Stability of solitary waves of the nonlinear Dirac equation

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ABSTRACT. The electron is known to be stable; perhaps it is the most stable particle. Wikipedia gives a one-line proof:

the electron is the least massive particle with non-zero electric charge, its decay would violate charge conservation.

On the mathematical side, the (classical) Dirac–Maxwell system is known to have localized solitary waves [*Lisi*[95]], [*Esteban, Georgiev, Séré*[96]], [*Abenda*[98]]. Their stability is still not known; unboundedness of the energy density from below does not immediately lead to instability. We can not claim whether these solutions have any relation to electrons, but at least we hope to understand their stability; no stability – no relation to electrons.

Our main result [*Boussaïd & Comech*[19]] is the spectral stability (absence of linear instability) of solitary waves in the nonlinear Dirac equation with the scalar self-interaction, known as the Soler model. The stability takes place when certain relation between powers of nonlinearity and the spatial dimension is satisfied.

History from the B.C. era

[*Earnshaw*[42]]: Any set of point charges is unstable

[*Thomson*[04]]: Plum-pudding model of the atom

[*Nagaoka*[04], *Rutherford*[12]]: Saturnian (planetary) model of the atom

Relativistic and nonrelativistic Schrödinger equations

- Bohr's atom, $L = mvr = n\hbar$, $n \in \mathbb{N}$ (1913): $\quad E_n = -\frac{m\alpha^2}{2n^2}, \quad n \in \mathbb{N}$

- [*Planck*[06]]: $E^2 = p^2 + m^2$ $\qquad$ [*Schrödinger*[25]]: $(\mathrm{i}\partial_t)^2\psi = (-\mathrm{i}\nabla)^2\psi + m^2\psi$

Energy levels in the Coulomb potential for relativistic Schrödinger [*Schiff*[49]]:

$$E_{\ell n} \approx m\Big[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\Big(\frac{n}{\ell + \frac{1}{2}} - \frac{3}{4}\Big)\Big], \qquad \ell = 0 \dots n-1$$

$E = \sqrt{m^2 + p^2} \approx m + \frac{p^2}{2m}$ $\qquad \Rightarrow \qquad$ [*Schrödinger*[26]]: $\mathrm{i}\partial_t\psi = \frac{1}{2m}(-\mathrm{i}\nabla)^2\psi$

- Energy levels in the Coulomb potential for Dirac equation:

$$E_{jn} \approx m\Big[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\Big(\frac{n}{j + \frac{1}{2}} - \frac{3}{4}\Big)\Big], \quad j = \Big|\ell \pm \frac{1}{2}\Big|, \quad \ell = 0 \dots n-1$$

First derived in [*Sommerfeld*[16]] via relativistic precession of elliptic orbits!

Dirac equation

[*Dirac*[28]]: Want to have the Hamiltonian linear in $E = \mathrm{i}\partial_t$, hence in $p = -\mathrm{i}\nabla$

$$E = \alpha^1 p_1 + \alpha^2 p_2 + \alpha^3 p_3 + \beta m \quad \text{implies} \quad E^2 = p^2 + m^2$$

$$\text{if} \quad \{\alpha^j, \alpha^k\} = 0 \ j \neq k; \quad \{\alpha^j, \beta\} = 0; \quad (\alpha^j)^2 = \beta^2 = I$$

Standard choice of self-adjoint Dirac matrices $\alpha^1,\ \alpha^2,\ \alpha^3,\ \beta$:

$$\alpha^j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} \text{ (with } \sigma_1, \sigma_2, \sigma_3 \text{ the Pauli matrices)}, \quad \beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$$

The Dirac equation:

$$\mathrm{i}\partial_t \psi = \underbrace{(-\mathrm{i}\alpha \cdot \nabla + \beta m)}_{D_m}\psi, \qquad \psi(t,x) \in \mathbb{C}^4, \qquad m > 0$$

Dirac–Maxwell system

$$\gamma^0 = \beta, \quad \gamma^j = \beta\alpha^j, \ 1 \le j \le 3$$

$$\gamma^\mu(\mathrm{i}\partial_\mu - A_\mu)\psi = m\psi, \quad \psi(t,x) \in \mathbb{C}^4; \qquad (\partial_t^2 - \Delta)A^\mu = \underbrace{\psi^*\beta\gamma^\mu\psi}_{J^\mu(t,x)}$$

[*Lisi*[95]], [*Esteban, Georgiev, Séré*[96]], [*Abenda*[98]]:

solitary waves $\psi(t,x) = \phi(x)e^{-\mathrm{i}\omega t} \in \mathbb{C}^4$, $\quad \omega \in (-m, m)$

[*Comech & Stuart*[18]]: nonrelativistic asymptotics for $\omega \gtrsim -m$

Solitary waves in Dirac-Maxwell? "Klein paradox" [*Lisi*[95]]

$$\mathrm{i}\partial_t\psi = -\mathrm{i}\boldsymbol{\alpha}\cdot\nabla\psi + m\beta\psi + \Phi\psi, \quad \psi(t,x) = e^{-\mathrm{i}\omega t}\begin{bmatrix} v(x) \\ u(x) \end{bmatrix} \in \mathbb{C}^4, \quad \omega \gtrsim -m$$

$$\omega\begin{bmatrix} v(x) \\ u(x) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\mathrm{i}\boldsymbol{\sigma}\cdot\nabla \\ -\mathrm{i}\boldsymbol{\sigma}\cdot\nabla & 0 \end{bmatrix}}_{-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla}\begin{bmatrix} v(x) \\ u(x) \end{bmatrix} + m\underbrace{\begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}}_{\beta}\begin{bmatrix} v(x) \\ u(x) \end{bmatrix} + \Phi(x)\begin{bmatrix} v(x) \\ u(x) \end{bmatrix}$$

$$\begin{cases} \omega v = -\mathrm{i}\boldsymbol{\sigma}\cdot\nabla u + mv + \Phi v & \Rightarrow \quad \mathrm{i}\boldsymbol{\sigma}\cdot\nabla u \approx 2mv \\ \omega u = -\mathrm{i}\boldsymbol{\sigma}\cdot\nabla v - mu + \Phi u \end{cases}$$

$$(m+\omega)u = \underbrace{-\mathrm{i}\boldsymbol{\sigma}\cdot\nabla\left(\frac{\mathrm{i}\boldsymbol{\sigma}\cdot\nabla u}{2m}\right)}_{(\boldsymbol{\sigma}\cdot\nabla)^2 u/2m = \Delta u/2m} + \Phi u$$

$$-(m+\omega)u(x) = -\tfrac{1}{2m}\Delta u(x) - \Phi(x)u(x)$$

Attractive potential; there are bound states! [*Lieb*[77], *Lions*[80], *Ma & Zhao*[10]]

“Derrick’s theorem”: localized solutions are unstable?

[*Derrick*[64]]: *Stationary localized states are unstable in* $\mathbb{R}^n$, $n \geq 3$

Consider $\quad -\partial_t^2 u = -\Delta u + g(u), \qquad u(t,x) \in \mathbb{R}, \quad x \in \mathbb{R}^n; \quad g(0) = 0$

Hamiltonian functional (“energy”):

$$H(u, \partial_t u) = \int_{\mathbb{R}^3} \Big(\frac{|\partial_t u|^2}{2} + \frac{|\nabla u|^2}{2} + G(u) \Big)\, dx, \quad G'(s) = g(s), \;\; G(0) = 0$$

If $u(t,x) = \theta(x)$ is a solution, $0 = -\Delta\theta(x) + g(\theta(x))$;

$$H(\theta, 0) = \underbrace{\frac{1}{2} \int_{\mathbb{R}^3} |\nabla\theta(x)|^2\, dx}_{T(\theta)} + \underbrace{\int_{\mathbb{R}^3} G(\theta(x))\, dx}_{V(\theta)}$$

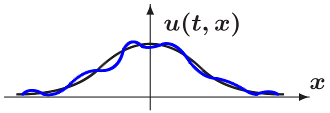
Let $\theta_\lambda(x) = \theta(x/\lambda)$; $\quad \Rightarrow \quad T(\theta_\lambda) = \lambda T(\theta), \;\; V(\theta_\lambda) = \lambda^3 V(\theta)$

$$\frac{d}{d\lambda}\Big|_{\lambda=1} \big(T(\theta_\lambda) + V(\theta_\lambda)\big) = 0 \quad \Rightarrow \quad \frac{d^2}{d\lambda^2}\Big|_{\lambda=1} \big(T(\theta_\lambda) + V(\theta_\lambda)\big) < 0 \text{ !!}$$

“Derrick’s theorem”. More rigorous approach: spectral stability

Consider $\quad -\partial_t^2 u = -\Delta u + g(u), \qquad u(t,x) \in \mathbb{R}, \quad x \in \mathbb{R}^n$

Linearization: let $u(t,x) = \theta(x) + \rho(t,x)$, $\qquad -\partial_t^2 \rho = -\Delta \rho + g'(\theta(x))\rho + \dots$



$$\partial_t \begin{bmatrix} \rho \\ \partial_t \rho \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\left(-\Delta + g'(\theta(x))\right) & 0 \end{bmatrix}}_{A} \begin{bmatrix} \rho \\ \partial_t \rho \end{bmatrix}$$

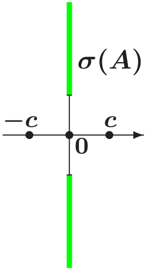
If $\sigma(A) \subset \mathrm{i}\mathbb{R}$: “spectral stability”

Note: $-\Delta\theta(x) + g(\theta(x)) = 0$, $(-\Delta + g'(\theta(x))\partial_{x_1}\theta = 0$...

$\lambda = 0$ is not the lowest eigenvalue! There is $\lambda = -c^2$, $c > 0$:

$$-c^2\varphi(x) = (-\Delta + g'(\theta(x))\varphi(x), \qquad \varphi(x) > 0$$

Then $\quad A \begin{bmatrix} \varphi(x) \\ c\varphi(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c^2 & 0 \end{bmatrix} \begin{bmatrix} \varphi(x) \\ c\varphi(x) \end{bmatrix} = c \begin{bmatrix} \varphi(x) \\ c\varphi(x) \end{bmatrix}$



No spectral stability!! Linear instability, in any dimension :):)

Spectral theory of operators in the Banach space X; σ_{d}

$A \in \mathscr{C}(\mathbf{X})$ closed operator with dense domain $\mathfrak{D}(A) \subset \mathbf{X}$:

if $x_j \in \mathfrak{D}(A),\ x_j \to x_0,\ Ax_j \to y_0,$ then $x_0 \in \mathfrak{D}(A)$ and $y_0 = Ax_0$

$z \in \rho(A)$ if $A - zI$ is invertible and $(A - zI)^{-1} \in \mathscr{B}(\mathrm{X})$ [with $\mathfrak{D} = \mathrm{X}$]

• Spectrum: $\sigma(A) = \mathbb{C} \setminus \rho(A)$ (F. Riesz [1918]; terminology: D. Hilbert [1904]))

Definition $\lambda \in \sigma_{\mathrm{d}}(A)$ if $\exists$ decomposition into invariant subspaces,

$\mathbf{X} = \mathbf{Y}_\lambda \oplus \mathbf{Z}_\lambda,$ with $\dim \mathbf{Y}_\lambda < \infty$ and $(A - \lambda I)|_{\mathbf{Z}_\lambda}$ having bounded inverse

As $\mathbf{Y}_\lambda$, one takes generalized eigenspace (if it is finite-dimensional),

$\mathfrak{L}_\lambda(A) = \left\{ x \in \mathfrak{D}(A) \mid \exists k \in \mathbb{N},\ (A-\lambda I)^j x \in \mathfrak{D}(A)\ \forall j < k,\ (A-\lambda I)^k x = 0 \right\}$

Spectral theory of operators in the Banach space X; σ_{d} 2

Equivalent characterizations of $\sigma_{\mathrm{d}}(A)$ [*Gohberg & Krein*[57], *Gohberg & Krein*[69]]:

1. $\exists$ decomposition into invariant subspaces, $\mathbf{X} = \mathbf{Y}_\lambda \oplus \mathbf{Z}_\lambda$,
   with $\dim \mathbf{Y}_\lambda < \infty$ and $(A - \lambda I)|_{\mathbf{Z}_\lambda}$ having bounded inverse;
2. λ is an isolated point in $\sigma(A)$, $A - \lambda I$ is semi-Fredholm;
3. λ is an isolated point in $\sigma(A)$, $A - \lambda I$ is Fredholm of index zero;
4. λ is an isolated point in $\sigma(A)$, $P_\lambda = -\frac{1}{2\pi \mathrm{i}} \oint\limits_{|z-\lambda|=\varepsilon} (A - zI)^{-1} dz$ of finite rank;
5. λ is an isolated point in $\sigma(A)$, $\dim \mathfrak{L}_\lambda(A) < \infty$, $\mathrm{Range}(A - \lambda I)$ is closed

Moreover, if $\lambda \in \sigma_{\mathrm{d}}(A)$, then $\mathfrak{L}_\lambda(A) = \mathrm{Range}(P_\lambda)$

Example: $A : e_j \mapsto e_{j+1}/j, \quad \mathfrak{L}_0(A) = \{0\}, \quad \sigma(A) = \{0\}, \quad 0 \notin \sigma_{\mathrm{d}}(A)$

"Quasinilpotent": $\|A^j\|^{1/j} \to 0$

Spectral theory of operators in the Banach space X; σ_{ess}

$\lambda \in \sigma_{\mathrm{ess},1}(A)$ [*Kato spectrum*] if either $\mathrm{Range}(A-\lambda I)$ is not closed

or $\dim\ker(A-\lambda I)=\infty$, $\dim\mathrm{coker}(A-\lambda I)=\infty$;

$\lambda \in \sigma_{\mathrm{ess},2}(A)$ if either $\mathrm{Range}(A-\lambda I)$ is not closed or $\dim\ker(A-\lambda I)=\infty$;

$\lambda \in \sigma_{\mathrm{ess},3}(A)$ [*Fredholm spectrum*] if either $\mathrm{Range}(A-\lambda I)$ is not closed

or $\dim\ker(A-\lambda I)=\infty$ or $\dim\mathrm{coker}(A-\lambda I)=\infty$;

$\lambda \in \sigma_{\mathrm{ess},4}(A)$ [*Weyl spectrum*] if either $\mathrm{Range}(A-\lambda I)$ is not closed

or $\mathrm{ind}(A-\lambda I):=\dim\ker(A-\lambda I)-\dim\mathrm{coker}(A-\lambda I)\neq 0$;

$\lambda \in \sigma_{\mathrm{ess},5}(A)$] [*Browder spectrum*]: union of $\sigma_{\mathrm{ess},1}(A)$ and components of

$\mathbb{C}\setminus\sigma_{\mathrm{ess},1}(A)$ which have no intersection with $\rho(A)$

Notes:

- $\sigma_{\mathrm{ess},1}(A)\subset\sigma_{\mathrm{ess},2}(A)\subset\sigma_{\mathrm{ess},3}(A)\subset\sigma_{\mathrm{ess},4}(A)\subset\sigma_{\mathrm{ess},5}(A)\subset\sigma(A)$;
- all essential spectra are equal if $\sigma(A)$ contains no open subsets of $\mathbb{C}$;
- $\sigma_{\mathrm{ess},5}(A)=\sigma(A)\setminus\sigma_{\mathrm{d}}(A)$, we will call it $\sigma_{\mathrm{ess}}(A)$

- We have $\lambda \in \sigma_{\mathrm{ess},2}(A)$ if there is a *Weyl sequence* $\psi_j \in \mathbf{X}$ for $A - \lambda I$:

  $\|\psi_j\| = 1$; no convergent subsequence; $(A - \lambda I)\psi_j \to 0$

Example: $\sigma_{\mathrm{ess},2}(-\partial_x^2) = [0, +\infty)$

Indeed, for $k \geq 0$, let $\psi_j(x) = \begin{cases} (2j)^{-1/2} e^{\mathrm{i}kx}, & |x| \leq j \\ 0, & |x| \geq j+1 \end{cases} \in C^\infty(\mathbb{R})$

Example: $\sigma_{\mathrm{ess},2}(-\partial_x^2 + V) = [0, +\infty)$ if $V \in C_{\mathrm{comp}}(\mathbb{R})$

Nonlinear Schrödinger equation: orbital stability

$$\mathrm{i}\partial_t\psi(t,x) = -\Delta\psi - |\psi|^{2\kappa}\psi, \qquad \psi(t,x)\in\mathbb{C}, \quad x\in\mathbb{R}^n, \quad \kappa>0$$

Schrödinger eigenstates: $\psi(t,x) = e^{-\mathrm{i}\omega t}\phi_\omega(x), \qquad \omega\phi_\omega(x) = -\Delta\phi_\omega - |\phi_\omega|^{2\kappa}\phi_\omega$

Conserved quantities:

Energy, $$H(\psi) = \int_{\mathbb{R}^n}\Big(\frac{|\nabla\psi(t,x)|^2}{2} - \frac{|\psi(t,x)|^{2\kappa+2}}{2\kappa+2}\Big)\,dx$$

Charge, $$Q(\psi) = \int_{\mathbb{R}^n}\frac{|\psi(t,x)|^2}{2}\,dx$$

If $\kappa < 2/n$, $\phi_\omega(x) > 0$ is a minimum of H under the charge constraint $Q =$ const

$\Rightarrow e^{-\mathrm{i}\omega t}\phi_\omega(x)$ is orbitally stable [*Cazenave & Lions*[82], *Weinstein*[85], *Grillakis et al.*[87]]:

$$\forall\varepsilon>0\ \ \exists\delta>0:\ \text{if}\ \ \|\psi(0,\cdot)-\phi\|\le\delta,\ \ \text{then}\ \ \sup_{t\ge 0}\ \underbrace{\inf_{s\in\mathbb{R}}\|\psi(t,\cdot)-\phi e^{\mathrm{i}s}\|}_{\text{distance to the orbit}}\le\varepsilon$$

Nonlinear Schrödinger equation: spectral stability

$$\psi(t,x) = e^{-\mathrm{i}\omega t}(\phi_\omega(x) + \rho(t,x)), \quad \partial_t \rho = A\rho; \qquad \sigma(A) \subset \mathrm{i}\mathbb{R}??$$

$$\mathrm{i}\partial_t \psi(t,x) = -\Delta\psi - |\psi|^{2\kappa}\psi, \qquad \psi(t,x) \in \mathbb{C}, \quad x \in \mathbb{R}^n, \quad \kappa > 0$$

Let $\psi(t,x) = e^{-\mathrm{i}\omega t}\big(\phi_\omega(x) + \underbrace{u(t,x) + \mathrm{i}v(t,x)}_{\rho(t,x)}\big), \quad \phi_\omega(x) > 0$

$$|\psi|^{2\kappa} = |(\phi_\omega + u)^2 + v^2|^\kappa = |\phi_\omega + u|^{2\kappa} + o(\dots) = |\phi_\omega|^{2\kappa} + 2\kappa\phi_\omega^{2\kappa-1}u + o(\dots)$$

Im: $\quad \partial_t u + \omega v = -\Delta v - \phi_\omega^{2\kappa} v$

Re: $\quad -\partial_t v + \omega u = -\Delta u - \phi_\omega^{2\kappa} u - 2\kappa\phi_\omega^{2\kappa} u$

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & \overbrace{-\Delta - \omega - \phi_\omega^{2\kappa}}^{L_0} \\ -(\underbrace{-\Delta - \omega - (1+2\kappa)\phi_\omega^{2\kappa}}_{L_1}) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$\sigma\left(\begin{bmatrix} 0 & L_0 \\ -L_1 & 0 \end{bmatrix}\right) =??$ Note: $\quad L_0\phi_\omega = 0, \quad L_1\partial_x\phi_\omega = 0, \quad L_1\partial_\omega\phi_\omega = \phi_\omega$

[*Zakharov*[67], *Kolokolov*[73]]

$$\lambda \begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & L_0 \\ -L_1 & 0 \end{bmatrix}}_{A_\omega} \begin{bmatrix} u \\ v \end{bmatrix}; \qquad \left. \begin{aligned} L_0\phi_\omega &= 0, \\ L_1\partial_x\phi_\omega &= 0, \\ L_1\partial_\omega\phi_\omega &= \phi_\omega \end{aligned} \right\}$$

$$L_0 = -\Delta - \phi_\omega^{2\kappa} - \omega, \quad L_1 = -\Delta - (1+2\kappa)\phi_\omega^{2\kappa} - \omega,$$

$$\sigma_{\mathrm{ess}}(L_0) = \sigma_{\mathrm{ess}}(L_1) = [|\omega|, +\infty), \quad \sigma_{\mathrm{ess}}(A_\omega) = \mathrm{i}(\mathbb{R} \setminus (-|\omega|, |\omega|))$$

$\lambda^2 u = -L_0 L_1 u$, so, $u \perp \phi$... $\mathrm{Span}(\phi) = \ker L_0$... $\Rightarrow \lambda^2 L_0^{-1} u = -L_1 u$

$\lambda^2 \langle u, L_0^{-1} u\rangle = -\langle u, L_1 u\rangle$, $\lambda^2 \in \mathbb{R}$, $\lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$...

$\lambda > 0$ (hence linear instability) as long as $\langle u, L_1 u\rangle < 0$ for $u \perp \phi$.

Nonlinear Schrödinger equation: spectral stability 3

[*Zakharov*[67], *Kolokolov*[73]]

$$L_0 = -\Delta - \phi_\omega^{2\kappa} - \omega,$$
$$L_1 = -\Delta - (1+2\kappa)\phi_\omega^{2\kappa} - \omega$$

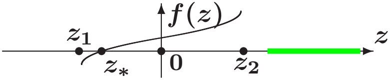
$$\lambda \begin{bmatrix} v \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & L_0 \\ -L_1 & 0 \end{bmatrix}}_{A_\omega} \begin{bmatrix} v \\ u \end{bmatrix}; \qquad \left.\begin{aligned} L_0\phi_\omega &= 0 \\ L_1\partial_x\phi_\omega &= 0 \\ L_1\partial_\omega\phi_\omega &= \phi_\omega \end{aligned}\right\}$$

$\lambda > 0$ (hence linear instability) as long as $\langle u, L_1 u\rangle < 0$ for $u \perp \phi$.

$$z_* = \inf_{u\perp\phi,\, \|u\|=1} \langle u, L_1 u\rangle, \quad L_1 u = zu + \mu\phi \quad \Rightarrow \quad (L_1 - z)u = \mu\phi,$$

$u = (L_1 - z)^{-1}\mu\phi \quad \Rightarrow \quad f(z) := \langle \phi, (L_1 - z)^{-1}\phi\rangle$ should vanish at $z = z_*$!

$f(z)$ defined for $z \in (z_0, z_1)$; $\quad z_0 < 0$, $z_1 > 0$ eigenvalues of $L_1|_{\text{even functions}}$



$f'(z) = \langle u, (L_1 - z)^{-2}u\rangle > 0$...

$z_* < 0$ [instability] $\Leftrightarrow$ $0 < f(0) = \langle \phi, L_1^{-1}\phi\rangle = \langle \phi, \partial_\omega\phi\rangle = \partial_\omega\langle \phi, \phi\rangle/2.$

If $\Phi(x) \in \mathbb{C}$ satisfies $\quad -\Phi = -\Delta\Phi - |\Phi|^{2\kappa}\Phi, \quad x \in \mathbb{R}^n,$

then $\phi_\omega(x) = |\omega|^{1/(2\kappa)}\Phi(|\omega|^{1/2}x)$, $\omega < 0$, satisfies

$$\omega\phi_\omega = -\Delta\phi_\omega - |\phi_\omega|^{2\kappa}\phi_\omega$$

Then

$$Q(\phi_\omega) = \frac{1}{2}\int_{\mathbb{R}^n} \underbrace{|\omega|^{1/\kappa}\Big|\Phi(|\omega|^{1/2}x)\Big|^2}_{|\phi_\omega(x)|^2} dx \cdot \frac{|\omega|^{n/2}}{|\omega|^{n/2}} = |\omega|^{1/\kappa}|\omega|^{-n/2},$$

$\Rightarrow \quad \partial_\omega Q(\phi_\omega) > 0$ [linear instability] if $\kappa > 2/n$

Spectral stability of solitary waves if $\kappa \le 2/n$

- Instability (blow-up) for $\kappa = 2/n$: [*Keraani*[06]] ($n \le 2$), [*Bégout & Vargas*[07]]
- Asymptotic stability for particular $n \in \mathbb{N}$, $\kappa < 2/n$ seems still unknown :)

Limiting Absorption Principle (LAP) [*Ignatowsky*[05], *Agmon*[70]]

$A \in \mathscr{C}(\mathbf{X}), \mathfrak{D}(A);\ \|(A-zI)^{-1}\|_{\mathbf{X}\to\mathbf{X}} \geq 1/\mathrm{dist}(z,\sigma(A)) \to \infty$ as $z \to \sigma(A)$

Definition A satisfies LAP at $z_0 \in \sigma_{\mathrm{ess}}(A)$ relative to $\mathbf{E} \subset \mathbf{X} \subset \mathbf{F},\ \Omega \subset \mathbb{C}\backslash\sigma(A)$, if

$$\exists \quad (A - z_0 I)^{-1}_{\mathbf{E},\mathbf{F},\Omega} := \lim_{z\to z_0,\, z\in\Omega} (A - zI)^{-1} : \mathbf{E} \to \mathbf{F}$$

Example [*Agmon*[70]]: $\partial_x : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\quad \sigma(\partial_x) = \mathrm{i}\mathbb{R}$

$(\partial_x - z)u = f \in L^2(\mathbb{R}),\ \mathrm{Re}\, z < 0 \quad \Rightarrow \quad u(x) = \int_{-\infty}^{x} e^{z(x-y)} f(y)\, dy$

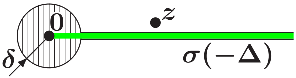
...If $\mathrm{Re}\, z \leq 0,\ f \in L^2_s(\mathbb{R}) \subset L^1(\mathbb{R})$, then $u \in L^\infty(\mathbb{R}) \subset L^2_{-s}(\mathbb{R}),\ s > 1/2$...

$$L^2_s(\mathbb{R}) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}) \mid (1+|x|)^s u \in L^2(\mathbb{R})\}$$

So, $(\partial_x - zI)^{-1} : L^2_s(\mathbb{R}) \to L^2_{-s}(\mathbb{R})$, $s > 1/2$, bounded uniformly in $\mathrm{Re}\, z \leq 0$

Example [*Agmon*[70]]: $(-\Delta - zI)^{-1} : L^2_s(\mathbb{R}^n) \to L^2_{-s}(\mathbb{R}^n)$, $s > 1/2$,

bounded uniformly in $z \in \mathbb{C} \setminus (\mathbb{D}_\delta \cup \mathbb{R}_+)$



LAP and virtual levels [*Jensen & Kato*[79], *Boussaïd & Comech*[21]]

Definition $z_0 \in \sigma(A)$ is a *virtual level* at relative to $\mathbf{E} \subset \mathbf{X} \subset \mathbf{F}$, $\Omega \in \mathbb{C} \setminus \sigma(A)$ if there is no LAP, yet there is $B \in \mathscr{B}_{00}(\mathbf{F}, \mathbf{E})$ [or B is A-compact] such that

$$\exists \quad (A + B - z_0 I)^{-1}_{\mathbf{E},\mathbf{F},\Omega} := \lim_{z \to z_0,\, z \in \Omega} (A + B - zI)^{-1} : \quad \mathbf{E} \to \mathbf{F}$$

$\Psi \in \mathbf{F}$ is a *virtual state* if $(A - z_0)\Psi = 0$, $\Psi = (A + B - z_0 I)^{-1}_{\Omega,\mathbf{E},\mathbf{F}}\phi$, $\phi \in \mathbf{E}$

Example: In $L^2(\mathbb{R})$, $(-\partial_x^2 - zI)^{-1} \sim \dfrac{e^{-|x-y|\sqrt{-z}}}{2\sqrt{-z}}$, no LAP as $z \to z_0 = 0$...

$(-\partial_x^2 - 0)1 = 0$, $(-\partial_x^2 + u - 0)1 = u$, $u \in C_0^\infty(\mathbb{R})$, $u \geq 0$

so $1 = (-\partial_x^2 + u - 0)^{-1}_{L^2_s, L^2_{-s'}} u$, $\qquad s > 3/2$, $s' > 1/2$

Example: In $L^2(\mathbb{R}^3)$, $(-\Delta - zI)^{-1} \sim \dfrac{e^{-|x-y|\sqrt{-z}}}{4\pi|x-y|}$, LAP as $z \to z_0 = 0$

Nonlinear Dirac equation [*Ivanenko*[38], *Soler*[70]]

$$\mathrm{i}\partial_t\psi = \underbrace{(-\mathrm{i}\alpha\cdot\nabla + m\beta)}_{D_m}\psi - |\psi^*\beta\psi|^\kappa\beta\psi, \qquad \psi(t,x)\in\mathbb{C}^4, \quad x\in\mathbb{R}^3, \quad \kappa>0$$

- Existence (numerical) of solitary waves in $\mathbb{R}^3$ [*Soler*[70], *Cazenave & Vázquez*[86]]:

$$\psi(t,x)=\phi_\omega(x)e^{-\mathrm{i}\omega t}, \qquad \omega\in(0,m), \qquad \phi_\omega\in H^1(\mathbb{R}^3)$$

- Attempts at stability: [*Bogolubsky*[79], *Alvarez & Soler*[86], *Strauss & Vázquez*[86]] ...
- Numerics suggest that (all?) solitary waves in 1D cubic Soler model are stable: [*Alvarez & Carreras*[81], *Alvarez & Soler*[83], *Berkolaiko & Comech*[12], *Lakoba*[18]]
- $\delta(x)f(\psi^*\beta\psi)\beta\psi$: [*Boussaid, Cacciapuoti, Carlone, Comech, Noja, Posilicano*[23]]
- Assuming spectral stability, one tries to prove asymptotic stability ("radial case"): [*Boussaïd*[06], *Boussaïd*[08]]

[*Pelinovsky & Stefanov*[12]] [*Boussaïd & Cuccagna*[12]] [*Comech, Phan, Stefanov*[17]]

Nonlinear Dirac equation: solitary waves with $\omega \lesssim m$

$$\mathrm{i}\partial_t\psi = \underbrace{(-\mathrm{i}\alpha\cdot\nabla + m\beta)}_{D_m}\psi - |\psi^*\beta\psi|^\kappa\beta\psi, \qquad \psi(t,x)\in\mathbb{C}^4, \quad x\in\mathbb{R}^3, \quad \kappa>0$$

Theorem 1 Let $0<\kappa<\frac{2}{n-2}$. Solitary waves in the nonrelativistic limit $\omega \lesssim m$:

$$\psi(t,x) \approx \underbrace{\begin{bmatrix} \epsilon^{\frac{1}{\kappa}} v(\epsilon x) \begin{bmatrix}1\\0\end{bmatrix} \\ -\epsilon^{1+\frac{1}{\kappa}} \frac{1}{2m} v'(\epsilon x) \frac{1}{|x|} x_j\sigma_j \begin{bmatrix}1\\0\end{bmatrix} \end{bmatrix}}_{\phi_\omega(x)} e^{-\mathrm{i}\omega t}, \qquad \epsilon=\sqrt{m^2-\omega^2}$$

where $v(x)>0$ solves $-\frac{1}{2m}v = -\frac{1}{2m}\Delta v - |v|^{2\kappa}v$

$n\geq 1$, $\kappa>0$: [*Ounaies*[00], *Guan*[08], *Boussaïd & Comech*[17]]

Nonlinear Dirac equation: one- and two-frequency solitary waves

[*Galindo*[77], *Boussaïd & Comech*[18]]: NLD has $\mathbf{SU(1,1)}$-symmetry!

$$\psi(t,x) \quad \text{solves NLD} \quad \Rightarrow \quad \underbrace{(a+bB\mathbf{K})}_{\mathbf{SU(1,1)}}\psi(t,x) \quad \text{solves NLD}$$

where $B = \mathrm{i}\gamma^2 = \mathrm{i}\beta\alpha^2 = \mathrm{i}\begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}$, $\mathbf{K}$ complex conjugation, $(B\mathbf{K})^2 = 1$,

$$a,\ b \in \mathbb{C}, \qquad |a|^2 - |b|^2 = 1$$

- $Q_\pm = \frac{1}{2}\int_{\mathbb{R}^3}\Big(|\psi_1 \pm \bar\psi_4|^2 + |\psi_2 \mp \bar\psi_3|^2\Big)\,dx; \qquad Q_+ + Q_- = Q = \int_{\mathbb{R}^3}|\psi|^2\,dx$

- $\psi_\omega(t,x) = \phi_\omega(x)e^{-\mathrm{i}\omega t} \quad \Rightarrow \quad \psi_{\omega,-\omega}(t,x) = (a+bB\mathbf{K})\phi_\omega(x)e^{-\mathrm{i}\omega t}$
  $= a\phi_\omega(x)e^{-\mathrm{i}\omega t} + b\chi_\omega(x)e^{\mathrm{i}\omega t}$

NLD has $\mathbf{SU(1,1)}$-symmetry; its solitary manifold has a larger symmetry group!

Nonlinear Dirac equation: one- and two-frequency solitary waves

$$\psi_\omega(t,x)=\underbrace{\begin{bmatrix} v_\omega(r)\vec{\mathrm{M}} \\ u_\omega(r)\,\frac{1}{|x|}x_j\sigma_j\vec{\mathrm{M}}\end{bmatrix}}_{\phi_{\omega,\vec{\mathrm{M}}}(x)}e^{-\mathrm{i}\omega t},\qquad \vec{\mathrm{M}}\in\mathbb{C}^2,\quad |\vec{\mathrm{M}}|=1,\quad r=|x|$$

$\Rightarrow$ more general solution [*Boussaïd & Comech*[18]]:

$$\psi_{\omega,-\omega}(t,x)=\underbrace{\begin{bmatrix} v_\omega(r)\vec{\mathrm{M}} \\ u_\omega(r)\,\frac{1}{|x|}x_j\sigma_j\vec{\mathrm{M}}\end{bmatrix}}_{\phi_{\omega,\vec{\mathrm{M}}}(x)}e^{-\mathrm{i}\omega t}+\underbrace{\begin{bmatrix} -u_\omega(r)\,\frac{1}{|x|}x_j\sigma_j\vec{\mathrm{N}} \\ v_\omega(r)\vec{\mathrm{N}}\end{bmatrix}}_{\chi_{\omega,\vec{\mathrm{N}}}(x)}e^{\mathrm{i}\omega t}$$

where $\vec{\mathrm{M}},\ \vec{\mathrm{N}}\in\mathbb{C}^2,\quad |\vec{\mathrm{M}}|^2-|\vec{\mathrm{N}}|^2=1$

Nonlinear Dirac equation: eigenvalues $\pm 2\omega\mathrm{i}$ of linearization

Consequence of existence of bi-frequency solitary waves [*Boussaïd & Comech*[18]]:

$$\psi_{\omega,-\omega}(t,x) = \phi(x)e^{-\mathrm{i}\omega t} + \chi(x)e^{\mathrm{i}\omega t}$$

$$\approx e^{-\mathrm{i}\omega t}\Big(\phi(x) + \underbrace{\chi(x)e^{2\mathrm{i}\omega t}}_{\rho(t,x)}\Big)$$

$$\Rightarrow \qquad \pm 2\omega\mathrm{i} \in \sigma_p(A_\omega), \text{ multiplicity } 2 \qquad (\text{for } \psi \in \mathbb{C}^4)$$

Embedded eigenvalues, like $\lambda = \pm 2\omega\mathrm{i}$, bad for proving asymptotic stability!

Need extra restrictions: [*Boussaïd & Cuccagna*[12]], [*Comech, Phan, Stefanov*[14]]

But this helps linear stability:

$\lambda = \pm 2\omega\mathrm{i}$ tells what happens near $\pm 2m\mathrm{i}$; no bifucating eigenvalues with $\mathrm{Re}\,\lambda \neq 0$!

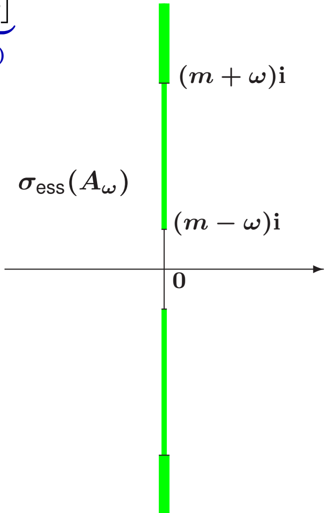
Nonlinear Dirac equation: linearization at a solitary wave

Given $\phi_\omega(x)e^{-\mathrm{i}\omega t}$, $\omega\in(-m,m)$, consider $\psi(t,x)=\big(\phi_\omega(x)+\rho(t,x)\big)e^{-\mathrm{i}\omega t}$

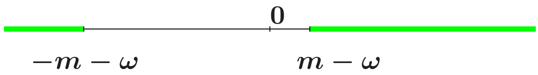
Linearized eqn: $\quad \mathrm{i}\partial_t\rho = D_m\rho-\omega\rho+V(x)\rho+W(x)\bar\rho \qquad\qquad V,W\sim\phi_\omega^{2\kappa}$

$$\partial_t\underbrace{\begin{bmatrix}\mathrm{Re}\,\rho\\ \mathrm{Im}\,\rho\end{bmatrix}}_{R(t,x)}=\underbrace{\begin{bmatrix}0 & D_m-\omega+\dots\\ -(D_m-\omega+\dots) & 0\end{bmatrix}}_{A_\omega}\underbrace{\begin{bmatrix}\mathrm{Re}\,\rho\\ \mathrm{Im}\,\rho\end{bmatrix}}_{R(t,x)}$$

$\partial_t R = A_\omega R$; $\ \sigma(A_\omega)\subset \mathrm{i}\mathbb{R}$??? (“spectral stability”)



$\sigma(D_m-\omega)$



Nonlinear Dirac equation: bifurcations from $\mathrm{i}\mathbb{R}$ when $\omega \to m$

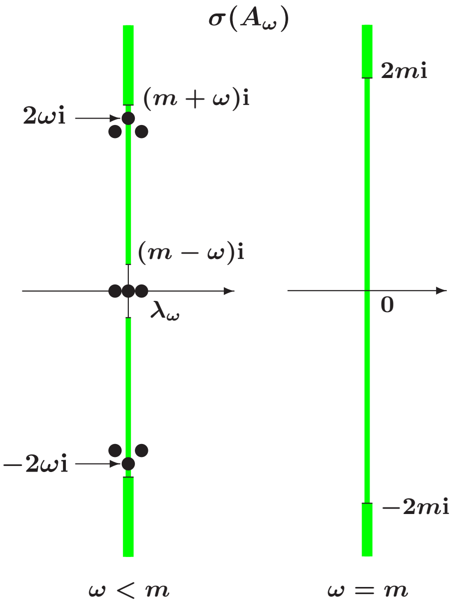
$$\mathrm{i}\partial_t\psi = D_m\psi - |\psi^*\beta\psi|^\kappa\beta\psi, \qquad x \in \mathbb{R}$$

$$\psi(t,x) = \phi_\omega(x)e^{-\mathrm{i}\omega t}$$

Theorem 2 ([*Boussaïd & Comech*[16]])

Assume: $\lambda_\omega \in \sigma_p(A_\omega)$, $\operatorname{Re}\lambda_\omega \neq 0$

Then $\lambda_\omega \underset{\omega\to m}{\longrightarrow} \{0, \pm 2m\mathrm{i}\}$



Nonlinear Dirac equation: spectral stability results

$$\mathrm{i}\partial_t\psi = D_m\psi - |\psi^*\beta\psi|^\kappa\beta\psi, \qquad x\in\mathbb{R}$$

$$\psi(t,x) = \phi_\omega(x)e^{-\mathrm{i}\omega t}$$

Theorem 3 ([*Boussaïd & Comech*[19]])

- If $\lambda_j \to 0$ as $\omega_j \to m$, $\mathrm{Re}\,\lambda_j \neq 0$,

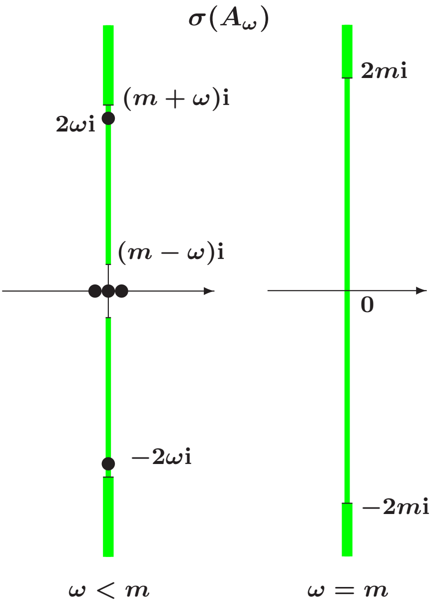
$$\frac{\lambda_j}{m^2-\omega_j^2} \to \sigma_{\mathrm{d}}\left(\begin{bmatrix} 0 & L_0 \\ -L_1 & 0\end{bmatrix}\right)\cap\mathbb{R};$$

limit is nonzero if $\kappa \neq 2/n$

- If $\lambda_j \to 2m\mathrm{i}$ as $\omega_j \to m$,

$$\frac{1}{\mathrm{i}}\frac{\lambda_j - 2\omega\mathrm{i}}{m^2-\omega_j^2} \to \sigma_{\mathrm{d}}(L_0)\cup\left\{\partial\sigma_{\mathrm{ess}}(L_0)\right\},$$

with $\partial\sigma_{\mathrm{ess}}(L_0)$ only when it is a virtual level



Corollary 4 ([*Boussaïd & Comech*[19]])

$\phi_\omega e^{-\mathrm{i}\omega t}$, $\omega \lesssim m$, spectrally stable if

$$\frac{2}{n} \ge \kappa > k_0(n) \approx \begin{cases} 1, & n = 1 \\ 0.621, & n = 2 \\ 0.461, & n = 3 \end{cases}$$

- Spectral stability for $\kappa = 2/n$ (unlike for NLS)
- Linear instability for $\kappa > 2/n$ (which disappears for $\omega \in (0, m)$ small enough!)
- Not known for $0 < \kappa \le k_0(n)$

Numerical confirmation: [*Berkolaiko & Comech*[12], *Cuevas-Maraver et al.*[16], *Lakoba*[18]]

M. Keldysh theory of characteristic roots [*Keldysh*[51]]

$M(z) \in \mathbf{End}(\mathbb{C}^N)$, analytic in $z \in \mathbb{C}$; z_0 is a characteristic root if $\det M(z_0) = 0$

Example: $M(z) = \begin{bmatrix} 0 & 1 \\ (z - z_0)^\alpha & 0 \end{bmatrix}$, $z \in \mathbb{C}$, $\alpha \in \mathbb{N}$; $z = z_0$ is a characteristic root

- At $z = z_0$, geometric multiplicity of $\lambda = 0$ is $g = 1$
- At $z = z_0$, algebraic multiplicity of $\lambda = 0$ is $\nu = 2$
- multiplicity of the characteristic root z_0: α, order of vanishing of $\det M(z)$

Note: $1 \le g \le \nu \le N$, $\quad 1 \le g \le \alpha$

How many characteristic roots bifurcate from z_0? The Rouché theorem for $\det M(z)$!

If $M(\epsilon, z) = \begin{bmatrix} \epsilon & 1 \\ (z - z_0)^\alpha & \epsilon \end{bmatrix}$, $\det M(\epsilon, z) = \epsilon^2 - (z - z_0)^\alpha = 0$ gives α families of characteristic roots $Z_j(\epsilon)$ bifurcating from z_0;

If $M(\epsilon, z) = \begin{bmatrix} 0 & 1 \\ (z - z_0 - \epsilon)^{\alpha-1}(z - z_0 + \epsilon) & 0 \end{bmatrix}$, two families: $Z_\pm(\epsilon) = z_0 \pm \epsilon$

M. Keldysh theory of characteristic roots [*Keldysh*[51]]

Assume: $A(z) \in \mathscr{C}(\mathbf{X})$, analytic in $z \in \Omega$; $0 \in \sigma_{\mathrm{d}}(A(z_0))$;

$A(z_0)$ is Fredholm; $A(z)$ has a bounded inverse for $z \neq z_0$ near z_0

Definition If $\ker A(z_0) \neq \{0\}$, we call z_0 a characteristic root of $A(z)$.

Its multiplicity $\alpha \in \mathbb{N}$ is the order of vanishing of $\det[A(z)P_0(z)]$ at z_0.

Above, $P_0(z) = -(2\pi \mathrm{i})^{-1} \oint_{|\zeta|=\delta} (A(z) - \zeta I)^{-1}\, d\zeta$, the Riesz projector.

Operator version of the Rouché theorem [*Gohberg & Sigal*[71]]:

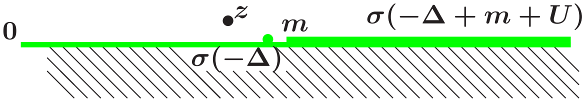
Sum of multiplicities of characteristic roots is stable under perturbations!

Assume:

- $A(\epsilon, z) \in \mathscr{C}(\mathbf{X})$, $\epsilon \geq 0$, $z \in \Omega$, $\mathfrak{D}(A(\epsilon, z)) = \mathfrak{D}$,
- $A(\epsilon, z)$ is analytic in $z \in \Omega$ and resolvent-continuous in ϵ, z:
  $(A(\epsilon, z) - \zeta I)^{-1}$ is continuous in ϵ, z [in the weak operator topology]

Theorem 5 Let z_0 be a characteristic root of $A(0, z)$ multiplicity α. $\exists\, \Omega_1 \subset \Omega$ (open), $z_0 \in \Omega_1$, such that the sum of multiplicities of all characteristic roots of $A(\epsilon, z)$ inside Ω_1 equals α if $\epsilon > 0$ is small.

Reduction from linear vector to nonlinear scalar case

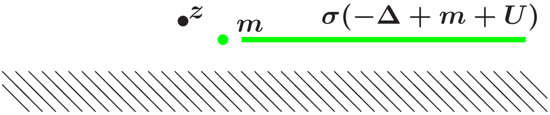


$$\begin{cases} (-\Delta + m + U(x) - z)\phi = V(x)\chi \\ (-\Delta - z)\chi = W(x)\phi \end{cases}, \qquad \operatorname{Im} z \geq 0,$$

with exponentially decaying $U(x)$, $V(x)$, $W(x)$

$$\Rightarrow \qquad (-\Delta + m + U(x) - z)\phi = V(x)(-\Delta - zI)^{-1}W(x)\phi, \qquad \operatorname{Im} z \geq 0$$

We can take $\operatorname{Im} z > -\varepsilon$ via analytic continuation of $V(x)(-\Delta - zI)^{-1}W(x)$,



Nonlinear Dirac equation: spectral stability of bi-frequency wave??

Given a perturbation of $\Phi(t,x) = \phi(x)e^{-\mathrm{i}\omega t} + \chi(x)e^{\mathrm{i}\omega t}$,

$$\psi(t,x) = \underbrace{\phi(x)e^{-\mathrm{i}\omega t} + \chi(x)e^{\mathrm{i}\omega t}}_{\Phi(t,x)} \; + \; R(t,x),$$

try to write it as

$$\psi(t,x) = \big(\phi(x) + R_1(t,x)\big)e^{-\mathrm{i}\omega t} + \big(\chi(x) + R_2(t,x)\big)e^{\mathrm{i}\omega t}$$

choosing R_1 and R_2 in such a way that $\chi(x)^*\beta R_1(t,x) + \phi(x)^*\beta R_2(t,x) = 0$,

then $\qquad \psi(t,x)^*\beta\psi(t,x) = \phi(x)^*\beta\phi(x) + \chi(x)^*\beta\chi(x) + O(R^2),$

hence nonlinearity $|\psi^*\beta\psi|^\kappa$ produces no new harmonics $e^{\pm 2\mathrm{i}\omega t}$, ... up to $O(R^2)$

This works for NLD for $n \le 2$ [*Boussaïd & Comech*[18]]

The result: stability of $\phi(x)e^{-\mathrm{i}\omega t} + \chi(x)e^{\mathrm{i}\omega t}$ reduces to stabililty of $\varphi(x)e^{-\mathrm{i}\omega t}$

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