bw(x) = Mw(x) - \psi(x)\Delta w(x). \tag{7}$$

So, the functions v_j are chosen so that the functions Bv_j form the complete system in the space X.

This method of construction of approximate solutions provides their required regularity, sufficient for taking the limit in the nonlinear operator $L(u) + V(u)$. The existence of the solution to the finite-dimensional system follows from simple topological considerations (degree of a continuous mapping). The required *a priori* estimates are obtained via the virtuoso work with embedding theorems.

Using an analogous approach, M. Vishik also studied the mixed initial-boundary value problem for parabolic systems of the form

$$\frac{\partial u}{\partial t} - L(u) = h(x,t), \tag{8}$$

where L is a strongly elliptic operator of order $2m$, the same as in (1), with the condition of the first boundary value problem on the boundary of the region M and with the initial condition $u|_{t=0} = u_0$. The solution is found as a limit of a (sub)sequence of the Galerkin approximations, that is,

$$u_k(x,t) = \sum_{j=1}^{k} C_{kj}(t)v_j(x).$$

Using the method analogous to the one described above, namely, acting on both sides of the equation (8) by the operator B^+, defined by

$$B^+w(x,t) = Mu(x,t) - \Delta(\psi(x)w(x,t)) - \frac{\partial}{\partial t}\Big[(T-t)\frac{\partial w}{\partial t}\Big], \tag{9}$$

we obtain for the coefficients $C_{kj}(t)$ the following system of the differential equations of the third (!) order:

$$\Big[B^+\Big(\frac{\partial u_k}{\partial t} - L(u_k)\Big), v_j\Big] = [B^+h, v_j] \quad (j = 1, \dots, k). \tag{10}$$

For the system (10) one can obtain a sufficiently strong *a priori* estimate, which allows one to take the limit in the weakly convergent subsequence u_{k_i} of approximate solutions, thus proving the existence of the solution (the uniqueness is proved separately).

Here we presented only one example of technical virtuosity shown by M. Vishik in his work; there are many other examples demonstrating his exceptional analytical level.

The articles of M. Vishik on quasilinear equations [50–58],[1] in part due to the generality of posed questions and in part due to the power of the obtained results, were a huge breakthrough and proved to be of great influence for the development of this subject, both in Russia and abroad.

[1] Numerical citations refer to Mark Vishik's bibliography in the end of the book. —*ed.*

Attractors for Nonlinear Nonautonomous Equations

Vladimir Chepyzhov

This is a review of the results obtained by Mark Vishik in collaboration with his former student Vladimir Chepyzhov in 1990–2012 that were published in about 50 their joint papers and two monographs (see the list of publications by Mark Vishik).

M. I. Vishik and V. V. Chepyzhov mostly have studied problems related to the theory of nonautonomous infinite-dimensional dynamical systems. They have constructed the attractors and studied their properties for various nonautonomous equations of mathematical physics: the 2D and 3D Navier–Stokes systems, reaction-diffusion systems, dissipative wave equations, the complex Ginzburg–Landau equations, and others. Since, as it has been discovered, the attractors of nonautonomous dynamical systems, being compact sets, usually have infinite dimension, the research has been focused on the Kolmogorov ε-entropy of the attractors. Upper estimates for the ε-entropy of uniform attractors of nonautonomous equations in terms of ε-entropy of time-dependent coefficients of the equation have been proved.

Another subject of joint researches by M. Vishik and V. Chepyzhov has been related to the constructions of attractors for those equations of mathematical physics for which the solution of the corresponding Cauchy problem is not unique or the uniqueness is not known (for example, for the non-homogeneous 3D Navier–Stokes system in a bounded domain). The theory of the trajectory attractors for these

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V. Chepyzhov (✉)
Moscow, Russia
e-mail: chep@iitp.ru

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Trends in Mathematics, https://doi.org/10.1007/978-3-031-33681-2_20

equations has been developed, that was used to construct attractors for equations without uniqueness. The method of trajectory attractors was also applied to the study of finite-dimensional approximations of attractors. The perturbation theory for trajectory and global attractors has been developed and used in the study of the attractors of equations containing terms that oscillate rapidly with respect to spatial and time variables.

One of the major problems in the study of evolution equations of mathematical physics is the investigation of the behaviour of the solutions of these equations when time is large or tends to infinity. The related important questions concern the stability of solutions as $t \to +\infty$ or the character of the instability if a solution is unstable. In the last decades considerable progress in this area have been achieved in the study of autonomous evolution partial differential equations. For a number of basic evolution equations of mathematical physics it was shown that the long time behaviour of their solutions is characterized by attractors. Attractors were constructed for the following equations and systems: the two-dimensional Navier–Stokes system, various classes of reaction-diffusion systems, nonlinear dissipative wave equations, complex Ginzburg–Landau equations and many other autonomous equations and systems. Mainly the global attractors of these equations were studied.

1 Global Attractors for Autonomous and Nonautonomous Equations

An autonomous evolution equation can be written in the following abstract form:

$$\partial_t u = A(u), \quad u|_{t=0} = u_0(x). \tag{1}$$

Here $u = u(x,t)$ is the solution of Eq. (1) and x, t denote the spatial and time variables, respectively. Corresponding to this equation is the *semigroup* of nonlinear operators $\{S(t)\} = \{S(t),\ t \geq 0\}$. The operator $S(t)$ maps the initial data $u_0(x)$ to the solution $u(x,t)$ of the Cauchy problem (1) at the time moment t:

$$S(t)u_0(x) = u(x,t), \quad t \geq 0.$$

One assumes that the Cauchy problem (1) has a unique solution for each initial data $u_0(x)$ that belongs, for example, to a certain Banach (or metric) space E. The space E is chosen in such a way that $u(x,t)$ belongs to E for all $t \geq 0$. Thus, the operator $S(t)$ maps E into E for all $t \geq 0$: $S(t): E \to E$. The operators $\{S(t)\}$ satisfy the semigroup properties:

$$S(t_1)S(t_2) = S(t_1 + t_2) \quad \text{for all } t_1, t_2 \geq 0;$$

$$S(0) = \mathrm{Id} \quad \text{is the identity operator.}$$

A set $\mathcal{A}$ from E is said to be a *global attractor* of the equation under consideration or, equivalently, of the corresponding semigroup $\{S(t)\}$ if it has the following properties:

1. the set $\mathcal{A}$ is compact in E;
2. $\mathcal{A}$ attracts each bounded set $B \subset E$: $\mathrm{dist}_E(S(t)B, \mathcal{A}) \to 0$ as $t \to +\infty$;
3. $\mathcal{A}$ is strictly invariant with respect to $\{S(t)\}$: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.

Here $\mathrm{dist}_E(\cdot, \cdot)$ denotes the Hausdorff semidistance in E:

$$\mathrm{dist}_E(U, V) = \sup_{u \in U} \inf_{v \in V} \|u - v\|_E.$$

It follows from the definition of the global attractor that the set $\mathcal{A}$ attracts solutions $u(x, t) = S(t)u_0(x)$ as $t \to +\infty$ uniformly with respect to bounded initial data $u_0(x)$. The global attractor $\mathcal{A}$ is unique if it exists. Thus the global attractor describes all the possible limits of solutions of Eq. (1).

It was shown that the Hausdorff and fractal dimension of the global attractors are finite for a number of equations and systems of mathematical physics. The estimates from above and from below for the Hausdorff and fractal dimension of attractors were found. For certain types of equations the structure of the global attractor $\mathcal{A}$ was completely described, for example, in the case where the equation has a global Lyapunov function. All these and other problems are treated in great detail in the books by R. Temam [Tem97], A. V. Babin and M. I. Vishik [BV92], J. K. Hale [Hal88], O. A. Ladyzhenskaya [Lad91] and in books of other authors (see [Chu02, SY02, Rob01, MZ08]).

The long-time behaviour of solutions of nonautonomous evolution equations of the form

$$\partial_t u = A(u, t), \quad u|_{t=\tau} = u_\tau(x)$$

and their attractors were studied in details by many authors for ordinary differential equations ($u \in \mathbb{R}^N$) and for some classes of operator and partial differential equations. The construction of the *skew product flow* of the *process* (the analog of the semigroup in autonomous case) played the main role in this theory; this allowed one to reduce the problem to the study of an attractor of some semigroup acting in an extended function phase space (see, for instance, R. K. Miller [Mil65], G. R. Sell [Sel67], R. K. Miller and G. R. Sell [MS76], J. K. Hale [Hal88]).

Dealing with evolution partial differential equations and especially with systems arising from mathematical physics it is a good idea to extend the phase space by using only the hull of the time-dependent coefficients of the equation under consideration. From this point of view the research was focused in the last decade on attractors for nonautonomous evolution equations of mathematical physics. It was assumed that external forces, interaction functions, and other coefficients in the equations explicitly depend on time. The dependence on time of these parameters can be periodic, quasiperiodic, or almost periodic. The spaces of these functions were studied in great detail in L. Amerio and G. Prouse [AP71], B. Levitan and V. Zhikov [LZ82].

In the present review we also consider the equations whose time-depending terms are *translation compact* functions in appropriate function spaces. The latter means, that, say, in the case of the external force $g(x,t)$ depending on time $t \in \mathbb{R}$, that all the translations $\{g(x, t+h),\ \ h \in \mathbb{R}\}$ form a precompact set in the space $L_2([t_1, t_2]; H)$ for every interval $[t_1, t_2] \subset \mathbb{R}$. Here H is a Hilbert (or more general) space corresponding to the physical nature of the function $g(x,t)$. Similarly, the translation compactness was defined for other terms of the equation, for example, for interaction functions of the form $f(u,t)$ and so on.

We denote by $\sigma(t)$ the collection of all time-dependent coefficients of a nonautonomous equation. The equation itself can be rewritten in the form

$$\partial_t u = A_{\sigma(t)}(u), \quad u|_{t=\tau} = u_\tau(x). \tag{2}$$

The parameter $\sigma(t)$ is said to be the *time symbol* (or just the *symbol*) of the equation. The values of $\sigma(t)$ belong to a metric or Banach space. For example, $\sigma(t) = (f(u,t), g(x,t))$ if the time-dependent terms of the equation are the interaction function $f(u,t)$ and the external force $g(x,t)$. Dealing with nonautonomous equations it is fruitful to study the entire family of Eq. (2) with time symbols $\sigma(t)$ belonging to a set Σ called the *symbol space*. A typical symbol space is as follows. We are given a fixed initial time symbol $\sigma_0(t)$ of the equation we want to study. Then we consider the set of all time translations of $\sigma_0(t)$, i. e., the set $\{\sigma_0(t+h),\ \ h \in \mathbb{R}\}$. Moreover, we add to the symbol space Σ all the functions $\sigma(t)$ that are the limits of the sequences of the form $\{\sigma_0(t+h_n)\}$ as $n \to \infty$. The limits are taken in a suitable function space. The resulting family of functions $\{\sigma(t)\}$ is called the *hull* of $\sigma_0(t)$ and is denoted by $\mathcal{H}(\sigma_0)$. For example, if $\sigma_0(t)$ is an almost periodic function with values in a metric space $\mathcal{M}$, then $\mathcal{H}(\sigma_0)$ is the hull of σ_0 in the space $C_b(\mathbb{R}; \mathcal{M})$. We now set $\Sigma = \mathcal{H}(\sigma_0)$ and study the family of Eq. (2) with symbols $\sigma \in \mathcal{H}(\sigma_0)$.

We start from the fact that the properly defined attractor $\mathcal{A}$ of the initial equation with symbol $\sigma_0(t)$ must simultaneously be the attractor of each Eq. (2) with symbol $\sigma(t) \in \mathcal{H}(\sigma_0)$ and, moreover, it must be the attractor of the entire family of these equations. This observation leads to the concept of the *uniform (with respect to* $\sigma \in \Sigma$) *global attractor* $\mathcal{A}_\Sigma$ of the family of Eq. (2) with symbols $\sigma \in \Sigma$.

The initial data $u_\tau(x)$ for (2) is taken in the Banach space E. We assume that the Cauchy problem (2) is uniquely solvable for every $u_\tau \in E$ and for all $\tau \in \mathbb{R}$. Corresponding to Eq. (2) is the *process* $\{U_\sigma(t,\tau)\} = \{U_\sigma(t,\tau)\colon t, \tau \in \mathbb{R}, t \geq \tau\}$ acting in the space E. Similarly to the autonomous Eq. (1) the operator $U_\sigma(t,\tau)$ maps the initial data $u_\tau(x) \in E$ to the solution $u(t,x)$ of the Cauchy problem (2) at the time moment t:

$$U_\sigma(t,\tau) u_\tau(x) = u(x,t), \quad t \geq \tau, \quad \tau \in \mathbb{R}.$$

We assume that $u(x, t)$ belongs to E for all $t \geq \tau$. Thus, the operators $U_\sigma(t, \tau)$ map E into E for all $t \geq \tau, \tau \in \mathbb{R}$: $U_\sigma(t, \tau) : E \to E$. The notion of process is a generalization of the notion of semigroup generated by an autonomous evolution equation. A process has the following characteristic properties

$$U(t, s)U(s, \tau) = U(t, \tau) \quad \text{for all } t \geq s \geq \tau, \quad \tau \in \mathbb{R};$$
$$U(\tau, \tau) = \mathrm{Id} \quad \text{for all } \tau \in \mathbb{R}.$$

We study the uniform attractor of the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ corresponding to the family of Eq. (2) with symbols $\sigma \in \Sigma$.

A set $\mathcal{A}_\Sigma$ from E is said to be a *uniform global attractor* of the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ if it has the following properties:

1. the set $\mathcal{A}_\Sigma$ is compact in E;
2. $\mathcal{A}_\Sigma$ attracts any bounded set $B = \{u_\tau(x)\} \subset E$ uniformly w.r.t. $\sigma \in \Sigma$:

$$\sup_{\sigma \in \Sigma} \mathrm{dist}_E(U_\sigma(t, \tau)B, \mathcal{A}_\Sigma) \to 0 \text{ as } t \to +\infty \text{ for every } \tau \in \mathbb{R};$$

3. $\mathcal{A}_\Sigma$ is the *minimal* set satisfying (1) and (2), that is, if a set $\mathcal{A}_1$ is compact in E and attracts any bounded set B uniformly w.r.t. $\sigma \in \Sigma$, then $\mathcal{A}_\Sigma \subseteq \mathcal{A}_1$.

Thus, the notion of uniform global attractor of a family of processes generalizes the notion of global attractor of a semigroup. The invariance property is replaced by the property of minimality.

We study uniform attractors of basic nonautonomous evolution equations of mathematical physics. The analysis of time symbols of these equations and systems is the key element in the theory of nonautonomous partial differential equations. The proposed method is quite simple and allows us to construct uniform attractors, to study their structure, and to estimate some important quantities related to attractors, such as the Hausdorff and fractal dimension and the Kolmogorov ε-entropy.

To describe the structure of uniform attractors we introduce the notion of a *kernel* of an equation. The kernel $\mathcal{K}_\sigma$ of Eq. (2) is the collection of all bounded (in E) solutions $u(t)$, $t \in \mathbb{R}$ of the equation that are defined on the entire time axis $\mathbb{R}$. The set

$$\mathcal{K}(t) = \{u(t) : u \in \mathcal{K}\} \subseteq E$$

is called the *kernel section* at the time moment $t \in \mathbb{R}$. We prove the following identity for the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ corresponding to problem (2):

$$\mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0). \tag{3}$$

Clearly, the right-hand side of (3) does not change if we replace 0 by an arbitrary time moment τ.

We construct uniform attractors for the nonautonomous two-dimensional Navier–Stokes system, nonautonomous reaction-diffusion systems, nonautonomous dissipative wave equations, nonautonomous Ginzburg–Landau equations and for other equations and systems. For each equation or system we describe in detail the function space to which the time symbol $\sigma_0(t)$ of this equation belongs. We present the conditions that provide the translation compactness of the symbol $\sigma_0(t)$ or, more precisely, the translation compactness of its components. We prove that the corresponding Cauchy problems have unique solutions in suitable function spaces. Using the property of dissipativity (specific to each problem) we establish the existence of uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) absorbing or attracting set for the corresponding family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$. We apply and develop various known methods for the investigation of various partial differential equations. We derive the corresponding *a priori* estimates for solutions $u(x, t)$ of these nonautonomous equations and systems. We also prove the necessary continuity properties of the processes. Then the general theorem implies the existence of a uniform attractor $\mathcal{A}_{\mathcal{H}(\sigma_0)}$ of a nonautonomous equation. In particular, identity (3) holds, that is, the global attractor is the union of all values of all bounded (in E) global solutions of all Eq. (2) with time symbols $\sigma \in \mathcal{H}(\sigma_0)$.

The notion of time symbol of a nonautonomous equation is also important in the study of the dimension of uniform attractors of the above equations and systems of mathematical physics. Using this approach we prove upper estimates for the Hausdorff and fractal dimension of uniform attractors of these systems. In a number of cases we are able to find lower estimates for the dimension of uniform attractors. For example, we prove such upper and lower estimates for the fractal dimension of the uniform attractor $\mathcal{A}_\Sigma$ of the 2D Navier–Stokes system with quasiperiodic (in time t) external force $g_0(x, t) = G(x, \alpha_1 t, \alpha_2 t, \ldots, \alpha_k t)$. Here $G(x, \omega_1, \omega_2, \ldots, \omega_k)$ is a function that is 2π-periodic in each variable $\omega_i \in \mathbb{R}$. The symbol space $\Sigma = \mathcal{H}(g_0)$ is diffeomorphic to the k-dimensional torus. We prove that the fractal dimension of the uniform attractor $\mathcal{A}_\Sigma$ of this system does not exceed the sum of two terms: the Grashof number Gr (the known parameter describing the number of "degrees of freedom" of a flow) and the number k of rationally independent frequencies of the external force $g(x, t)$. The number k is the dimension of the symbol space. Thus for $k = 0$ (autonomous case) we obtain the known estimate for the fractal dimension of the global attractor of the autonomous 2D Navier–Stokes system. Examples show that the fractal dimension of $\mathcal{A}_\Sigma$ can be greater than k. These facts reflect the importance of the number k in the estimates of the dimension of uniform attractors for the Navier–Stokes system. Moreover, if $g_0(x, t)$ is a general almost periodic function in time t, then examples show that the fractal dimension of the uniform attractor can be infinite. Similar facts are proved for other nonautonomous equations of mathematical physics having quasiperiodic or almost periodic time symbols.

In the cases where the fractal dimension of uniform attractors is equal to infinity it is natural to study other characteristics and quantities of uniform attractors of nonautonomous equations. The famous work A. N. Kolmogorov and V. M. Tikhomirov [KT59] is devoted to the systematic study of the ε-entropy of compact sets in various function spaces. Notice that the uniform attractor $\mathcal{A}_\Sigma$ is a compact set in E. Then it is reasonable to investigate the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ of the uniform attractor. It is well-known that the number $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ is equal to $\log_2 N_\varepsilon(\mathcal{A}_\Sigma)$, where $N_\varepsilon(\mathcal{A}_\Sigma)$ is the minimal number of balls in E with radius ε covering the set $\mathcal{A}_\Sigma$. Since $\mathcal{A}_\Sigma$ is compact, $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ is finite for every $\varepsilon > 0$. The problem arises to study the rate of growth of the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ as $\varepsilon \to 0+$. We have proved upper estimates for the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ of uniform attractors of nonautonomous evolution equations with translation compact symbols $\sigma_0(t)$ in the corresponding spaces. These estimates are optimal in some sense and generalize the well-known estimates for the fractal dimension of the corresponding autonomous equations and systems. In particular the ε-entropy of the uniform (w.r.t. $g \in \Sigma = \mathcal{H}(\sigma_0)$) attractor $\mathcal{A}_\Sigma$ of the 2D Navier–Stokes system does not exceed the sum of two terms: the first term is the Grashof number Gr multiplied by $\log_2\left(\frac{1}{\varepsilon}\right)$ and the second is the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{H}(g_0))$ of the hull of the external force $g_0(x,t)$ measured on the finite time interval $[0,l]$, where $l = O\left(\log_2\left(\frac{1}{\varepsilon}\right)\right)$ (in the quasiperiodic case this term has the form $k \cdot \log_2\left(\frac{1}{\varepsilon}\right)$, where k is the number of rationally independent frequencies of $g_0(x,t)$). In particular, the functional dimension of the uniform attractor does not exceed the functional dimension of the hull $\mathcal{H}(g_0)$. We prove similar results for other nonautonomous equations of mathematical physics. In particular the estimates for the ε-entropy of the uniform attractors imply the estimates for the fractal dimension of the uniform attractors if the symbols of the equations are quasiperiodic functions.

Notice that the book [Har91] by A. Haraux contains chapters that are devoted to the study of attractors of processes generated by nonautonomous partial differential equations with almost periodic in time coefficients and terms. This book played a stimulating role for the authors in the study of nonautonomous evolution equations.

2 Trajectory Attractors

We study attractors of equations of mathematical physics for which the solution of the corresponding Cauchy problem exists on any time interval but, maybe, is not unique or the uniqueness theorem is not proved yet. The classical example is the 3D Navier–Stokes system. It is known from the works of J. Leray and E. Hopf that the Cauchy problem for this system has a weak solution on an arbitrary time interval, but it is not known whether this weak solution is unique. Another important example is the wave equation with nonlinear interaction term of fast polynomial growth. This hyperbolic equation appears in many branches of modern physics, for example, in relativistic quantum mechanics. The existence of a weak solution (in the sense of distributions) of the Cauchy problem for this equation is known, whereas the uniqueness theorem is proved only for a moderate growth of the interaction function (see J.-L. Lions [Lio69]). Even though the complex Ginzburg–Landau equations playing a central role in the theory of amplitude equations, the global existence and uniqueness of solutions are not established for all values of the dispersion parameters. For all these equations and systems the theory of global attractors of semigroups descried above is not applicable. To overcome this difficulty we have developed the theory of so-called *trajectory attractors* which enables us to study the limiting behaviour of solutions of equations of mathematical physics without uniqueness. Moreover, it is also possible to construct generalized global attractors for such equations using the trajectory attractors. In particular, this theory covers all the above problems of mathematical physics.

Let us briefly explain the idea of the construction of a trajectory attractor using as an example the 3D Navier–Stokes system

$$\partial_t u + \nu L u + P(u, \nabla)u = Pg(x), \qquad (\nabla, u) = 0, \qquad u|_{\partial\Omega} = 0, \qquad t \geq 0, \tag{4}$$

where $x = (x_1, x_2, x_3) \in \Omega \Subset \mathbb{R}^3$, $u = u(x,t) = (u^1, u^2, u^3)$. Here L is the Stokes operator, $g(x) = (g^1, g^2, g^3)$ is the external force, $\nu > 0$ is the viscosity coefficient, and P denotes the orthogonal projection onto the space H of divergence free vector fields with finite L_2-norm. We study weak solutions $u(x,t)$, $t \geq 0$ of system (4) that satisfy the known energy inequality (see J.-L. Lions [Lio69]). Notice that all the weak solutions resulting from the Galerkin approximation method always satisfy this energy inequality. Therefore the stock of such weak solution is reasonably large. The collection of all these solutions is denoted by $\mathcal{K}^+$.

The traditional theory of global attractors uses the set of initial data $\{u_0(x)\}$ of the Cauchy problem (1) as the phase space E on which the corresponding semigroup $\{S(t)\}$ acts. Now the phase space corresponding to system (4) is the set

$$\mathcal{K}^+ = \{u(x,t),\ t \geq 0\}$$

of weak solutions defined on the entire time semiaxis $\mathbb{R}_+$. The elements of the phase space are functions depending on time. The set $\mathcal{K}^+$ is called the *trajectory space* of the 3D Navier–Stokes system and the elements of $\mathcal{K}^+$ are called *trajectories*. We consider the translation operators $\{T(h),\ h \geq 0\}$ acting on $\mathcal{K}^+$ by the formula $T(h)u(x,t) = u(x, t+h)$. The translation $T(h)$, $h \geq 0$ maps any function $u(x,t)$, $t \geq 0$ onto the shifted function $u(x, t+h)$, $t \geq 0$. It follows from the definition of $\mathcal{K}^+$ that $u(x, t+h) \in \mathcal{K}^+$ if $u(x,t) \in \mathcal{K}^+$. It is clear that the translations

$$\{T(h)\} = \{T(h),\ h \geq 0\}$$

form a semigroup acting on $\mathcal{K}^+$: $T(h)\colon \mathcal{K}^+ \to \mathcal{K}^+$ for $h \geq 0$. We study the global attractor of the translation semigroup $\{T(h)\}$ on $\mathcal{K}^+$.

In the trajectory space $\mathcal{K}^+$ we consider a weak convergence topology. It follows easily that the space $\mathcal{K}^+$ is closed in this topology and the translation semigroup $\{T(h)\}$ is continuous in $\mathcal{K}^+$. We define bounded sets in $\mathcal{K}^+$ and prove the existence of a bounded absorbing set B_0 of the semigroup $\{T(h)\}$ in $\mathcal{K}^+$, that is, for any bounded set $B \subset \mathcal{K}^+$ there exists $h_1 = h_1(B) > 0$ such that $T(h)B \subset B_0$ for all $h \geq h_1$. Since the set B_0 is bounded, it is compact in the weak topology of the space $\mathcal{K}^+$. From this facts it follows that the semigroup $\{T(h)\}$ has a global attractor $\mathfrak{A} \subset \mathcal{K}^+$, that is, $\mathfrak{A}$ is compact in the weak topology, strictly invariant with respect to $\{T(h)\} : T(h)\mathfrak{A} = \mathfrak{A}$ for all $h \geq 0$, and for every bounded set B of trajectories from $\mathcal{K}^+$ the set $T(h)B$ tends to $\mathfrak{A}$ in the weak topology as $h \to +\infty$. The set $\mathfrak{A}$ is called the *trajectory attractor* of the 3D Navier–Stokes system (4).

Notice that the weak topology in $\mathcal{K}^+$ is stronger than the local strong convergence topology of the spaces $L_2^{loc}(\mathbb{R}_+; H^{1-\delta})$ and $C^{loc}(\mathbb{R}_+; H^{-\delta})$, where $0 < \delta \leq 1$. Therefore for any bounded set B from $\mathcal{K}^+$ and for every $M > 0$

$$\operatorname{dist}_{L_2(0,M;H^{1-\delta})}\big(T(h)B,\, \mathfrak{A}\big) \to 0,$$

$$\operatorname{dist}_{C([0,M];H^{-\delta})}\big(T(h)B,\, \mathfrak{A}\big) \to 0 \text{ as } h \to \infty. \tag{5}$$

From (5) we deduce that the set $\mathcal{A} = \mathfrak{A}|_{t=0} \subset H$ is the *global attractor* of the 3D Navier–Stokes system (4). More precisely, $\mathcal{A}$ is bounded and closed in H and satisfies the following attracting property: the restriction $B|_t$ at time t of any bounded set of solutions $B \subset \mathcal{K}^+$ tends to $\mathcal{A}$ as $t \to \infty$ in the space $H^{-\delta}$:

$$\operatorname{dist}_{H^{-\delta}}\big(B|_t,\, \mathcal{A}\big) \to 0 \quad \text{as} \quad t \to \infty, \quad 0 < \delta \leq 1. \tag{6}$$

Moreover, $\mathcal{A}$ is the minimal closed (in H) set that satisfies (6). Thus, the set $\mathcal{A}$ has all the properties known for the global attractor of the semigroup corresponding to the Cauchy problem for which the uniqueness theorem holds (for example, the 2D Navier–Stokes system).

Using this scheme we construct trajectory attractors and global attractors for other autonomous equations and systems of mathematical physics of the form (1) for which the uniqueness theorem of the Cauchy problem is not proved or does not hold. For example, we construct trajectory attractors and global attractors and study their properties for the dissipative wave equation with arbitrary polynomial growth of the interaction function.

Notice that in a number of cases it is also reasonable to study trajectory attractors for the equations for which the uniqueness theorem holds. In this case the trajectory attractor $\mathfrak{A}$ consists of all trajectories $u(t), \ t \geq 0$, that lie on the usual global attractor $\mathcal{A}$:

$$\mathfrak{A} = \{u(t) = S(t)u_0,\ t \geq 0 \mid u_0 \in \mathcal{A}\}$$

and $\mathfrak{A}$ attracts bounded sets of trajectories in a stronger topology.

The methods of trajectory attractors is also fruitful in the theory of perturbation of attractors and in the study of attractors of equations containing rapidly oscillating terms. For example, we prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the wave equation

$$\varepsilon \partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) + g(x)$$

depending on a positive small parameter ε converges as $\varepsilon \to 0+$ in the corresponding space to the trajectory attractor $\mathfrak{A}_0$ of the limiting parabolic equation

$$\gamma \partial_t u = \Delta u - f(u) + g(x).$$

Here $f(u)$ is a function with arbitrary polynomial growth with respect to u. Since for the limiting parabolic equation the uniqueness theorem holds, it has the usual global attractor $\mathcal{A}_0$ and the trajectory attractor $\mathfrak{A}_0$ consists of all solutions $u(t)$ of this equation lying on $\mathcal{A}_0$ for all $t \geq 0$. Besides this case of a singular perturbation we consider other problems of the theory of perturbation of partial differential equations. These results reflect the following general property of the trajectory attractors of equations of mathematical physics: the trajectory attractors of perturbed equations depend upper semicontinuously on the perturbation parameters.

Similarly to autonomous equations we study *uniform trajectory attractors* and *global attractors* for nonautonomous equations of mathematical physics of the form (2) with terms depending on time. We assume that the time symbols $\sigma(t)$ are translation compact in the corresponding spaces. To begin with we consider the equations for which the existence of the Cauchy problem is not proved or does not hold. We construct the uniform trajectory attractor for the nonautonomous 3D Navier–Stokes system with translation compact (in time t) external force $g = g(x,t)$. We also study the dissipative hyperbolic equation containing the interaction function $f(u,t)$ with arbitrary polynomial growth with respect to u. We also consider other nonautonomous equations of mathematical physics. Separately we study the trajectory attractors for nonautonomous equations with uniqueness. This leads to stronger attraction of trajectories to the uniform trajectory attractor.

The trajectory attractors also satisfy the following important property known in the theory of global attractors. For example, we study the 3D Navier–Stokes system (4). We consider the corresponding Galerkin approximation system of order m, that is, the system of ordinary differential equations in m-dimensional Euclidean space. Using the above scheme we construct the trajectory attractor $\mathfrak{A}^{(m)}$ of this system. Recall that $\mathfrak{A}^{(m)}$ consists of all solutions $u_m(x,t),\ t \geq 0$ of the Galerkin system that lie on the global attractor (in $\mathbb{R}^m$) $\mathcal{A}^{(m)}$ of this system. We prove that $\mathfrak{A}^{(m)}$ converges to $\mathfrak{A}$ in the weak topology as $m \to +\infty$. In particular, $\mathrm{dist}_{H^{-\delta}}\big(\mathcal{A}^{(m)}, \mathcal{A}\big) \to 0\ (m \to \infty),\ \ 0 < \delta \leq 1$. Here $\mathfrak{A}$ and $\mathcal{A}$ are the trajectory attractor and the global attractor of the Navier–Stokes system (4), respectively. This property of upper semicontinuity of attractors holds for all equation and systems considered in this review. No matter whether the corresponding uniqueness theorem holds or not.

We investigate the attractors of evolution equations with terms that oscillate rapidly with respect to the spatial or time variable. The parameter $\varepsilon^{-1}, \varepsilon > 0$ characterizes the oscillation frequency. We assume that rapidly oscillating terms and coefficients have averages in a weak sense as $\varepsilon \to 0+$ in the corresponding function spaces. The equation with averaged terms and coefficients is called the *averaged equation*. We prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the equation with rapidly oscillating terms converges as $\varepsilon \to 0+$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the averaged equation in a suitable weak sense. Moreover, the global attractors $\mathcal{A}_\varepsilon$ of the original equations with rapidly oscillating terms converge as $\varepsilon \to 0+$ to the global attractor $\overline{\mathcal{A}}$ of the averaged equation as $\varepsilon \to 0+$ in the corresponding function space. We apply these results to the 3D and 2D Navier–Stokes systems with external force of the form $g\left(x, \frac{x}{\varepsilon}\right)$ (or $g\left(x, t, \frac{t}{\varepsilon}\right)$). We assume that the function $g\left(x, \frac{x}{\varepsilon}\right)$ has the average $\bar{g}\,(x)$ as $\varepsilon \to 0+$, for example, in the space H_w. (The space H_w is the space H endowed with the weak topology.) Then the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to $\overline{\mathfrak{A}}$ in the following sense: for every $M > 0$

$$\begin{aligned} \operatorname{dist}_{L_2(0,M;H^{1-\delta})}\big(\mathfrak{A}_\varepsilon, \overline{\mathfrak{A}}\big) &\to 0, \\ \operatorname{dist}_{C([0,M];H^{-\delta})}\big(\mathfrak{A}_\varepsilon, \overline{\mathfrak{A}}\big) \to 0 \quad \text{as} \quad \varepsilon &\to 0+, \quad 0 < \delta \le 1. \end{aligned} \tag{7}$$

For the corresponding global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ we have the following relation:

$$\operatorname{dist}_{H^{-\delta}}\big(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\big) \to 0 \quad \text{as} \quad \varepsilon \to 0+\,. \tag{8}$$

We prove similar results for the reaction-diffusion systems and for the dissipative hyperbolic equations with rapidly oscillating terms. If the corresponding Cauchy problem is uniquely solvable, then we prove that relations (7)–(8) hold in more regular space with stronger topology. For example, for the 2D Navier–Stokes system we have that

$$\operatorname{dist}_{H^{1-\delta}}\big(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\big) \to 0 \text{ as } \varepsilon \to 0+\,.$$

For perturbed potential reaction-diffusion systems with rapidly oscillating terms it has been shown that the distance between the global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ is at most $C\varepsilon^\gamma$, where $\gamma > 0$ (see B. Fiedler and M. I. Vishik [FV03]).

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Rigorous Results in Space-Periodic Two-Dimensional Turbulence

Sergei Kuksin and Armen Shirikyan

Abstract We survey the recent advance in the rigorous qualitative theory of the 2d stochastic Navier–Stokes system that are relevant to the description of turbulence in two-dimensional fluids. After discussing briefly the initial-boundary value problem and the associated Markov process, we formulate results on the existence, uniqueness and mixing of a stationary measure. We next turn to various consequences of these properties: strong law of large numbers, central limit theorem, and random attractors related to a unique stationary measure. We also discuss the Donsker–Varadhan and Freidlin–Wentzell type large deviations, as well as the inviscid limit and asymptotic results in 3d thin domains. We conclude with some open problems.

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Keywords 2d Navier–Stokes system · Stationary measure · Mixing · Strong law of large numbers · Central limit theorem · Random attractors · Large deviations · Inviscid limit

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S. Kuksin (✉)
Université Paris-Diderot (Paris 7), Paris, France
e-mail: sergei.kuksin@imj-prg.fr

A. Shirikyan
Department of Mathematics, CY Cergy Paris University, Cergy–Pontoise, France
e-mail: Armen.Shirikyan@u-cergy.fr

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1 Introduction

The main subject of this article is the two-dimensional Navier–Stokes system in $\mathbb{R}^2$ subject to periodic boundary conditions, perturbed by a random force. We thus consider the equations

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \tag{1}$$

where $u = (u_1, u_2)$ and p are the velocity field and pressure of a fluid, $\nu > 0$ is the kinematic viscosity, f is an external (random) force, and $\langle u, \nabla \rangle = u_1\partial_1 + u_2\partial_2$. All the functions are assumed to be 2π-periodic in the spacial variables x_1 and x_2. Equations (1) are supplemented with the initial condition

$$u(0,x) = u_0(x), \tag{2}$$

where u_0 is a given divergence-free vector field that is 2π-periodic and locally square-integrable. Our aim is to review rigorous results on the qualitative behaviour of solutions for (1) as $t \to \infty$ and/or $\nu \to 0$. These two limits are important for the mathematical description of the space–periodic 2d turbulence (for a physical treatment of this topic see e.g. [BV12]). In our review we avoid detailed discussion, related to the relevance of the 2d Navier–Stokes system (1) for physics, referring the reader to [Bat82, Fri95, Gal02, BV12]. But we mention that a number of equations, closely related to the 2d Navier–Stokes system, are used in meteorology and oceanography, so the methods, developed for Navier–Stokes equations, can be applied in the mathematical theory of climate and the statistical description of the ocean. A result, presented in Sect. 7, gives a very basic explanation for the relevance of the 2d models for statistical description of the 3d phenomena.

Most of the results, discussed in our work, were obtained in this century. Complete proofs and discussion of many of them can be found in [KS12]. New material, not treated in that book, includes the theory of large deviations and recent progress concerning the mixing in Eq. (1).

2 Equations and Random Forces

Cauchy Problem

As was mentioned in the Introduction, we consider Eq. (1) with periodic boundary conditions. Before describing the class of random forces f we deal with, let us recall a general result on the existence, uniqueness, and regularity of solutions to the deterministic Cauchy problem (1), (2). We begin with the definition of a solution on an arbitrary time interval $J_T = [0, T]$. The definitions of all the functional spaces used here and henceforth can be found in the list of frequently used notations at the end of the paper.

Definition 1 Let $f(t,x)$ be the time derivative of a piecewise continuous function[1] $g : J_T \to L^2(\mathbb{T}^2, \mathbb{R}^2)$, vanishing at $t = 0$: $f = \partial_t g(t,x)$. A function $u(t,x)$ defined on $J_T \times \mathbb{T}^2$ is called a *weak solution* for (1) if it belongs to the space

$$\mathcal{X}_T = C(J_T, L^2_\sigma) \cap L^2(J_T, H^1(\mathbb{T}^2, \mathbb{R}^2) \cap L^2_\sigma),$$

and satisfies the relation

$$\big(u(t), \varphi\big) + \int_0^t \big(\langle u, \nabla\rangle u - \nu\Delta u, \varphi\big)\mathrm{d}s = \big(u(0), \varphi\big) + \big(g(t), \varphi\big), \quad t \in J_T, \tag{3}$$

where φ is an arbitrary divergence-free smooth vector field on $\mathbb{T}^2$, and the term with the Laplacian under the integral is understood in the weak sense: $(\Delta u, \varphi) = -(\nabla u, \nabla\varphi)$. In what follows, when discussing Eq. (1), we often say *solution* rather than *weak solution*.

Note that the Navier–Stokes system contains three unknown functions, u_1, u_2, and p, whereas the definition of a solution specifies only the velocity field u. This is due to the fact that, once $u(t,x)$ satisfying (3) is constructed, the Leray decomposition (see Theorem 1.5 in [Tem79, Chapter 1]) may be used to find a unique (up to an additive function of t) distribution $p(t,x)$ such that the first equation in (1) holds in a weak sense. A proof of the following theorem is essentially contained in [Tem79, Chapter 3].

Theorem 2.1 *Let $T > 0$ and let f be the sum of a square-integrable function $h(x)$ and the time derivative of a piecewise continuous function with range in $H^1(\mathbb{T}^2, \mathbb{R}^2)$. Then, for any $u_0 \in L^2_\sigma$, Eq.* (1) *has a unique weak solution $u \in \mathcal{X}_T$ satisfying the initial condition* (2)*.*

If the function $g(t,x)$ is such that its mean value (in x) vanishes identically in t, then the mean value of the solution $u(t,x)$ is time-independent. Below we always assume that the mean values of the forces we apply to the Navier–Stokes system and of its solutions which we consider, both vanish identically in time.

Theorem 2.1 allows one to construct a unique solution of Eq. (1) for the three important classes of random forces f, specified below. Namely, let H be the space of divergence free square-integrable vector fields on $\mathbb{T}^2$ with zero mean value. We shall deal with random forces of the form

$$f(t,x) = h(x) + \eta(t,x), \tag{4}$$

[1] Here and in similar situations below, it means that g has at most countably many points of discontinuity, where it has left and right limits. Traditionally we normalise such functions to be right-continuous everywhere. In particular, $g(0)$ is well defined.

where $h \in H$ is a deterministic function, and η is one of the following three random processes:

Spatially Regular White Noise Let $\mathbb{Z}_*^2$ be the set of non-zero integer vectors $j = (j_1, j_2)$ and let $\{e_j, j \in \mathbb{Z}_*^2\}$ be a trigonometric basis in H defined by

$$e_j(x) = \frac{j^\perp}{\sqrt{2}\pi|j|} \begin{cases} \cos\langle j, x\rangle & \text{if } j_1 > 0 \text{ and if } j_1 = 0,\ j_2 > 0, \\ \sin\langle j, x\rangle & \text{if } j_1 < 0 \text{ and if } j_1 = 0,\ j_2 < 0, \end{cases} \tag{5}$$

where $j^\perp = (-j_2, j_1)$. Let us fix numbers $\{b_j, j \in \mathbb{Z}_*^2\}$ such that

$$\mathfrak{B}_1 < \infty,$$

where for $k \geq 0$ we denote

$$\mathfrak{B}_k = \sum_{j \in \mathbb{Z}_*^2} b_j^2 |j|^{2k} \leq \infty, \tag{6}$$

and define

$$\eta(t, x) = \frac{\partial}{\partial t}\zeta(t, x), \quad \zeta(t, x) = \sum_{j \in \mathbb{Z}_*^2} b_j \beta_j(t) e_j(x), \tag{7}$$

where $\{\beta_j, j \in \mathbb{Z}_*^2\}$ is a family of independent standard Brownian motions. Then $\eta(t, x) = \sum b_j \eta_j(t) e_j(x)$, where $\{\eta_j = \dot{\beta}_j, j \in \mathbb{Z}_*^2\}$, are standard independent *white noises*. It follows from the Doob–Kolmogorov inequality (see Theorem 3.8 in [KS91, Chapter 1]) that, with probability 1, the series in (7) converges in H^1 uniformly in $t \in [0, T]$ for any $T < \infty$, so that $\eta(t)$ is the time derivative of a continuous vector function with range in the space $V = H \cap H^1(\mathbb{T}^2, \mathbb{R}^2)$.

Random Kicks Let $\{\eta^k\}$ be a sequence of i.i.d. random variables (the kicks) with range in V. Define

$$\eta(t, x) = \frac{\partial}{\partial t}\zeta(t, x), \quad \zeta(t, x) = \sum_{k=1}^{\infty} \eta^k(x)\theta(t - k), \tag{8}$$

where $\theta(t) = 0$ for $t < 0$ and $\theta(t) = 1$ for $t \geq 0$. Since η has jumps only at positive integers, the trajectories of η are the time derivative of piecewise continuous functions.

Piecewise Independent Process Let $\{\eta^k\}$ be a sequence of i.i.d. random variables in $L^2(J_1, H)$. We define a random process of the form

$$\eta(t, x) = \sum_{k=1}^{\infty} I_{[k-1,k)}(t)\eta^k(t - k + 1, x), \tag{9}$$

where $I_{[k-1,k)}$ is the indicator function of the interval.

Example 1 Let us take H-valued processes $\{\eta^k(t, \cdot), 0 \le t \le 1\}$ of the form

$$\eta^k(t, x) = \sum_{j \in \mathbb{Z}_*^2} b_j \, \eta_j^k(t) \, e_j(x) \,, \qquad \mathfrak{B}_1 < \infty \,,$$

where $\{\eta_j^k : k \ge 1, j \in \mathbb{Z}_*^2\}$ are real-valued independent random processes, distributed as a fixed process $\tilde{\eta} : [0, 1] \to \mathbb{R}$. Taking for $\tilde{\eta}$ a random series with respect to the Haar basis of the space $L_2(0, 1)$ (see Section 21 in [Lam96]), we arrive at a random process η as in (9) that has the form

$$\eta(t, x) = \sum_{j \in \mathbb{Z}_*^2} b_j \, \eta_j(t) \, e_j(x) \,. \tag{10}$$

Here $\{\eta_j\}$ are independent random processes, distributed as the process

$$\eta_0(t) = c \sum_{l=1}^{\infty} \xi_l I_{[l-1,l)}(t) + \sum_{N=0}^{\infty} \sum_{l=0}^{\infty} c_n \xi_l^n H_l^n(t) \,, \tag{11}$$

where c and c_n are real constants, ξ_l and ξ_l^n are i.i.d. real-valued random variables with a law λ, and H_l^n are the Haar functions:

$$H_l^n(t) = \begin{cases} 0 & \text{if } \ t < l2^{-n} \text{ or } t \ge (l+1)2^{-n}, \\ 1 & \text{if } \ l2^{-n} \le t < (l+1/2)2^{-n}, \\ -1 & \text{if } \ (l+1/2)2^{-n} \le t < (l+1)2^{-n}. \end{cases}$$

Thus, $\eta_0(t)$ is a random wavelet series, and $\eta(t, x)$ is an H-valued process whose expansion in the trigonometric basis $\{e_j\}$ has independent random wavelet coefficients. Notice that a.s. the set of discontinuities of η is the family of dyadic numbers and, hence, is dense on the positive half-line. Besides, all trajectories of η are continuous at non-dyadic points and right-continuous everywhere.

If $c = 1, c_n = 2^{n/2}$, and λ is the centred normal law with unit variance, then $\eta_0(t)$ is white noise (see [Lam96]). If $c_n \ll 2^{n/2}$, then $\eta_0(t)$ is a *red noise*. In particular, if $|c_n| \le Cn^{-q}$ for some $q > 1$, and λ has a bounded support, then the red noise $\eta_0(t)$ is bounded uniformly in t and ω.

For reasons of space, we shall usually state the results for the case of spatially regular white noise. However, suitable reformulations of most of them remain valid in the two other cases.

Markov Process and a Priori Estimates

The family of solutions, corresponding to the three classes of random forces considered in the foregoing subsection, give rise to Markov processes in the space H. This allows us to apply to the study of the Navier–Stokes system (1) with such random forces well-developed probabilistic methods. We start with Eqs. (1), (7) (recalling that always $h \in H$ and $\mathfrak{B}_1 < \infty$), and for a random initial data $u_0 = u_0^\omega(x)$, independent from the force f, denote by $u(t; u_0)$ a solution of (1), (2), (7). For a non-random $u_0 = v \in H$ denote

$$P_t(v, \cdot) = \mathcal{D}(u(t; v)) .$$

This is a measure in H, depending on (t, v) in a measurable way, and satisfying the Kolmogorov–Chapman relation. So P_t is the transition function of a Markov process in H. The latter is the Markov process generated by solutions of Eqs. (1), (7). It defines the *Markov semigroups* on functions and on measures by the relations

$$\mathfrak{P}_t : C_b(H) \to C_b(H), \qquad (\mathfrak{P}_t g)(u) = \int_H P_t(u, \mathrm{d}z) g(z), \tag{12}$$

$$\mathfrak{P}_t^* : \mathcal{P}(H) \to \mathcal{P}(H), \qquad (\mathfrak{P}_t^* \mu)(\Gamma) = \int_H P_t(z, \Gamma) \mu(\mathrm{d}z). \tag{13}$$

The two semigroups are instrumental to study the equation since the former defines the evolution of mean values of observables: for any $g \in C_b(H)$ and any $v \in H$, we have

$$\mathbb{E}\, g(u(t; v)) = (\mathfrak{P}_t g)(v), \quad t \geq 0. \tag{14}$$

On the other hand, the latter defines the evolution of the laws since

$$\mathcal{D}(u(t; u_0)) = \mathfrak{P}_t^* \mu, \quad t \geq 0, \tag{15}$$

if u_0 is a random variable independent from f, and its law equals μ.

The white in time structure of the noise allows not only to prove the Markovian character of evolution, but also to derive *a priori* estimates by an application of Ito's formula. The theorem below summarises some of them. For any integer $k \geq 0$, we set

$$\mathcal{E}_u(k,t) = t^k \|u(t)\|_k^2 + \int_0^t s^k \|u(s)\|_{k+1}^2 \mathrm{d}s, \quad t \geq 0.$$

In the case $k = 0$, we write $\mathcal{E}_u(t)$. For $k \in \mathbb{N}$ we denote $H^k = H \cap H^k(\mathbb{T}^2, \mathbb{R}^2)$ (so $H^1 = V$).

Theorem 2.2 *Consider Eqs.* (1), (7). *The following properties hold for any $\nu > 0$ and any H-valued random variable u_0, independent from ζ.*

Energy and Enstrophy Balances *If $\mathbb{E}|u_0|_2^2 < \infty$, then*

$$\mathbb{E}\,|u(t)|_2^2 + 2\nu\mathbb{E}\int_0^t |\nabla u(s)|_2^2 ds = \mathbb{E}\,|u_0|_2^2 + \mathfrak{B}_0 t + 2\,\mathbb{E}\int_0^t (u,h)\mathrm{d}s. \tag{16}$$

If, in addition, $\mathbb{E}\,\|u_0\|_1^2 < \infty$, then

$$\mathbb{E}\,\|u(t)\|_1^2 + 2\nu\mathbb{E}\int_0^t |\Delta u(s)|_2^2 ds = \mathbb{E}\,\|u_0\|_1^2 + \mathfrak{B}_1 t + 2\,\mathbb{E}\int_0^t (\nabla u, \nabla h)\mathrm{d}s. \tag{17}$$

Time Average *There is $\gamma > 0$ depending only on $\{b_j\}$ such that*

$$\mathbb{P}\left\{ \sup_{t\geq 0}\bigl(\mathcal{E}_u(t) - (\mathfrak{B}_0 + 2\nu^{-1}|h|_2^2)\,t\,\bigr) \geq |u_0|_2^2 + \rho \right\} \leq e^{-\gamma\nu\rho}, \quad \rho > 0.$$

Exponential Moment *There is $c > 0$ not depending on ν, h and $\{b_j\}$ such that, if $\varkappa > 0$ and u_0 satisfy the inequalities*

$$\varkappa \sup_{j\geq 1} b_j^2 \leq c, \quad \mathbb{E}\exp\bigl(\varkappa\nu|u_0|_2^2\bigr) < \infty,$$

then, for some number $K = K(\nu, \varkappa, \mathfrak{B}_0, h)$, we have

$$\mathbb{E}\exp\bigl(\varkappa\nu|u(t)|_2^2\bigr) \leq e^{-\varkappa\nu^2 t}\mathbb{E}\exp\bigl(\varkappa\nu|u_0|_2^2\bigr) + K, \quad t \geq 0. \tag{18}$$

Higher Sobolev Norms *Suppose that $h \in H^k$ and $\mathfrak{B}_k < \infty$ for some integer $k \geq 1$. Then, for any $m \geq 1$ and $T \geq 1$, there is $C(k,m,T) > 0$ such that*

$$\mathbb{E}\sup_{0\leq t\leq T} \mathcal{E}_u(k,t)^m \leq C(k,m,T)\bigl(1 + \nu^{-m(7k+2)}\bigl(\mathbb{E}\,|u_0|_2^{4m(k+1)} + 1\bigr)\bigr). \tag{19}$$

A straightforward consequence of the energy balance (16) and Grönwall's inequality is the exponentially fast stabilisation of the L^2 norms of solutions:

$$\mathbb{E}\,|u(t)|_2^2 \le e^{-\nu t}\mathbb{E}\,|u_0|_2^2 + \nu^{-1}\mathfrak{B}_0 + \nu^{-2}|h|_2^2, \quad t \ge 0.$$

Combining this with (18) and (19), we conclude that, if all moments of $|u_0|_2^2$ are finite, then

$$\mathbb{E} \sup_{s\le t\le s+T} \mathcal{E}_u(k,t)^m \le C'(k,m,T) \quad \text{for all } s \ge 0,\ m \in \mathbb{N}. \tag{20}$$

A proof of all these results, as well as of their counterparts for random kick–forces, can be found in Chapter 2 of [KS12].

The Markov process defined by solutions of Eqs. (1), (7) is time-homogeneous: the law at time t_2 of a solution $u(t)$ which takes a prescribed deterministic value v at $t = t_1 < t_2$ depends only on v and $t_2 - t_1$. Solutions of the kick-forced equation (1), (8) define an inhomogeneous Markov process, but its restriction to integer values of time $t \in \mathbb{Z}$ is a homogeneous Markov chain, and when studying Eqs. (1), (8) we usually restrict ourselves to integer t's, see in [KS12]. Finally, solutions $u(t)$ of Eqs. (1), (9) do not define a Markov process, but their restrictions to $t \in \mathbb{Z}$ form a homogeneous Markov chain, which is the subject of our study when dealing with that equation, see [Shi15, KNS20].

3 Mixing

In the previous section, we discussed the existence, uniqueness, and regularity of the flow for the Navier–Stokes system subject to an external random force. Our next goal is to study its large-time asymptotics. As before, we shall mostly concentrate on the case of spatially regular white noise.

Existence of a Stationary Measure

We consider the Navier–Stokes system (1) with the right-hand side of the form (4), where $h \in H$ is a deterministic function and η has the form (7). Since $\mathfrak{B}_1 < \infty$, then ζ is a continuous functions of time with range in V.

Definition 2 A measure $\mu \in \mathcal{P}(H)$ is said to be *stationary* for the Navier–Stokes system if $\mathfrak{P}_t^*\mu = \mu$ for all $t \ge 0$.

By (14), (15), a measure $\mu \in \mathcal{P}(H)$ is stationary if and only if there is an H-valued random variable u_0 independent from ζ such that $\mathcal{D}(u_0) = \mu$, and one of the following equivalent properties is satisfied for the corresponding solution $u(t; u_0)$:

(a) for all $g \in C_b(H)$ and $t \geq 0$, we have $\mathbb{E}\, g(u(t)) = \mathbb{E}\, g(u_0)$;
(b) the measure $\mathcal{D}(u(t))$ coincides with μ for all $t \geq 0$.

A solution $u(t)$ of (1) as in (a) (or (b)) is called a *stationary solution*.

The existence of a stationary measure for the 2d Navier–Stokes system can be established with the help of the Borolyugov–Krylov argument, even though the first works dealing with both 2d and 3d cases used a different approach; see [VF88]. We refer the reader to the article [Fla94] a for self-contained and simple proof of the existence of a stationary measure. In addition, Theorem 2.2 jointly with Fatou's lemma imply *a priori* estimates for any stationary measure of (1). A detailed proof of the following result can be found in [KS12, Chapter 2].

Theorem 3.3 *The Navier–Stokes system* (1)*,* (7) *has at least one stationary measure* μ_ν*. It is supported by the space* H^2 *and satisfies the energy and enstrophy balance relations*

$$\nu \int_H |\nabla u|_2^2 \mu_\nu(\mathrm{d}u) = \frac{1}{2}\mathfrak{B}_0 + \int_H (h, u)\mu_\nu(\mathrm{d}u), \tag{21}$$

$$\nu \int_H |\Delta u|_2^2 \mu_\nu(\mathrm{d}u) = \frac{1}{2}\mathfrak{B}_1 + \int_H (\nabla h, \nabla u)\mu_\nu(\mathrm{d}u). \tag{22}$$

If, in addition, $h \in C^\infty$ *and* $\mathfrak{B}_k < \infty$ *for all* $k \geq 0$*, then every stationary measure* $\mu_\nu \in \mathcal{P}(H)$ *is concentrated on infinitely smooth functions,*[2] *and there are positive numbers* $\varkappa$*,* C*, and* C_{km} *not depending on* ν *such that, for any integers* $m \geq 1$ *and* $k \geq 2$*, we have*

$$\int_H \exp\bigl(\varkappa\nu\|u\|_1^2\bigr)\mu_\nu(\mathrm{d}u) \leq C, \tag{23}$$

$$\nu^{m(7k+2)} \int_H \|u\|_k^{2m} \mu_\nu(\mathrm{d}u) \leq C_{km}. \tag{24}$$

Besides, for stationary solutions $u(t, x)$ of Eq. (1), estimate (19) implies uniform in $s \geq 0$ bounds of the form

$$\mathbb{E} \sup_{s \leq t \leq s+T} \|u(t)\|_k^{2m} \leq C(k, m, T)\, \nu^{-m(7k+2)}\,;$$

see Corollary 2.4.13 in [KS12].

[2] This means that the μ_ν-measure of the space of C^∞ functions is equal to 1.

Uniqueness and Exponential Stability

In contrast to the existence of a stationary measure (which is established by rather soft tools), its uniqueness is a deep result that was proved thanks to the contribution of various research groups. It was first established in the case of spatially irregular white noise by Flandoli and Maslowski [FM95] and then extended to various types of regular noises in [KS00] and next in [EMS01, BKL02, KS01, Kuk02b, Mat02, KS02, KS03, Shi05, Oda08] (see Chapter 3 in [KS12] for more references). The following theorem summarises those results in the case of spatially regular white noise.

Theorem 3.4 *If the random process ζ in (7) satisfies*

$$b_j \neq 0 \quad \textit{for all } j \in \mathbb{Z}^2_*\,, \tag{25}$$

then the problem (1), (7) has a unique stationary distribution $\mu_\nu \in \mathcal{P}(H)$. This measure possesses the following properties.

Exponential mixing. *There are positive numbers γ_ν and $\varkappa_\nu$ such that, for any locally Hölder-continuous function $g : V \to \mathbb{R}$ with at most exponential growth at infinity and any H-valued random variable u_0 independent from ζ, we have*

$$\big|\mathbb{E}\, g(u(t;u_0)) - \langle g, \mu_\nu\rangle\big| \le C(\nu, g)e^{-\gamma_\nu t}\,\mathbb{E}\, e^{\varkappa_\nu |u_0|_2^2}, \quad t \ge 1, \tag{26}$$

where $C(\nu, g)$ depends on ν and a specific norm of g, but not on u_0.

Convergence for observables. *If, in addition, $h \in H^k$ and $\mathfrak{B}_k < \infty$ for all k, then a similar convergence holds for any Hölder-continuous function g that is defined on the Sobolev space H^s of any finite order and has at most polynomial growth at infinity.*

Space homogeneity. *If, in addition to (25), $b_s \equiv b_{-s}$, then the measure μ_ν is space homogeneous (i.e. the space translations $H \ni u(x) \mapsto u(x+y)$, $y \in \mathbb{T}^2$, do not change it).*

The first assertion of the theorem implies that the Markov process in H which we discuss is *exponentially mixing in the Lipschitz–dual distance in the space H*, i.e. for each measure $\rho \in \mathcal{P}(H)$ with a finite second exponential moment we have

$$\|\mathfrak{P}_t^*\rho - \mu_\nu\|^*_{\mathrm{Lip}(H)} \le C_\nu e^{-\gamma'_\nu t} \quad \text{for} \quad t \ge 0 \tag{27}$$

with some $C_\nu, \gamma'_\nu > 0$, where

$$\|\rho_1 - \rho_2\|^*_{\mathrm{Lip}(H)} = \sup\langle \rho_1 - \rho_2, f\rangle\,, \quad \langle \rho, f\rangle = \int_H f(v)\,\rho(dv)\,, \tag{28}$$

and the supremum is taken oven all Lipschitz functions f on H whose norm and Lipschitz constant are bounded by one. If (27) holds, we also say that the stationary measure μ_ν and Eq. (1) are exponentially mixing (in H).

Inequality (26) expresses the property of convergence of the ensemble average of an observable g to its mean value with respect to the stationary measure. It applies to various physically relevant quantities, such as the energy $\frac{1}{2}|u|_2^2$, the enstrophy $\frac{1}{2}|\nabla u|_2^2 = \frac{1}{2}|\operatorname{rot} u|_2^2$, and the correlation tensors $u_i(x)u_j(y)$, where $x, y \in \mathbb{T}^2$ are arbitrary points.

Similar results hold for solutions of Eqs. (1), (8) if in (26) we replace $t \geq 1$ by $t \in \mathbb{N}$.

In Theorem 3.4, the rate of convergence depends on the viscosity ν: when the latter decreases, the attractor of the unperturbed problem becomes larger and more chaotic, and it seems to be a very complicated task to establish a uniform convergence to the limiting measure. On the other hand, it is not difficult to prove that, when $\nu > 0$ is fixed, the non-degeneracy condition (25) can be relaxed, requiring only that the noise should act directly on the (finitely many[3]) determining modes of the dynamics. A challenging problem, important, in particular, for numerical simulations, is to prove that the mixing property remains true under a weaker hypothesis on the noise, allowing for the noise's localisation in a part of either the physical or the Fourier spaces, so that the determining modes of the dynamics do not necessarily belong to the region of the phase space affected by the noise. Propagation of the randomness then may take place due to the "mixing" properties of the (deterministic) Navier–Stokes flow. Uniqueness and mixing of the random flow in this situation are mostly established in the case when the deterministic component h of the random force is zero.

We begin with the case when the random force is localised in the Fourier space. In [HM06, HM11] Hairer and Mattingly obtained the following result:

Theorem 3.5 *Let the random force f have the form (4), (7) with $h = 0$ and with η satisfying*

$$b_j \neq 0 \quad \textit{if and only if } j \in \mathcal{I}, \tag{29}$$

where $\mathcal{I}$ is a finite subset of $\mathbb{Z}_*^2$ such that any vector in $\mathbb{Z}^2$ can be represented as an integer linear combination of the elements of $\mathcal{I}$, and $\mathcal{I}$ contains at least two vectors of different length. Then Eq. (1) is exponentially mixing in H.

A key ingredient of the proof is an infinite-dimensional version of the Malliavin calculus, which uses the white noise structure of the random perturbation. Note that the result above does not imply the assertion of Theorem 3.4 since now the set of modes $j \in \mathbb{Z}_*^2$, excited by the random force, must be finite.

[3] The fact that dissipative parabolic-type PDEs have only finitely many determining modes goes back to the paper [FP67] and implies, in particular, that the global attractor is finite-dimensional.

The recent paper [KNS20] deals with the case when the noise is a piecewise independent random process of the form (9). The main result of [KNS20] is an abstract theorem, establishing the mixing property for a large class of nonlinear PDE, perturbed by bounded random forces of the form (4), (9); its proof relies on the method of optimal control and a variant of the Nash–Moser scheme. In particular, the theorem in [KNS20] applies to the Navier–Stokes system, perturbed by a bounded red noise as in Example 1:

Theorem 3.6 *Let the random force f have the form* (4)*, in which $h = 0$ and η is given* (10)*,* (11)*, with the set of coefficients b_j satisfying* (29)*. Assume that $|c_n| \le Cn^{-q}$ for all $n \ge 1$ with some $q > 1$, and that the law of the random variables ξ_l and ξ_l^n has the form $\lambda = p(r)\,\mathrm{d}r$ with $p \in C_0^1(-1, 1)$, $p(0) \ne 0$. Then Eq.* (1) *is exponentially mixing in H.*

The situation in which the random perturbation is localised in the physical space was studied in [Shi15] (see also [Shi21] for the case of a boundary noise). To formulate the corresponding result, we fix a non-empty open set $Q \subset \mathbb{R} \times \mathbb{T}^2$ whose closure is contained in $(0, 1) \times \mathbb{T}^2$ and denote by $\{\varphi_j\} \subset H^1(Q, \mathbb{R}^2)$ an orthonormal basis in $L^2(Q, \mathbb{R}^2)$. Considering again the random force (4), we assume that $h \equiv 0$, and η is a piecewise independent random process of the form (9), with

$$\eta^k(t, x) = \sum_{j=1}^{\infty} b_j \xi_{jk} \psi_j(t, x).$$

Here $\psi_j = \chi\varphi_j$, where $\chi \in C_0^\infty(Q)$ is a nonzero function, $\{b_j\}$ are real numbers such that $\sum_j b_j \|\psi_j\|_1 < \infty$, and ξ_{jk} are independent random variables whose laws have have the form $\lambda_j = p_j(r)\,\mathrm{d}r$, where $p_j \in C_0^1(-1, 1)$ and $p_j(0) \ne 0$. Thus, the random force entering the right-hand side of (1) is bounded and space-time localised in Q. The following theorem is the main result of [Shi15].

Theorem 3.7 *In addition to the above hypotheses, assume that $b_j \ne 0$ for all $j \ge 1$. Then, for any $\nu > 0$, Eq.* (1) *is has a unique stationary measure, which is exponentially mixing in H.*

4 Consequences of Mixing

The results of the previous section concern the evolution of the mean values of observables and the laws of solutions under the stochastic Navier–Stokes flow. This section is devoted to studying the typical behaviour of individual trajectories. We will assume that the random force in Eq. (1) is such that the equation is exponentially mixing, either for $t > 0$, or for $t \in \mathbb{N}$. That is, either the assumptions of Theorems 3.4, 3.5, 3.6 or 3.7 hold, or f is a kick-force of the form (4), (8), where (25) holds. We state the results for the case when Theorem 3.4 applies. Situation with the other cases is very similar; e.g., see [KS12] for the case of the kick-forced equations.

Ergodic Theorems

We consider the Navier–Stokes system (1), (4) in which $h \in H$ is a fixed function and η is given by (7). Let us denote[4] by $\mathcal{H}$ the space of locally Hölder-continuous functions $g : V \to \mathbb{R}$ with at most exponential growth at infinity. The following result established in [Kuk02a, Shi06] (see also Section 4.1.1 in [KS12]) shows that the time average of a large class of observables converges to their mean value with respect to the stationary measure.

Theorem 4.8 (Strong Law of Large Numbers) *Under the hypotheses of Theorem 3.4, for any $\gamma \in [0, 1/2)$, any non-random $u_0 \in H$ and $g \in \mathcal{H}$, with probability* 1 *we have*

$$\lim_{t\to\infty} t^{\gamma}\left(\frac{1}{t}\int_0^t g(u(s;u_0))\,\mathrm{d}s - \langle g, \mu_\nu\rangle\right) = 0\,. \tag{30}$$

Convergence (30) remains true for random initial functions that are independent from η and have a finite exponential moment. Moreover, some further analysis shows that the law of iterated logarithm (LIL) is also valid. In particular, the number γ in (30) characterizing the rate of convergence to the mean value cannot be taken to be equal to $1/2$; see Section 4.1.2 in [KS12].

We now turn to the central limit theorem (CLT). To this end, given an observable $g \in \mathcal{H}$ with zero mean value with respect to μ_ν, we denote

$$\sigma_g^2 = 2\int_H \int_0^\infty (\mathfrak{P}_t g)(v)\mathrm{d}t\, g(v)\mu_\nu(\mathrm{d}v).$$

In view of inequality (26) and the assumption that $\langle g, \mu_\nu\rangle = 0$, the function $\mathfrak{P}_t g$ decays exponentially to zero as $t \to \infty$, and it is not difficult to prove that, under the hypotheses of Theorem 3.4, the number σ_g^2 is well defined and positive for any non-constant g; see Proposition 4.1.4 in [KS12]. A proof of the following result can be found in [Kuk02a, Shi06] (see also Section 4.1.3 in [KS12]).

[4] To avoid unimportant complications, we do not give an exact definition of the space $\mathcal{H}$, referring the reader to Section 4.1 in [KS12].

Theorem 4.9 (Central Limit Theorem) *Under the hypotheses of Theorem 3.4, for any non-random $u_0 \in H$ and any non-constant $g \in \mathcal{H}$ satisfying $\langle g, \mu_\nu \rangle = 0$, we have*

$$\mathcal{D}\biggl(\frac{1}{\sqrt{t}} \int_0^t g(u(s; u_0))\,\mathrm{d}s\biggr) \rightharpoonup \mathcal{N}_{\sigma_g} \quad \textit{as } t \to \infty, \tag{31}$$

where $\mathcal{N}_\sigma$ stands for the centred normal law on $\mathbb{R}$ with a variance $\sigma^2 > 0$, and $\rightharpoonup$ stands for the weak convergence of measures.

Let us emphasise that σ_g is expressed in terms of the stationary measure μ_ν and does not depend on u_0. Furthermore, convergence (31) is equivalent to the relation

$$\lim_{t\to\infty} \mathbb{P}\biggl\{\frac{1}{\sqrt{t}} \int_0^t g(u(s))\,\mathrm{d}s \in \Gamma\biggr\} = \mathcal{N}_{\sigma_g}(\Gamma), \tag{32}$$

where $\Gamma \subset \mathbb{R}$ is an arbitrary Borel set whose boundary has zero Lebesgue measure.

Random Attractors

Another important object characterising the large-time behaviour of trajectories, is the random attractor. There are many definitions of this object, and in the context of random dynamical systems most of them deal with the concept of pullback attraction. The meaning of the latter is that, for some fixed observation time, the distance between trajectories of the system and the attractor decreases when the moment of beginning of the observation goes to $-\infty$. In this section, we discuss a concept of attractor that possesses an attraction property forward in time and is closely related to the unique stationary measure constructed in Theorem 3.4. To simplify notation, we fix the viscosity $\nu > 0$ and do not follow the dependence of various objects on it. Moreover, we assume that the Brownian motions $\{\beta_j\}$ entering (7) are two-sided[5] and denote by $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ the canonical filtered probability space associated with the process ζ; see Section 2.4 in [KS91]. In particular, the measurable space $(\Omega, \mathcal{F})$ may be chosen to coincide with the Fréchet space $C(\mathbb{R}, V)$ of continuous functions from $\mathbb{R}$ to V, with the topology of uniform convergence on bounded intervals and the corresponding Borel sigma-algebra.

Let us denote by $\varphi_t(\omega) : H \to H$ the random flow generated by the Navier–Stokes system with spatially regular white noise (7). Thus, for any initial function $u_0 \in H$, the solution $u(t; u_0)$ of (1), (2), (7) is given by $\varphi_t(\omega)u_0$. The family $\{\varphi_t(\omega), \omega \in \Omega, t \geq 0\}$ possesses the perfect co-cycle property. Namely, let $\theta_t : \Omega \to \Omega$ be the shift operator taking $\omega(\cdot)$ to $\omega(\cdot + t)$. Then, there is a set of full

[5] That is, they are defined for all $t \in \mathbb{R}$ and vanish at $t = 0$.

measure $\Omega_0 \in \mathcal{F}$ such that, for any $\omega \in \Omega_0$, we have

$$\varphi_{t+s}(\omega)u_0 = \varphi_t(\theta_s(\omega))\varphi_s(\omega)u_0 \quad \text{for all } t, s \geq 0, u_0 \in H.$$

The following result is established in [Led86, LeJ87, Cra91] (see also Theorem 4.2.9 in [KS12]) in the context of general random dynamical systems.

Theorem 4.10 *Under the hypotheses of Theorem 3.4, for any sequence $\{t_k\}$ going to $+\infty$ the limit*

$$\mu_\omega = \lim_{k\to\infty} \varphi_{t_k}(\theta_{-t_k}(\omega))_*\mu \tag{33}$$

exists in the weak topology of $\mathcal{P}(\Omega)$ for almost every $\omega \in \Omega$. Moreover, the following properties hold:

Uniqueness. *If $\{t_k'\}$ is another sequence going to $+\infty$ and $\{\mu_\omega'\}$ is the corresponding limit, then $\mu_\omega = \mu_\omega'$ almost everywhere.*
Reconstruction. *For any $\Gamma \in \mathcal{B}(H)$, the mapping $\omega \mapsto \mu_\omega(\Gamma)$ is measurable, and μ can be reconstructed by the formula $\mu(\Gamma) = \mathbb{E}\mu_\cdot(\Gamma)$.*

We now describe a random attractor associated with μ. To this end, we fix any sequence $\{t_k\}$ going to $+\infty$ and denote by Ω_0 the set of full measure on which limit (33) exists. The almost sure existence of the limit in (33) implies that Ω_0 is $\mathbb{P}$-invariant under θ_t; that is, $\mathbb{P}(\Omega_0 \triangle (\theta_t(\Omega_0)) = 0$ for any $t \in \mathbb{R}$, where $\triangle$ stands for the symmetric difference of two sets. We define μ_ω by (33) on the set of full measure Ω_0 and denote by $\mathcal{A}_\omega$ the support of μ_ω for $\omega \in \Omega_0$, while we set $\mathcal{A}_\omega = \varnothing$ on the complement of Ω_0. The measurability property of $\mu_\cdot$ mentioned in Theorem 4.10 implies that $\{\mathcal{A}_\omega, \omega \in \Omega\}$ is also measurable in the sense that, for any $u \in H$, the function $\omega \mapsto d_H(u, \mathcal{A}_\omega)$ is measurable. The following result is established in [KS04] (see also Section 4.2 in [KS12]).

Theorem 4.11 *Under the hypotheses of Theorem 3.4, the following properties hold:*

Invariance. *For any $t \geq 0$ and almost every $\omega \in \Omega$, we have $\varphi_t(\omega)\mathcal{A}_\omega = \mathcal{A}_{\theta_t(\omega)}$.*
Attraction. *For any $u \in H$, the functions $\omega \mapsto d_H(\varphi_t(\omega)u, \mathcal{A}_{\theta_t(\omega)})$ converge to zero in probability as $t \to \infty$. That is, for any $\varepsilon > 0$, we have*

$$\mathbb{P}\big\{d_H(\varphi_t(\omega)u, \mathcal{A}_{\theta_t(\omega)}) \geq \varepsilon\big\} \to 0 \quad \text{as } t \to \infty.$$

Minimality. *If $\{\mathcal{A}_\omega', \omega \in \Omega\}$ is another measurable family of closed subsets that satisfies the first two properties, then $\mathcal{A}_\omega \subset \mathcal{A}_\omega'$ for almost every $\omega \in \Omega$.*

Dependence on Parameters and Stability

We now investigate how the laws of trajectories of (1) vary with parameters. To simplify the presentation, we shall consider only the dependence on the random forcing, assuming that it is a spatially regular white noise. However, similar methods can be used to study the dependence on other parameters, such as the viscosity, as well as the relationship between stationary measures corresponding to various types of random forcings.

We thus assume that X is a metric space and that the right-hand side in (1) has the form (4), (7), where $h \in H$ is fixed and b_j's are continuous functions of $a \in X$, satisfying

$$\sup_{a\in X} \sum_{j\in\mathbb{Z}_*^2} b_j(a)^2|j|^2 < \infty.$$

A proof of the following result can be found in Section 4.3.1 of [KS12].

Theorem 4.12 *In addition to the above hypotheses, let*

$$b_j(\hat{a}) \neq 0 \quad \text{for all } j \in \mathbb{Z}_*^2, \tag{34}$$

where $\hat{a} \in X$ is a fixed point, and let $\{\mu^a\}$ be a family of stationary measures[6] *of Eq.* (1) *with the right-hand side corresponding to the value $a \in X$ of the parameter. Then $\mu^a \to \mu^{\hat{a}}$ as $a \to \hat{a}$, and for any compact subset $\Lambda \subset \mathcal{P}(H)$ there exists a continuous function $A_\Lambda(\rho) > 0$ going to zero with ρ such that*

$$\sup_{t\geq 0}\big\|\mathfrak{P}_t^*(a)\lambda_1 - \mathfrak{P}_t^*(\hat{a})\lambda_2\big\|_L^* \leq A_\Lambda\big(\|\lambda_1-\lambda_2\|_L^* + d_X(a,\hat{a})\big) \quad \text{for } \lambda_1,\lambda_2\in\Lambda,\ a\in X,$$

where $\mathfrak{P}_t^*(a)$ denotes the Markov semigroup for (1) *corresponding to the parameter a.*

We emphasise that by this result the law of a solution for (1) depends on the parameters of the random force f *continuously and uniformly in time*; cf. Theorem 7.17 below.

5 Large Deviations

Having discussed the typical behaviour of trajectories, we now turn to a description of probabilities of rare events. Two different asymptotics will be studied: deviations of the time-average of observables from their ensemble average as $t \to \infty$, and

[6] Existence of a measure μ^a is guaranteed by Theorem 3.3, and Theorem 3.4 implies that the measure $\mu^{\hat{a}}$ is uniquely determined.

deviations from the limiting dynamics in the small noise regime. In the latter setting, we shall only discuss the behaviour of a stationary distribution, since in the case of an additive noise the asymptotics of trajectories with given initial data is a simple result that follows immediately from the large deviations principle (LDP) for Gaussian random variables. We refer the reader to the paper [CM10] and the references therein for this type of results for stochastic PDEs with multiplicative noise.

Donsker–Varadhan Type Large Deviations

We first describe the general idea. To this end, let us note that, for an arbitrary function $g \in \mathcal{H}$ (whose mean value is not necessarily zero), we can rewrite the convergence (32) in the form

$$\lim_{t\to\infty} \mathbb{P}\left\{ \frac{1}{t}\int_0^t g(u(s))\,\mathrm{d}s \in \langle g, \mu_\nu\rangle + \frac{\Gamma}{\sqrt{t}} \right\} = \mathcal{N}_{\sigma_g}(\Gamma), \tag{35}$$

Thus, the CLT can be interpreted as a description of the probabilities of small deviations of the time average of observables from their mean value. The goal of the theory of large deviations is to describe the probabilities of order 1 deviations (when $\Gamma/\sqrt{t}$ in (35) is replaced by Γ). This type of results were first obtained by Donsker and Varadhan in the case of finite-dimensional diffusion processes [DV75] and later extended to many other situations. In the context of the Navier–Stokes equations, the theory was developed in the case of random kicks [JNPS15, JNPS18] (see also the recent papers [MN18, Ner19] devoted to spatially regular white noise), and we now describe the main achievements.

Let us consider the Navier–Stokes system (1), (4), where $h \in H$ is a deterministic function, and the random forcing η is given by (8). In this case, the trajectories of (1) have jumps at integer times, and we normalise them to be right-continuous in time. Setting $u_k = u(k)$, we see that the random sequence $\{u_k\}$ satisfies the relation

$$u_k = S(u_{k-1}) + \eta^k, \quad k \geq 1, \tag{36}$$

where $S : H \to H$ denotes the time-1 shift along trajectories of Eq. (1) with $f = h$. Equation (36) defines a discrete-time Markov process in H, and we use the notation introduced in Sect. 2 to denote the related objects, replacing t with k. We thus write $P_k(v, \Gamma)$, $\mathfrak{P}_k$, and $\mathfrak{P}_k^*$.

To formulate the result on LDP, we shall need some hypotheses on the kicks η^k. Namely, we shall assume that they satisfy the following condition, in which $\{e_j\}$ is the trigonometric basis in H defined by (5).

Structure of the noise. *The random kicks η^k have the form*

$$\eta^k(x) = \sum_{j \in \mathbb{Z}_*^2} b_j \xi_{jk} e_j(x),$$

where b_j are some non-zero numbers satisfying $\mathfrak{B}_1 < \infty$ and $\{\xi_{jk}\}$ are independent random variables whose laws possess C^1-smooth positive densities ρ_j with respect to the Lebesgue such that $\int_{\mathbb{R}} |\rho_j'(r)| \mathrm{d}r \leq 1$ for any j.

This condition ensures that the Markov process $\{u_k\}$ associated with (1) has a unique stationary measure μ_ν for any $\nu > 0$, and the CLT holds for any Holder-continuous functional $f : H \to \mathbb{R}$ with at most exponential growth at infinity (cf. Theorems 3.4 and 4.9).

We now introduce the *occupation measures*

$$\mu_n^\omega = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{u_k},$$

where $\delta_v \in \mathcal{P}(H)$ stands for the Dirac mass at $v \in H$, and $\{u_k\}$ is a trajectory of (36). Let us recall that the space $\mathcal{P}(H)$ is endowed with the topology of weak convergence. We shall say that a mapping $I : \mathcal{P}(H) \to [0, +\infty]$ is a *good rate function* if its sub-level set $\{\lambda \in \mathcal{P}(H) : I(\lambda) \leq c\}$ is compact for any $c \geq 0$. The following result is established in Section 1 of [JNPS18] (see also [JNPS15] for the case of bounded kicks).

Theorem 5.13 *Suppose that the above hypothesis on the structure of the noise is satisfied. Then the following assertions hold for any $\nu > 0$:*

Pressure. *For any $g \in C_b(H)$ and any deterministic initial function $u_0 \in H$, there is a finite limit*

$$Q(g) = \lim_{n \to \infty} n^{-1} \log \mathbb{E} \exp\left\{ \sum_{k=0}^{n-1} g(u_k) \right\}$$

that is independent from u_0. Moreover, Q is a 1-Lipschitz function satisfying the relation $Q(g + C) = Q(g) + C$ for any $C \in \mathbb{R}$.

Rate function. *The Legendre transform $I : \mathcal{P}(H) \to [0, +\infty]$ of Q defined by*

$$I(\lambda) = \sup_{g \in C_b(H)} \big(\langle g, \lambda \rangle - Q(g) \big)$$

is a convex good rate function that can be represented by the Donsker–Varadhan relation

$$I(\lambda) = \sup_{g \geq 1} \int_H \log \frac{g}{\mathfrak{P}_1 g} \, d\lambda,$$

where the supremum is taken over all functions $g \in C_b(H)$ minorised by 1.

LDP. *For any random initial function u_0 independent from $\{\eta^k\}$ such that $\mathbb{E}\exp(\delta|u_0|_2^2) < \infty$ for some $\delta > 0$, and any Borel set $\Gamma \subset \mathcal{P}(H)$, we have*

$$-I(\dot{\Gamma}) \leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{\mu_n \in \Gamma\} \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{\mu_n \in \Gamma\} \leq -I(\overline{\Gamma}),$$

where $\dot{\Gamma}$ and $\overline{\Gamma}$ denote the interior and closure of Γ, respectively, and $I(A)$ is the infimum of I on the set A.

Assuming faster decay for the coefficients b_j and considering the Navier–Stokes system in higher Sobolev spaces, it is not hard to show that a similar result holds in the case when H is replaced by H^s. Next, application of the standard techniques of the theory of large deviations implies that, for any Hölder-continuous function $g : H^s \to \mathbb{R}$ with moderate growth at infinity, the time average $n^{-1}\sum_{k=0}^{n-1} g(u_k)$ satisfies the LDP with a good rate function $I_g : \mathbb{R} \to \mathbb{R}$ that can be expressed in terms of I by the relation (cf. Section 1.3 in [JNPS15])

$$I_g(r) = \inf\{I(\lambda) : \lambda \in \mathcal{P}(H), \langle g, \lambda \rangle = r\}.$$

Vanishing Noise Limit

We now go back to the Navier–Stokes system (1) with a spatially regular white noise and discuss the behaviour of the unique stationary measure as the stochastic component of the noise goes to zero. Namely, let us assume that the external force in (1) has the form

$$f(t,x) = h(x) + \sqrt{\varepsilon}\,\eta(t,x), \tag{37}$$

where $h \in H$ is a fixed function, $\varepsilon > 0$ is a small parameter, and η is given by (7). We assume that the coefficients b_j are non-zero (and satisfy the inequality $\mathfrak{B}_1 < \infty$), so that for any $\nu > 0$ and $\varepsilon > 0$ there is a unique stationary measure $\mu_\nu^\varepsilon \in \mathcal{P}(H)$. We are interested in the behaviour of μ_ν^ε for a fixed ν, so we drop ν from the notation in this section and write simply μ^ε. In what follows, we assume that the following hypothesis is satisfied.

Global asymptotic stability. *The flow of the unperturbed Navier–Stokes system (1), corresponding to $f = h$, has a unique fixed point $\hat{u} \in H$, which is* *globally asymptotically stable in the sense that any other trajectory converges to it as $t \to +\infty$.*

Notice that this condition is satisfied with $\hat{u} = 0$ if the deterministic part of the force (4) vanishes. A simple argument based on the uniqueness of a stationary distribution for the limiting equation implies that $\{\mu^\varepsilon\}$ converges weakly to the Dirac mass concentrated at $\hat{u}$. In fact, much more detailed information about that convergence is available. Namely, let us denote by $S_t(u_0, f)$ the solution of (1), (2) and introduce a *quasi-potential* by the relation

$$\mathcal{V}(v) = \lim_{r\to 0} \inf\left\{\frac{1}{2}\int_0^T \|f(s)\|_b^2 \mathrm{d}s : T > 0, f \in L^2(J_T, H), |S_T(\hat{u}, f) - v|_2 \le r\right\},$$

where the infimum is taken over all T and f for which the inequality holds, and we set $\|g\|_b^2 = \sum_j b_j^{-2}(g, e_j)^2$. A proof of the following result can be found in [Mar18] (see also [BC17] for the case of spatially irregular noise).

Theorem 5.14 *Suppose that global asymptotic stability for the limiting dynamics holds, and the coefficients $\{b_j\}$ entering (7) are non-zero. Then the function $\mathcal{V} : H \to [0, +\infty]$ has compact sub-level sets in H, vanishes only at the point $\hat{u}$, and controls the LDP for the family $\{\mu^\varepsilon\}$; that is, for any $\Gamma \in \mathcal{B}(H)$, we have*

$$-\inf_{v\in\dot{\Gamma}} \mathcal{V}(v) \le \liminf_{\varepsilon\to 0^+} \varepsilon \log \mu^\varepsilon(\Gamma) \le \limsup_{\varepsilon\to 0^+} \varepsilon \log \mu^\varepsilon(\Gamma) \le -\inf_{v\in\overline{\Gamma}} \mathcal{V}(v).$$

The facts that $\mathcal{V}$ vanishes only at $\hat{u}$ and has compact sub-level sets imply that $\mu^\varepsilon(H \setminus B) \sim e^{-c(B)/\varepsilon}$ as $\varepsilon \to 0$, where $B \subset H$ is an arbitrary ball around $\hat{u}$, and $c(B) > 0$ is a number. Hence, the LDP gives an estimate for the rate of concentration of μ^ε around $\hat{u}$. Let us also note that, in the case when the limiting dynamics is not globally asymptotically stable, it is still possible to prove that the family $\{\mu^\varepsilon\}$ is exponentially tight, but the validity of the LDP is not known to hold (see Open problem 3 in Sect. 8). However, more detailed information on the limiting dynamics would be sufficient to get the LDP. For instance, this is the case when there are finitely many stationary points, and the unperturbed dynamics possesses a global Lyapunov function. We refer the reader to [Mar17] for details.

6 Inviscid Limit

If the random force f in Eq. (1) has the form (37) with $h = 0$ and with η as in (7), where $\nu > 0$ is fixed and $\varepsilon \to 0$, then the corresponding stationary measure μ_ν^ε converges to the delta-measure at $0 \in H$, and the results of Sect. 5 describe this convergence in more detail. Now assume that ε and ν both go to zero in such a way that $\varepsilon = \nu^a$ with some $a > 0$. Applying to $\mu_\nu^{\nu^a}$ relation (21) with $h = 0$ and $b_j := \sqrt{\varepsilon}\, b_j$, $j \in \mathbb{Z}_*^2$, we get

$$\int_H |\nabla u|_2^2\, \mu_\nu^{\nu^a}(du) = \frac{\mathfrak{B}_0}{2}\, \nu^{a-1}.$$

From this we conclude that the measure $\mu_\nu^{\nu^a}$ may have a non-trivial limit as $\nu \to 0$ only if $a = 1$. Then Eq. (1) becomes

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \sqrt{\nu}\, \eta(t, x), \quad \operatorname{div} u = 0\,. \tag{38}$$

Assume that (25) holds. Then a stationary measure μ_ν^ν of (38) is unique. For short, we re-denote it as μ^ν.[7]

Properties of μ^ν, Independent from ν

In this subsection, we discuss a number of properties of the stationary measures μ^ν which hold uniformly in ν. They depend only on the quantities $\mathfrak{B}_0$ and $\mathfrak{B}_1$ and indicate certain universal properties of the statistical equilibria of Eq. (38).

Relations (21) and (22) with $h = 0$ and $\mathfrak{B}_0 =: \nu \mathfrak{B}_0$, $\mathfrak{B}_1 =: \nu \mathfrak{B}_1$ imply that, uniformly in ν,

$$\mathbb{E}^{\mu^\nu} |\nabla u|_2^2 = \frac{1}{2} \mathfrak{B}_0, \quad \mathbb{E}^{\mu^\nu} |\Delta u|_2^2 = \frac{1}{2} \mathfrak{B}_1\,. \tag{39}$$

Since $|\nabla u|_2^2 \le |u|_2 |\Delta u|_2$, then $\mathbb{E}^{\mu^\nu} |\nabla u|_2^2 \le (\mathbb{E}^{\mu^\nu} |u|_2^2)^{1/2} (\mathbb{E}^{\mu^\nu} |\Delta u|_2^2)^{1/2}$. It follows that $\mathfrak{B}_0^2 / 2\mathfrak{B}_1 \le \mathbb{E} |u^\nu|_2^2 \le \frac{1}{2} \mathfrak{B}_0$, so that the averaged kinetic energy is bounded below and above. Moreover, uniformly in ν, the measures μ^ν satisfy (23) with $\nu := 1$ (this follows immediately from the proof in [KS12]). Consider a stationary solution $u^\nu(t, x)$, corresponding to μ^ν. Then $\mathbb{E}\, |u^\nu|_2^2 = \mathbb{E}^{\mu^\nu} |u|_2^2$, and the Reynolds number of u^ν is

$$\mathrm{Re}_\nu = \frac{[u^\nu][x]}{\nu} = \frac{\left(\mathbb{E}\, |u^\nu|_2^2\right)^{1/2} \cdot 1}{\nu} \sim \nu^{-1}$$

($[\,\cdot\,]$ stands for the characteristic size of a variable), while the averaged kinetic energy $\frac{1}{2}(\mathbb{E}\, |u^\nu|_2^2)$ is of order one. So when $\nu \to 0$ the solutions u^ν describe space-periodic stationary 2d turbulence.

[7] Since now f has the form (4) with $h = 0$, then the uniqueness of the stationary measure also follows from (29), but we need (25) for the validity of some results in this section.

Assume that $b_s \equiv b_{-s}$, so the measures μ^ν are space-homogeneous, and that b_s decay sufficiently fast when $|s| \to \infty$. In this case, as it is shown in [KP05, KS12], the measures μ^ν possess additional properties. Namely, let $g(r)$ be any continuous function, having at most a polynomial growth at infinity. Then, denoting $v = \operatorname{rot} u$, we have the following *balance relation*, valid for all $\nu > 0$:

$$\mathbb{E}^{\mu^\nu} g(v(t,x))|\nabla v(t,x)|^2 = \frac{1}{2}(2\pi)^{-2}\mathfrak{B}_1 \mathbb{E}^{\mu^\nu} g(v(t,x)) \tag{40}$$

(by the homogeneity the relation does not depend on x). Since $|\Delta u|_2^2 = |\nabla v|_2^2$, then relations (39) and the translational invariance of μ^ν imply that

$$\mathbb{E}^{\mu}|\nabla v(t,x)|^2 = \frac{1}{2}(2\pi)^{-2}\mathfrak{B}_1.$$

So (40) means that the random variables $|\nabla v(t,x)|^2$ and $g(v(t,x))$ are uncorrelated, for any continuous function g as above and any (t,x).

The balance relations (40) admit a surprising reformulation. For any $\tau \in \mathbb{R}$ denote by $\Gamma_\tau(\omega)$ the random curve $\{x \in \mathbb{T}^2 : v^\omega(t,x) = \tau\}$ (it is well defined for a.a. τ and ω, if b_s decays fast enough). Then

$$\mathbb{E}^{\mu^\nu} \int_{\Gamma_\tau(\omega)} |\nabla v^\omega| \, d\ell = \frac{1}{2}(2\pi)^{-1}\mathfrak{B}_1 \mathbb{E}^{\mu^\nu} \int_{\Gamma_\tau(\omega)} |\nabla v^\omega|^{-1} \, d\ell\,, \qquad \text{for a.a. } \tau\,, \tag{41}$$

where $d\ell$ is the length element on $\Gamma_\tau(\omega)$, and the existence of the integrals in the l.h.s. and the r.h.s. for a.a. τ is a part of the assertion. This is the *co-area form of the balance relations*. Besides, relations (40) imply the following point-wise exponential estimates

$$\mathbb{E}^{\mu^\nu}\left(e^{\sigma|v(t,x)|} + e^{\sigma|u(t,x)|} + e^{\sigma|\nabla u(t,x)|^{1/2}}\right) \le K \quad \forall x\,, \tag{42}$$

valid uniformly in ν, where the positive constants σ and K depend only on the first few numbers $\mathfrak{B}_j$; see [KS12].

Inviscid Limit

Since estimates (39) hold uniformly in ν, the family of measures $\{\mu_\nu, 0 < \nu \le 1\}$ is tight in $H^{2-\epsilon}$ and, by Prokhorov's theorem, relatively compact in the space $\mathcal{P}(H^{2-\epsilon})$, for any $\epsilon > 0$. So any sequence of measures $\{\mu^{\nu'_j}, \nu'_j \to 0\}$, contains a weakly converging subsequence:

$$\mu^{\nu_j} \to \mu \quad \text{weakly in} \quad \mathcal{P}(H^{2-\epsilon})\,. \tag{43}$$

Relations (39) immediately imply that

$$\mathbb{E}^{\mu}|\nabla u|_2^2 = \frac{1}{2}\mathfrak{B}_0, \quad \mathbb{E}^{\mu}|\Delta u|_2^2 \le \frac{1}{2}\mathfrak{B}_1, \quad \mathfrak{B}_0^2/2\mathfrak{B}_1 \le \mathbb{E}^{\mu}|u|^2 \le \frac{1}{2}\mathfrak{B}_0, \tag{44}$$

so μ is supported by the space H^2. More delicate analysis of the convergence (43) shows that

$$\mu(K) = 1, \quad \text{where} \quad K = \{u \in H^1 : \operatorname{rot} u \in L_\infty\}, \tag{45}$$

see [GSV15]. Moreover, the measure μ is invariant for the deterministic Eq. $(38)|_{\nu=0}$, i.e. for the free 2d Euler equation

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0. \tag{46}$$

See [Kuk04, KS12].[8] The limit (43) is the *inviscid limit* for the (properly scaled) stochastic 2d NSE. In view of what has been said at the beginning of the previous subsection, the inviscid limit measures μ describe the statistics of space-periodic stationary 2d turbulence. We summarise the results concerning this limit in a theorem:

Theorem 6.15

(1) *Any inviscid limit measure μ satisfies* (44), (45) *and is invariant for Eq.* (46).
(2) *If $b_s \equiv b_{-s}$ and $|b_s|$ decay sufficiently fast as $|s| \to \infty$, then the measure μ is space-homogeneous and satisfies* (42)*, where $\mathbb{E}^{\mu^\nu}$ should be replaced by $\mathbb{E}^{\mu}$.*

The last assertion follows from (42), convergence (43) and Fatou's lemma. Convergence (43) does not allow to pass to the limit in (40) and (41), and we do not know if the balance relations hold for the inviscid limit measures μ. Relations (44) imply that the limiting measures μ are non-trivial in the sense that they do not equal the delta-measure in the origin. In fact, they are non-degenerate in a much stronger sense. To state the corresponding result, we denote

$$E(u) = \frac{1}{2}|u|_2^2, \quad E_1(u) = \frac{1}{2}|\operatorname{rot} u|_2^2 = \frac{1}{2}|\nabla u|_2^2,$$

and call a real analytic function $f(r)$ *admissible* if $f''(r)$ has at most a polynomial growth as $|r| \to \infty$ and is bounded from below (e.g., f is a polynomial of the form $f = r^{2m} + \cdots$, or any trigonometric polynomial).

[8] The measure μ is supported by the space K, on which the flow of the Euler equation is well defined in the sense of Yudovich and is continuous in the weak* topology of that set, see in [GSV15]. Before (45) was obtained, an additional construction, suggested in [Kuk04], was used to explain in which sense the limiting measure μ is invariant for (46).

Theorem 6.16 ([Kuk08, KS12])

(1) *The inviscid limits μ are such that the push-forward measures $E_*\mu$ and $(E_1)_*\mu$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

(2) *If $\mathfrak{B}_2 < \infty$ and $b_s \equiv b_{-s}$, then for any $d \in \mathbb{N}$ and any admissible functions $f_1, \dots, f_d$ such that their derivatives $f_1', \dots, f_d'$ are linearly independent modulo constants,* [9] *the push-forward of μ under the mapping*

$$u(\cdot) \mapsto \Big(\int_{\mathbb{T}^2} f_k(\operatorname{rot} u(x))\, \mathrm{d}x,\ 1 \le k \le d \Big) \in \mathbb{R}^d\,,$$

is absolutely continuous with respect to the Lebesgue measure.

Due to this result, the Hausdorff dimension of supp μ is infinite. Indeed, if this is not the case, then choosing d bigger than the Hausdorff dimension, we see that the push-forward measure in item (2) of the theorem is supported by a set of dimension $< d$, which contradicts the assertion.

7 3d Navier–Stokes System in Thin Domains

In this section we consider a thin layer around the torus $\mathbb{T}^2$ and the 3d Navier–Stokes system, perturbed by a random kick-force with a sufficiently small vertical component. We show that when the width of the layer goes to zero, statistical characteristics of the 3d flow converge to those of the 2d flow (1), (4), (8), where the kicks η^k are the horizontal components of the 3d kicks. Moreover, this convergence *is uniform in time*. Since Earth's atmosphere is a thin spherical layer, this result gives a good support to the belief that suitably chosen 2d stochastic meteorological models can be successfully used to model the climate. Usually these 2d models are related to Eq. (1); see the works [Var13, Kle17] and the references therein.

Let $Q_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon) = \{x = (x_1, x_2, x_3)\}$. Consider the 3d Navier–Stokes system in Q_ε under the free boundary conditions:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0, \quad x \in Q_\varepsilon\,, \tag{47}$$

$$u_3 = \partial_3 u_{1,2} = 0 \quad \text{for} \quad x_3 = 0 \text{ and } x_3 = \varepsilon\,, \tag{48}$$

$$u(0, x) = u_0(x)\,, \tag{49}$$

[9] That is, if $c_1 f_1' + \cdots + c_d f_d' \equiv \text{const}$, then all c_k vanish.

where $u = (u_1, u_2, u_3)$. Denote by H_ε (by V_ε) the L_2-space (the H^1-space) of divergence-free vector fields $(u_1, u_2, u_3)(x)$ on Q_ε such that u_1 and u_2 have zero mean. Denote by $|\cdot|_\varepsilon$ the L_2-norm on Q_ε, i.e. the norm in H_ε (note that $|\mathbf{1}|_\varepsilon = \varepsilon$), by $(\cdot,\cdot)_\varepsilon$ the corresponding L_2 scalar product, and denote by $\|\cdot\|_\varepsilon$ the homogeneous norm in V_ε, $\|u\|_\varepsilon = |\nabla u|_\varepsilon$. The space H as in Sect. 2 is naturally embedded in H_ε:

$$i : H \ni (u_1, u_2) \mapsto (u_1(x_1, x_2), u_2(x_1, x_2), 0) \in H_\varepsilon ,$$

and the norm of this embedding equals $\sqrt{\varepsilon}$. Introduce in H_ε two orthogonal projections:

$$M_\varepsilon u = (\varepsilon^{-1}\int_0^\varepsilon u_1(x', y)\,\mathrm{d}y, \varepsilon^{-1}\int_0^\varepsilon u_2(x', y)\,\mathrm{d}y, 0), \quad N_\varepsilon = \mathrm{id} - M_\varepsilon ,$$

where $x' = (x_1, x_2)$. Then $M_\varepsilon H_\varepsilon = iH$. The 3d Stokes operator $L_\varepsilon = -\Delta|_{H_\varepsilon}$ preserves the spaces $M_\varepsilon H_\varepsilon$ and $N_\varepsilon H_\varepsilon$, and its eigenfunctions are of two kinds:

$$e_s(x_1, x_2) \in M_\varepsilon H_\varepsilon,\ s \in \mathbb{Z}^2_\varepsilon, \quad \text{and} \quad e^\varepsilon_j(x_1, x_2, x_3) \in N_\varepsilon H_\varepsilon,\ j \in \mathbb{N},$$

where $|e_s|_\varepsilon = |e^\varepsilon_j|_\varepsilon = \sqrt{\varepsilon}$ for all s and j, and

$$L_\varepsilon e_s = |s|^2 e_s \ \ \forall s, \qquad L_\varepsilon e^\varepsilon_j = \Lambda^\varepsilon_j e^\varepsilon_j \ \ \forall\, j ,$$

so that

$$\|e_s\|_\varepsilon = (Le_s, e_s)_\varepsilon^{1/2} = |s|\sqrt{\varepsilon}, \quad \|e^\varepsilon_j\|_\varepsilon = \sqrt{\Lambda^\varepsilon_j\,\varepsilon}\,.$$

The vectors $e_s \in iH$ will be identified with the eigenvectors of the 2d Stokes operator (which is the opposite of the 2d Laplacian, restricted to the space H). Each vector e^ε_j has components $(e^\varepsilon_j)_l = C^l_j(\sin/\cos)(s^l_j \cdot x')\cos(\frac{\pi}{\varepsilon}n^l_j x_3)$, $l = 1, 2, 3$, where $s^l_j \in \mathbb{Z}^2_*$, $n^l_j \in \mathbb{N}\cup\{0\}$ and at least one of the numbers $n^1_j, \dots, n^3_j$ is non-zero. Therefore the eigenvalue Λ^ε_j has the form

$$\Lambda^\varepsilon_j = A_j + B_j\pi^2\varepsilon^{-2}, \quad A_j \in \mathbb{N}\cup\{0\},\ B_j \in \mathbb{N}. \tag{50}$$

Assume that the force $\eta(t, x)$ in (47) is a kick-process of the form (8) with the kicks

$$\eta^k_\varepsilon(x) = \sum_{s\in\mathbb{Z}^2_*} b_s \xi^k_s e_s(x) + \sum_{j=1}^\infty d^\varepsilon_j \zeta^k_j e^\varepsilon_j(x)\,.$$

Here the constants $\{b_s\}$ and $\{d_j^\varepsilon\}$ are such that

$$\mathfrak{B}_1 := \sum b_s^2|s|^2 < \infty, \quad b_s \neq 0 \,\forall\, s, \quad \mathcal{D}_1^\varepsilon := \sum (d_j^\varepsilon)^2 \Lambda_j^\varepsilon < \infty\,, \tag{51}$$

and $\{\xi_s^k\}$, $\{\zeta_j^k\}$ are i.i.d. random variable with law $p(r)\mathrm{d}r$, where $p \in C_0^1(-1, 1)$ satisfies the conditions

$$p(0) \neq 0, \quad \int_{\mathbb{R}} r p(r)\,\mathrm{d}r = 0.$$

We shall compare solutions of Eqs. (47)–(49) with those of the kick-forced 2d Navier–Stokes system (1), (4), (8), where the kicks η^k are

$$\eta^k = M_\varepsilon \eta_\varepsilon^k = \sum_s b_s \zeta_s^k e_s(x)\,.$$

Under the assumptions (51), this 2d equation is exponentially mixing in the space H in the sense that there exists a measure $\mu \in \mathcal{P}(H)$ such that for every solution $u(t)$ of (1), (4), (8) its law, evaluated at integer points $t \in \mathbb{N}$, converges to μ exponentially fast in the dual-Lipschitz norm in $\mathcal{P}(H)$; see [KS12]. The mixing property for Eqs. (47)–(49), claimed in the theorem below, is understood in a similar way.

Theorem 7.17 *Assume that*

$$\mathcal{D}_1^\varepsilon \leq \varepsilon^{-1}\gamma^2(\varepsilon), \quad \textit{where} \quad \gamma(\varepsilon) \to 0 \;\; \textit{as} \;\; \varepsilon \to 0\,. \tag{52}$$

Then there exist $c_0, \varepsilon_0 > 0$ such that the following properties hold.

Exponential mixing. *If $0 < \varepsilon \leq \varepsilon_0$, then the set*

$$\mathcal{O}_\varepsilon = \{u : \|M_\varepsilon u\|_\varepsilon \leq c_0\sqrt{\varepsilon},\ \|N_\varepsilon u\|_\varepsilon \leq c_0\gamma(\varepsilon)\} \subset V_\varepsilon$$

is invariant for Eqs. (47)–(49) *and the dynamics on $\mathcal{O}_\varepsilon$ is exponentially mixing with invariant measure $\mu_\varepsilon \in \mathcal{P}(\mathcal{O}_\varepsilon)$.*

Convergence. *As $\varepsilon \to 0$, the measure $(M_\varepsilon)_*\mu_\varepsilon$ converges to μ weakly in H.*

Stability. *Let $v_0 \in H$ be such that $\|v_0\|_1 < c_0$, let $u(t)$ be a solution of* (1), (4), (8)*, equal v_0 at $t = 0$, and let $u_\varepsilon(t)$ be a solution of Eqs.* (47)–(49) *with $u_0 = i v_0 \in \mathcal{O}_\varepsilon$. Then*

$$\|\mathcal{D}(M_\varepsilon u_\varepsilon(t)) - \mathcal{D}(u(t))\|_{L(H)}^* \to 0 \quad as \quad \varepsilon \to 0,$$

uniformly in $t \geq 0$.

For a random vector field u on Q_ε its averaged normalised kinetic energy is $\mathcal{E}_\varepsilon(u) = \frac{1}{2}\mathbb{E}\,|u|_\varepsilon^2/\mathrm{Vol}(Q_\varepsilon) = \frac{1}{2\varepsilon}\mathbb{E}\,|u|_\varepsilon^2$. By (52) and (50), for a kick η_ε^k, we have

$$\mathcal{E}_\varepsilon(M_\varepsilon\eta_\varepsilon^k) = \frac{1}{2}\kappa^2\sum_s b_s^2 \sim 1, \quad \mathcal{E}_\varepsilon(N_\varepsilon\eta_\varepsilon^k) = \frac{1}{2}\kappa^2\sum_j (d_j^\varepsilon)^2 \lesssim \varepsilon\gamma^2(\varepsilon)$$

where $\kappa^2 = \int r^2 p(r)\,\mathrm{d}r$. For the averaged normalised dissipation of energy $\mathcal{D}_\varepsilon(u) = \frac{1}{2\varepsilon}\mathbb{E}\,\|u\|_\varepsilon^2$, we have

$$\mathcal{D}_\varepsilon(M_\varepsilon\eta_\varepsilon^k) = \frac{1}{2}\kappa^2\mathfrak{B}_1 \sim 1, \quad \mathcal{D}_\varepsilon(N_\varepsilon\eta_\varepsilon^k) = \tfrac{1}{2}\kappa^2\mathcal{D}_1^\varepsilon \le \frac{1}{2}\kappa^2\varepsilon^{-1}\gamma^2(\varepsilon).$$

Therefore the vertical component of the random force in (47) should be small in terms of the energy, but not in terms of the dissipation of energy.

8 Open Problems

Open problem 1 (Mixing of Pipe Flow) *Let us consider the Navier–Stokes system* (1) *in the strip*

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, |x_2| < 1\}.$$

The initial-boundary value problem for (1) *is well posed in L^∞ spaces on D, with no decay conditions at infinity (see [AM05, AZ14]). Moreover, the problem is dissipative and possesses a global attractor (in the deterministic setting). It follows that, at least in the case of a bounded stochastic forcing, the random dynamics has a stationary distribution. A challenging problem is to prove its uniqueness and mixing.*

Open problem 2 (Mixing in the Whole Space) *The 2d Navier–Stokes system considered on the whole space $\mathbb{R}^2$ is well posed in L^∞ spaces (see [GMS01, LR02]). However, it is not known if the dynamics is dissipative, and it seems to be a hopeless task to prove any kind of regular behaviour of solutions under stochastic perturbations. On the other hand, the Navier–Stokes system with the Ekman damping*

$$\partial_t u + \gamma u + \langle u, \nabla\rangle u - \nu\Delta u + \nabla p = f(t,x), \quad \mathrm{div}\, u = 0, \tag{53}$$

where $\gamma > 0$ is a number, is dissipative (see [Zel13]). It is a natural question to investigate the existence of stationary measure and its mixing properties under various types of random perturbation.

Open problem 3 (Vanishing Noise Limit) *The vanishing noise limit of stationary measures described in Theorem 5.14 concerns a rather particular situation: the unperturbed dynamics should be globally asymptotically stable. This is a very restrictive hypothesis, and removing it is an important question. In the finite-dimensional case this problem is rather well understood, and one can establish the so-called Freidlin–Wentzell asymptotics for stationary measures; see Section 6.4 in [FW12]. As for stochastic PDE's, similar results can be proved, provided that the global attractor for the unperturbed dynamics has a regular structure. The latter means that the attractor consists of finitely many steady states and the heteroclinic orbits joining them. Such a result is proved in [Mar17] for the case of a damped nonlinear wave equation. The attractor of the Navier–Stokes system is not likely to possess that property, and the validity of the Freidlin–Wentzell type asymptotics for stationary distributions remains an open problem.*

Open problem 4 (Inviscid Limit)

(a) *Does the limiting measure μ in (43) depend on the sequence $\{\mu_{\nu_j}\}$? (We believe that it does).*

(b) *Consider the energy spectrum of a mesure μ as above:*

$$E_k(\mu) = Z^{-1} \sum_{\{s\in\mathbb{Z}^2_*: M^{-1}k\le|s|\le Mk\}} \mathbb{E}^\mu |u_s|^2,$$

where Z is the number of terms in the sum and $M > 1$ is a suitable constant. Do there exist positive constants a, b, C such that

$$C^{-1}k^{-a} \le E_k(\mu) \le Ck^{-b} \tag{54}$$

for all $k \ge 1$? Do the exponents a and b depend on the inviscid limit μ?

(c) *Do the stationary measures μ^ν satisfy (54) for $C_1\nu^{-\alpha} \le k \le C_2\nu^{-\beta}$, for suitable $0 \le \alpha < \beta$ and $C_1, C_2 > 0$? See [Bor13] for an affirmative answer to this question with $a = b = 2$ and $\alpha = 0$, $\beta = 1$, when u is a solution of the 1d stochastic Burgers equation.*

Open problem 5 (Thin 3d Domains) *Improve the result of Theorem 7.17 by replacing condition (52) with a weaker restrain, thus allowing in (47) for random forces with bigger vertical components (this seems to be possible to achieve by making better use of the stochastic nature of the force η). Obtain similar results for 3d stochastic models of Earth's atmosphere and their suitable 2d approximations.*

Frequently Used Notations

C, C_i	Positive numbers that may depend on the parameters mentioned in brackets
$J \subset \mathbb{R}$	Closed interval
$C(J, E)$	Space of continuous functions on J with range in a Banach space E
$L^p(J, E)$	Space of Borel-measurable functions $f : J \to E$ such that $\int_J \|f(t)\|_E^p \,\mathrm{d}t < \infty$, with obvious modification for $p = \infty$
X	Complete separable metric space with a distance d_X
$\mathcal{B}(X)$	Borel σ-algebra of X
$C_b(X)$	Space of bounded continuous functions $g : X \to \mathbb{R}$ with the norm $\|g\|_\infty = \sup_{u\in X} \|g(u)\|$
$d_X(u, B)$	Minimal distance from a point $u \in X$ to a subset $B \subset X$
$\mathcal{P}(X)$	Space of Borel probability measures on X endowed with the topology of weak convergence; the latter can be metrised by the Lipschitz-dual metric $\|\cdot\|^*_{\mathrm{Lip}(X)}$, see (28)
$\mathcal{D}(\xi)$	The law of a random variable ξ
$F_*\mu$	The image of a measure μ under a measurable mapping F
$L^p(D), H^s(D)$	Lebesgue and Sobolev spaces on a domain $D \subset \mathbb{R}^d$ with standard norms $\|\cdot\|_p$ and $\|\cdot\|_s$, respectively; sometimes we write $L^p(D, \mathbb{R}^m)$ and $H^s(D, \mathbb{R}^m)$ to denote the corresponding spaces of $\mathbb{R}^m$-valued functions, and $(\cdot, \cdot)$ stands for L_2-scalar product
H^s	Space of divergence-free vector functions with zero mean that belong to $H^s(\mathbb{T}^2, \mathbb{R}^2)$, $s \in \mathbb{N} \cup \{0\}$.
H, V	Abbreviations for the spaces H^0 and H^1
L^2_σ	The set of divergence-free functions in $L^2(\mathbb{T}^2, \mathbb{R}^2)$
$\mathbb{R}_+$, $\mathbb{Z}_+$	Sets of real numbers and non-negative integers, respectively
$\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$	Two-dimensional torus with sides 2π
$(\Omega, \mathcal{F}, \mathbb{P})$	Complete probability space

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Attractors of Hamiltonian Nonlinear Partial Differential Equations

Andrew Comech, Alexander Komech, and Elena Kopylova

To the memory of Mark Vishik

Abstract We survey the theory of attractors of nonlinear Hamiltonian partial differential equations since its appearance in 1990. These are results on global attraction to stationary states, to solitons and to stationary orbits, on adiabatic effective dynamics of solitons and their asymptotic stability. Results of numerical simulations are also given. Based on these results, we propose a new general hypothesis on attractors of G-invariant nonlinear Hamiltonian partial differential equations. The obtained results suggest a novel dynamical interpretation of basic quantum phenomena: Bohr's transitions between quantum stationary states, wave-particle duality, and probabilistic interpretation.

1 Introduction

Theory of attractors of nonlinear PDEs originated from the seminal paper of Landau [B9] published in 1944, where he suggested the first mathematical interpretation of turbulence as the growth of the dimension of attractors of the Navier–Stokes equations when the Reynolds number increases. Modern development of the theory

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A. Comech (✉)
Department of Mathematics, Texas A&M University, College Station, TX, USA
e-mail: comech@tamu.edu

A. Komech
Faculty of Mathematics, University of Vienna, Vienna, Austria

E. Kopylova
Faculty of Mathematics, University of Vienna, Vienna, Austria

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of attractors for general *dissipative systems*, i.e., systems with friction (the Navier–Stokes equations, nonlinear parabolic equations, reaction-diffusion equations, wave equations with friction, etc.) originated in the 1975–1985's in the works of Foiaş, Hale, Henry, Temam, and others [B5, B6, B7, B8, B9, B10]. M.I. Vishik together with his collaborators worked on global attractors of dissipative nonlinear PDEs from 1980 until 2012. These investigations significantly advanced this theory [B1, B2, B3, B4].

A typical result of this theory, in the absence of external excitation, is a global convergence to stationary states: for any finite energy solution to dissipative *autonomous* equation in a region $\Omega \subset \mathbb{R}^n$, there is a convergence

$$\psi(x,t) \to S(x), \qquad t \to +\infty. \tag{1}$$

Here $S(x)$ is a stationary solution, depending on the initial state $\psi(x,0)$. As a rule, the convergence holds in the $L^2(\Omega)$-metric. In particular, the relaxation to an equilibrium regime in chemical reactions is due to the energy dissipation.

A development of a similar theory for *Hamiltonian PDEs* looked unmotivated and even impossible since the convergence (1) seemed to contradict the energy conservation and time reversibility admitted by these equations. On the other hand, such a theory of global attractors is indispensable for mathematical foundation of quantum physics since all fundamental PDEs of quantum theory are Hamiltonian: the Maxwell, Schrödinger, Klein–Gordon, Dirac, Yang–Mills equations, etc. Even the shape of the theory is already suggested by fundamental postulates of quantum physics. In particular, the global attraction of type (1) is suggested by Bohr's postulates of 1913 on transitions between quantum stationary states. Namely, the postulate can be interpreted as the global attraction of all quantum trajectories to an attractor formed by quantum stationary states. Such a global convergence also allows us to give a mathematical interpretation to the L. de Broglie wave particle duality and to the M. Born probabilistic interpretation; for more details, see [E7, A13, A14, A15]. We note that in 1961 W. Heisenberg began developing a nonlinear theory of elementary particles [P5, P6].

Thus, the basic postulates of quantum physics suggest the type of long-time behavior of solutions of the related fundamental dynamical equations: Maxwell–Schrödinger, Maxwell–Dirac, Yang–Mills–Dirac, and other equations and systems. All these coupled equations are nonlinear Hamiltonian systems of partial differential equations, and it seems unlikely that these and other fundamental dynamical equations of quantum physics are exceptional among generic Hamiltonian PDEs. This situation leads to the conjecture that the global convergence to a proper attractor is an inherent feature of general nonlinear Hamiltonian PDEs.

The first results on attractors for the Hamiltonian equations were obtained by Morawetz, Segal, and Strauss in the case when the attractor consists of the single point zero. In this case the attraction is called *local energy decay* [C1, C2, C3, C4, C5, C6, C7].

It was not until 1990 when the case of a nontrivial attractor was considered, when the hypothesis on the global attractors was set forth by one of the authors. Since then, this hypothesis has been developed in collaboration with the international group of researchers: with M. Kunze, H. Spohn, and B. Vainberg and later with V.S. Buslaev, A. Comech, V. Imaykin, E. Kopylova, D. Stuart, and others.

The investigations since 1990 suggest that such long-time behavior of solutions is not merely the peculiarity of the dynamical equations of quantum physics, but rather a characteristic property of *generic* nonlinear Hamiltonian PDEs; see the surveys [E7, E8, E9]. Below, we sketch these results. All these results were discussed in many talks by the authors at the seminars of M.I. Vishik in the Department of Mechanics and Mathematics of Moscow State University.

This theory differs significantly from the theory of attractors of dissipative systems where the attraction to stationary states is due to an energy dissipation caused by a friction. For Hamiltonian equations the friction and energy dissipation are absent, and the attraction is caused by radiation which irrevocably brings the energy to infinity. It is exactly this radiation that is addressed in the Bohr postulate on transitions between quantum stationary states.

2 Bohr's Postulates: Quantum Jumps

In 1913, Bohr formulated the following two fundamental postulates of the quantum theory of atoms [A4]:

1. *Each electron lives on one of the quantum stationary orbits, and sometimes it jumps from one stationary orbit to another: in the Dirac notation,*

$$|E_n\rangle \mapsto |E_{n'}\rangle. \tag{2}$$

2. *On a stationary orbit, the electron does not radiate, while every jump is followed by the radiation of an electromagnetic wave of frequency*

$$\omega_{nn'} = \frac{E_{n'} - E_n}{\hbar} = \omega_{n'} - \omega_n, \qquad \omega_n := E_n/\hbar. \tag{3}$$

Both these postulates were inspired by stability of atoms, by the Rydberg–Ritz *Combination Principle*, and by the Einstein theory of the photoeffect.

In 1926, with the discovery of the Schrödinger theory [A24], the question arose about the implementation of the above Bohr axioms in the new theory based on partial differential equations. This and other questions have been frequently addressed in the 1920s and 1930s in heated discussions by N. Bohr, E. Schrödinger, A. Einstein, and others [A5]. However, a satisfactory solution was not achieved, and a rigorous dynamical interpretation of these postulates is still unknown. This lack of theoretical clarity hinders a further progress in the theory (e.g., in superconductivity and in nuclear reactions) and in numerical simulation of many engineering processes (e.g., laser radiation and quantum amplifiers) since a computer can solve dynamical equations but cannot take into account postulates.

3 Schrödinger's Identification of Stationary Orbits

Besides the equation for the wave function, the Schrödinger theory contains a highly nontrivial definition of stationary orbits (or *quantum stationary states*) in the case when the Maxwell external potentials do not depend on time: in this case, the Schrödinger equation reads as

$$i\hbar\dot{\psi}(x,t) = H\psi(t) := \frac{1}{2\mathrm{m}}\Big[-i\hbar\nabla - \frac{e}{c}\mathbf{A}_{\mathrm{ext}}(x)\Big]^2\psi(x,t) + eA^0_{\mathrm{ext}}(x)\psi(x,t), \qquad (4)$$

where m and e denote the electron's mass and charge, c is the speed of light in vacuum, $\mathbf{A}_{\mathrm{ext}}(x)$ is the external magnetic potential, and $A^0_{\mathrm{ext}}(x)$ is the external scalar potential. In the *unrationalized*[1] Gaussian units, also called the Heaviside–Lorentz units [A12, p. 781], the values of the physical constants (the electron charge and mass, the speed of light in vacuum, the Boltzmann constant, and the Planck constant) are approximately equal to

$$\left.\begin{array}{c} e = -4.8\times 10^{-10}\mathrm{esu}, \qquad \mathrm{m} = 9.1\times 10^{-28}\mathrm{g}, \quad c = 3.0\times 10^{10}\mathrm{cm/s} \\ k = 1.38\times 10^{-16}\mathrm{erg/K}, \qquad \hbar = 1.1\times 10^{-27}\mathrm{erg}\cdot\mathrm{s} \end{array}\right| . \qquad (5)$$

In Schrödinger's theory [A24], stationary orbits are identified with finite energy solutions of the form

$$\psi(x,t) = \varphi_\omega(x)e^{-i\omega t}, \qquad \omega\in\mathbb{R}. \qquad (6)$$

Substitution of this Ansatz into the Schrödinger equation (6) leads to the famous eigenvalue problem

$$\omega\varphi_\omega = H\varphi_\omega, \qquad (7)$$

which determines the corresponding frequencies ω and amplitudes φ_ω; the latter are subject to the normalization condition

$$\int |\varphi_\omega(x)^2|\,dx = 1. \qquad (8)$$

This condition means that the wave function describes one electron, as discussed below.

Such definition is rather natural since then $|\psi(x,t)|$ does not depend on time. Most likely, this definition was suggested by the de Broglie wave function for *free*

[1] *Rationalization* refers to the choice of units aimed at suppression of the explicit factors of 4π in the Maxwell equations.

particles $\psi(x,t) = Ce^{i(kx-\omega t)}$, which factorizes as $Ce^{ikx}e^{-i\omega t}$. Indeed, in the case of *bound particles*, it is natural to change the spatial factor Ce^{ikx} since the spatial properties have changed and ceased to be homogeneous. On the other hand, the homogeneous time factor $e^{-i\omega t}$ must be preserved since the external potentials are independent of time. However, these "algebraic" arguments do not withdraw the question of agreement of the Schrödinger definition with the Bohr postulate (2)!

Thus, the problem of the mathematical interpretation of the Bohr postulate (2) in the Schrödinger theory arises. One of the simplest interpretations of the jumps (2), (3) is the long-time asymptotics

$$\psi(x,t) \sim \psi_{\pm}(x)e^{-i\omega_{\pm}t}, \qquad t \to \pm\infty \tag{9}$$

for each finite energy solution, where $\omega_- = \omega_n$ and $\omega_+ = \omega_{n'}$. However, *for the linear Schrödinger equation* (4), such asymptotics are obviously wrong due to the *superposition principle*: for example, such asymptotics can not hold for solutions of the form $\psi(x,t) \equiv \phi_1(x)e^{-i\omega_1 t} + \phi_2(x)e^{-i\omega_2 t}$ with $\omega_1 \neq \omega_2$ and with nonzero ϕ_1 and ϕ_2. It is exactly this contradiction that shows that the linear Schrödinger equation alone cannot serve as the basis for the theory compatible with the Bohr postulates.

Remark 3.1 The Schrödinger definition (6) has been formalized by P. Dirac [A6] and J. von Neumann [A22]: they have identified quantum states with the rays in the Hilbert space, so the entire circular orbit (6) corresponds to one quantum state. Our main conjecture is that this formalization is indeed a *theorem*: namely, we suggest that the asymptotics (9) is an inherent property of the nonlinear Maxwell–Schrödinger system (see (16)–(17) below). This conjecture is confirmed by perturbative arguments below.

4 Coupled Maxwell–Schrödinger Equations

As we have seen, the linear Schrödinger equation cannot explain the Bohr postulates. Thus, one should look for a nonlinear theory. Fortunately, we do not need to invent anything artificial since such a theory is well established since 1926: it is the system of the Schrödinger equation coupled to the Maxwell equations which was essentially introduced in the first of Schrödinger's articles [A24]. The corresponding second-quantized system is the main subject of nonrelativistic Quantum Electrodynamics.

This coupled system arises in the following way. The wave function $\psi(x,t)$ defines the corresponding charge and current densities, which, in turn, generate their own Maxwell field. The corresponding potentials of this own field should be added to the external Maxwell potentials in the Schrödinger equation (4). This modification implies the nonlinear interaction between the wave function and the

Maxwell field. More precisely, the Schrödinger theory associates to the wave function $\psi(x,t)$ the corresponding charge and current densities

$$\begin{aligned} \rho(x,t) &= e|\psi(x,t)|^2, \\ \mathbf{j}(x,t) &= \frac{e}{\mathrm{m}}\,\mathrm{Re}\Big(\overline{\psi}(x,t)\Big[-i\hbar\nabla - \frac{e}{c}\mathbf{A}_{\mathrm{ext}}(x,t)\Big]\psi(x,t)\Big). \end{aligned} \tag{10}$$

For any solution of the Schrödinger equation (4), these densities satisfy the charge continuity equation

$$\dot{\rho}(x,t) + \operatorname{div}\mathbf{j}(x,t) = 0. \tag{11}$$

Note that the normalization condition (8) means that the total charge corresponding to the stationary orbit (6) is

$$\int \rho(x)\,dx = e, \tag{12}$$

i.e., we have exactly one electron on this orbit. The charge and current densities (10) generate the Maxwell field according to the Maxwell equations

$$\begin{cases} \operatorname{div}\mathbf{E}(x,t) = \rho(x,t), & \operatorname{curl}\mathbf{E}(x,t) = -\frac{1}{c}\dot{\mathbf{B}}(x,t), \\ \operatorname{div}\mathbf{B}(x,t) = 0, & \operatorname{curl}\mathbf{B}(x,t) = \frac{1}{c}\big(\mathbf{j}(x,t) + \dot{\mathbf{E}}(x,t)\big). \end{cases} \tag{13}$$

The second and third equations imply the Maxwell representations

$$\mathbf{B}(x,t) = \operatorname{curl}\mathbf{A}(x,t), \qquad \mathbf{E}(x,t) = -\frac{1}{c}\dot{\mathbf{A}}(x,t) - \nabla A^0(x,t). \tag{14}$$

We will assume the Coulomb gauge

$$\operatorname{div}\mathbf{A}(x,t) \equiv 0. \tag{15}$$

Then the Maxwell equations (13) are equivalent to the system

$$\frac{1}{c^2}\ddot{\mathbf{A}}(x,t) = \Delta\mathbf{A}(x,t) + \frac{1}{c}P\mathbf{j}(x,t), \quad \Delta A^0(x,t) = -\rho(x,t), \qquad x \in \mathbb{R}^3, \tag{16}$$

where P is the *orthogonal projection* in the real Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{R}^3$ onto divergence-free vector fields.

Finally, one should add these "own" Maxwell potentials $\mathbf{A}(x,t)$ and $A^0(x,t)$ to the external potentials in the Schrödinger equation (4), so we obtain the modified Schrödinger equation

$$\begin{aligned} i\hbar\dot{\psi}(x,t) &= H(t)\psi(t) \\ &:= \frac{1}{2\mathrm{m}}\Big[-i\hbar\nabla - \frac{e}{c}(\mathbf{A}_{\mathrm{ext}}(x)+\mathbf{A}(x,t))\Big]^2\psi(x,t) \\ &\quad + e(A^0_{\mathrm{ext}}(x)+A^0(x,t))\psi(x,t). \end{aligned} \tag{17}$$

Equations (16)–(17) form the nonlinear Maxwell–Schrödinger system.

5 Bohr's Postulates via Perturbation Theory

The remarkable success of the Schrödinger theory was the explanation of Bohr's postulates via asymptotics (9) by means of *perturbation theory* applied to the *coupled Maxwell–Schrödinger equations* (16)–(17) in the case of *static external potentials*

$$\mathbf{A}_{\mathrm{ext}}(x,t)\equiv\mathbf{A}_{\mathrm{ext}}(x), \qquad A^0_{\mathrm{ext}}(x,t)\equiv A^0_{\mathrm{ext}}(x). \tag{18}$$

For instance, in the case of an atom, $A^0_{\mathrm{ext}}(x)$ is the Coulomb potential of the nucleus, while $\mathbf{A}_{\mathrm{ext}}(x)$ is the vector potential of the magnetic field of the nucleus. Namely, as the first approximation, the fields $\mathbf{A}(x,t)$ and $A^0(x,t)$ in Eq. (17) can be neglected, so we obtain Eq. (4) with the Schrödinger operator $H(t)\equiv H$. For "sufficiently good" external potentials and initial conditions, each finite energy solution can be expanded into eigenfunctions:

$$\psi(x,t)=\sum_n C_n\psi_n(x)e^{-i\omega_n t}+\psi_c(x,t), \qquad \psi_c(x,t)=\int C(\omega)e^{-i\omega t}\,d\omega, \tag{19}$$

where the integration is performed over the continuous spectrum of the operator H, and for any $R>0$ we have

$$\int_{|x|<R}|\psi_c(x,t)|^2\,dx\to 0, \quad t\to\pm\infty; \tag{20}$$

see, for example, [N16, Theorem 21.1]. The substitution of the expansion (19) into the expression for currents (10) gives

$$\mathbf{j}(x,t)=\sum_{n,n'}\mathbf{j}_{nn'}(x)e^{-i\omega_{nn'}t}+c.c.+\mathbf{j}_c(x,t), \tag{21}$$

where $\mathbf{j}_c(x,t)$ has a continuous frequency spectrum. Thus, the current in the right-hand side of the Maxwell equation (16) contains, besides the continuous spectrum, only *discrete frequencies* $\omega_{nn'}$. Hence, the *discrete spectrum* of the corresponding Maxwell field also contains only these frequencies $\omega_{nn'}$. This justifies the Bohr rule (3) *only in the first order of perturbation theory* since this calculation ignores the feedback action of the radiation onto the atom.

The same arguments also suggest treating the jumps (2) as the *single-frequency asymptotics* (9) for solutions of the Maxwell–Schrödinger system (16)–(17). Indeed, the currents (21) on the right of the Maxwell equation (16) produce radiation when nonzero frequencies $\omega_{nn'} \neq 0$ are present. This is due to the fact that $\mathbb{R} \setminus 0$ is the absolutely continuous spectrum of the Maxwell equations. However, this radiation cannot last forever since it irrevocably carries the energy to infinity while the total energy is finite. Therefore, in the long-time limit only $\omega_{nn'} = 0$ survives, which means exactly that we have *single-frequency asymptotics* (9) in view of (20).

6 Quantum Jumps as Global Attraction

Of course, the perturbation arguments above cannot provide a rigorous justification of the long-time asymptotics (9) for the coupled Maxwell–Schrödinger equations. We suggest that this asymptotics is a manifestation of a more general fact which we will discuss in the next section. Note that the system (16)–(17) admits the symmetry group $\mathbf{U}(1)$:

$$(\mathbf{A}(x), A^0(x), \psi(x)) \mapsto (\mathbf{A}(x), A^0(x), \psi(x)e^{i\theta}), \qquad \theta \in [0, 2\pi]. \tag{22}$$

That is, if $(\mathbf{A}(x,t), A^0(x,t), \psi(x,t))$ is a solution to the system (16)–(17), then, for any $\theta \in \mathbb{R}$, $(\mathbf{A}(x,t), A^0(x,t), \psi(x,t)e^{i\theta})$ is also a solution.

Definition 6.1 *Stationary orbits* of the Maxwell–Schrödinger nonlinear system (16)–(17) are finite energy solutions of the form

$$\big(\mathbf{A}(x),\ A^0(x),\ e^{-i\omega t}\phi(x)\big). \tag{23}$$

The existence of such stationary orbits for the system (16), (17) was proved in [D3] in the case of external potentials

$$\mathbf{A}_{\rm ext}(x,t) \equiv 0, \qquad A^0_{\rm ext}(x,t) = -\frac{eZ}{|x|}. \tag{24}$$

The perturbation arguments of previous section suggest that the Bohr postulate (2) can be treated as the long-time asymptotics

$$\big(\mathbf{A}(x,t),\ A^0(x,t),\ \psi(x,t)\big) \sim \big(\mathbf{A}_\pm(x),\ A^0(x),\ e^{-i\omega_\pm t}\phi_\pm(x)\big), \qquad t \to \pm\infty \tag{25}$$

for all finite-energy solutions of the Maxwell–Schrödinger equations (16), (17) in the case of static external potentials (18). We conjecture that these asymptotics hold in the H^1-norm on every bounded region of $\mathbb{R}^3$. The asymptotics (25) means that there is a *global attraction* to the set of stationary orbits. We expect that a similar attraction takes place for Maxwell–Dirac, Maxwell–Yang–Mills, and other coupled equations. This suggests the interpretation of quantum stationary states as the points and orbits that constitute the *global attractor* of the corresponding quantum dynamical equations.

The asymptotics (25) have not been proved for the Maxwell–Schrödinger system (16)–(17). On the other hand, similar asymptotics are now being proved in [H1]–[H14] for a number of model nonlinear PDEs with the symmetry group $\mathbf{U}(1)$. In the next section we state a general conjecture which reduces to the asymptotics (25) in the case of the Maxwell–Schrödinger system.

Remark 6.2 Experiments show that the transition time of quantum jumps (2) is of the order of 10^{-8}s, although the asymptotics (25) require infinite time. We suppose that this discrepancy can be explained by the following arguments:

i) 10^{-8}s is the transition time between small neighborhoods of initial and final states;
ii) during this time, the atom emits the major part of the radiated energy.

7 Relation to Linear Quantum Mechanics

The linear eigenvalue problem (7) together with Bohr's second rule (3) give very efficient description of the atomic and molecular spectra. This efficiency is presumably due to the smallness of the fraction $\frac{e}{c} \sim 10^{-20}$. Namely, substituting the stationary orbit (23) into the Maxwell–Schrödinger system (16)–(17), we obtain the corresponding *nonlinear eigenvalue problem* for the frequencies ω, which play the role of *nonlinear eigenvalues*, and for the corresponding *nonlinear eigenfunctions* $(\mathbf{A}(x), A^0(x), \phi(x))$. After such a substitution, we can neglect the terms with $\mathbf{A}(x)$ and $A^0(x)$ in Eq. (17); this corresponds to the first order approximation of the perturbation series in $\frac{e}{c}$. Then the nonlinear eigenvalue problem reduces to the linear one, represented by (7).

8 Conjecture on Attractors of G-Invariant Equations

In this section, we formulate the general conjecture on attractors introduced in [E7, E8, E9]. Let us consider general G-invariant *autonomous* Hamiltonian nonlinear PDEs in $\mathbb{R}^n$ of the form

$$\dot{\Psi}(t) = F(\Psi(t)), \qquad t \in \mathbb{R}, \tag{26}$$

with a Lie symmetry group G acting on a suitable Hilbert or Banach phase space $\mathcal{E}$ via a linear representation T. The Hamiltonian structure means that

$$F(\Psi) = J D\mathcal{H}(\Psi), \qquad J^* = -J, \tag{27}$$

where $\mathcal{H}$ denotes the corresponding Hamiltonian functional. The G-invariance means that

$$F(T(g)\Psi) = T(g)F(\Psi), \qquad \Psi \in \mathcal{E}, \tag{28}$$

for all $g \in G$. In that case, for any solution $\Psi(t)$ to Eq. (26), the trajectory $T(g)\Psi(t)$ is also a solution, so the representation commutes with the dynamical group $U(t)$: $\Psi(0) \mapsto \Psi(t)$; that is,

$$T(g)U(t) = U(t)T(g). \tag{29}$$

Let us note that the theories of elementary particles deal systematically with the symmetry groups $\mathbf{SU}(2)$, $\mathbf{SU}(3)$, $\mathbf{SU}(5)$, $\mathbf{SO}(10)$ and so on, like e.g. the group

$$\mathbf{SU}(4) \times \mathbf{SU}(2) \times \mathbf{SU}(2)$$

from the Pati–Salam model [P1, P8], one of the candidates for the "Grand Unified Theory".

Conjecture A (On Attractors) For "generic" G-invariant autonomous Eq. (26), any finite energy solution $\Psi(t)$ admits a long-time asymptotics

$$\Psi(t) \sim e^{\hat{\lambda}_\pm t}\Psi_\pm, \qquad t \to \pm\infty, \tag{30}$$

in the appropriate topology of the phase space $\mathcal{E}$. Here $\hat{\lambda}_\pm = T'(e)\lambda_\pm$, where $e \in G$ is the identity element and $\lambda_\pm$ belong to the corresponding Lie algebra $\mathfrak{g}$, while $\Psi_\pm$ are some *limiting amplitudes* depending on the solution $\Psi(t)$.

In other words, all solutions of the form $e^{\hat{\lambda}t}\Psi$ with $\hat{\lambda} = T'(e)\lambda$, where $\lambda \in \mathfrak{g}$, constitute a global attractor for *generic* G-invariant Hamiltonian nonlinear PDEs of the form (26). This conjecture suggests that we define *stationary G-orbits* for Eq. (26) as solutions of the form

$$\Psi(t) = e^{\hat{\lambda}t}\Psi, \qquad t \in \mathbb{R}; \qquad \hat{\lambda} = T'(e)\lambda, \quad \lambda \in \mathfrak{g}. \tag{31}$$

This definition leads to the corresponding *nonlinear eigenvalue problem*

$$\hat{\lambda}\Psi = F(\Psi). \tag{32}$$

In particular, in the case of the linear Schrödinger equation with the symmetry group $\mathbf{U}(1)$, stationary orbits are solutions of the form $e^{i\omega t}\phi(x)$, where $\omega \in \mathbb{R}$ is an eigenvalue of the Schrödinger operator and $\phi(x)$ is the corresponding eigenfunction. In the case of the symmetry group $G = \mathbf{SU}(3)$, the generator ("eigenvalue") λ is a 3×3-matrix, and solutions (31) can be, in particular, quasiperiodic in time.

Note that the conjecture (30) fails for linear equations: that is, linear equations are exceptional, not "generic"!

Remark 8.3 The existence of the stationary orbits of the form (31) was proved for nonlinear wave equations and the Maxwell–Schrödinger equations [D1, D2, D3, D5]. The proofs of the existence for the Maxwell–Dirac and Klein–Gordon–Dirac coupled nonlinear equations in [D4] essentially relies on the topological methods of Lusternik–Schnirelman [A19, A20].

Empirical Evidence The conjecture (30) agrees with the Gell-Mann–Ne'eman theory of baryons [P3, P7]. Indeed, in 1961, Gell-Mann and Ne'eman suggested using the symmetry group $\mathbf{SU}(3)$ for the strong interaction of baryons relying on the discovered parallelism between empirical data for the baryons and the "Dynkin diagram" of the Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ with 8 generators (the famous "eightfold way"). This theory resulted in the scheme of quarks in quantum chromodynamics [P4] and in the prediction of a new baryon with prescribed values of its mass and decay products. This particle (the Ω^--hyperon) was promptly discovered experimentally [P2].

On the other hand, the elementary particles seem to describe long-time asymptotics of quantum fields. Hence the empirical correspondence between elementary particles and generators of the Lie algebras presumably gives an evidence in favor of our general conjecture (30) for equations with Lie symmetry groups.

9 Results on Global Attractors for Nonlinear Hamiltonian PDEs

Here we give a brief survey of rigorous results [E3]–[H2] obtained since 1990 that confirm the conjecture (30) for a list of model equations of the form (26). The details can be found in the surveys [E7, E8, E9].

The results confirm the existence of finite-dimensional attractors in the Hilbert or Banach phase spaces and demonstrate the explicit correspondence between the long-time asymptotics and the symmetry group G of equations. The results obtained so far concern Eq. (26) with the following four basic groups of symmetry: the trivial symmetry group $G = \{e\}$, the translation group $G = \mathbb{R}^n$ for translation-invariant equations, the unitary group $G = \mathbf{U}(1)$ for phase-invariant equations, and the orthogonal group $\mathbf{SO}(3)$ for "isotropic" equations. In these cases, the asymptotics (30) reads as follows.

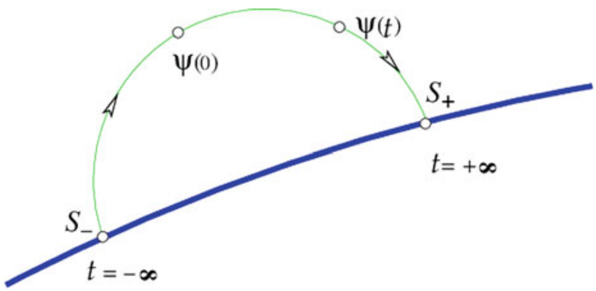

Fig. 1 Convergence to stationary states

Equations with the Trivial Symmetry Group G = {e}

For such *generic equations*, the conjecture (30) means the attraction of *all finite energy solutions* to stationary states:

$$\psi(x,t) \to S_\pm(x), \qquad t \to \pm\infty, \tag{33}$$

as illustrated on Fig. 1. Here the states $S_\pm = S_\pm(x)$ depend on the trajectory $\Psi(t)$ under consideration, while the convergence holds in local seminorms of type $L^2(|x| < R)$ with any $R > 0$. This convergence cannot hold in global norms (i.e., in norms corresponding to $R = \infty$) due to energy conservation. The asymptotics (33) can be symbolically written as the transitions

$$S_- \mapsto S_+, \tag{34}$$

which can be considered as the mathematical model of Bohr's "quantum jumps" (2). Such an attraction was proved for a variety of model equations in [E1]–[E16]. In [E2]–[E10], the convergence was proved

(1) for a string coupled to nonlinear oscillators:

$$\ddot\psi(x,t) = \psi''(x,t) + f(x,\psi(x,t)), \qquad x \in \mathbb{R}, \tag{35}$$

with $f(x,\psi(x,t)) = \sum_k \delta(x - x_k)F_k(\psi(x_k,t))$ and with $f(x,\psi(x,t)) = \chi(x)F(\psi(x,t))$;

(2) for a three-dimensional wave equation coupled to a charged particle

$$\left\{ \begin{array}{l} \ddot\psi(x,t) = \Delta\psi(x,t) - \rho(x - q(t)), \qquad x \in \mathbb{R}^3 \\ \dot p(t) = -\nabla V(q) - \displaystyle\int \nabla\psi(x,t)\rho(x - q(t))\,dx, \end{array} \right| \tag{36}$$

where

$$p(t) = \frac{m\dot{q}(t)}{\sqrt{1 - \dot{q}^2(t)}}$$

is the relativistic momentum of the particle. This is Hamiltonian system with the Hamilton functional

$$\begin{aligned} H(\psi, \pi, q, p) = \frac{1}{2} \int [|\pi(x)|^2 + |\nabla\psi(x)|^2]\, dx \\ + \int \psi(x)\rho(x-q)\, dx + \sqrt{1+p^2} + V(q) \end{aligned} \tag{37}$$

The global attraction to solitons was extended (1) to the wave equation, the Dirac equation, and the Klein–Gordon equation with concentrated nonlinearities, and (2) to the Maxwell–Lorentz system, which is the system of type (36) with the Maxwell equations instead of the wave equation and the Lorentz equation instead of the Newton equation:

$$\left\{ \begin{array}{ll} \dot{\mathbf{E}}(x,t) = \operatorname{curl} \mathbf{B}(x,t) - \dot{q}(t)\rho(x-q(t)), & \dot{\mathbf{B}}(x,t) = -\operatorname{curl} \mathbf{E}(x,t) \\ \operatorname{div} \mathbf{E}(x,t) = \rho(x-q(t)), & \operatorname{div} \mathbf{B}(x,t) = 0 \\ \dot{p}(t) = -\nabla V(q) + \displaystyle\int [\mathbf{E}(x,t) + \dot{q}(t) \times \mathbf{B}(x,t)]\rho(x-q(t))\, dx & \end{array} \right| . \tag{38}$$

This model of the electrodynamics with *extended electron* was introduced by Abraham to avoid the infinite mass and energy of the charged point particle, known as the *ultraviolet divergence*, [A1, A2].

All the proofs of the attraction (33) rely on the analysis of radiation which irreversibly carries a portion of energy to infinity. The details can be found in the surveys [E7, E8, E9].

In all the problems considered, the convergence (33) implies, by the Fatou theorem, the inequality

$$\mathcal{H}(S_\pm) \le \mathcal{H}(Y(t)) \equiv \text{const}, \quad t \in \mathbb{R}, \tag{39}$$

where $\mathcal{H}$ is the corresponding Hamiltonian (energy) functional. This inequality is an analog of the well-known property of weak convergence in Hilbert and Banach spaces. Simple examples show that the strict inequality in (39) is possible, which means that an irreversible scattering of energy to infinity occurs.

Example 1 (The d'Alembert Waves) In particular, the asymptotics (33) with the strict inequality (39) can easily be demonstrated for the d'Alembert equation $\ddot{\psi}(x,t) = \partial_x^2 \psi(x,t)$ with general solution

$$\psi(x,t) = f(x-t) + g(x+t). \tag{40}$$

Indeed, the convergence $\psi(\cdot, t) \to 0$ in $L^2_{\rm loc}(\mathbb{R})$ obviously holds for all $f, g \in L^2(\mathbb{R})$. On the other hand, the convergence to zero in *global norms* obviously fails if $f(x) \not\equiv 0$ or $g(x) \not\equiv 0$.

Example 2 (Nonlinear Huygens Principle) Consider solutions of 3D wave equation with a unit propagation velocity and initial data with support in a ball $|x| < R$. The corresponding solution is concentrated in spherical layers $|t| - R < |x| < |t| + R$. Therefore, the energy localized in any bounded region converges to zero as $t \to \pm\infty$, although its total energy remains constant. This convergence to zero is known as the *strong Huygens principle*. Thus, global attraction to stationary states (33) is a generalization of the strong Huygens principle to nonlinear equations. The difference is that for the linear wave equation, the limit is always zero, while for nonlinear equations the limit can be any stationary solution.

Remark 9.4 The proofs in [E10] and [E14] rely on the relaxation of acceleration

$$\ddot{q}(t) \to 0, \qquad t \to \pm\infty \tag{41}$$

for solutions to the coupled system (36) and similarly for solutions to the Maxwell–Lorentz system. This relaxation was discovered about 100 years ago in classical electrodynamics and is known as the *radiation damping*; see [A12, Chapter 16]. The detailed account on early investigations of this problem by A. Sommerfeld, H. Poincaré, P. Dirac and others can be found in Chapter 3 of [E15], see also [A7].

However, a rigorous proof of the relaxation (41) is not so obvious. It first appeared in [E10] and [E14] under the Wiener condition on the particle charge density

$$\hat{\rho}(k) := \int e^{ikx} \rho(x)\, dx \neq 0, \qquad k \in \mathbb{R}^3. \tag{42}$$

This condition is an analogue of the "Fermi Golden Rule" first introduced by Sigal in the context of nonlinear wave and Schrödinger equations [J34]. The proof of the relaxation (41) relies on a novel application of the Wiener Tauberian theorem.

Group of Translations $G = \mathbb{R}^n$

For *generic translation-invariant equations*, the conjecture (30) means the attraction of *all finite-energy solutions* to solitons:

$$\psi(x, t) \sim \psi_\pm(x - v_\pm t), \qquad t \to \pm\infty, \tag{43}$$

where the convergence holds in local seminorms of type $L^2(|x - v_\pm t| < R)$ with any $R > 0$, i.e., *in the comoving reference frame*. A trivial example is provided by the d'Alembert equation $\ddot{\psi}(x, t) = \partial_x^2 \psi(x, t)$ with general solution (40) corresponding to the asymptotics (43) with $v_+ = \pm 1$ and $v_- = \pm 1$.

Such soliton asymptotics was first proved for *completely integrable equations* (Korteweg–de Vries equation, etc.); see [F1, F8]. Moreover, for the Korteweg–de Vries equation, more accurate soliton asymptotics in *global norms* with several solitons were discovered by Kruskal and Zabusky in 1965 by numerical simulation: it is the decay to solitons

$$\psi(x,t) \sim \sum_k \psi_\pm(x - v_\pm^k t) + w_\pm(x,t), \qquad t \to \pm\infty, \tag{44}$$

where $w_\pm$ are some dispersive waves.

Later on, such asymptotics were proved by the method of *inverse scattering problem* for nonlinear completely integrable Hamiltonian translation-invariant equations (Korteweg–de Vries, etc.) in the works of Ablowitz, Segur, Eckhaus, van Harten, and others [F1, F8].

For equations which are not completely integrable, the global attraction (43) was established in [F2]–[F7]. The first result was obtained in [F6] for the translation-invariant system (36) with zero external potential $V = 0$. In this case the total momentum is conserved:

$$P := p - \int \dot\psi(x,t)\nabla\psi(x,t)\,dx = \text{const}. \tag{45}$$

In [F2], the result was extended to the Maxwell–Lorentz system (38) with $V = 0$. The proofs in [F6] and [F2] rely on variational properties of solitons and their orbital stability, as well as on the relaxation of the acceleration (41) under the Wiener condition (42). The case of small charge was considered in [F3, F4].

Let us mention that for non-completely-integrable equations the multi-soliton asymptotics (44) were observed numerically for 1D *relativistically-invariant* nonlinear wave equations in [F7].

Unitary Symmetry Group $G = \mathbf{U}(1)$

For *generic* $\mathbf{U}(1)$*-invariant equations*, the conjecture (30) means the attraction of *all finite-energy solutions* to "stationary orbits":

$$\psi(x,t) \sim \phi_\pm(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{46}$$

where $\omega_\pm \in \mathbb{R}$. Such asymptotics are similar to the Bohr transitions between stationary orbits (25) of the coupled Maxwell–Schrödinger equations. This asymptotics means that there is a global attraction to the solitary manifold formed by all *stationary orbits* (6). The asymptotics is considered in the local seminorms $L^2(|x| < R)$ with any $R > 0$. The global attractor is a smooth manifold formed

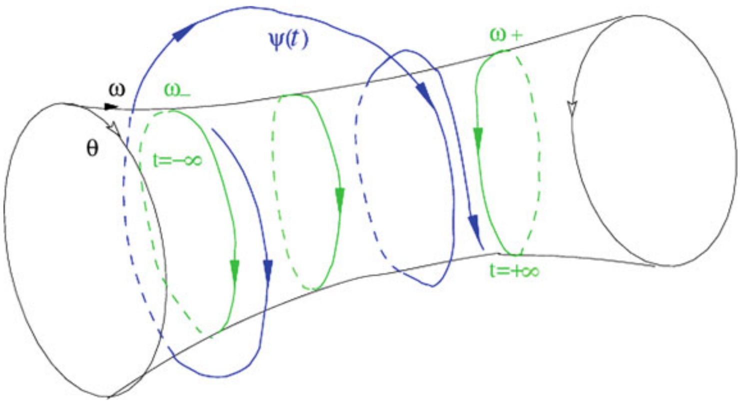

Fig. 2 Convergence to stationary orbits

by the circles that are the orbits of the action of the symmetry group $\mathbf{U}(1)$, as is illustrated on Fig. 2.

Such an attraction was proved for the first time

(1) in [H1] and [H3]–[H9] for the Klein–Gordon and Dirac equations coupled to $\mathbf{U}(1)$-invariant nonlinear oscillators,

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi + \sum_{k=1}^{N} \rho(x-x_k) F_k(\langle\psi(\cdot,t), \rho(\cdot - x_k)\rangle), \tag{47}$$

$$i\dot{\psi}(x,t) = \big(-i\alpha\cdot\nabla + \beta m\big)\psi + \sum_{k=1}^{N} \rho(x-x_k) F_k(\langle\psi(\cdot,t), \rho(\cdot - x_k)\rangle), \tag{48}$$

under the Wiener condition (42), where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = \alpha_0$ are the Dirac matrices;

(2) in [H2] for a discrete approximations of such coupled systems, i.e. for the corresponding finite difference schemes; and

(3) in [H11]–[H14] for the wave, Klein–Gordon, and Dirac equations with concentrated nonlinearities. More precisely, we have proved global attraction to the *solitary manifold* of all stationary orbits, although global attraction to particular stationary orbits (orbits with fixed $\omega_\pm$) is still an open problem.

All these results were proved under the assumption that the equations are "strictly nonlinear". For linear equations, the global attraction obviously fails if the discrete spectrum consists of at least two different eigenvalues.

The proofs of all these results rely on (1) a nonlinear analog of the Kato theorem on the absence of embedded eigenvalues; (2) a new theory of multiplicators in the space of quasimeasures; (3) a novel application of the Titchmarsh convolution theorem [A10, A18, A25]. For application to the Klein–Gordon equation in discrete space-time (a numerical scheme) in [H2], the appropriate version of the Titchmarsh theorem is obtained in [H10].

The asymptotics (46) is closely related to the *limiting amplitude principle* [A16, A21] and to the theory of waveguides [A17].

Orthogonal Group $G = \mathbf{SO}(3)$

For *generic rotation-invariant equations*, the conjecture (30) takes the form of the long-time asymptotics

$$\Psi(t) \sim e^{-i\hat{\Omega}_{\pm}t}\Psi_{\pm}, \qquad t \to \pm\infty \tag{49}$$

for *all finite-energy solutions*, where $\hat{\Omega}_{\pm}$ are suitable representations of $\Omega_{\pm} \in \mathfrak{so}(3)$. This means that global attraction to "stationary $\mathbf{SO}(3)$-orbits" takes place. Such asymptotics for the Maxwell–Lorentz equations with a rotating particle are proved in [G5].

10 On Generic Equations

We must still specify the meaning of the term *generic* in our conjecture (30). In fact, this conjecture means that the asymptotics (30) hold for all solutions from an open dense set of G-invariant equations.

1. In particular, the asymptotics (33), (43), (46), and (49) hold under appropriate conditions, which define some "open dense subset" of G-invariant equations with the four types of the symmetry group G. This asymptotic expression may break down if these conditions fail: this corresponds to some "exceptional" equations. For example, the global attraction (46) breaks down for linear Schrödinger equations with at least two different eigenvalues. Thus, linear equations are exceptional, not generic!
2. The general situation is as follows. Let the Lie group G_1 be a (proper) subgroup of some larger Lie group G_2. Then G_2-invariant equations form an "exceptional subset" among all G_1-invariant equations, and the corresponding asymptotics (30) may be completely different. For example, the trivial group $\{e\}$ is a proper subgroup in $\mathbf{U}(1)$ and in $\mathbb{R}^n$, and the asymptotic expressions (43) and (46) may differ significantly from (33).
3. Examples of solitary wave solutions violating asymptotics (30) in particular *non-generic* models were constructed in [H8, H6] (two-frequency solitary waves in models based on the Klein–Gordon equation with $\mathbf{U}(1)$-symmetry group), in [H2] (two- and four-frequency solitary waves in the discrete time-space Klein–Gordon equation with $\mathbf{U}(1)$-symmetry group), and in [I1] (two-frequency solitary

waves in the Soler model with **SU**(1, 1)-symmetry group, when the solitary manifold is larger than the orbit of the action of **SU**(1, 1)).

11 Adiabatic Effective Dynamics of Solitons

The system (36) admits soliton solutions

$$\phi_v(x - vt), \qquad q = vt \tag{50}$$

in the case of identically zero external potential $V(q) \equiv 0$. However, even in the case of nonzero external potential, *soliton-like solutions* of the form

$$\psi(x,t) \approx \psi_{v(t)}(x - q(t)) \tag{51}$$

may exist if the potential is slowly varying:

$$|\nabla V(q)| \leq \varepsilon \ll 1. \tag{52}$$

In this case, the total momentum (45) is generally not conserved, but its slow evolution together with evolution of the parameters $q(t)$, $v(t)$ in (51) can be described in terms of an appropriate finite-dimensional Hamiltonian dynamics.

Indeed, denote by $P = P_v \in \mathbb{R}^3$ the total momentum (45) of the soliton (50). It is important that the map $\mathbf{P} : v \mapsto P_v$ is an isomorphism of the ball $|v| < 1$ on $\mathbb{R}^3$. Therefore, we can consider q, P as global coordinates on the *solitary manifold*

$$\mathcal{S} := \{S_{q,v} = (\phi_v(x - q), -v \cdot \nabla\phi_v(x - q), a, v) : q, v \in \mathbb{R}^3, \quad |v| < 1\}.$$

The effective Hamiltonian functional is defined by

$$H_{\text{eff}}(q, P_v) \equiv H(S_{q,v}), \qquad q,\ P_v \in \mathbb{R}^3, \tag{53}$$

where H is the Hamiltonian (37) with $V(q) \equiv 0$. This functional can be represented as

$$H_{\text{eff}}(Q, \Pi) = E(\Pi) + V(Q),$$

since the first integral in (37) does not depend on Q, while the last integral vanishes on the solitons. Hence, the corresponding Hamiltonian equations have the form

$$\dot{Q}(t) = \nabla E(\Pi(t)), \qquad \dot{\Pi}(t) = -\nabla V(Q(t)). \tag{54}$$

The main result in [G6] is the following theorem. Let us denote by $\|\cdot\|_R$ the norm in $L^2(B_R)$, where B_R is the ball $\{x \in \mathbb{R}^3 : |x| < R\}$, and denote $\pi_v(x) := -v\nabla\phi_v(x)$.

Theorem 11.1 *Let the condition (52) hold, and assume that* $(\psi(0), \dot{\psi}(0), q(0), p(0)) \in \mathcal{S}$ *is a soliton with the total momentum* $P(0)$. *Then the corresponding solution* $(\psi(x,t), \dot{\psi}(x,t), q(t), p(t))$ *of the system (36) with* $V = 0$ *admits the "adiabatic asymptotics"*

$$|q(t) - Q(t)| \le C_0, \quad |P(t) - \Pi(t)| \le C_1\varepsilon \quad \text{for} \quad |t| \le C\varepsilon^{-1},$$

$$\sup_{t\in\mathbb{R}}\Big[\|\nabla\psi(q(t)+y,t) - \nabla\psi_{v(t)}(y)]\|_R + \|\dot{\psi}(q(t)+y,t) - \pi_{v(t)}(y)\|_R\Big] \le C\varepsilon,$$

where $P(t)$ *denotes the total momentum (45),* $v(t) = \mathbf{P}^{-1}(\Pi(t))$, *and* $(Q(t), \Pi(t))$ *is the solution of the effective Hamiltonian equations (54) with initial conditions*

$$Q(0) = q(0), \qquad \Pi(0) = P(0).$$

We note that such relevance of the effective dynamics (54) is due to the consistency of Hamiltonian structures:

1. The effective Hamiltonian (53) is the restriction of the Hamiltonian functional (37) with $V = 0$ to the solitary manifold $\mathcal{S}$.
2. As shown in [G6], the canonical differential form of the Hamiltonian system (54) is also the restriction to $\mathcal{S}$ of the canonical differential form of the system (36): formally,

$$P \cdot dQ = \Big[p \cdot dq + \int \pi(x)\, d\psi(x)\, dx\Big]\Big|_{\mathcal{S}}.$$

   Therefore, the total momentum $\mathbf{P}$ is canonically conjugate to the variable Q on the solitary manifold $\mathcal{S}$. This fact justifies the definition (53) of the effective Hamiltonian as a function of the total momentum P_v, and not of the particle momentum p_v.

One of the important results in [G6] is the following "effective dispersion relation":

$$E(\Pi) \sim \frac{\Pi^2}{2(1+m_e)} + \text{const}, \qquad |\Pi| \ll 1. \tag{55}$$

It means that the nonrelativistic mass of a slow soliton increases, due to the interaction with the field, by the amount

$$m_e = -\frac{1}{3}\langle\rho, \Delta^{-1}\rho\rangle. \tag{56}$$

This increment is proportional to the field energy of the soliton at rest,

$$E_{\rm own} = H(\Delta^{-1}\rho, 0, 0, 0) = -\frac{1}{2}\langle\rho, \Delta^{-1}\rho\rangle, \tag{57}$$

which agrees with the Einstein mass-energy equivalence principle (see Sect. 11 below).

Remark 11.5 The relation (55) suggests that m_e is an increment of the effective mass. The true *dynamical justification* for such an interpretation is given by Theorem 11.1 which demonstrates the relevance of the effective dynamics (54).

Generalizations In [G7], Kunze and Spohn extended Theorem 11.1 to solitons of the Maxwell–Lorentz equations (38) with slowly varying potential (52). Following the articles [G6, G7], suitable adiabatic effective dynamics was obtained in [G3] and [G4] for nonlinear Hartree and Schrödinger equations with slowly varying external potentials, and in [G2, G8] and [G9] for the nonlinear equations of Einstein–Dirac, Chern–Simon–Schrödinger and Klein–Gordon–Maxwell with small external fields. In [G5], the adiabatic effective dynamics was established for the Maxwell–Lorentz equations with a rotating particle. Similar adiabatic effective dynamics was established in [G1] for an electron in the second-quantized Maxwell field in the presence of a slowly varying external potential. The results of numerical simulation [F7] confirm the adiabatic effective dynamics of solitons for relativistic 1D nonlinear wave equations (see Sect. 14).

Mass-Energy Equivalence

In [G7], Kunze and Spohn have established that in the case of the Maxwell–Lorentz equations (38) with slowly varying potential (52), the increment of nonrelativistic mass also turns out to be proportional to the energy of the static soliton's own field. Such an equivalence of the self-energy of a particle with its mass was first discovered in 1902 by M. Abraham: he showed by direct calculation that the energy $E_{\rm own}$ of electrostatic field of an electron at rest adds

$$m_e = \frac{4}{3}E_{\rm own}/c^2 \tag{58}$$

to its nonrelativistic mass (see [A1, A2], and also [A13, pp. 216–217]). By (57), this self-energy is infinite for a point particle with the charge density $\rho(x) = \delta(x)$. This means that the field mass for a point electron is infinite, which contradicts experiments. That is why M. Abraham introduced the model of electrodynamics with *extended electron* (38), whose self-energy is finite.

At the same time, M. Abraham conjectured that the *entire mass* of an electron is due to its own electromagnetic energy, i.e., $m = m_e$: "matter disappeared, only energy remains", as philosophically-minded contemporaries wrote [A11, pp. 63, 87, 88]. (smile :))

In 1905, the formula (58) was corrected by A. Einstein, who discovered the famous universal relation $E = m_0c^2$ which follows from the Special Theory of Relativity [A8]. Thus, Abraham's discovery of the mass-energy equivalence anticipated Einstein's theory. The doubtful factor $\frac{4}{3}$ in Abraham's formula is due to the nonrelativistic character of the system (38). According to the modern view, about 80% of the electron mass is of electromagnetic origin [A9].

12 Stability of Stationary Orbits and Solitons

In 1987–1990, Grillakis, Shatah and Strauss developed general theory of orbital stability of the solitary waves [I7]. In [J2], Bambusi and Galgani established orbital stability of solitons for the Maxwell–Lorentz system (38).

Asymptotic stability of solitary manifold has the meaning of the local attraction, i.e., the convergence to this manifold of states that were sufficiently close to it. The first results of such type were obtained by Morawetz, Segal, and Strauss in the case when the attractor consists of the single point zero. In this case, the attraction is called *local energy decay* [C1]–[C7].

In the case of a continuous attractor, the key peculiarity of this convergence is the instability of the dynamics *along the manifold*. The instability follows directly from the fact that solitons move with different speeds and therefore run away for large times. Analytically, this instability is caused by the presence of the Jordan block corresponding to eigenvalue $\lambda = 0$ in the spectrum of the generator of linearized dynamics. The tangent vectors to the soliton manifold are eigenvectors and generalized eigenvectors corresponding to this Jordan block. Thus, Lyapunov's theory is not applicable to this case.

Remark 12.6 According to Arnold (see e.g. [A3] and references therein), a similar instability mechanism may be responsible for the mixing and ergodicity of the turbulent flows.

In a series of articles published during 1985–2003, Weinstein, Soffer, Buslaev, Perelman, and Sulem discovered an original strategy for proving asymptotic stability of solitary manifolds. This strategy relies on (1) special projection of a trajectory onto the solitary manifold, (2) modulation equations for parameters of the projection, and (3) time decay of the transversal component. This approach is a far-reaching development of the Lyapunov stability theory.

Spectral Stability of Solitons

Spectral (or linear) stability of solitary waves can be traced back to 1967, to the work of Zakharov [I19] on nonlinear Schrödinger equation

$$i\dot{\psi} = -\Delta\psi - |\psi|^2\psi, \qquad \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}^n, \quad n \geq 1, \tag{59}$$

which was later developed by Kolokolov [I8]. This is the "weakest" type of stability: one considers the perturbation of a solitary wave solution $\phi(x)e^{-i\omega t}$, $\phi(x) \in \mathbb{R}$, in the form of the Ansatz

$$\psi(x,t) = (\phi(x) + \chi(x,t))e^{-i\omega t},$$

with complex-valued perturbation $\chi(x,t) = u(x,t) + iv(x,t)$ (with u, v real-valued), which is assumed to be small at $t = 0$. Then one derives the linear approximation to the evolution of χ, of the form

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix}, \tag{60}$$

and studies the spectrum of the operator $A = \begin{bmatrix} 0 & L_0 \\ -L_1 & 0 \end{bmatrix}$, where

$$L_0 = -\Delta - \phi^2 + \omega, \qquad L_1 = -\Delta - 3\phi^2 + \omega$$

are the Schrödinger-type operators. One can show that in the case $n \leq 2$ one has $\sigma(A) \subset i\mathbb{R}$ (and the corresponding solitary wave is called spectrally stable), while for $n \geq 3$ the intersection $\sigma(A) \cap (0, +\infty)$ contains a positive eigenvalue (the corresponding solitary wave is called linearly unstable). We note that the system (60) can not be written as a $\mathbb{C}$-linear equation on χ: in the Physics terminology, the $\mathbf{U}(1)$-symmetry is broken by the choice $\phi(x) \in \mathbb{R}$.

In the context of the nonlinear Schrödinger and Klein–Gordon equations and similar systems the spectral stability has been extended to orbital stability; see [I6, I7, I10, I18]. At the same time, the orbital stability for the nonlinear Dirac equation, and likewise for the Dirac–Maxwell and Dirac–Klein–Gordon systems, does not seem possible except via the proof of asymptotic stability, with the exception of the completely integrable massive Thirring model in one spatial dimension [I13]. Let us mention that the spectral stability for the nonlinear Dirac equation in the charge-subcritical cases has been proved in [I2] (see also [I3]).

Asymptotic Stability of Stationary Orbits: Orthogonal Projection

This strategy was formed in 1985–1992 in the pioneering work of Soffer and Weinstein [I11, I12, I18]; see the review [I17]. The results concern nonlinear $\mathbf{U}(1)$-invariant Schrödinger equations with real-valued potential $V(x)$,

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t) + \lambda|\psi(x,t)|^p\psi(x,t), \qquad x \in \mathbb{R}^n, \tag{61}$$

where $\lambda \in \mathbb{R}$, $p = 3$ or 4, $n = 2$ or $n = 3$, and $\psi(x,t) \in \mathbb{C}$. The corresponding Hamiltonian functional is given by

$$\mathcal{H} = \int \left[\frac{1}{2}|\nabla\psi(x)|^2 + \frac{1}{2}V(x)|\psi(x)|^2 + \frac{\lambda}{p}|\psi(x)|^p\right]dx.$$

For $\lambda = 0$, equation (61) is linear. Let $\phi_*(x)$ denote its ground state corresponding to the minimal eigenvalue $\omega_* < 0$. Then $C\phi_*(x)e^{-i\omega_* t}$ are periodic solutions for any complex constant C. Corresponding phase curves are circles filling the complex plane. For nonlinear equations (61) with a small real $\lambda \neq 0$, it turns out that a wonderful *bifurcation* occurs: small neighborhood of zero in the complex plane turns into an analytic soliton manifold $\mathcal{S}$ invariant with respect to the dynamics group corresponding to (61), which is still filled with invariant circles $\phi_\omega(x)e^{-i\omega t}$ whose frequencies ω are close to ω_*.

The main result of [I11, I12] (see also [I14]) is long-time attraction to one of these circles for any solution with sufficiently small initial data:

$$\psi(x,t) = \phi_\pm(x)e^{-i\omega_\pm t} + r_\pm(x,t), \tag{62}$$

where the remainder decay in weighted norms: for any $\sigma > 2$,

$$\|\langle x\rangle^{-\sigma} r_\pm(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0, \qquad t \to \pm\infty,$$

where $\langle x\rangle := (1+|x|)^{1/2}$. The proof relies on linearization of the dynamics and decomposition of solutions into two components,

$$\psi(t) = e^{-i\Theta(t)}(\phi_{\omega(t)} + \chi(t)),$$

satisfying the following orthogonality condition [I11, (3.2) and (3.4)]:

$$\langle \phi_{\omega(t)}, \chi(t)\rangle = 0. \tag{63}$$

This orthogonality and dynamics (61) imply the *modulation equations* for $\omega(t)$ and $\gamma(t)$, where $\gamma(t) := \Theta(t) - \int_0^t \omega(s)ds$ (see [I11, (3.9a)–(3.9b)]). The orthogonality (63) implies that the component $\phi(t)$ lies in the continuous spectral space of the

Schrödinger operator

$$H(\omega_0) := -\Delta + V + \lambda|\phi_{\omega_0}|^p,$$

which leads to the time decay of $\chi(t)$ (see [I11, (4.2a) and (4.2b)]). Finally, this decay implies the convergence $\omega(t) \to \omega_\pm$ and the asymptotics (62).

These results and methods were subsequently widely developed for the nonlinear Schrödinger, wave and Klein–Gordon equations with potentials under various spectral assumptions on linearized dynamics; see, for instance, [I15, I16].

Asymptotic Stability of Solitons: Symplectic Projection

Next breakthrough in the theory of asymptotic stability was achieved in 1990–2003 by Buslaev, Perelman and Sulem [J5, J6, J7], who first proved asymptotics of type (62) for translation-invariant 1D Schrödinger equations

$$i\dot{\psi}(x,t) = -\psi''(x,t) - F(\psi(x,t)), \qquad x \in \mathbb{R}, \tag{64}$$

which are also assumed to be $\mathbf{U}(1)$-invariant, i.e., $F(\psi) = -\nabla U(\psi)$, and

$$F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \qquad \psi \in \mathbb{C}, \;\; \theta \in \mathbb{R}. \tag{65}$$

Moreover, they assume that

$$U(\psi) = \mathcal{O}(|\psi|^{10}), \qquad \psi \to 0. \tag{66}$$

Under some simple conditions on the potential U, there exist finite energy solutions of the form

$$\psi(x,t) = \phi_0(x)e^{i\omega_0 t}, \tag{67}$$

with $\omega_0 > 0$. The amplitude $\phi_0(x)$ satisfies the corresponding stationary equation

$$-\omega_0\phi_0(x) = -\phi_0''(x) - F(\phi_0(x)), \qquad x \in \mathbb{R}, \tag{68}$$

which implies the "conservation law"

$$\frac{|\phi_0'(x)|^2}{2} + U_e(\phi_0(x)) = E, \tag{69}$$

where the effective potential is given by $U_e(\psi) = U(\psi) + \omega_0\frac{|\psi|^2}{2} \sim \omega_0\frac{|\psi|^2}{2}$ as $\psi \to 0$ by (66). For the existence of a finite energy solution, the graph of "effective

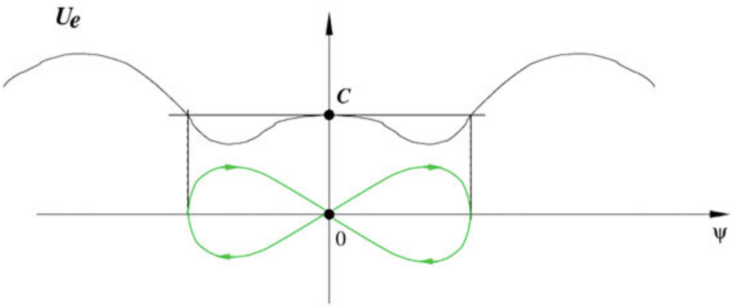

Fig. 3 Reduced potential and soliton

potential" $U_e(\psi)$ should be similar to Fig. 3. The finite energy solution $\phi_0(x)$ is defined by (69) with the constant $E = U_e(0)$ since for other E the solutions to (69) do not converge to zero as $|x| \to \infty$. Equation (69) with $E = U_e(0)$ implies that

$$\frac{|\phi_0'(x)|^2}{2} = U_e(0) - U_e(\phi_0(x)) \sim \frac{\omega_0}{2}\phi_0^2(x). \tag{70}$$

Hence, for finite energy solutions, one has

$$\phi_0(x) \sim e^{-\sqrt{\omega_0}|x|}, \qquad |x| \to \infty. \tag{71}$$

It is easy to verify that the following functions are also solutions (*moving solitons*):

$$\psi_{\omega,v,a,\theta}(x,t) = \phi_\omega(x - vt - a)e^{i(\omega t + kx + \theta)}, \qquad \omega = \omega_0 - v^2/4, \qquad k = v/2. \tag{72}$$

The set of all such solitons with parameters ω, v, a, θ forms a 4-dimensional smooth submanifold $\mathcal{S}$ in the Hilbert phase space $\mathcal{X} := L^2(\mathbb{R})$. Solitons (72) are obtained from (67) by the Galilean transformation

$$G(a, v, \theta) : \psi(x,t) \mapsto \varphi(x,t) = \psi(x - vt - a, t)e^{i(-\frac{v^2}{4}t + \frac{v}{2}x + \theta)}; \tag{73}$$

the Schrödinger equation (64) is invariant with respect to this group of transformations.

Linearization of the Schrödinger equation (64) at the soliton (67) is obtained as in Sect. 12, by substitution $\psi(x,t) = (\phi_0(x) + \chi(x,t))e^{i\omega_0 t}$ and retaining terms of the first order in χ. The resulting equation is not linear over the field of complex numbers, since it contains both χ and $\overline{\chi}$. This follows from the fact that the nonlinearity $F(\psi) = a(|\psi|^2)\psi$ is not complex-analytic due to the $\mathbf{U}(1)$-invariance (65). Complexification of this linearized equation reads as follows:

$$\dot{\Psi}(x,t) = A_0\Psi(x,t), \qquad A_0 = -jH_0, \tag{74}$$

where j is a real 2×2 matrix, representing the multiplier i, $\Psi(x,t) \in \mathbb{C}^2$, and $H_0 = -d^2/dx^2 + \omega_0 + V(x)$, where $V(x)$ is a real, matrix-valued potential which decreases exponentially as $|x| \to \infty$ due to (71). Note that the operator A_0 in (74) corresponds to the linearization at the soliton (72) with parameters $\omega = \omega_0$ and $a = v = \theta = 0$. Similar operators $A(\omega, a, v, \theta)$ corresponding to linearization at solitons (72) with various parameters ω, a, v, θ are also connected by the linear Galilean transformation (73). Therefore, their spectral properties completely coincide. In particular, their continuous spectra coincide with $(-i\infty, -i\omega_0] \cup [i\omega_0, i\infty)$.

Main results of [J5, J6, J7] are asymptotics of type (62) for solutions with initial data close to the solitary manifold $\mathcal{S}$:

$$\psi(x,t) = \phi_\pm(x - v_\pm t)e^{-i(\omega_\pm t + k_\pm x)} + W(t)\Phi_\pm + r_\pm(x,t), \qquad \pm t > 0, \tag{75}$$

where $W(t)$ is the dynamical group of the free Schrödinger equation, $\Phi_\pm$ are some scattering states of finite energy, and $r_\pm$ are remainder terms which decay to zero in the global norm:

$$\|r_\pm(\cdot, t)\|_{L^2(\mathbb{R})} \to 0, \qquad t \to \pm\infty. \tag{76}$$

These asymptotics were obtained under the following assumptions on the spectrum of the generator B_0:

U1. The discrete spectrum of the operator C_0 consists of exactly three eigenvalues 0 and $\pm i\lambda$, and

$$\lambda < \omega_0 < 2\lambda. \tag{77}$$

This condition means that the discrete mode interacts with the continuous spectrum already in the first order of approximation.

U2. The edge points $\pm i\omega_0$ of the continuous spectrum are neither eigenvalues, nor virtual levels (threshold resonances) of C_0.

U3. Furthermore, it is assumed the condition [J7, (1.0.12)], which means a strong coupling of discrete and continuous spectral components, providing energy radiation, similarly to the Wiener condition [A14, (6.24)], [E9, (1.5.13)]. It ensures that the interaction of discrete component with continuous spectrum does not vanish already in the first order of perturbation theory.

This condition is a nonlinear version of the Fermi Golden Rule [A23], which was introduced by Sigal in the context of nonlinear PDEs [J34].

Examples of potentials satisfying all these conditions are constructed in [J19].

In 2001, Cuccagna extended results of [J5, J6, J7] to translation-invariant Schrödinger equations in $\mathbb{R}^n$, $n \geq 2$ [J9]. Similar approach to the asymptotic stability of solitons was developed for the Korteweg–de Vries equation in [J30], for the regularized long-wave equation in [J29], and for the nonlinear Schrödinger equation in $\mathbb{R}^3$ with concentrated nonlinearity in [J1]. In the context of the nonlinear

Dirac equation with a potential, stability with respect to perturbations of solitary waves in particular directions was considered in [I5].

Method of Symplectic Projection in the Hilbert Phase Space Novel approach [J5, J6, J7] relies on *symplectic projection* P of solutions onto the solitary manifold. This means that

$$Z := \psi - S \quad \text{symplectic-orthogonal to the tangent space} \quad \mathcal{T} := T_S\mathcal{S}$$

for the projection $S := P\psi$. This projection is correctly defined in a small neighborhood of $\mathcal{S}$. For this it is important that $\mathcal{S}$ is a *symplectic manifold*, i.e. the corresponding symplectic form is non-degenerate on the tangent spaces $T_S\mathcal{S}$.

In particular, the approach [J5, J6, J7] allowed to get rid of the assumption that the initial data are small.

Now a solution $\psi(t)$ for each $t > 0$ decomposes onto *symplectic-orthogonal* components $\psi(t) = S(t) + Z(t)$, where $S(t) := P\psi(t)$, and the dynamics is linearized on the soliton $S(t)$.

The Hilbert phase space $\mathcal{X} := L^2(\mathbb{R})$ is split as $\mathcal{X} = \mathcal{T}(t) \oplus \mathcal{Z}(t)$, where $\mathcal{Z}(t)$ is **symplectic-orthogonal** complement to the tangent space $\mathcal{T}(t) := T_{S(t)}\mathcal{S}$. The corresponding equation for the *transversal component* $Z(t)$ reads

$$\dot{Z}(t) = A(t)Z(t) + N(t),$$

where $A(t)Z(t)$ is the linear part, and $N(t) = \mathcal{O}(\|Z(t)\|^2)$ is the corresponding nonlinear part. The main difficulties in studying this equation are as follows (1) it is *non-autonomous*, and (2) the generators $A(t)$ *are not self-adjoint* (see Appendix in [J17]). It is important that $A(t)$ are *Hamiltonian operators*, for which the existence of spectral decomposition is provided by the Krein–Langer theory of J -selfadjoint operators [J24, J25]. In [J17, J18] we have developed a special version of this theory providing the corresponding eigenfunction expansion which is necessary for the justification of the approach [J5, J6, J7].

The main steps of the strategy [J5, J6, J7] are as follows:

- **Modulation equations.** The parameters of the soliton $S(t)$ satisfy **modulation equations**: for example, for the speed $v(t)$ we have

  $$\dot{v}(t) = M(\psi(t)),$$

  where $M(\psi) = \mathcal{O}(\|Z\|^2)$ for small norms $\|Z\|$. Therefore, the parameters change "super-slow" near the soliton manifold, like adiabatic invariants.
- **Tangent and transversal components.** The transversal component $Z(t)$ in the splitting $\psi(t) = S(t) + Z(t)$ belongs to the transversal space $\mathcal{Z}(t)$. The tangent

space $\mathcal{T}(t)$ is the root space of the generator $A(t)$ and corresponds to the "unstable" spectral point $\lambda = 0$. The key observation is that

1. symplectic-orthogonal subspace $\mathcal{Z}(t)$ does not contain "unstable" tangent vectors, and, moreover,
2. the subspace $\mathcal{Z}(t)$ is **invariant** with respect to the generator $A(t)$ since the subspace $\mathcal{T}(t)$ is invariant and $A(t)$ is the Hamiltonian operator.

- **Continuous and discrete components.** The transversal component allows further splitting $Z(t) = z(t) + f(t)$, where $z(t)$ and $f(t)$ belong, respectively, to discrete and continuous spectral subspaces $\mathcal{Z}_d(t)$ and $\mathcal{Z}_c(t)$ of $A(t)$ in the space $\mathcal{Z}(t) = \mathcal{Z}_d(t) + \mathcal{Z}_c(t)$.
- **Poincare normal forms and Fermi Golden Rule.** The component $z(t)$ satisfies a nonlinear equation, which is reduced to Poincare normal form up to higher order terms [J7, Equations (4.3.20)]. (For the relativistic-invariant Ginzburg–Landau equation, a similar reduction done in [J23, Equations (5.18)].) The normal form allowed to derive some "conditional" decay for $z(t)$ using the Fermi Golden Rule [J7, (1.0.12)].
- **Method of majorants.** A skillful combination of the conditional decay for $z(t)$ with super-slow evolution of the soliton parameters allows us to prove the decay for $f(t)$ and $z(t)$ by the method of majorants. Finally, this decay implies the asymptotics (75)–(76).

Generalizations and Applications

N-Soliton Solutions The methods and results of [J7] were developed in [J26, J27, J28] and [J31, J32, J33] for N-soliton solutions for translation-invariant nonlinear Schrödinger equations.

Multiphoton Radiation In [J10], Cuccagna and Mizumachi extended methods and results of [J7] to the case when the inequality (77) is changed to

$$N\lambda < \omega_0 < (N+1)\lambda,$$

with some natural $N > 1$, and the corresponding analogue of condition U3 holds. It means that the interaction of discrete modes with a continuous spectrum occurs only in the N-th order of perturbation theory. The decay rate of the remainder term (76) worsens with growing N.

Linear Equations Coupled to Nonlinear Oscillators and Particles In [I4, I9] methods and results of [J7] were extended to (1) the Schrödinger equation interacting with a nonlinear $\mathbf{U}(1)$– invariant oscillator, (2) in [J14, J15]– to translation-invariant systems of relativistic particle coupled to the wave and Maxwell equations, and (3) in [J13, J16, J20]– to similar translation-invariant systems of relativistic particle coupled to the Klein–Gordon, Schrödinger and Dirac equations. The review of the results [J13, J14, J15] can be found in [J12].

Relativistic Equations In [J3, J4, J21, J22, J23] methods and results [J7] were extended for the first time to *relativistic-invariant* nonlinear equations. Namely, (1) in [J3] and [J21, J22, J23] asymptotics of type (75) were obtained for 1D relativistic-invariant nonlinear wave equations with potentials of the Ginzburg–Landau type,

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + f(\psi(x,t)), \tag{78}$$

and (2) in [J4] and [J8] for relativistic-invariant nonlinear Dirac equations. In [J19] we have constructed examples of potentials providing all spectral properties of the linearized dynamics imposed in [J21, J22, J23].

The Justification of the Eigenfunction Expansions In [J17, J18] we have justified the eigenfunction expansions for nonselfadjoint Hamiltonian operators which were used in [J21, J22, J23]. For the justification we have developed a special version of the Krein–Langer theory of J -selfadjoint operators [J24, J25]

Vavilov–Cherenkov Radiation The article [J11] concerns the nonrelativistic particle coupled to the Schrödinger equation (system (1.9)–(1.10) in [J11]). This system is considered as a model of the Cherenkov radiation. The main result of [J11] is long-time convergence to a soliton with the sonic speed for initial solitons with a supersonic speed in the case of a weak interaction ("Bogoliubov limit") and small initial field.

Asymptotic stability of solitons for similar system was established in [J16].

Further Generalizations

The results on asymptotic stability of solitons were developed in different directions.

Systems with Several Bound States The case of many simple eigenvalues of linearization at a soliton (74) was first investigated in [K1, K2, K3, K4, K5] for the nonlinear Schrödinger equation

$$i\dot{\psi}(x,t) = (-\Delta + V(x))\psi(x,t) \pm |\psi(x,t)|^2\psi(x,t), \qquad x \in \mathbb{R}^3.$$

Asymptotic stability and long-time asymptotics of solutions were established under the following assumptions:

1 the endpoint of the essential spectrum is neither an eigenvalue nor a virtual level (threshold resonance) for linearized equation;
2. the eigenvalues of the linearized equation satisfy the corresponding non-resonance condition;
3. a new version of the Fermi Golden Rule.

The main obtained result: any solution with small initial data converges to some ground state at $t \to \infty$ with speed $t^{-1/2}$ in $L^2_{\rm loc}(\mathbb{R}^3)$. There are different long-time regimes depending on relative contributions of eigenfunctions into initial data.

One of the difficulties is the possible existence of invariant tori corresponding to the resonances. Great efforts have been applied to show that the role of metastable tori decays like $t^{-1/2}$ as $t \to \infty$.

This result was extended in [K1] to nonlinear Klein–Gordon equations

$$\ddot{\psi}(x,t) = (\Delta - V(x) - m^2)\psi(x,t) + f(\psi(x,t)), \qquad x \in \mathbb{R}^3.$$

Namely, under conditions (1)–(3) above, any sufficiently small solution for large times asymptotically is a free wave in the norm $H^1(\mathbb{R}^3)$. The proofs largely rely on the theory of the Birkhoff normal forms. The main novelty is the transition to normal forms without loss of the Hamiltonian structure.

In [K2] the results of [J10, K1] are extended to nonlinear Schrödinger equation

$$i\dot{\psi}(x,t) = (-\Delta + V(x))\psi(x,t) + \beta(|\psi(x,t)|)\psi(x,t), \qquad x \in \mathbb{R}^3.$$

Main results are long-time asymptotics "ground state + dispersive wave" in the norm $H^1(\mathbb{R}^3)$ for solutions close to the ground state.

The corresponding linearized equation can have many eigenvalues, which must satisfy the non-resonant conditions [K1], and suitable modification of the Fermi Golden Rule holds. However, for nonlinear Schrödinger equations methods of [K1] required a significant improvement: now the canonical coordinates are constructed using the Darboux theorem.

General Theory of Relativity The article [L4] concerns so-called "kink instability" of self-similar and spherically symmetric solutions of the equations of the general theory of relativity with a scalar field, as well as with a "hard fluid" as sources. The authors constructed examples of self-similar solutions that are unstable to the kink perturbations.

The article [L2] examines linear stability of slowly rotating Kerr solutions for the Einstein equations in vacuum. In [L5] a pointwise damping of solutions to the wave equation is investigated for the case of stationary asymptotically flat space-time in the three-dimensional case.

In [L1] the Maxwell equations are considered outside slowly rotating Kerr black hole. The main results are: (1) boundedness of a positive definite energy on each hypersurface $t = \text{const}$ and (2) convergence of each solution to a stationary Coulomb field.

In [L3] the pointwise decay was proved for linear waves against the Schwarzschild black hole.

Method of Concentration Compactness In [M4] the concentration compactness method was used for the first time to prove global well-posedness, scattering and blow-up of solutions to critical focusing nonlinear Schrödinger equation

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) - |\psi(x,t)|^{\frac{4}{n-2}}\psi(x,t), \qquad x \in \mathbb{R}^n$$

in the radial case. Later on, these methods were extended in [M1, M3, M5, M9] to general non-radial solutions and to nonlinear wave equations of the form

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) + |\psi(x,t)|^{\frac{4}{n-2}}\psi(x,t), \qquad x \in \mathbb{R}^n.$$

One of the main results is splitting of the set of initial states, close to the critical energy level, into three subsets with certain long-term asymptotics: either a blow-up in a finite time, or an asymptotically free wave, or the sum of the ground state and an asymptotically free wave. All three alternatives are possible; all nine combinations with $t \to \pm\infty$ are also possible. Lectures [M11] give excellent introduction to this area. The articles [M2, M6] concern super-critical nonlinear wave equations.

Recently, these methods and results were extended to critical wave mappings [M7, M8, M9, M10]. The "decay onto solitons" is proved: every 1-equivariant finite-energy wave mapping of exterior of a ball with Dirichlet boundary conditions into three-dimensional sphere exists globally in time and dissipates into a single stationary solution of its own topological class.

13 Linear Dispersion

In all results on long-time asymptotic for nonlinear Hamiltonian PDEs, the key role is played by dispersive decay of solutions of the corresponding linearized equations. In this section, we choose most important or recent articles out of a huge number of publications.

Dispersion Decay in Weighted Sobolev Norms Dispersion decay for wave equations was first proved in linear scattering theory [N25]. For the Schrödinger equation with a potential, a systematic approach to dispersive decay was proposed by Agmon [N1] and Jensen and Kato [N12]. This theory was extended by many authors to wave, Klein–Gordon and Dirac equations and to the corresponding discrete equations, see [N3]–[N11], [N13]–[N24] and references therein.

L^1-L^∞ *decay* for solutions of the linear Schrödinger equation was proved for the first time by Journé, Soffer, and Sogge [N13]. Namely, it was shown that the solutions to

$$i\dot{\psi}(x,t) = H\psi(x,t) := (-\Delta + V(x))\psi(x,t), \qquad x \in \mathbb{R}^n, \qquad n \geq 3 \tag{79}$$

satisfy

$$\|P_c\psi(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-n/2}\|\psi(0)\|_{L^1(\mathbb{R}^n)}, \qquad t > 0, \tag{80}$$

provided that $\lambda = 0$ is neither an eigenvalue nor a virtual level of H. Here P_c is an orthogonal projection onto continuous spectral space of the operator H. It is assumed that the potential $V(x)$ is real-valued, sufficiently smooth, and rapidly decays as $|x| \to \infty$. This result was generalized later by many authors; see below.

The decay of type (80) and Strichartz estimates were established in [N27] for 3D Schrödinger equations (79) with "rough" and time-dependent potentials $V = V(x,t)$. Similar estimates were obtained in [N3] for 3D Schrödinger and wave equations with (stationary) Kato class potentials.

In [N7], the 4D Schrödinger equations (79) are considered in the case when there is a virtual level or an eigenvalue at zero energy. In particular, in the case of an eigenvalue at zero energy, there is a time-dependent operator F_t of rank 1 such that $\|F_t\|_{L^1\to L^\infty} \le 1/\log t$ for $t > 2$, and

$$\|e^{itH}P_c - F_t\|_{L^1\to L^\infty} \le Ct^{-1}, \qquad t > 2.$$

Similar dispersive estimates were also proved for solutions to 4D wave equation with a potential.

In [N8, N9], the Schrödinger equation (79) is considered in $\mathbb{R}^n$ with $n \ge 5$ with the assumption that zero is an eigenvalue (the Schrödinger operator in dimension $n \ge 5$ could only have zero eigenvalues but no genuine virtual level at zero). It is shown, in particular, that there is a time-dependent rank one operator F_t such that $\|F_t\|_{L^1\to L^\infty} \le C|t|^{2-n/2}$ for $|t| > 1$, and

$$\|e^{itH}P_c - F_t\|_{L^1\to L^\infty} \le C|t|^{1-n/2}, \qquad |t| > 1.$$

With a stronger decay of the potential, the evolution admits an operator-valued expansion

$$e^{itH}P_c(H) = |t|^{2-n/2}A_{-2} + |t|^{1-n/2}A_{-1} + |t|^{-n/2}A_0,$$

where A_{-2} and A_{-1} are finite rank operators $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$, while A_0 maps weighted L^1 spaces to weighted L^∞ spaces. The operators A_{-2} and A_{-1} are equal to zero under certain conditions of the orthogonality of the potential V to the eigenfunction which corresponds to zero eigenvalue. Under the same orthogonality conditions, the remainder term $|t|^{-n/2}A_0$ also maps $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, and therefore the group $e^{itH}P_c(H)$ has the same dispersion decay as the free evolution, despite its eigenvalue at zero.

L^p-L^q decay was first established in [N26] for solutions of the free Klein–Gordon equation $\ddot\psi = \Delta\psi - \psi$ with initial state $\psi(0) = 0$:

$$\|\psi(t)\|_{L^q} \le Ct^{-d}\|\dot\psi(0)\|_{L^p}, \qquad t > 1, \tag{81}$$

where $1 \le p \le 2$, $1/p + 1/q = 1$, and $d \ge 0$ is a piecewise-linear function of $(1/p, 1/q)$. The proofs use the Riesz interpolation theorem.

In [N2], the estimates (81) were extended to solutions of the perturbed Klein–Gordon equation

$$\ddot{\psi} = \Delta\psi - \psi + V(x)\psi$$

with $\psi(0) = 0$. It is shown that (81) holds for $0 \le 1/p - 1/2 \le 1/(n+1)$. The smallest value of p and the fastest decay rate d occurs when $1/p = 1/2 + 1/(n+1)$, $d = (n-1)/(n+1)$. The result is proved under the assumption that the potential V is smooth and small in a suitable sense. For example, the result holds true when $|V(x)| \le c(1 + |x|^2)^{-\sigma}$, where $c > 0$ is sufficiently small. Here $\sigma > 2$ for $n = 3$, $\sigma > n/2$ for odd $n \ge 5$, and $\sigma > (2n^2 + 3n + 3)/4(n+1)$ for even $n \ge 4$. The results also apply to the case when $\psi(0) \ne 0$.

The seminal article [N13] concerns L^p-L^q -decay of solutions to the Schrödinger equation (79). It is assumed that $(1 + |x|^2)^\alpha V(x)$ is a multiplier in the Sobolev spaces H^η for some $\eta > 0$ and $\alpha > n + 4$, and the Fourier transform of V belongs to $L^1(\mathbb{R}^n)$. Under this conditions, the main result of [N13] is the following theorem: if $\lambda = 0$ is neither an eigenvalue nor a virtual level of H, then

$$\|P_c\psi(t)\|_{L^q} \le Ct^{-n(1/p-1/2)}\|\psi(0)\|_{L^p}, \qquad t > 1, \tag{82}$$

where $1 \le p \le 2$ and $1/p + 1/q = 1$. Proofs are based on L^1-L^∞-decay (80) and the Riesz interpolation theorem.

In [N28], under suitable conditions on the spatial decay of $V(x)$, the estimates (82) were proved for all $1 \le p \le 2$ if $\lambda = 0$ is neither an eigenvalue nor a virtual level of H, and for all $3/2 < p \le 2$ otherwise.

The Strichartz estimates were extended in [N6] to the Schrödinger equations with the magnetic field in $\mathbb{R}^n$ with $n \ge 3$; in [N5]—to the wave equations with a magnetic potential in $\mathbb{R}^n$ for $n \ge 3$; in [N4]—to the wave equation in $\mathbb{R}^3$ with potentials of the Kato class.

14 Numerical Simulation of Soliton Asymptotics

Here we describe the results of Arkady Vinnichenko (1945–2009) on numerical simulation of (1) global attraction to solitons (43) and (44), and (2) adiabatic effective dynamics of solitons for relativistic-invariant one-dimensional nonlinear wave equations [F7].

Kinks of Relativistic-Invariant Ginzburg–Landau Equations

First, we simulated numerically solutions to relativistic-invariant 1D nonlinear wave equations with polynomial nonlinearity

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \tag{83}$$

where $F(\psi) = -\psi^3 + \psi$. Since $F(\psi) = 0$ for $\psi = 0, \pm 1$, there are three equilibrium states: $S(x) \equiv 0,\ +1,\ -1$.

The corresponding potential $U(\psi) = \frac{\psi^4}{4} - \frac{\psi^2}{2} + \frac{1}{4}$ has minima at $\psi = \pm 1$ and a local maximum at $\psi = 0$, therefore two finite energy solutions $\psi = \pm 1$ are stable, and the solution $\psi = 0$ with infinite energy is unstable. Such potentials with two wells are called Ginzburg–Landau potentials.

Besides the constant stationary solutions $S(x) \equiv 0, +1, -1$, there is also a non-constant one, $S(x) = \tanh \frac{x}{\sqrt{2}}$, called a "kink". Its shifts and reflections $\pm S(\pm x - a)$ are also solutions, as well as their Lorentz transforms

$$\pm S(\gamma(\pm x - a - vt)), \qquad \gamma = 1/\sqrt{1 - v^2}, \quad |v| < 1.$$

These are uniformly moving "traveling waves" (i.e. solitons). The kink is strongly compressed when the velocity v is close to ± 1. Equation (83) is formally equivalent to the Hamiltonian system with the Hamiltonian

$$\mathcal{H}(\psi, \pi) = \int \Big[\frac{1}{2}|\pi(x)|^2 + \frac{1}{2}|\psi'(x)|^2 + U(\psi(x))\Big]\,dx. \tag{84}$$

This Hamiltonian is finite for functions (ψ, π) from the Hilbert space $\mathcal{E} = H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, for which the convergence

$$\psi(x) \to \pm 1 \quad \text{as} \quad |x| \to \pm\infty$$

is sufficiently fast.

Numerical Simulation Our numerical experiments show a decay of finite energy solutions to a finite set of kinks and dispersive waves that corresponds to the asymptotics of (44). One of the experiments is shown on Fig. 4: a finite energy solution to equation (83) decays into three kinks. Here the vertical line is the time axis, and the horizontal line is the space axis; the spatial scale redoubles at $t = 20$ and at $t = 60$. Red color corresponds to $\psi > 1 + \varepsilon$ values, blue color to $\psi < -1 - \varepsilon$ values, and the yellow one to intermediate values $-1 + \varepsilon < \psi < 1 - \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Thus, the yellow stripes represent the kinks, while the blue and red zones outside the yellow stripes are filled with dispersive waves. For $t = 0$, the solution begins with a rather chaotic behavior, when there are no visible kinks. After 20 seconds, three separate kinks appear, which subsequently move almost uniformly.

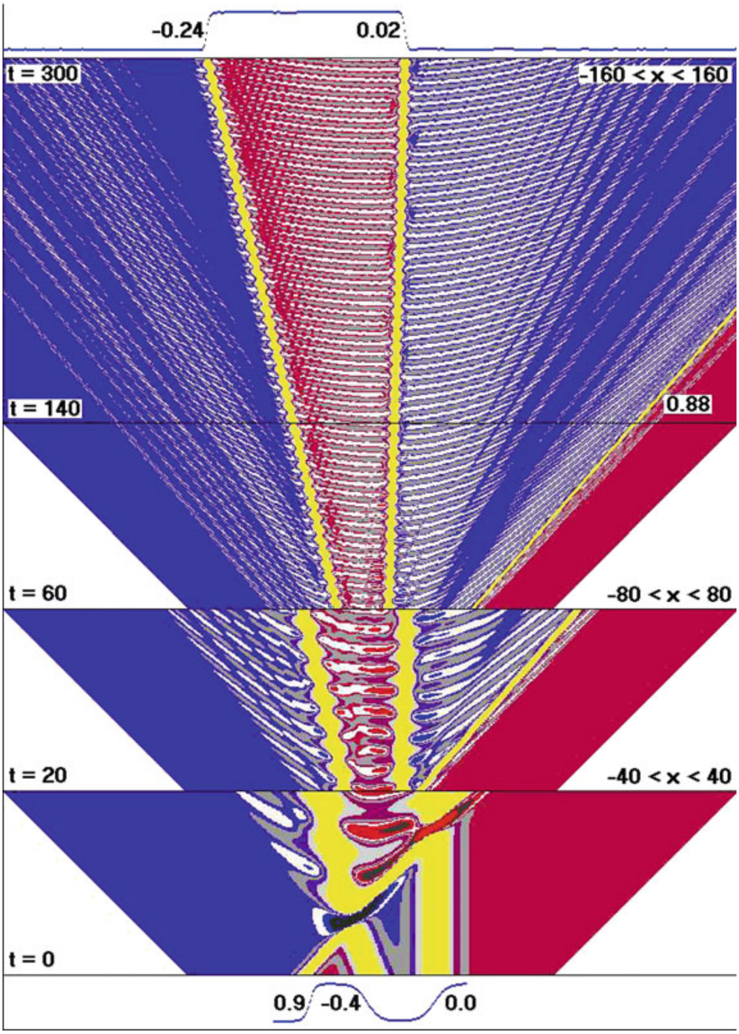

Fig. 4 Decay into three kinks

The Lorentz Contraction The left kink moves to the left at a low speed $v_1 \approx 0.24$, the central kink moves with a small velocity $v_2 \approx 0.02$, and the right kink moves very fast with the speed $v_3 \approx 0.88$. The Lorentz spatial contraction $\sqrt{1 - v_k^2}$ is clearly visible on this picture: the central kink is wide, the left one is a bit narrower, and the right one is very narrow.

The Einstein Time Delay Also, the Einstein time delay is also very prominent. Namely, all three kinks oscillate due to the presence of nonzero eigenvalue in the linearized equation at the kink: substituting $\psi(x,t) = S(x) + \varepsilon\varphi(x,t)$ in (83), we get in the first order of ε the linearized equation

$$\ddot{\varphi}(x,t) = \varphi''(x,t) - 2\varphi(x,t) - V(x)\varphi(x,t), \tag{85}$$

where the potential $V(x) = 3S^2(x) - 3 = -\frac{3}{\cosh^2 x/\sqrt{2}}$ decays exponentially for large $|x|$. It is of great convenience that for this potential the spectrum of the corresponding *Schrödinger operator* $H := -\frac{d^2}{dx^2} + 2 + V(x)$ is well known [F8]. Namely, the operator H is non-negative, and its continuous spectrum is given by $[2, \infty)$. It turns out that H still has a two-point discrete spectrum: the points $\lambda = 0$ and $\lambda = \frac{3}{2}$. Exactly this nonzero eigenvalue is responsible for the pulsations that we observe for the central slow kink, with the frequency $\omega_2 \approx \sqrt{\frac{3}{2}}$ and period $T_2 \approx 2\pi\sqrt{\frac{2}{3}}$. On the other hand, for fast kinks, the ripples are much slower, i.e., the corresponding period is longer. This time delay agrees numerically with the Lorentz formulas. These agreements qualitatively confirm the relevance of our numerical simulation.

Dispersive Waves The analysis of dispersive waves provides additional confirmation. Namely, the space outside the kinks on Fig. 4 is filled with dispersive waves, whose values are very close to ± 1, with accuracy 0.01. These waves satisfy with high accuracy the linear Klein–Gordon equation which is obtained by linearization of the Ginzburg–Landau equation (83) at the stationary solutions $\psi_\pm \equiv \pm 1$:

$$\ddot{\varphi}(x,t) = \varphi''(x,t) - 2\varphi(x,t).$$

The corresponding dispersion relation $\omega^2 = k^2 + 2$ determines the group velocities of high-frequency wave packets:

$$\omega'(k) = \frac{k}{\sqrt{k^2+2}} = \pm\frac{\sqrt{\omega^2 - 2}}{\omega}. \tag{86}$$

These wave packets are clearly visible on Fig. 4 as straight lines whose propagation speeds converge to ± 1. This convergence is explained by the high-frequency limit $\omega'(k) \to \pm 1$ as $\omega \to \pm\infty$. For example, for dispersive waves emitted by central kink, the frequencies $\omega = \pm n\omega_2 \to \pm\infty$ are generated by the polynomial nonlinearity in (83), see [E8] for details.

The nonlinearity in (83) is chosen exactly because of well-known spectrum of the linearized Eq. (85). In numerical experiments [F7] we have also considered more general nonlinearities of the Ginzburg–Landau type. The results were qualitatively the same: for "any" initial data, the solution decays for large times into a sum of kinks and dispersive waves. Numerically, this is clearly visible, but rigorous justification remains an open problem.

Numerical Observation of Soliton Asymptotics

Besides the kinks, our numerical experiments [F7] have also resulted in the soliton-type asymptotics (44) and adiabatic effective dynamics for solutions to the 1D relativistically-invariant nonlinear wave Eq. (83) with $F(\psi) = -U'(\psi)$ for the polynomial potentials

$$U(\psi) = a|\psi|^{2m} - b|\psi|^{2n}, \tag{87}$$

where $a,\ b > 0$ and $m > n = 2,\ 3,\ \dots$. Respectively,

$$F(\psi) = -2am|\psi|^{2m-2}\psi + 2bn|\psi|^{2n-2}\psi. \tag{88}$$

The parameters $a,\ m,\ b,\ n$ were taken as follows: We have considered various

a	m	b	n
1	3	0.61	2
10	4	2.1	2
10	6	8.75	5

"smooth" initial functions $\psi(x, 0)$, $\dot\psi(x, 0)$ with supports on the interval $[-20, 20]$. The second order finite-difference scheme with $\Delta x, \sim 0.01$, $\Delta t \sim 0.001$ was employed. In all cases we have observed the asymptotics of type (44) with the numbers of solitons 0, 1, 3, 5 for $t > 100$.

Adiabatic Effective Dynamics of Relativistic Solitons

In the numerical experiments [F7] we also observed the adiabatic effective dynamics for soliton-like solutions of the 1D equations with a slowly varying external potential $V(x)$:

$$\ddot\psi(x,t) = \psi''(x,t) - \psi(x,t) + F(\psi(x,t)) - V(x)\psi(x,t), \qquad x \in \mathbb{R}. \tag{89}$$

This equation is formally equivalent to the Hamiltonian system with the Hamiltonian functional

$$\mathcal{H}_V(\psi,\pi) = \int \left[\frac{1}{2}|\pi(x)|^2 + \frac{1}{2}|\psi'(x)|^2 + U(\psi(x)) + \frac{1}{2}V(x)|\psi(x)|^2\right] dx. \tag{90}$$

The soliton-like solutions are of the form

$$\psi(x,t) \approx e^{i\Theta(t)}\phi_{\omega(t)}(\gamma_{v(t)}(x - q(t))). \tag{91}$$

Let us describe our numerical experiments which qualitatively confirm the adiabatic effective Hamiltonian dynamics for the parameters Θ, ω, q, and v, although its rigorous justification is still missing. The effective dynamics of such type is proved in [G1]–[G9] for several Hamiltonian models of PDEs coupled to a particle.

Figure 5 represents solutions to Eq. (89) with the potential (87) with $a = 10$, $m = 6$, $b = 8.75$, and $n = 5$. We choose

$$V(x) = -0.2\cos(0.31x) \tag{92}$$

and the following initial conditions:

$$\psi(x,0) = \phi_{\omega_0}(\gamma_{v_0}(x - q_0)), \qquad \dot{\psi}(x,0) = 0, \tag{93}$$

where $v_0 = 0$, $\omega_0 = 0.6$, and $q_0 = 5.0$. We note that the initial state does not belong to solitary manifold. The effective width (half-amplitude) of the solitons is in the range $[4.4, 5.6]$. It is quite small compared to the spatial period of the potential $2\pi/0.31 \sim 20$, which is confirmed by numerical simulations shown on Fig. 5. Namely,

- Blue and green colors represent a dispersive wave with values $|\psi(x,t)| < 0.01$, while red color represents a soliton with values $|\psi(x,t)| \in [0.4, 0.8]$.
- The soliton trajectory (a thick red meandering curve) corresponds to oscillations of a classical particle in the potential $V(x)$.
- For $0 < t < 140$, the solution is far from the solitary manifold; the radiation is intense.
- For $3020 < t < 3180$, the solution approaches the solitary manifold; the radiation weakens. The oscillation amplitude of the soliton is almost unchanged for a long time, confirming Hamilton-type dynamics.
- However, for $5260 < t < 5420$, the amplitude of the soliton oscillation is halved. This suggests that at a large time scale the deviation from the Hamiltonian effective dynamics becomes essential. Consequently, the effective dynamics gives a good approximation only on the adiabatic time scale $t \sim \varepsilon^{-1}$.
- The deviation from the Hamiltonian dynamics is due to radiation, which plays the role of dissipation.
- The radiation is realized as dispersive waves bringing the energy to the infinity. The dispersive waves combine into uniformly moving bunches with discrete set of group velocities, as on Fig. 4. The magnitude of solutions is of order ~ 1 on the trajectory of the soliton, while the values of the dispersive waves is less than 0.01 for $t > 200$, so that their energy density does not exceed 0.0001. The amplitude of the dispersive waves decays at large times. In the limit $t \to \pm\infty$, the soliton converges to a limit position which corresponds to a local minimum of the potential (92).

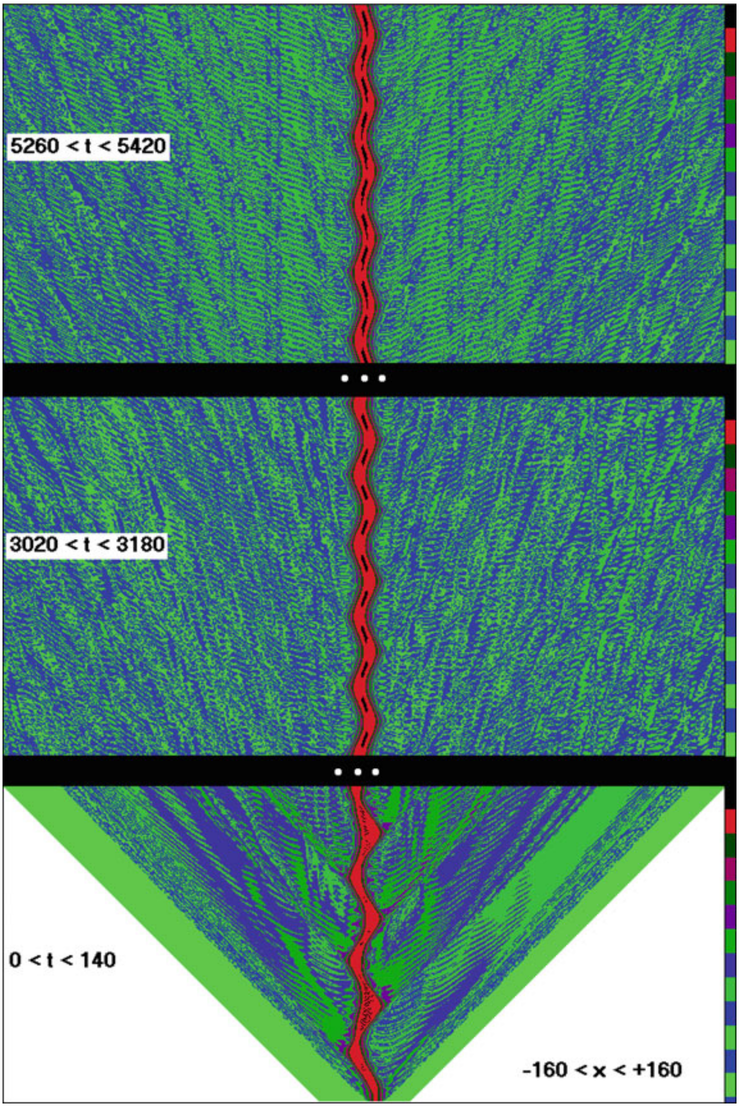

Fig. 5 Adiabatic effective dynamics of relativistic solitons

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The True Story of Quantum Ergodic Theorem

Alexander Shnirelman

Let M be a smooth compact Riemannian manifold, and Δ the Laplace–Beltrami operator on M. Let u_n and λ_n be the eigenfunctions and eigenvalues of Δ, i.e. $\Delta u_n + \lambda_n u_n = 0$. Let $\Omega = \{(x, \xi) \in T^*M, |\xi| = 1\}$ be the bundle of unit covectors, and $g_t : T^*M \to T^*M$ be the geodesic flow in T^*M. Then Ω is invariant under $\{g_t\}$, and $\omega = \dfrac{dx \wedge d\xi}{d|\xi|}$ is the $(2n-1)$-form on Ω invariant under $\{g_t\}$ and thus defining the g_t-invariant (Liouville) measure μ.

Theorem 0.1 ([Shn74, Shn93, Zel87, CdV85]) *Suppose the geodesic flow $\{g_t\}$ is ergodic with respect to the measure μ. Then there exists a subsequence u_{n_k} of eigenfunctions having density 1 such that for any smooth function $\varphi \in C^\infty(M)$,*

$$\lim_{k\to\infty} \int_M \varphi(x)|u_{n_k}(x)|^2\, dx = \int_M \varphi(x)\, dx.$$

Here I'll tell the discovery story of this theorem. This story consists of a number of steps, some of them being not so obvious.

1. My teacher Mark Iosifovich Vishik was one of the founders of the Microlocal Analysis and Pseudodifferential Operators. He developed his original version of it, which in some respects (PDO in bounded domains, factorization) went

A. Shnirelman (✉)
Department of Mathematics and Statistics, Concordia University, Montreal, QC, Canada
e-mail: alexander.shnirelman@concordia.ca

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far beyond the standard theory. His Mehmat seminar was a crucible of the new concepts in this domain. No wonder that it was in M.I.'s seminar where Egorov explained for the first time his celebrated theorem, and I could learn it firsthand (though I appreciated its importance much later).

2. When I was in the 3rd year of Mehmat studies, M.I. assigned me the first research topic, namely the solvability (or Fredholmness) of singular integral (i.e. 1-dimensional pseudodifferential of order zero) equations degenerating at one point. To my surprise, I managed to find a satisfactory solution of this problem (Fredholmness condition and appropriate functional spaces). In the process I discovered for myself a (rather primitive) version of the wavelet decomposition of a function, i.e. its microlocalization in the phase space.

3. In the 4th year student work I studied the difference equations in the bounded domain. Using the discrete version of factorization, I found a correct formulation of Boundary Value Problems in the convex domains.

In my 5th year graduate thesis (which became my first published work) I found new classes of Fredholm convolution equations with constant symbol in half-space.

My PhD. thesis was devoted to topological methods in nonlinear problems of complex analysis, and had nothing to do with eigenfunctions. It had no resonance during the next 50 years, though some ideas may be interesting even today.

4. While in the graduate school, I bumped into the work of Babich and Lazutkin [БЛ67] and of Lazutkin [Лаз68] (translated in [BL68, Laz68]). In these works, they constructed asymptotic solutions to the Helmholtz equation on a closed surface concentrated near a closed and stable geodesic (Fig. 1a) and in a bounded domain concentrated near a closed billiard trajectory (for example, the forth-and-back trajectory on Fig. 1b). It should be noted that at that time

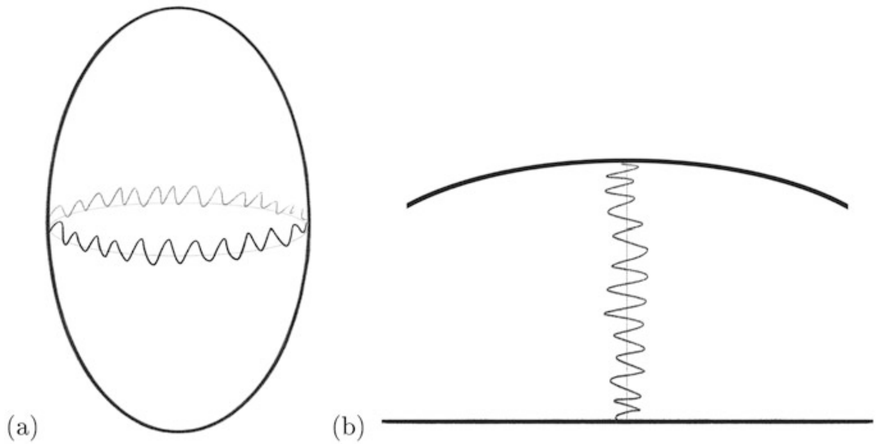

Fig. 1 (**a**) An eigenfunction on a closed surface concentrated near a closed (dynamically stable) geodesic. (**b**) An eigenfunction in a bounded domain concentrated near a bouncing trajectory

the foreign journals were almost unavailable for us. It was in part because the subscription was quite limited, and in part because of our poor English and other languages except Russian. Practically we were confined to the Russian journals, and a few foreign articles given to us by our teachers. However, we were able to read all the Russian journals on display in the Mehmat library, including obscure ones. The works of Babich and Lazutkin were published in such unassuming places. But they were true gems! The first consequence of my reading of those articles was related to the famous work of Arnold, "Modes and Quasimodes" [Arn72].

Once Arnold told in his seminar about his new results on the spectrum of a linear oscillating system with symmetries. He argued that if, for example, the system has a symmetry of the 3rd order, then a part of the eigenvalues have a stable multiplicity 2, and the remaining eigenvalues are simple. In general, there are no eigenvalues of stable multiplicity 3: they split into simple and double ones (according to the dimensions of irreducible representations of the group of order 3). At this point I stood up and boldly declared that this statement is wrong, and that I have a counterexample. Arnold asked me to produce it. So, I drew on the blackboard an equilateral triangle with rounded corners, and said that according to the results of Lazutkin, such a membrane (with fixed boundary) has "laser" eigenfunctions concentrated near three altitudes of the triangle (Fig. 2). They do not feel one another, and are transformed one into another by a rotation by 120°. Hence the eigenvalue is stably triple. Arnold scratched his head, and did not find what to answer. However, when I met him in a week, he was shining with joy. "I know what happens here! These eigenfunctions are just approximate! In fact they overlap a little, and so anyway there is a small (exponential) splitting into the simple and double eigenvalues". (In fact I'd pretty much confused things. The works of Babich and Lazutkin

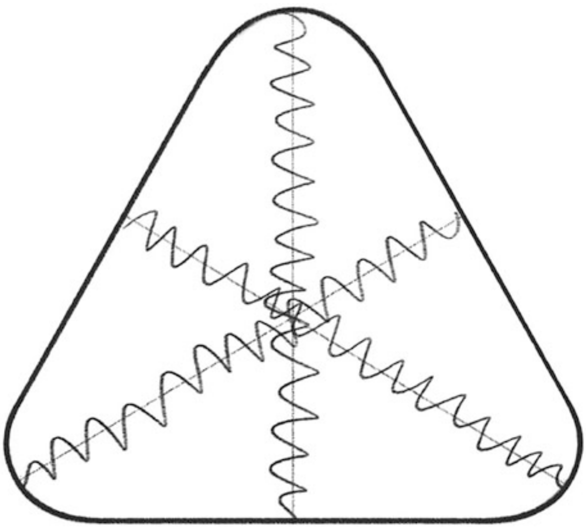

Fig. 2 Laser-type eigenfunctions in a domain with the 3rd order symmetry; an almost-counterexample to Arnold's conjecture

are rather difficult, and there are some arithmetic conditions deep inside them making an obstacle to the continuous deformation of eigenfunctions in the course of a change of a domain.) Afterwards Arnold described his theory in his famous article “Modes and Quasimodes” [Arn72]. In this paper he describes this story in detail. Nevertheless, this example is known as “Arnold’s example”, in spite of the explicit attribution in his article.

5. Since my first reading of the work of Babich and Lazutkin [BL68, Laz68], I could not think of anything else but the high-frequency eigenfunctions. In [BL68] and [Laz68], the construction hinges on the assumption that the billiard trajectory is dynamically stable. And what about the unstable trajectories? I tried to reproduce their construction (which, in the first approximation, reduced to some ansatz resulting in the quantum harmonic oscillator equation in the transverse direction to the trajectory); of course, I failed. The same scheme resulted in the Weber equation describing the scattering of waves on the potential barrier; it could not produce any discrete eigenvalues and corresponding asymptotic eigenfunctions concentrated near an unstable periodic trajectory.

6. Gradually I came to the following general formulation: “How do the high-frequency eigenfunctions look like in general?” I did not have a clear idea what the word “look” exactly means. And there was a good reason for such fuzziness. My teacher Mark Iosifovich Vishik, before his deep works on the microlocal analysis, devoted several years to the joint work with Lazar Aronovich Lyusternik on the asymptotic behavior of solutions to elliptic equations with a small parameter at the higher order terms (what is called “singular perturbations”) [VL57]. Of course, I was aware of this activity. The problem looked quite similar to the high-frequency eigenfunction problem which, too, can be regarded as a problem with a small parameter (namely, inverse of the eigenvalue) at the higher order term (Laplacian). The difference was the sign of the small parameter: it was negative for the boundary layer situation, and positive for the eigenfunction problem. Hence, we have a clear and simple picture of asymptotic behavior in one case (a smooth core and thin boundary layers [VL57]; later internal layers were added to the picture after the work of Ventsel and Freidlin [VF70]), and a hell in the other. The (now classical) asymptotic methods used by Vishik and Lyusternik gave nothing for the study of high-frequency eigenfunctions.

7. There was a well-developed asymptotic theory of eigenfunctions for the classically integrable systems, i.e. for Riemannian manifold M such that the geodesic flow on the surface $\Omega \subset T^*M$ is completely integrable [Kel58, FM76]. A typical example is the high-frequency eigenfunction in a disk with Dirichlet’s condition on the boundary (Fig. 3). Later this theory was generalized by Lazutkin to the Riemannian manifolds whose geodesic flow is a small perturbation of the integrable one. The book [Laz93] contains exhaustive results in this direction. However, this theory gives no clue to the understanding of high-frequency eigenfunctions in the general, non-integrable case.

8. I decided to consider the case which is maximally remote from the classically integrable cases, like the one studied by Lazutkin. Of course, it was the

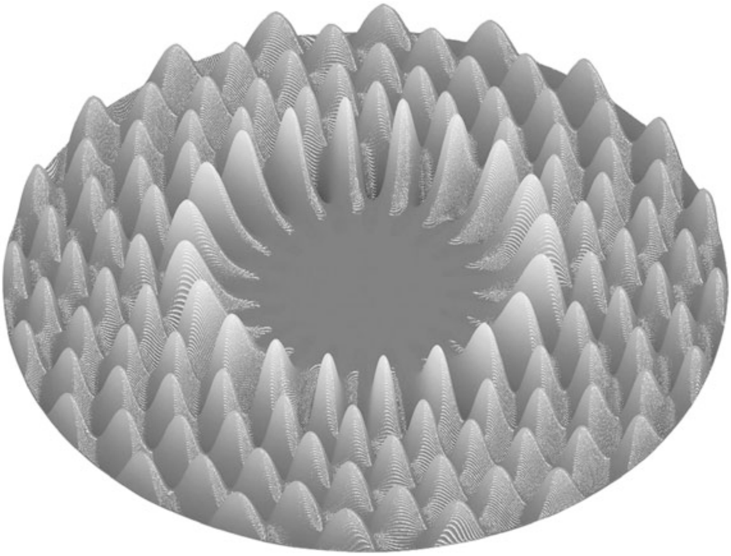

Fig. 3 High-frequency eigenfunction ($J_n(j_{n,k}r)\sin(n\theta)$ with $n = 20$ and $k = 6$) in a unit disk with Dirichlet's condition on the boundary

Laplacian on a compact surface of constant negative curvature. Why constant? Because, I thought, it may give us some extra tools to study the problem. In this I was right, and at the same time profoundly wrong; after all, this restriction was superfluous, and hence misleading. So, I considered the surface which was a factor of the action of a Fuchsian group Γ of discrete transformation of the hyperbolic plane $\mathbb{H}$: $M = \mathbb{H}/\Gamma$ (Fig. 4). If u_n is an eigenfunction on M, $\Delta u_n + \lambda_n u_n = 0$, then it can be lifted to a Γ-invariant (or automorphic) eigenfunction v_n on $\mathbb{H}$: $v_n(x) = v_n(gx)$ $(x \in \mathbb{H},\ g \in \Gamma)$.

9. My first idea was to find a representation of an automorphic eigenfunction v_n similar to the Poincare series for automorphic forms and functions. Soon I realized that the natural building blocks of such representation are the "horospheric" eigenfunctions $w_n(x, y)$, i.e. such that their level curves are the horospheres touching the absolute at one point y (Fig. 5). If ρ is the distance from a horosphere to a fixed "zero" one, then $w_n(x, y) = e^{(-\frac{1}{2}+ik_n)\rho}$, and $\lambda_n = k_n^2 + \frac{1}{2}$.

10. My first achievement was the proof that for every automorphic eigenfunction $v_n(x)$ there exists a distribution $\Phi_n(y)$ of order not exceeding 1 on the absolute A such that

$$v_n(x) = \int_A \Phi_n(y) w_n(x, y)\, dy. \tag{1}$$

Later on I learned that this statement is known under the name "Helgason's Theorem" [Hel18], and is the central fact of the harmonic analysis on locally symmetric spaces.

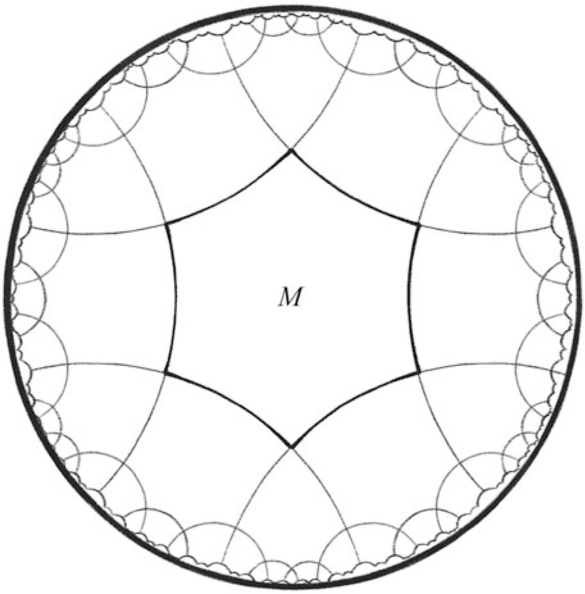

Fig. 4 The surface M as a fundamental domain of the discrete group Γ

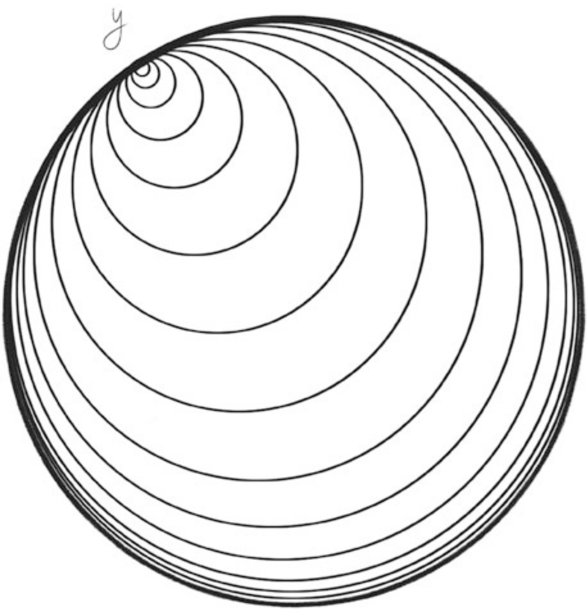

Fig. 5 Level curves of a horospheric eigenfunction $W_n(x, y)$ $(x \in \mathbb{H}, y \in A)$

11. The next step was the functional (or homological) equation for the function $\Phi_n(x, y)$. Namely, for any $g \in \Gamma$,

$$\Phi_n(g(y)) = |g'(y)|^{-1/2} e^{ik_n g'(y)} \Phi_n(y). \tag{2}$$

Thus, we have an overdetermined system of functional equations on one single function $\Phi_n(y)$. Fortunately, we do not need to care about its solvability: we know a priori that it has solutions for the discrete sequence of numbers $k_n = \sqrt{\lambda_n - 1/2}$. So, we can concentrate on the study of properties of the function $\Phi_n(y)$.

12. The action of the group Γ on the absolute A is quite complicated; all the orbits are dense, and there is no invariant measure. So, it is worthwhile to lift the action of Γ onto the cotangent space T^*A. This action already preserves the Liouville measure $dy \wedge d\eta$, but (a) for almost all points $(y, \eta) \in T^*A$ their orbits are dense, and (b) the volume of the phase space T^*A is infinite.

13. My goal was to prove that the functions Φ_n for all, or at least almost all n look like typical realizations of the white noise. This property is best expressed in terms of the *Wigner measures* μ_n corresponding to the functions Φ_n. For every function Φ_n, we can define the *Wigner measure* μ_n in T^*A. This measure describes the distribution of the energy of the function $\Phi_n(y)$ in the phase space. It is defined in the following way. For any smooth function (symbol) $a(y, \eta)$ with compact support, let $\hat{a}$ be the corresponding (Weyl) pseudodifferential operator with the symbol a [Hör94a]. Now consider the bilinear expression $(\hat{a}\Phi_n, \Phi_n)$. This expression linearly depends on the symbol a, i.e. it has the form

$$(\hat{a}\Phi_n, \Phi_n) = \int_{T^*A} U_n(y, \eta)\, a(y, \eta)\, dy\, d\eta.$$

It turns out that the distribution $U_n(y, \eta)$ is asymptotically (as $n \to \infty$) a non-negative measure which we denote by μ_n; this measure is called *Wigner measure*.

14. There is a physical device which produces the Wigner measure of a signal (i.e. function of one variable; say, the time). It consists of a number of filters cutting a narrow frequency band from the signal; each filter is characterized by the median frequency. If we plot the squared filter outputs as a function of time and the median frequency, we get a positive and highly oscillating function (sometimes called the Husimi function). After some mollifying we get a smooth positive density, which is the density of the Wigner measure. It is conjectured that our actual hearing has a similar mechanism, i.e. our brain converts the incoming sound into its Wigner measure which is further processed to extract the meaningful information.

Some cases of the Wigner measure have been known since long ago: I mean the music scores. Consider a musical sound, say a song played by some instrument or sung by a human voice. The corresponding music score is a 2-dimensional domain ***Score*** endowed with two coordinates (time and frequency) with the notes which we regard as points. Let us put into each such point a mass proportional to the intensity of the corresponding note; then we get a measure on the ***Score*** which is a coarse approximation of the true Wigner measure μ in the time-frequency plane.

15. The significance of the Wigner measure for our problem was based on the following observation. The quantity $I = \int_M f(x)|v_n(x)|^2\,dx$ can be transformed (after some manipulations) into $I = (B_f\Phi_n, \Phi_n)$ where B_f is a pseudodifferential operator with a smooth and compactly supported symbol $b_f(y,\eta)$ depending on f (I do not write this symbol explicitly, but its most important property is that $\int_{T^*A} b_f(y,\eta)\,dy\,d\eta = \int_M f(x)\,dx$). Thus,

$$\int_M f(x)|v_n(x)|^2\,dx \sim (B_f\Phi_n, \Phi_n) \sim \int_{T^*A} b_f\,d\mu_n.$$

So, in order to prove the asymtotic equidistribution of v_n, we have to prove that the measures μ_n are asymptotically equivalent to the Liouville measure $\Lambda = dy \wedge d\eta$ on T^*A.

16. Then I was able to prove that the "arithmetic mean" of the measure μ_n is equal to the Liouville measure Λ. The exact meaning of this result is that $4\pi\tau\sum_n e^{-\lambda_n\tau}\mu_n \sim \Lambda$ as $\tau \to 0$. It was done in the traditional way, with the use of the heat equation on M (Carleman's method [BGV04]). I have to confess that at that time I did not master the Tauberian theorem, and therefore did not make the next step, and only proved that $\frac{1}{N}\sum_{n=1}^{N}\mu_n \sim \Lambda$ for $N \to \infty$. This ignorance can be seen in the first publications of my results; later this gap was filled [Shn93].

17. The relations (2) imply that for any $g \in \Gamma$, the measure μ_n is asymptotically (as $n \to \infty$) invariant under the transformation

$$(y,\eta) \mapsto F_g(y,\eta) = \big(g(y), (g'(y))^{-1}\eta + k_n g''(y)\big). \tag{3}$$

18. Then I proved that the transformations F_g defined above have the property of equidistribution (in the sense of Kazhdan [Kaz65]). To define this property, introduce the word distance $d(g,h)$ in the group Γ as the minimal length p of the word $i_1, \ldots, i_p$ such that $g \cdot h^{-1} = \gamma_{i_1}^{\pm 1} \cdot \ldots \cdot \gamma_{i_p}^{\pm 1}$, where γ_i are generators of the group Γ. Let $B_R = \{g \in \Gamma \mid d(g,e) \le R\}$. The family of transformations $\{F_g\}$, $g \in \Gamma$, possesses the equidistribution property if for any $z_0 \in T^*A$ and any two bounded domains $U, V \subset T^*A$,

$$\lim_{R\to\infty} \frac{|\{g \in B_R | F_g(z_0) \in U\}|}{|\{g \in B_R | F_g(z_0) \in V\}|} = \frac{|U|}{|V|}. \tag{4}$$

19. The above properties of the functions Φ_n imply that for a sequence n_k of density 1, the Wigner measures μ_{n_k} tend weakly to the Liouville measure $\Lambda = dy \wedge d\eta$ in T^*A (I've used the version of the Birkhoff ergodic theorem). And this implies the conclusion of Theorem 1, i.e. the asymptotic equidistribution of a subsequence v_{n_k} of density 1. This completed the proof for the eigenfunctions of the Laplacian on the compact surface $M = \mathbb{H}/\Gamma$ having constant negative curvature.

20. At this point I realized that it was possible to define the Wigner measures for the functions v_k themselves as measures on T^*M, and to work with them, thus skipping a big part of the proof. The Wigner measures are defined in a manner similar to the above 1-d definition. Namely, for any smooth, compactly supported function (symbol) $a(x,\xi)$ ($(x,\xi) \in T^*M$), let $\hat{a} = \mathrm{op}(a)$ be the pseudodifferential operator with the symbol a. Consider the bilinear expression $(\hat{a}v_n, v_n)$. Using the Gårding inequality [Hör94a], we show that there exists a sequence of nonnegative measures μ_n on T^*M of mass 1 such that $(\hat{a}v_n, v_n) \sim \int_{T^*M} a\, d\mu_n$.

21. Here I have to make some remarks on the Gårding inequality [Hör94a]. Let $a(x,\xi)$ be a symbol of the Hörmander class $S^0_{1,0}$, i.e. $|\partial_x^\alpha \partial_\xi^\beta a| \le C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|}$; let $\hat{a}$ be the pseudodifferential operator with the symbol $a(x,\xi)$. Then the Gårding inequality (in fact, a pair of inequalities) says that if the symbol $a(x,\xi) \in S^0_{1,0}$ is real and nonnegative, then

$$\mathrm{Re}(\hat{a}u, u) \ge -C\|u\|^2_{H^{-1/2}}, \tag{5}$$

$$|\mathrm{Im}(\hat{a}u, u)| \le C\|u\|^2_{H^{-1/2}}, \tag{6}$$

where $H^{-1/2}$ is the Sobolev space.

Here the constant C is common for all symbols belonging to a bounded set in the space $S^0_{1,0}$. Examples of symbols of class $S^0_{1,0}$ are (a) symbols homogeneous in ξ of degree zero, and (b) symbols of the form $a(x,\xi) = \alpha(x,\varepsilon\xi)$, where $\alpha \in C_0^\infty$ and $\alpha(x,\xi) = 0$ for ξ close to 0 (such symbols belong to a bounded set in $S^0_{1,0}$ uniformly for all $0 < \varepsilon < 1$). So, when I refer to the Gårding inequality, I mean the symbols of type (b).

22. There exists a natural device which recovers the Wigner measure for oscillating functions in any dimension. This device is our eye! If the eye is put at some point in the space, then it will see some brightness distribution on the "sky", and in every direction it will see some colour. Thus we have a measure (the energy distribution) depending on the position, direction, and frequency, i.e. exactly in the phase space. A closer analysis of the work of the eye (or any similar optical device) shows that, in fact, the visible picture seen by the eye on the "sky" is nothing but the Wigner measure. For example, suppose that the eigenfunction u_n is "quasiclassical", i.e. if locally

$$u_n(x) = \sum_{k=1}^{K} a_k(x) e^{i\lambda_n^{1/2}\varphi_k(x)}. \tag{7}$$

Here the phase functions $\varphi_k(x)$ satisfy the equation $|\nabla\varphi_k| \equiv 1$, and the amplitudes $a_k(x)$ satisfy the transport equation $\nabla\varphi_k \cdot \nabla|a_k|^2 + \Delta\varphi_k \cdot |a_k|^2 = 0$.

The points $(x, \nabla\varphi_k(x)) \in T^*M$ form locally a Lagrangian manifold Λ_k; for different k, these manifolds are called *Lagrangian sheets*. The "eye" put at the point x will "see" K discrete stars on the dark "sky" in the directions $\nabla\varphi_k,\ \ k = 1, \ldots, K$.

23. Some pairs of Lagrangian sheets can merge along an $(n-1)$-dimensional manifold called *caustic*. These two sheets, together with the natural projection $\pi : T^*M \to M$, form a singularity called a *fold* in the singularity theory. If two Lagrangian sheets of the eigenfunction (7), $a_k(x)e^{i\lambda_n^{1/2}\varphi_k(x)}$ and $a_m(x)e^{i\lambda_n^{1/2}\varphi_m(x)}$, merge together along a caustic, and the "eye" is moving and its trajectory crosses the caustic, it will "see" two "stars" approaching one another, and then merging together, forming a single, much brighter "star". However, upon further motion of the "eye", it will see that the "star" almost immediately fades down and disappears. This exactly happens for the high-frequency eigenfunctions of the Laplacian inside the disk showed on Fig. 3. There is a central domain where the eigenfunction is of order $e^{-C\sqrt{\lambda_n}}$ (the shadow domain). It is bounded by the concentric circle, the caustic. On this circle, the amplitude of the eigenfunction attains its maximum. In the annulus between the caustic and the boundary of the disk the eigenfunction is "quasiclassical", i.e. has the form

$$u_n(x) = \sum_{k=1}^{2} b_k(x)\cos(\sqrt{\lambda_n}\varphi_k(x)). \tag{8}$$

So, if the eye is in the "quasiclassical" domain, it sees two pairs of stars (one pair is opposite to the other) on the dark "sky" (here the "sky" is a circle); see Fig. 6a. As the eye moves and approaches the caustics, these pairs of stars merge and form two stars at the opposite positions on the sky, whose brightness becomes much higher (Fig. 6b). But if the eye moves further into the shadow domain, the stars fade away, and the eye enters the darkness (Fig. 6c).

24. Further, we prove that

1. Each measure μ_n in T^*M is concentrated near the hypersurface $|\xi| = \sqrt{\lambda_n}$;
2. $\sum_n \mu_n \sim dx \wedge d\xi$;
3. The measures μ_n are asymptotically invariant under the geodesic flow (at this point I used the hyperbolic equation $u_t = i\sqrt{-\Delta}u$ and the Egorov Theorem [Hör94a, Hör94b]);
4. If the measures μ_n have the properties (1)–(3), then there exists a subsequence μ_{n_k} of density 1 such that the measures μ_{n_k} are asymptotically equidistributed on the energy surfaces $|\xi| = \lambda_{n_k}^{1/2}$; the proof is based on the Birkhoff ergodic theorem.

25. Returning to our optical interpretation, our result means that if we consider the typical high-frequency eigenfunction u_{n_k} (where $\{n_k\}$ is the subsequence of density 1 for which the equidistribution of the Wigner measure holds), and

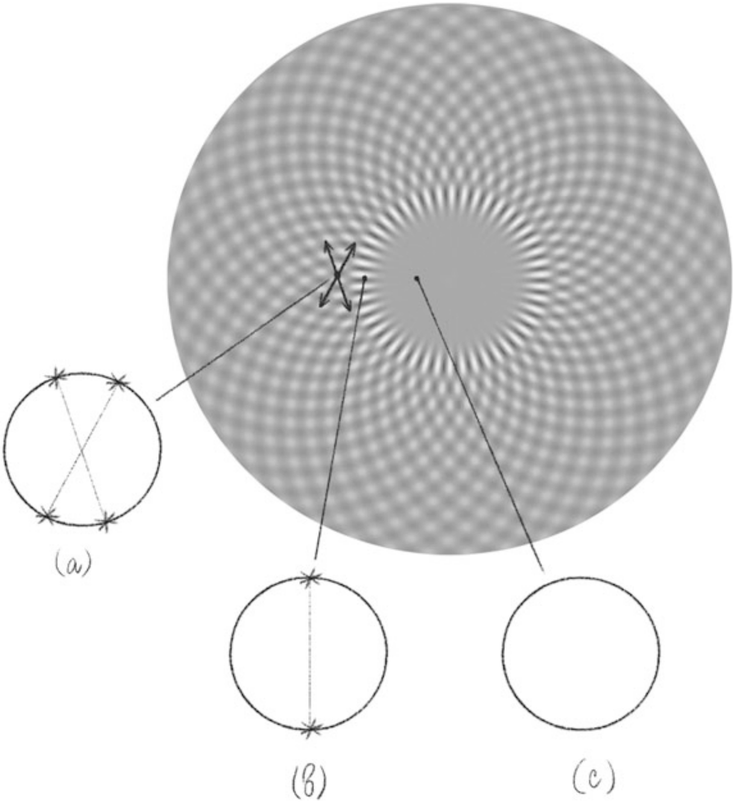

Fig. 6 What the eye sees on the sky from the quasiclassical domain (**a**), from a point on the caustic (**b**), and from the shadow domain (**c**). The directions to the four starts on firmament (**a**) are given by normals to the two wave fronts of the wave function at a given point of the domain (these directions are shown on the picture). On the caustic, four directions degenerate into two

put the eye at any point of the manifold M, it will see the sky uniformly glowing; the intensity of this glow does not depend on the eye's position and direction of its sight. Thus, the properties of the eigenfunctions are quite opposite to the quasiclassical ones: in the quasiclassical case, an eigenfunction is concentrated on an n-dimensional Lagrangian manifold, while in the ergodic case, an eigenfunction is uniformly distributed over a finite energy submanifold of dimension $2n-1$ (this value can be interpreted as the dimension of the space of light rays in the cotangent bundle).

26. This proof of quantum ergodicity appears quite different from the previous one, sketched above in paragraphs **9–19**. In place of the discrete group Γ acting on the measures μ_n by (3) we have a continuous one-parameter group of sympectic transformations of the phase space (the phase flow). Thus, in the first proof, we have a discrete non-commutative group of symplectic transformations of the space T^*A, the cotangent bundle of the absolute A of the Lobachevsky plane, while in the second proof we have a continuous 1-parameter group of symplectic transformations of the space T^*M, the cotangent bundle of the surface M. The properties of the Wigner measures in these two proofs are quite different, too: in the first proof they are spread over the whole phase space, for the functions $\Phi_n(y)$ are singular even for small n, and in the second proof each measure μ_n is concentrated on the energy surface $|\xi| = \lambda_n^{1/2}$. These (and other) differences hint at the possibility that in these two proofs different structures were used, and this difference can result in future interesting results. This opinion is confirmed by the excellent achievement of Rudnik and Sarnak [RS94] who proved that on the arithmetic surface *all* eigenfunctions are asymptotically equidistributed (this property is called Quantum Unique Ergodicity). These authors used an additional structure present in the arithmetic case, namely the existence of the Hecke correspondences, providing extra symmetries to the Wigner measures.

27. Upon the proof of the Quantum Ergodicity Theorem it immediately became clear that

1. The curvature of the surface M can be non-constant (we need only ergodicity of the phase flow) [Zel87];
2. The dimension of M may be arbitrary [CdV85];
3. We can consider any elliptic operator, not only Laplacian [CdV85];
4. The manifold M may have a boundary [ZZ96];
5. We can consider not only scalar, but also vector- and bundle-valued functions and matrix operators on them [Gér91, GL93];
6. There appear the first impressive results in the direction of Quantum Unique Ergodicty [Ana08].

28. This was the Past. Now comes the Future.

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Photographs

Asya Guterman. Around 1930

A. Comech et al. (eds.), *Partial Differential Equations and Functional Analysis*,
Trends in Mathematics, https://doi.org/10.1007/978-3-031-33681-2

Asya Guterman. 1938

The First International Topological Conference. Moscow, 4–10 September 1935. Top row: Eduard Čech, Hassler Whitney, Kazimierz Zarankiewicz, Albert Tucker, Solomon Lefschetz, Hans Freudenthal, Felix Frankl, Jacob Nielsen, Karol Borsuk, Dmitrii Sintsov, Lev Abramovich Tumarkin, Maria Antonovna Nikolaenko, Vyacheslav Vassilievich Stepanoff, Egbert van Kampen, Andrey Nikolaevich Tychonoff. Bottom row: Kazimierz Kuratowski, Juliusz Schauder, Stefan Cohn-Vossen, Poul Heegaard, Julia Antonovna Rozanska, James Alexander, Heinz Hopf, Pavel Sergeevich Alexandroff, Pavel Aleksandrovich Solovyov

Bronisław Knaster

Stanisław Mazur

Władysław Orlicz

Stanisław Saks

Hugo Steinhaus

Edward Szpilrajn

Miron Zaritskii

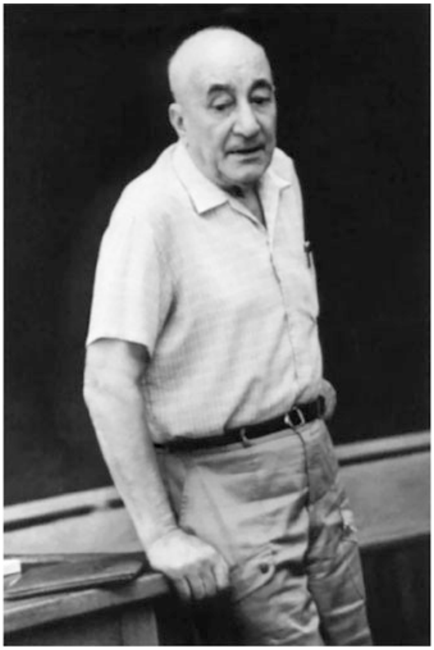
Stephan Banach

Stefan Banach and Stanisław Mazur. Reproduction from the newspaper "Вільна Україна", 13 December 1940

Stefan Banach. Lwów, 1944

Nikolai Ivanovich
Muskhelishvili

Karen Avetovich
Ter-Martirosyan



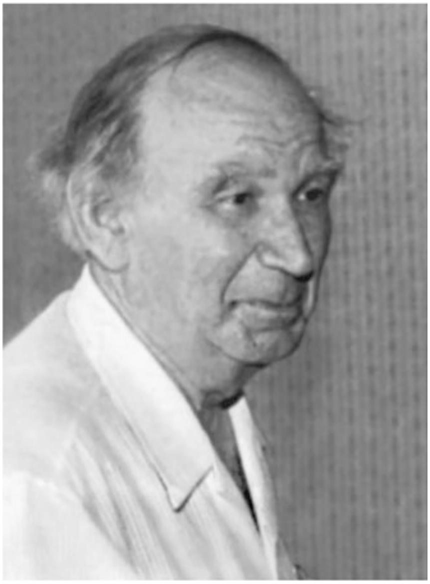
Ilia Nesterovich Vekua

Arnold Walfisz

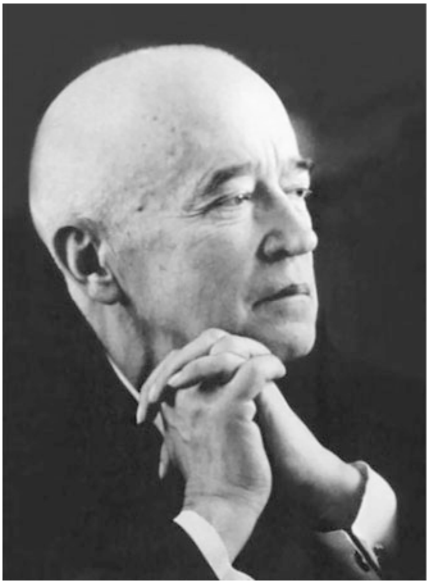
Mark Iosifovich Vishik. Around 1945

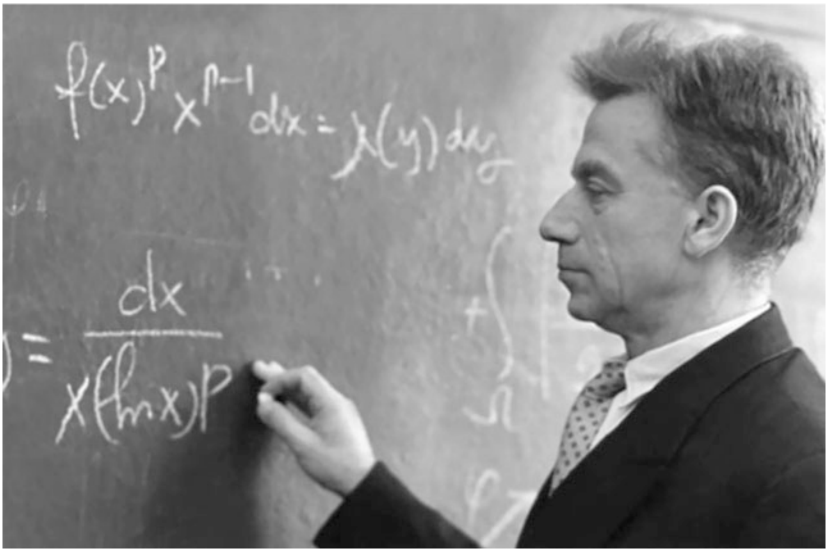
In a sanatorium in Essentuki, 1949

Evgenii Mikhailovich Landis



Ivan Georgievich Petrovsky





Sergei Lvovich Sobolev

With son Senya. Around 1950

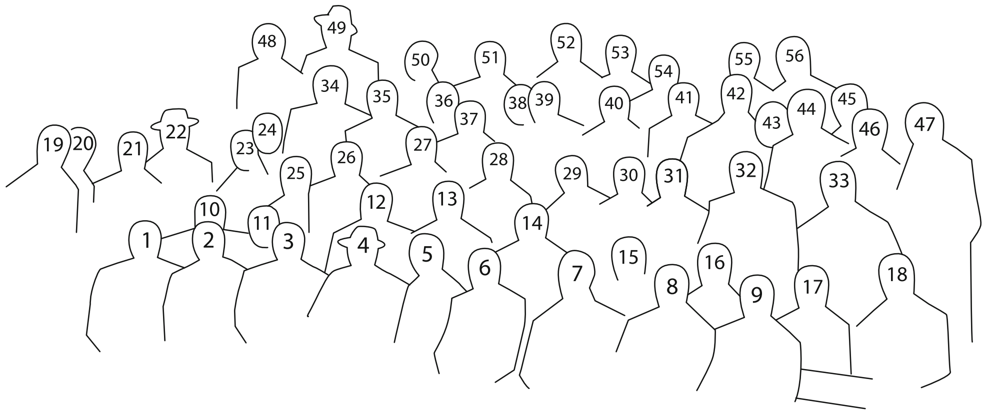
At dacha. On a bicycle: Senya Vishik. July 1951

Mark Iosifovich Vishik with his son Senya. Around 1955

Mark Iosifovich Vishik with his son Misha. Around 1955

Władysław Lyantse with his wife Olga. 18 July 1947

With Lazar Aronovich Lyusternik. 1957



Interuniversity Conference on Functional Analysis and Its Applications. Odessa, garden of the House of Scientists, 25 October 1958. 1. Mark Iosifovich Vishik; 2. Moisei Aronovich Rutman; 3. Mark Aronovich Naimark; 4. Mark Grigorievich Krein; 5. Yahya Dzhafarovich Mamedov; 6. Farman S. Aliev; 7. Leonid Pavlovich Nizhnik; 8. Mordukhai Moiseevich Vainberg; 9. Mikhail Iosifovich Kadets; 10. Arif Aligeydar-ogly Babaev; 11. Gashim Nizamovich Agaev; 12. Jalal Eyvaz Allakhverdiyev; 13. R.M. Sultanov; 14. Koshkar Teymur Akhedov; 15. Boris Aronovich Vertgeim; 16. ?; 17. Eliazar Khaimovich Gokhman; 18. ?; 19. Anatoliy Ivanovich Perov; 20. Anatolii Dmitrievich Lyashko; 21. Mikhail Solomonovich Birman; 22. Abraham Wilhelmovich Strauss; 23. Ilia Yakovlevich Bakelman; 24. Israel Tsudikovich Gohberg; 25. Tofik A. Zamanov; 26. Daoxing Xia; 27. Faramaz Gazanfar Maksudov; 28. Evgeny Vitalievich Maikov; 29. Dmitry Petrovich Zhelobenko; 30. Yakov Bronislavovich Rutitskii; 31. Vladimir Ivanovich Sobolev; 32. Selim Grigorievich Krein; 33. Yurii Lvovich Daletskii; 34. Mark Iosifovich Graev; 35. Boris Nikolaevich Panaioti; 36. Alexander Semyonovich Dynin; 37. Dmitrii Abramovich Raikov; 38. ?; 39. David Pinhusovich Milman; 40. Yurii Lvovich Shmulyan; 41. Alexander Yakovlevich Povzner; 42. Yurii Iosifovich Domshlak; 43. Mikhail Vasilievich Fedoryuk; 44. Mark Aleksandrovich Krasnoselskii; 45. Iosif Semyonovich Iokhvidov; 46. ?; 47. Samuil Davidovich Eidelman; 48. Alexander Semyonovich Markus; 49. Ludwig Dmitrievich Faddeev; 50. Vladimir Andreevich Yakubovich; 51. Boris Samuilovich Mityagin; 52. Karl Moritsovich Fishman; 53. Yurii Nikolaevich Valitskii; 54. Grigorii Yakovlevich Lyubarskii; 55. Gleb Pavlovich Akilov; 56. Aleksandr Aleksandrovich Kirillov

Interuniversity Conference on Functional Analysis and Its Applications. Odessa, garden of the House of Scientists, 25 October 1958. First row, left to right: Mark Iosifovich Vishik, Moisei Aronovich Rutman, Mark Aronovich Naimark (in a black hat), Mark Grigorievich Krein (in a black hat), Yahya Dzhafarovich Mamedov. Standing, left to right: Mikhail Solomonovich Birman, Abraham Wilhelmovich Strauss, Tofik A. Zamanov (lower, with black moustache, in a hat), Daoxing Xia, Jalal Eyvaz Allakhverdiyev (in a black hat, in front of Daoxing Xia), R.M. Sultanov (in a hat), Koshkar Teymur Akhedov (in a black hat), Faramaz Gazanfar Maksudov (slightly above, in a dark coat), Evgeny Vitalievich Maikov (in a beret), Dmitry Petrovich Zhelobenko (the rightmost).

Seminar at Moscow Power Engineering Institute. First row (left to right): Mark Iosifovich Vishik, Boris Veniaminovich Kutuzov, Yurii Isaevich Grosberg, and Lev Zinovievich Rumshiskii. Second row: Mikhail Leontievich Krasnov, Vladimir Valentinovich Zaitsev, Sergei Aleksandrovich Lomov. Third row, on the left: Almaz Guseinbekova (PhD. student of M.I. Vishik from Baku); on the right: Ljudmila Koltsova. Fourth row: Yulii Andreevich Dubinskii. Spring 1961

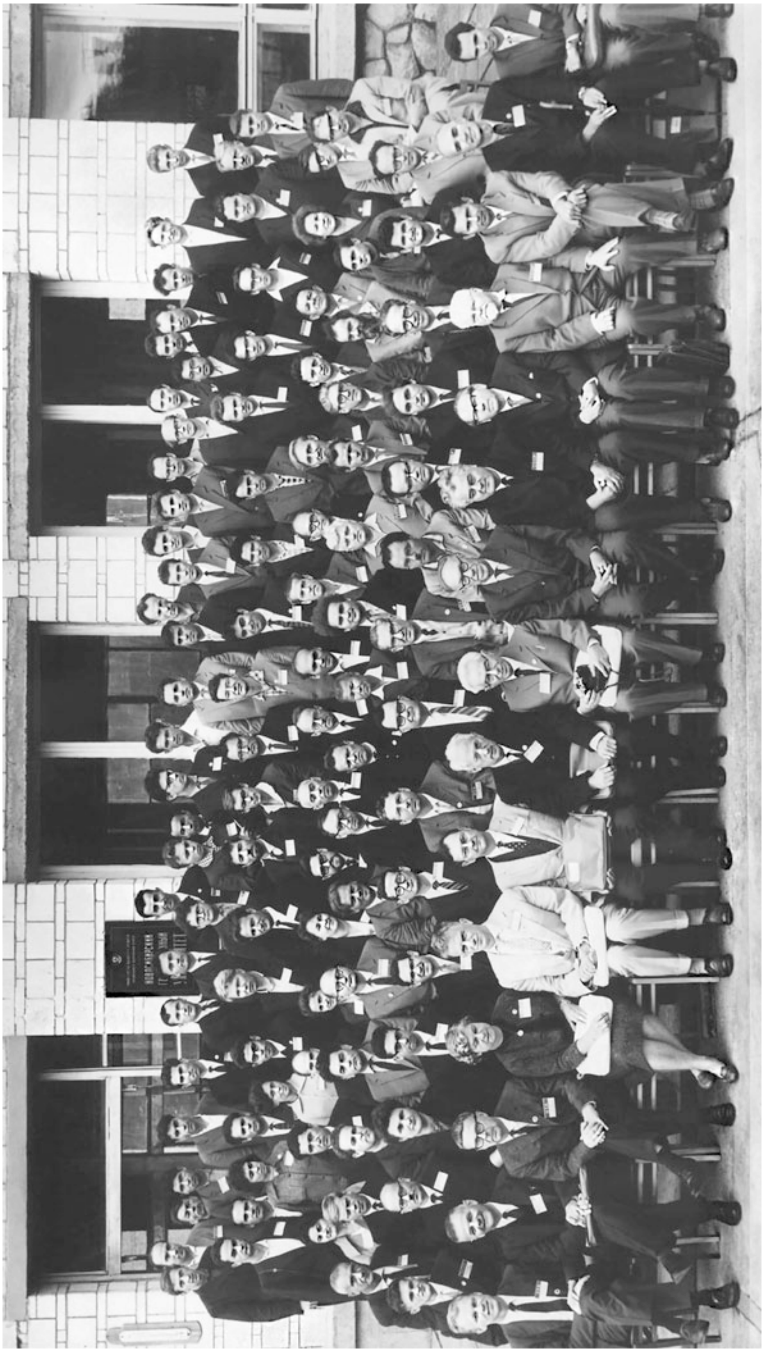
Same seminar. Moscow Power Engineering Institute, Spring 1961

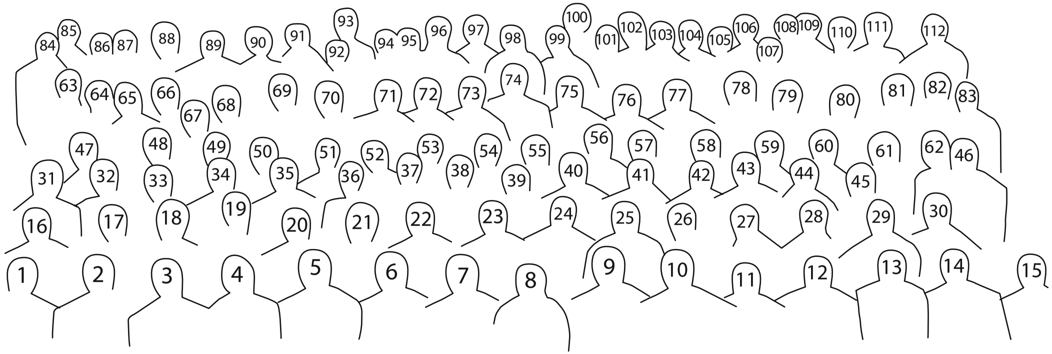
With Yulii Andreevich Dubinskii. Around 1960

With Yulii Andreevich Dubinskii and Leonid Dmitrievich Pokrovskii. Around 1960

With Grigory Ivanovich Makarenko and his family. Dubna, 1961

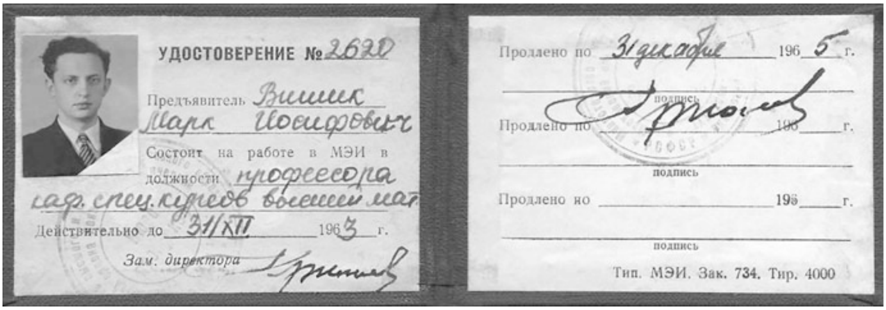
With Grigory Ivanovich Makarenko





Joint Soviet-American Symposium on Partial Differential Equations. Novosibirsk, August 1963. 1. J. Moser; 2. A. Zygmund; 3. L.V. Ahlfors; 4. N.N. Brunswick (Jasny); 5. S.L. Sobolev; 6. Ch.B. Morrey; 7. C. Loewner; 8. R. Courant; 9. M.A. Lavrent'ev; 10. I.N. Vekua; 11. S. Bergman; 12. A.N. Tikhonov; 13. G.I. Marchuk; 14. D.C. Spencer; 15. A.S. Dynin; 16. T.I. Zelenyak; 17. M.G. Krein; 18. O.A. Oleinik; 19. H.F. Weinberger; 20. H. Grad; 21. M. Schechter; 22. J. Douglas; 23. F.E. Browder; 24. M.H. Protter; 25. A.D. Myshkis; 26. Yu.M. Berezanskii; 27. V.A. ilyin; 28. A.P. Calderón; 29. P.D. Lax; 30. E.B. Dynkin; 31. A.Ya. Povzner; 32. B.L. Rozhdestvsnskii; 33. P.E. Sobolevskii; 34. G.N. Agaev; 35. B.N. Panaioti; 36. B.V. Shabat; 37. L.D. Kudryavtsev; 38. G.D. Suvorov; 39. T.I. Amanov; 40. P.P. Belinskii; 41. S.G. Krein; 42. I.D. Safronov; 43. P.D. Richtmyer; 44. A.A. Lyapunov; 45. S.K. Godunov; 46. L.I. Kamynin; 47. E.I. Obolashvili; 48. M.S. Salakhitdinov; 49. R. Finn; 50. translator; 51. translator; 52. L. Nirenberg; 53. G.M. Komladze; 54. M.K. Fage; 55. B.R. Vainberg; 56. L.V. Ovsyannikov; 57. L.I. Volkovyskii; 58. I.I. Danilyuk; 59. M.I. Vishik; 60. S.M. Nikolskii; 61. V.N. Maslennikova; 62. Yu.V. Egorov; 63. M.S. Agranovich; 64. L.P. Nizhnik; 65. N.D. Vvedenskaya; 66. L.R. Volevich; 67. T.D. Venttsel; 68. A.M.Ilyin; 69. A.F. Sidorov; 70. Ya.A. Roitberg; 71. T.G. Golenpolskii; 72. I.A. Shishmarev; 73. Ya.S. Bugrov; 74. ?; 75. Yu.L. Rodin; 76. A.V. Sychev; 77. V.S. Ryabenkii; 78. O.V. Besov; 79. S.V. Uspenskii; 80. V.G. Dulov; 81. P.T. Dybov; 82. V.K. Ivanov; 83. S.N. Kruzhkov; 84. V.A. Solonnikov; 85. M.M. Lavrent'ev; 86. A.K. Gerasimov; 87. ?; 88. N.E. Tovmasyan; 89. A.D. Dzhuraev; 90. P.I. Lizorkin; 91. T.D. Dzhuraev; 92. ?; 93. G.N. Salikhov; 94. V.S. Buslaev; 95. ?; 96. V.M. Babich; 97. L.D. Faddeev; 98. A.I. Koshesyov; 99. M.S. Birman; 100. S.I. Pokhozhaev; 101. P.A. Biluta; 102. V.P. Didenko; 103. A.I. Prilepko; 104. V.A. Kondrat'ev; 105. A.M. Molchanov; 106. B.I. Zaslavskii; 107. ?; 108. V.R. Portnov; 109. Yu.I. Gilderman; 110. I.I. Pyatetskii-Shapiro; 111. L.G. Mikhailov; 112. Yu.V. Sidorov

Ilia Iosifovich Pyatetskii-Shapiro, Alexander Semyonovich Dynin, Leonid Romanovich Volevich, Mark Iosifovich Vishik, Aleksandr Aleksandrovich Kirillov. Novosibirsk, 1962



УДОСТОВЕРЕНИЕ № 2620

Предъявитель Вишик Марк Иосифович

Состоит на работе в МЭИ в должности профессора каф. спец. курсов высшей мат

Действительно до 31/XII 1963 г.

Зам. директора

Продлено по 31 декабря 1965 г.

подпись

Продлено по ______ 196__ г.

подпись

Продлено но ______ 196__ г.

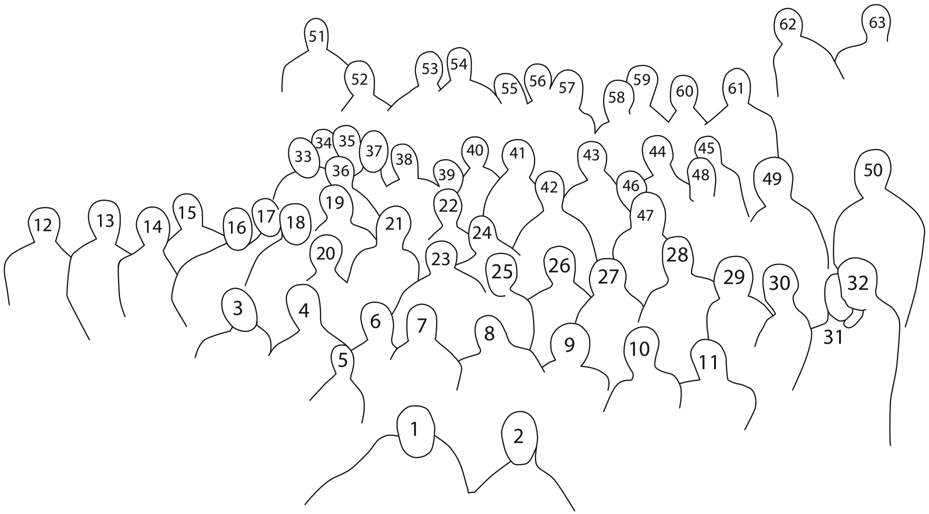
подпись

Тип. МЭИ. Зак. 734. Тир. 4000

Professor of Moscow Power Engineering Institute


Горьковская топологическая школа
(17-28 июня 1964 г.)
в доме отдыха „Волга"

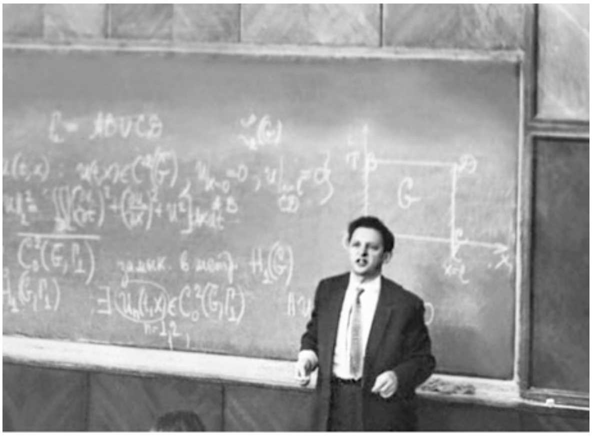

Горьковская топологиче
школа
(17–28 июня 1964 г.)
в доме отдыха "Волга"



Gorky Topological School. Resort "Volga" near Nizhny Novgorod, 17–28 June 1964. 1. S.N. Gindikin; 2. A.A. Kirillov; 3. D.V. Anosov; 4. B.Yu. Sternin; 5. M.S. Alber; 6. S.I. Alber; 7. M.S. Agranovich; 8. M.I. Vishik; 9. M.M. Postnikov; 10. A.S. Dynin; 11. S.P. Novikov; 12. A.Yu. Neimark; 13. M.L. Gromov; 14. E.L. Gerlovin; 15. V.P. Palamodov; 16. V.S. Itenberg; 17. Yu.A. Dubinskii; 18. M.I. Graev; 19. A.V. Chernavskii; 20. I.M. Dektyarev; 21. V.N. Voronkov; 22. K.A. Sitnikov; 23. R.V. Belova; 24. L.V. Rodygin; 25. I. Gluskina (Esther Sokolinsky); 27. E.A. Leontovich-Andronova; 28. L.P. Shilnikov; 29. Yu.I. Neimark; 30. E.Yu. Alber; 31. L.S. Kirillova; 32. Ya.G. Sinai; 33. M.Ya. Antonovskii; 34. A. Matuzevičius; 35. Ya.A. Roitberg; 36. G.I. Eskin; 37. A.A. Mal'tsev; 38. R.V. Plykin; 40. R.L. Frum-Ketkov; 41. B.S. Mityagin; 43. D.A. Gudkov; 44. A.A. Rosenblum; 45. L.E. Kaplan; 46. A.B. Sossinsky; 47. L.D. Mdzinarishvili; 48. G.L. Laitadze; 50. S.Kh. Aranson; 52. A.P. Savin; 53. V.L. Golo; 54. D.B. Fuchs; 55. A.M. Leontovich; 56. M.S. Kushel'man; 57. A.I. Fet; 59. G.A. Utkin; 61. A.D. Yunakovsky; 62. Ya.I. Al'ber; 63. V.A. Brusin

At vacation. Around 1964

On a lecture



Lazar Aronovich Lyusternik

At the new building of
Moscow State University
3 March 1972

With Andrei Vladimirovich
Fursikov after his PhD.
defence. 3 March 1972

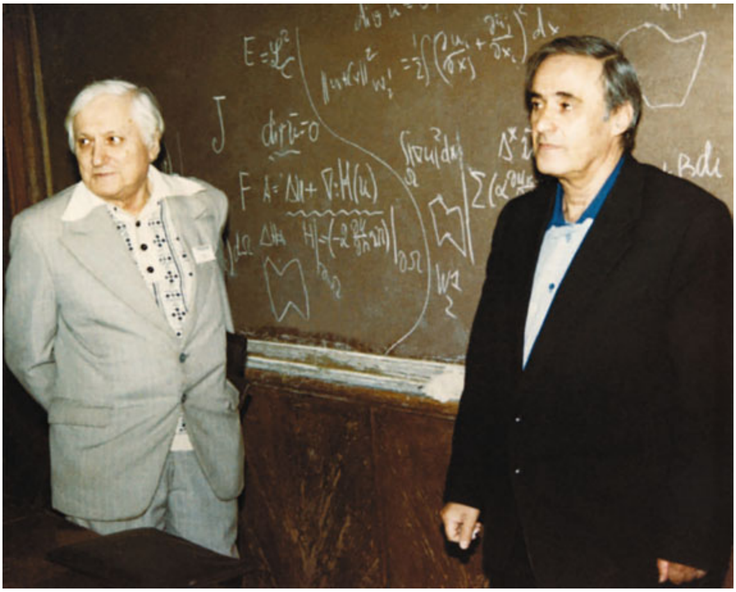
Jacques-Louis Lions

Gerard Tronel

Roger Temam

Alain Haraux

Yurii Makarovich Berezanskii, Erhard Meister, Zinovii Grigorievich Sheftel, Andrei Andreevich Shkalikov. International Conference on Differential Equations “Ivan Petrovsky – 90”. Moscow State University, Moscow, 1991



Selim Grigorievich Krein and Mikhail Solomonovich Birman. Moscow State University, Moscow, 1991

Doina Cionarescu and Lars Gårding. Moscow, 1991

Sergei Yurievich Dobrokhotov and Arlen Mikhailovich Ilyin. Moscow State University, Moscow, 1991

Willi Jäger, Yakov Abramovich Roitberg, Haïm Brezis, Cathleen Morawetz, Zinovii Grigorievich Sheftel, Peter Lax. Moscow, 1991

Willi Jäger, Yakov Abramovich Roitberg, Lars Gårding, Peter Lax, Cathleen Morawetz, Haïm Brezis. Moscow, 1991

Alexander Ilyich Komech, Sergei Yurievich Dobrokhotov, Yulii Andreevich Dubinskii. Moscow State University, Moscow, 1991

Yuri Kordyukov, Cathleen Morawetz, Haïm Brezis. Moscow, 1991

On the left: Mark Malamud. Moscow State University, Moscow, 1991

Cathleen Morawetz, Sergei Yurievich Dobrokhotov, Zinovii Grigorievich Sheftel, Maria Grigorieva, Willi Jäger. Moscow, 1991

Alexander Yakovlevich Povzner, Alexander Ilyich Komech, Haïm Brezis. Moscow State University, Moscow, 1991

Boris Ruvimovich Vainberg, Leonid Romanovich Volevich, Aizik Isaakovich Volpert. Moscow State University, Moscow, 1991

With Haïm Brezis. Moscow State University, Moscow, 1991

With Vladimir Tikhomirov. Moscow State University, Moscow, 1991

Vasilii Vasilievich Zhikov. In the last row: Elena Georgievna Sitnikova and Alexander Arkadievich Kosmodemyanskii. Moscow State University, Moscow, 1991

Boris Ruvimovich Vainberg, Mikhail Aleksandrovich Shubin, Leonid Romanovich Volevich, Mikhail Semyonovich Agranovich. Moscow, 1990

Potsdam, 1995

With Vladimir Viktorovich Chepyzhov. Potsdam, 1995

With Nikolai Nikolaevich Tarkhanov and Vladimir Viktorovich Chepyzhov. Potsdam, 1995

With Vladimir Gilelevich Mazya. Stuttgart, 1996

M.A. Shubin's talk at Vishik's seminar, around 1996. First row: N.D. Vvedenskaya, T.D. Venttsel, M.I. Vishik. Second row: A.A. Ilyin, A.R. Shirikyan, L.R. Volevich, A.V. Fursikov, S.B. Kuksin. In the back, to the left of the blackboard: V.V. Chepyzhov

After the talk of M.A. Shubin. A.R. Shirikyan, M.A. Shubin, M.I. Vishik, A.V. Fursikov, A.S. Demidov, L.R. Volevich, A.A. Shkalikov

On the Humboldt Prize ceremony. Bamberg, 23 March 1997

Leonid Romanovich Volevich, Mark Iosifovich Vishik, Armen Rafikovich Shirikyan. Rostock, 1998

Asya Moiseevna

Berlin. Around 1998

At the Symposium in Honor of Professor Mark Vishik. Free University of Berlin, 20 December 2001

Bernold Fiedler, Asya Vishik, Faouzi Lakrad. Berlin, 2001

Mark and Asya Vishik. In the back: Louis Nirenberg and Eberhard Zeidler. Berlin, 2001

Mark and Asya Vishik, Eberhard Zeidler. Berlin, 2001

7 May 2004

Vishik's seminar, 29 March 2004. Left to right: ?, Nikita Dmitrievna Vvedenskaya, Mikhail Semyonovich Agranovich, Leonid Romanovich Volevich, Vladimir Viktorovich Chepyzhov, Alexey Andreevich Ilyin, Mark Iosifovich Vishik, Elena Georgievna Sitnikova (in the back), Andrei Vladimirovich Fursikov

Asya and Mark Vishik with their son Mikhail and granddaughter Inna. Irvine, California, 2004

Asya with Claire Vishik.
Irvine, California, 2004

Vishik's seminar, 25 April 2005. First row: Leonid Romanovich Volevich, Mark Iosifovich Vishik, Andrei Vladimirovich Fursikov. Second row: Nikita Dmitrievna Vvedenskaya, Arkadii Petrovich Vinnichenko, Alexey Andreevich Ilyin. Third row: Evgenii Vladimirovich Radkevich, Andrey Yurievich Goritsky

Vishik’s seminar, 12 September 2005

Asya Moiseevna

Mark Iosifovich Vishik

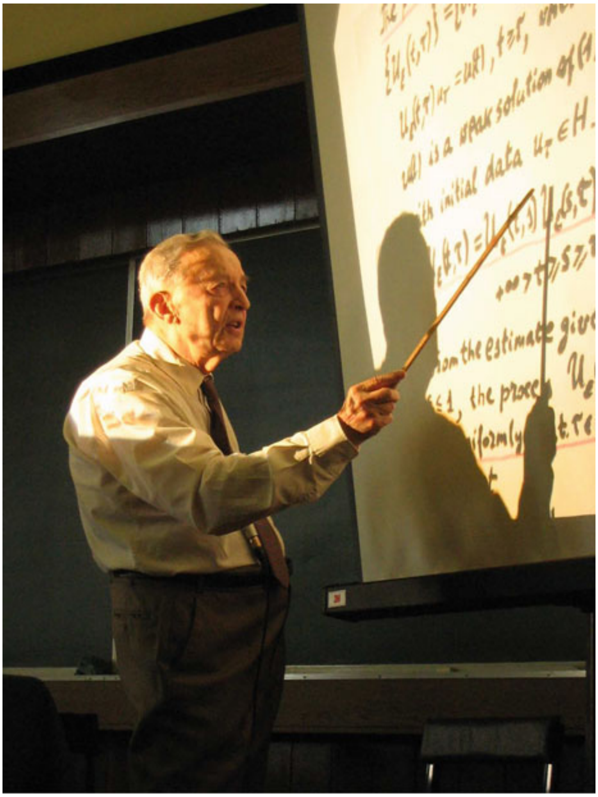
Vishik's seminar, 13 February 2006. First two rows, left to right: Mark Iosifovich Vishik, Vladimir Viktorovich Chepyzhov, Andrei Vladimirovich Fursikov, Andrey Yurievich Goritsky, Elena Kopylova, Valery Marsovich Imaykin

With Vladimir Viktorovich Chepyzhov. 13 February 2006

With Alexander Iosifovich Shnirelman and his wife Alla Davidovna. 29 June 2006

7 October 2006

1 November 2007



21 May 2007

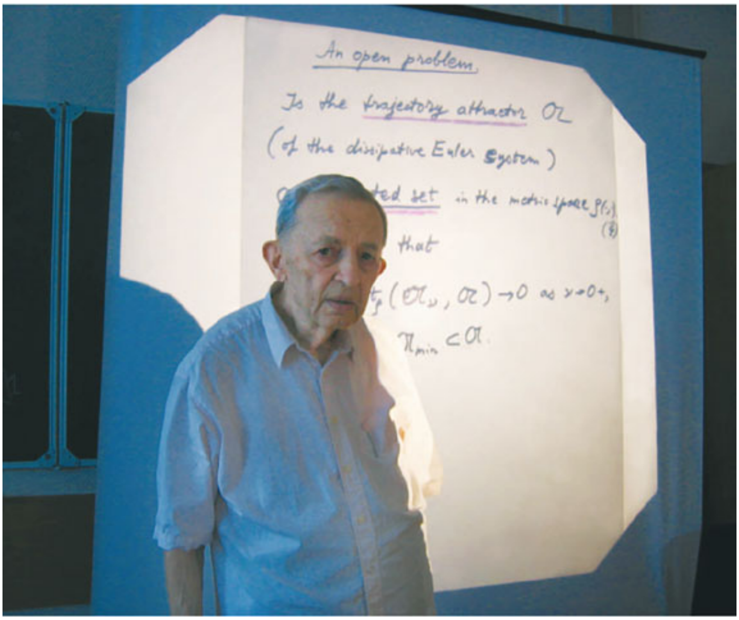
With Armen Rafikovich Shirikyan. 2007

With Louis Nirenberg. May 2007

30 December 2007

With Elena Andreevna Kopylova and Robert Adolfovich Minlos. 2008

With Alexander Sergeevich Demidov. 11 March 2009



16 July 2009

With Mikhail Semyonovich Agranovich. 28 March 2012

Bogdan Bojarski, Louis Nirenberg, and Peter Lax. New York, 27 September 2012

With Vladimir Viktorovich Chepyzhov. Mark Vishik's 90th Anniversary Conference, Moscow, 4 June 2012

With Alexander Petrovich Kuleshov. 4 June 2012

With Vsevolod Alekseevich Solonnikov. 4 June 2012

Gregory Ilyich Eskin, Andrew Comech, Mark Iosifovich Vishik, Claude Bardos, Louis Nirenberg.
7 June 2012