Bayesian Analysis of ARA Imperfect Repair Models

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Résumé

This paper proposes a Bayesian analysis of a class of imperfect repair models, the ARA models. The choice of prior distributions and the computation of posterior distributions are discussed. A numerical study on the quality of the Bayesian estimators is presented, as well as a comparison with the maximum likelihood estimators. Finally, the approach is applied to a real data set.

Keywords : Bayesian Inference, Imperfect Repair, Repairable Systems Reliability, Maintenance Efficiency, Virtual age, MCMC algorithms.

1 Introduction

The reliability of a repairable system depends on both the ageing process and the efficiency of repair. The basic assumptions on repair efficiency are known as minimal repair or As Bad As Old (ABAO) and perfect repair or As Good As New (AGAN). The corresponding stochastic models for the failure process are respectively the Non Homogeneous Poisson Processes (NHPP) and the Renewal Processes (RP). The reality is generally between these two extreme cases : standard repair is better than minimal but not necessarily perfect. This is known as imperfect repair.

Many imperfect repair models have been proposed. The most usual are Kijima's virtual age models [15]. Brown and Proschan [5] assumed that repair is perfect with probability p and minimal with probability $1-p$. Doyen and Gaudoin [9] have proposed two classes of models, the ARA and ARI models, based on a reduction of virtual age or failure intensity.

Statistical inference is needed in order to estimate model parameters and to compute reliability indicators from failure data. Many results exist in the NHPP and RP cases, but only a few papers deal with the statistical inference in

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imperfect repair models. For virtual age models, empirical studies on maximum likelihood estimators (MLE) has been presented by Lim [18], Shin et al [28], Kaminskiy and Krivtsov [14], Gasmi et al [10], Doyen and Gaudoin [9]. Theoretical results on these estimators have been derived more recently by Peña et al [21] and Doyen [8].

In practice, for reliable systems, only a few failures occur. So the asymptotic results on MLE cannot be used. Moreover, the engineers knowledge about the degradation and failure process could be very helpful to improve the estimations of the model parameters. Thus, a Bayesian analysis of these models is an interesting alternative to usual frequentist methods.

The Bayesian inference has been extensively studied in the NHPP case : Guida et al. [11, 12, 6, 13], Bar-Lev et al [2], Campodónico and Singpurwalla [7], Kuo and Yang [16], Beiser and Rigdon [3], Sen [26], Ryan [25], Pievatolo and Ruggeri [22]. Conversely, Bayesian analysis for imperfect repair models has been seldom studied : Lim et al [17] for the Brown-Proschan model, Pan and Rigdon [20] for the ARA_1 and ARI_1 models. With a different purpose, Sethuraman and Hollander [27] derived theoretical results for general virtual age models when the virtual ages are known.

The aim of this paper is to present a Bayesian analysis of the class of ARA imperfect repair models. This class involves the ARA_1 model studied by Pan and Rigdon [20], and also many others including the ARA_∞ model.

The paper is organized as follows. The ARA models are defined in section 2. The Bayesian analysis is developed in section 3, with a discussion on the choice of prior distributions and the computation of posterior distributions. Section 4 studies the properties of the estimators by means of Monte-Carlo simulations. Finally, an application to real data is presented.

2 The ARA imperfect repair models

Let $\{T_i\}_{i\geq 1}$ be the successive failure times of a repairable system, starting from $T_0 = 0$. We assume that a repair task is performed after each failure and that repair times are negligible. Let $\{X_i\}_{i\geq 1}$ be the times between failures $(X_i = T_i - T_{i-1})$ and N_t be the number of failures observed up to time t.

Then, the failure process is a random point process. Its distribution is completely given by the failure intensity, defined as :

$$\forall\, t \geq 0, \quad \lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(N_{t+\Delta t} - N_{t-} = 1 | \mathcal{H}_{t-}) \tag{1}$$

where $\mathcal{H}_t$ is the history of the failure process at time t, i.e. the set of all events occurred before t. N_{t-} denotes the left-hand limit of N_t. In most cases, the failure process is a self-excited process, i.e. λ_t is a function of the number of failures and the failure times before t : $\lambda_t = \lambda_t(N_{t-}, T_1, \ldots, T_{N_{t-}})$.

We assume that before the first failure, the intensity is a deterministic continuous function of time, denoted $\lambda(t)$, and called the *initial intensity*. The initial intensity characterizes the intrinsic behaviour of a new system. λ is the hazard rate of the first failure time $T_1 = X_1$. In the following, we assume that the initial intensity is the intensity of a Power Law Process (PLP) :

$$\lambda(t) = \alpha\beta t^{\beta-1},\ \alpha > 0, \beta > 0 \tag{2}$$

In general, the system is ageing intrinsically, so $\lambda(t)$ is increasing and $\beta > 1$.

For the two basic cases, the failure intensity of a NHPP (ABAO) is a function of time :

$$\lambda_t = \lambda(t) \tag{3}$$

and the failure intensity of a RP (AGAN) is a function of the time elapsed since last failure :

$$\lambda_t = \lambda(t - T_{N_{t-}}) \tag{4}$$

The idea of the virtual age models proposed by Kijima [15], is to assume that there exists a sequence of random variables $\{A_i\}_{i\geq 1}$, with $A_0 = 0$, such that after the ith repair, the system behaves like a new one having survived without failure until A_i. This property can be written :

$$P(X_{i+1} > x | A_i, X_1, \dots, X_i) = P(Y > A_i + x | Y > A_i, A_i) \text{ for all } x \geq 0, \tag{5}$$

where Y denotes a random variable with the same distribution as the first failure time X_1. Then, it can easily be proved that the failure intensity is :

$$\lambda_t = \lambda(A_{N_{t-}} + t - T_{N_{t-}}) \tag{6}$$

The *virtual age* of the system at time t is $A_{N_{t-}} + t - T_{N_{t-}}$. A_i is the virtual age just after the i^{th} repair and is called the i^{th} *effective age*. The NHPP is a virtual age model with $\forall i, A_i = T_i$ and the RP is a virtual age model with $\forall i, A_i = 0$.

This paper is dedicated to the Arithmetic Reduction of Age (ARA) class of imperfect repair models, defined in [9], which are particular virtual age models. The Arithmetic Reduction of Age model with memory m, denoted ARA$_m$, is defined by its failure intensity :

$$\lambda_t = \lambda\left(t - \rho \sum_{j=0}^{\min(m-1, N_{t-}-1)} (1-\rho)^j T_{N_{t-}-j}\right) \tag{7}$$

m reflects a Markovian property : it is the maximal number of previous failure times involved in the failure intensity [9]. ρ is a parameter which will be described later.

Two values of m are particularly interesting :

– $m = 1$. The failure intensity of the ARA$_1$ model is :

$$\lambda_t = \lambda\left(t - \rho T_{N_{t-}}\right) \tag{8}$$

In this model, the effect of a repair is to reduce the virtual age just before repair of a quantity proportional to the time elapsed since last repair :

$$A_i = A_{i-1} + X_i - \rho X_i \Longrightarrow A_i = (1-\rho)T_i$$

The ARA_1 model is similar to Kijima's type I model [15] with deterministic repair effects, and is also the same as the Proportional Reduction of Age proposed by Malik [19].

– $m = +\infty$. The failure intensity of the ARA_∞ model is :

$$\lambda_t = \lambda\left(t - \rho \sum_{j=0}^{N_{t-}-1} (1-\rho)^j T_{N_{t-}-j}\right) \tag{9}$$

In this model, the effect of a repair is to reduce the virtual age just before repair of a quantity proportional to this virtual age :

$$A_i = (1-\rho)(A_{i-1} + X_i)$$

The ARA_∞ model is similar to Kijima's type II model [15] with deterministic repair effects, and is also the same as the model proposed by Brown, Mahoney and Sivazlian [4].

In these models, the repair efficiency is measured by the value of ρ. Particular values of ρ are :

– $\rho = 1$: perfect repair (AGAN)
– $0 < \rho < 1$: efficient repair
– $\rho = 0$: minimal repair (ABAO)
– $\rho < 0$: harmful repair

For the sake of simplicity, we will assume that repair cannot be harmful, so $\rho \in [0, 1]$.

In this paper, we derive most of our results under the general framework of the ARA_m model. The interest is that these results will be valid, among others, for both the ARA_1 and ARA_∞ models. For the ARA_1 model, we will be able to compare our results with those obtained by Pan and Rigdon [20]. The results for the ARA_∞ model will be new.

The ARA_∞ model has several advantages compared to the ARA_1 model :

– From an industrial point of view, it has a more natural interpretation.
– In the ARA_1 model, there exists an increasing minimal wear intensity [9]. Thus, the repair effect cannot completely compensate the ageing of the system. This problem does not exist in the ARA_∞ model.
– Monte-Carlo simulations show that the MLE have a better behavior in the ARA_∞ model than in the ARA_1 model [9].

For a point process model with parameter θ (here $\theta = (\alpha, \beta, \rho)$), the general expression of the likelihood function associated with the observation of n failures up to time t of a single repairable system is :

$$L_t(\theta; n, \mathbf{t_n}) = \left[\prod_{i=1}^{n} \lambda_{t_i} \right] \exp\left(- \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} \lambda_s \, ds \right) \tag{10}$$

where $\mathbf{t_n} = (t_1, \ldots, t_n)$ denotes the vector of observed failure times, $t_0 = 0$ and $t_{n+1} = t$. In this situation, the process is time-truncated at time t. For a process failure-truncated after n failures, (10) is still valid by taking $t_{n+1} = t_n$.

For the ARA$_m$ model with PLP initial intensity, the likelihood becomes, with the same kind of notations as in Pulcini [23] :

$$L_t(\alpha, \beta, \rho; n, \mathbf{t_n}) = \alpha^n \beta^n \, V(\rho)^{\beta-1} \exp\left(-\alpha Z(\beta, \rho)\right) \tag{11}$$

where

$$V(\rho) = \prod_{i=1}^{n} \left[t_i - S_i(\rho) \right] ,$$

$$Z(\beta, \rho) = \sum_{i=1}^{n+1} \left[(t_i - S_i(\rho))^{\beta} - (t_{i-1} - S_i(\rho))^{\beta} \right] ,$$

$$S_1(\rho) = 0 \ \text{ and } \ \forall \, i \geq 2, \ S_i(\rho) = \rho \sum_{j=0}^{min(m-1, i-2)} (1-\rho)^j t_{i-j-1}.$$

Let $\hat{\alpha}_t$, $\hat{\beta}_t$ and $\hat{\rho}_t$ be the MLE of α, β and ρ, obtained by maximizing (11). It is easy to show that :

$$\hat{\alpha}_t = \frac{N_t}{Z(\hat{\beta}_t, \hat{\rho}_t)} \tag{12}$$

Numerical optimization procedures are necessary to compute $\hat{\beta}_t$ and $\hat{\rho}_t$.

3 Bayesian inference

In this paper, we consider a single repairable system, which is so reliable that only very few failures occur. Then, the quality of the MLE can be very poor. The Bayesian analysis improves the accuracy of parameter estimations by adding the expert knowledge to operation feedback data. In the Bayesian framework, the parameters are considered as random variables. Their prior distributions reflect the expert knowledge on the system ageing and repair efficiency.

The ARA models have 3 parameters $\alpha > 0$, $\beta > 0$ and $\rho \in [0, 1]$. For a given prior density $\pi_{(\alpha,\beta,\rho)}(\alpha, \beta, \rho)$, the posterior density is obtained by :

$$\pi(\alpha, \beta, \rho | n, \mathbf{t_n}) \propto L_t(\alpha, \beta, \rho; n, \mathbf{t_n}) \, \pi_{(\alpha,\beta,\rho)}(\alpha, \beta, \rho) \tag{13}$$

The choice of prior distributions for this kind of parameters have been discussed in many papers. Non informative priors are used when no particular information is known on the failure and repair process. When such information is available, it is translated into informative priors. In practice, experts give an average behaviour and a degree of uncertainty on this information, which can be converted into prior mean and variance.

Since α and β are the parameters of a PLP, the prior distributions used in the Bayesian analysis of the PLP can be chosen here. Non informative priors for α and β are used by Guida et al [11, 6], Bar-Lev et al [2], and Sen [26]. Informative priors are used either by considering expert knowledge on the failure mechanism, or by using conjugate distributions. For α, the gamma distribution has been used by Kuo and Yang [16], Pievatolo and Ruggeri [22], Pan and Rigdon [20]. For β, the uniform distribution has been used by Guida et al [11], Sen [26] and Pan and Rigdon [20]. The beta distribution has been used by Sen [26].

For the repair efficiency parameter of the Brown-Proschan model, Lim et al [18] used the beta distribution. For the repair efficiency parameters of the ARA_1 and ARI_1 models, Pan and Rigdon [20] used also the beta distribution.

In this paper we will choose informative, non informative and semi informative priors on the model parameters. Informative priors can also be chosen on reliability indicators.

3.1 Informative priors on the parameters

For the scale parameter α, we choose the gamma distribution, denoted $\Gamma(a, b)$, with density function :

$$\pi_\alpha(\alpha) = \frac{b^a}{\Gamma(a)}\, \alpha^{a-1} \exp(-b\alpha), \tag{14}$$

where $\Gamma(a) = \int_0^{+\infty} t^{a-1}\, e^{-t}\, dt$. The expectation and variance of this distribution are respectively a/b and a/b^2. We will see later that this is a conjugate prior for α.

For the shape parameter β, we choose the uniform distribution on $[\beta_1, \beta_2]$, denoted $\mathcal{U}[\beta_1, \beta_2]$:

$$\pi_\beta(\beta) = \frac{1}{\beta_2 - \beta_1}\, \mathbf{1}_{[\beta_1,\beta_2]}(\beta). \tag{15}$$

β is linked to the system ageing in a Weibull framework. So experts are likely to give plausible values for β. β_1 and β_2 can be understood as respectively the lower and upper bounds for these values. It is possible to take $\beta_1 = 1$ in order to assume that, without repair, the system wears out.

For the repair efficiency parameter ρ, since we have assumed that ρ belongs to $[0, 1]$, we choose the beta distribution, denoted $\beta(c, d)$, with density function :

$$\pi_\rho(\rho) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\, \rho^{c-1}(1-\rho)^{d-1}\, \mathbf{1}_{[0,1]}(\rho). \tag{16}$$

The prior expectation is $\dfrac{c}{c+d}$ and the prior variance is $\dfrac{cd}{(c+d)^2(c+d+1)}$.

Finally, the priors are supposed to be independent, so the joint prior density is :

$$\begin{aligned} \pi_{(\alpha,\beta,\rho)}(\alpha,\beta,\rho) &= \pi_\alpha(\alpha)\pi_\beta(\beta)\pi_\rho(\rho) \\ &\propto \alpha^{a-1}\exp(-b\alpha)\,\rho^{c-1}(1-\rho)^{d-1}\,\mathbf{1}_{[\beta_1,\beta_2]}(\beta) \end{aligned} \tag{17}$$

3.2 Non informative priors

The usual non informative priors for α and β in the PLP case are :

$$\pi_\alpha(\alpha) \propto \frac{1}{\alpha} \text{ and } \pi_\beta(\beta) \propto \frac{1}{\beta} \tag{18}$$

For non harmful repair, ρ belongs to $[0,1]$. So a non informative prior in this case is the uniform distribution $\mathcal{U}[0,1]$.

Then, with the independence of priors, the joint non informative prior is :

$$\pi_{(\alpha,\beta,\rho)}(\alpha,\beta,\rho) \propto \frac{1}{\alpha\beta} \tag{19}$$

3.3 Unified priors

Following Pulcini [23], we can unify the informative and non informative cases by making the following remarks.

- For ρ, the $\mathcal{U}[0,1]$ distribution is the same as the $\beta(1,1)$ distribution, so it is a particular case of the $\beta(c,d)$ distribution.
- For α, the non informative prior is a limit case of the $\Gamma(a,b)$ distribution, where a and b tend to zero.
- For β, the $\mathcal{U}[\beta_1,\beta_2]$ density and the non informative prior can both be written :
$$\pi_\beta(\beta) \propto \beta^{-\gamma}\,\mathbf{1}_{[\beta_1,\beta_2]}(\beta) \tag{20}$$
where $\gamma = 0$ corresponds to the uniform distribution, and $\gamma = 1$, $\beta_1 = 0$ and $\beta_2 = +\infty$ corresponds to the non informative prior.

So we can define a unified prior as :

$$\pi_{(\alpha,\beta,\rho)}(\alpha,\beta,\rho) \propto \alpha^{a-1}\exp(-b\alpha)\,\beta^{-\gamma}\rho^{c-1}(1-\rho)^{d-1}\,\mathbf{1}_{[\beta_1,\beta_2]}(\beta) \tag{21}$$

The informative prior (17) is obtained for $\gamma = 0$ and the non informative prior (19) is obtained for $a = b = 0$, $\gamma = 1$, $c = d = 1$, $\beta_1 = 0$ and $\beta_2 = +\infty$.

It is also possible to use semi-informative priors. Indeed, the 3 parameters of the ARA models have well identified meanings. β is linked to the intrinsic ageing, ρ characterizes repair efficiency and α is a scale parameter. In practice, it is much easier to obtain prior information on β and ρ than on α. So one can consider, as in Pulcini [23], to use the non informative prior for α and the informative priors for β and ρ. This can be done by taking only $a = b = 0$ in (21).

3.4 Posterior distributions

For the unified prior, the joint posterior distribution is derived using (13), (11) and (21). The posterior density of (α, β, ρ) is :

$$\begin{aligned} \pi(\alpha, \beta, \rho | n, \mathbf{t_n}) \quad \propto \quad & \alpha^{n+a-1} \exp\left(-[b + Z(\beta, \rho)]\alpha\right) \\ & \beta^{n-\gamma} V(\rho)^{\beta-1} \rho^{c-1} (1-\rho)^{d-1} \, \mathbf{1}_{[\beta_1, \beta_2]}(\beta) \end{aligned} \tag{22}$$

The conditional posterior densities of each parameter can also be derived as :

$$\pi(\alpha | \beta, \rho, n, \mathbf{t_n}) \quad \propto \quad \alpha^{n+a-1} \exp\left(-[b + Z(\beta, \rho)]\alpha\right) \tag{23}$$

$$\pi(\beta | \alpha, \rho, n, \mathbf{t_n}) \quad \propto \quad \beta^{n-\gamma} V(\rho)^{\beta-1} \exp\left(-\alpha Z(\beta, \rho)\right) \mathbf{1}_{[\beta_1, \beta_2]}(\beta) \tag{24}$$

$$\pi(\rho | \alpha, \beta, n, \mathbf{t_n}) \quad \propto \quad V(\rho)^{\beta-1} \exp\left(-\alpha Z(\beta, \rho)\right) \rho^{c-1} (1-\rho)^{d-1} \tag{25}$$

(23) proves that the posterior distribution of α given β and ρ is the gamma distribution $\Gamma\left(n + a, b + Z(\beta, \rho)\right)$. Then the gamma distribution is a conjugate prior for α.

Because the parameter (α, β, ρ) is multidimensional and there are no conjugate priors for β and ρ, the posterior distributions will be derived by means of the Gibbs sampling algorithm, with Metropolis-Hastings steps [24]. This Markov Chain Monte-Carlo (MCMC) algorithm produces samples from the joint posterior distribution, from which it is possible to compute all the features of the posterior distributions. For instance, the posterior mean, median and mode are point Bayesian estimators of the parameters. With the posterior distribution, it is also possible to give credibility intervals for the parameters.

We propose two versions of the algorithm, one for which a prior is proposed for α, and one for which α is derived from (12).

Algorithm 1 (Algorithm with a prior for α) .

1. $k \leftarrow 0$. *Choose initial values of* α^0, β^0 *and* ρ^0.
2. $k \leftarrow k + 1$.
3. *Sample* α^k *given* β^{k-1} *and* ρ^{k-1} *from (23).*
4. *Sample* β^k *given* α^k *and* ρ^{k-1} *from (24).*
5. *Sample* ρ^k *given* α^k *and* β^k *from (25).*
6. *While* $k < K$, *go to step 2.*

Since it is not possible to sample directly from (24) and (25) we use the Metropolis-Hastings algorithm in steps 4 and 5. For instance, in step 4, we sample $\beta^\star$ from an instrumental distribution easy to simulate, with density q. Let

$$r = \frac{\pi(\beta^\star | \alpha^k, \rho^{k-1}, n, \mathbf{t_n}) \, q(\beta^{k-1} | \beta^\star)}{\pi(\beta^{k-1} | \alpha^k, \rho^{k-1}, n, \mathbf{t_n}) \, q(\beta^\star | \beta^{k-1})} \tag{26}$$

Then, $\beta^k \leftarrow \beta^\star$ with probability $\min(r,1)$ and $\beta^k \leftarrow \beta^{k-1}$ otherwise. For a symmetric transition, $q(\beta^{k-1}|\beta^\star) = q(\beta^\star|\beta^{k-1})$, so (26) becomes

$$r = \frac{\pi(\beta^\star|\alpha^k, \rho^{k-1}, n, \mathbf{t_n})}{\pi(\beta^{k-1}|\alpha^k, \rho^{k-1}, n, \mathbf{t_n})} \tag{27}$$

Here we take for q a Gaussian distribution centered on the given value of β.

As said before, it is not easy to give a prior on the scale parameter α. A first possibility is to choose a non informative prior by taking $a = b = 0$ in (21). Another possibility is to use the fact that, in (12), the MLE is computed as a function of the MLE of β and ρ. Then, we propose a second version of the algorithm, for which priors are given only for β and ρ, and α is computed as a function of β and ρ, according to (12).

Algorithm 2 (Algorithm with no prior for α) .

1. *$k \leftarrow 0$. Choose initial values of β^0 and ρ^0.*
2. *Compute α^0 from (12).*
3. *$k \leftarrow k+1$.*
4. *Sample β^k given α^{k-1} and ρ^{k-1} from (24).*
5. *Sample ρ^k given α^{k-1} and β^k from (25).*
6. *Compute α^k as a function of β^k and ρ^k from (12).*
7. *While $k < K$, go to step 3.*

3.5 Informative prior on reliability indicators

Giving a prior is facilitated if the considered quantity has a clear physical meaning, such as a reliability indicator. In the PLP case, Guida et al [11, 6] and Sen [26] proposed a gamma prior for the expected number of failures $E[N_t]$. With an informative prior on β, it is possible to derive a prior for (α, β).

For the ARA model, the mean number of failures takes into account both the intrinsic ageing (through α and β) and repair efficiency (through ρ). Since these parameters can have compensating effects, it is not so easy to give a prior value for $E[N_t]$. Then, we propose to give a prior on an indicator which is linked only to the intrinsic ageing, the mean time to the first failure :

$$\mu = E[T_1] = \frac{1}{\alpha^{1/\beta}}\Gamma(1 + \frac{1}{\beta}) \tag{28}$$

Since α appears at the denominator of μ, we choose an inverse gamma prior, denoted $\mathrm{I}\Gamma(A, B)$, for μ. Then, the prior density is :

$$\pi_\mu(\mu) = \frac{B^A}{\Gamma(A)}\,\mu^{-A-1}\exp\left(-\frac{B}{\mu}\right) \tag{29}$$

For $A > 2$, the expectation and variance of this distribution are respectively $\dfrac{B}{A-1}$ and $\dfrac{B^2}{(A-1)^2(A-2)}$.

The conditional prior density of α is derived thanks to a change of variables :

$$\pi(\alpha|\beta) = \frac{B^A\,\alpha^{A/\beta-1}}{\Gamma(A)\,\Gamma(1+\frac{1}{\beta})^A}\,\exp\left(-\frac{B\alpha^{1/\beta}}{\Gamma(1+\frac{1}{\beta})}\right) \tag{30}$$

If we keep the informative priors $\mathcal{U}[\beta_1,\beta_2]$ for β and $\beta(c,d)$ for ρ, the joint prior distribution is :

$$\begin{aligned}\pi_{\alpha,\beta,\rho}(\alpha,\beta,\rho) &= \pi(\alpha|\beta)\,\pi(\beta,\rho) \qquad (31)\\ &\propto \frac{\alpha^{A/\beta-1}}{\beta\,\Gamma(1+\frac{1}{\beta})^A}\,\exp\left(-\frac{B\alpha^{1/\beta}}{\Gamma(1+\frac{1}{\beta})}\right)\rho^{c-1}(1-\rho)^{d-1}\,\mathbf{1}_{[\beta_1,\beta_2]}(\beta),\end{aligned}$$

and the joint posterior distribution is :

$$\begin{aligned}\pi(\alpha,\beta,\rho|n,\mathbf{t_n}) \;\propto\; & \frac{\alpha^{n+A/\beta-1}}{\Gamma(1+\frac{1}{\beta})^A}\,\beta^{n-1}\,\rho^{c-1}(1-\rho)^{d-1}V(\rho)^{\beta-1}\\ & \exp\left(-\alpha Z(\beta,\rho)-\frac{B\,\alpha^{1/\beta}}{\Gamma(1+\frac{1}{\beta})}\right)\mathbf{1}_{[\beta_1,\beta_2]}(\beta).\end{aligned} \tag{32}$$

3.6 Posterior distributions of reliability indicators

With the posterior distribution of the parameters, it is possible to determine the posterior distribution of any quantity of interest, function of these parameters, such as reliability indicators. Following Pulcini [23], we derive the posterior distribution of the failure intensity and the expected number of failures.

Ce serait bien de rajouter un calcul de MTTF prédictif, estimateur bayésien de la durée moyenne d'attente de la prochaine panne.

3.6.1 Posterior distribution of the failure intensity

The failure intensity at a time $\tau \in]t_i, t_{i+1}]$ is

$$\lambda_\tau = \alpha\beta[\tau - S_{i+1}(\rho)]^{\beta-1}.$$

Thanks to a change of variables in the joint posterior distribution for the unified prior (22), the posterior distribution of λ_τ is found to be :

$$\begin{aligned}\pi(\lambda_\tau|n,\mathbf{t_n}) \propto \int_{\beta_1}^{\beta_2}\int_0^1 \quad & \frac{V(\rho)^{\beta-1}\,\rho^{c-1}(1-\rho)^{d-1}}{\beta^{a+\gamma}[\tau-S_{i+1}(\rho)]^{(n+a)(\beta-1)}}\,\lambda_\tau^{n+a-1}\\ & \exp\left(-\frac{b+Z(\beta,\rho)}{\beta[\tau-S_{i+1}(\rho)]^{\beta-1}}\,\lambda_\tau\right)d\rho\,d\beta\end{aligned} \tag{33}$$

The Bayesian point estimator of λ_τ is the posterior mean :

$$E[\lambda_\tau|n,\mathbf{t_n}] = C\int_{\beta_1}^{\beta_2}\int_0^1 \frac{V(\rho)^{\beta-1}\,\rho^{c-1}(1-\rho)^{d-1}\,\beta^{n-\gamma+1}[\tau-S_{i+1}(\rho)]^{\beta-1}}{[b+Z(\beta,\rho)]^{n+a+1}}\,d\rho\,d\beta \tag{34}$$

where C is a normalizing constant. In practice, the integrals involved in (34) are computed by Monte Carlo simulations.

Credibility intervals for λ_τ can also be obtained from (33). Then, computing these values for each τ, it is possible to draw a plot of the Bayesian point and interval estimators of the failure intensity.

3.6.2 Posterior distribution of the expected number of failures

A usual result of the theory of point processes is that the expected number of failures at any time τ is linked to the failure intensity by :

$$E[N_\tau] = E\left[\int_0^\tau \lambda_s\,ds\right] \tag{35}$$

Having observed failure times $t_1 < \ldots < t_n$, for $\tau \in]t_i, t_{i+1}]$, $1 \le i \le n$, we have :

$$\Lambda_\tau = \int_0^\tau \lambda_s\,ds = \sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\lambda_s\,ds + \int_{t_i}^{\tau}\lambda_s\,ds \tag{36}$$

For the ARA models, we obtain :

$$\Lambda_\tau = \alpha Z_{i,\tau}(\beta,\rho) \tag{37}$$

where

$$Z_{i,\tau}(\beta,\rho) = \sum_{j=1}^{i}\left[(t_j - S_j(\rho))^\beta - (t_{j-1}-S_j(\rho))^\beta\right] + (\tau - S_{i+1}(\rho))^\beta - (t_i - S_{i+1}(\rho))^\beta \tag{38}$$

Of course, $Z_{n,t}(\beta,\rho) = Z(\beta,\rho)$.

Then, it is possible to make a Bayesian prediction of the expected number of failures at time τ by computing the posterior distribution of Λ_τ. For a given data set, comparing the prediction with the observed value $N_\tau = i$ will give an indication of the goodness-of-fit of the ARA model.

Note that the expression of Λ_τ is similar to that of $\lambda_\tau = \alpha\beta[\tau - S_{i+1}(\rho)]^{\beta-1}$. In fact, the results for Λ_τ can be derived from those on λ_τ by replacing $\beta[\tau - S_{i+1}(\rho)]^{\beta-1}$ by $Z_{i,\tau}(\beta,\rho)$.

Then, the posterior distribution of Λ_τ is :

$$\pi(\Lambda_\tau|n,\mathbf{t_n}) \propto \int_{\beta_1}^{\beta_2}\int_0^1 \frac{V(\rho)^{\beta-1}\,\rho^{c-1}(1-\rho)^{d-1}\,\beta^{n-\gamma}}{Z_{i,\tau}(\beta,\rho)^{n+a}}\,\Lambda_\tau^{n+a-1} \exp\left(-\frac{b+Z(\beta,\rho)}{Z_{i,\tau}(\beta,\rho)}\,\Lambda_\tau\right)d\rho\,d\beta \tag{39}$$

and the Bayesian point estimator of Λ_τ is the posterior mean :

$$E[\Lambda_\tau|n,\mathbf{t_n}] = C\int_{\beta_1}^{\beta_2}\int_0^1 \frac{V(\rho)^{\beta-1}\,\rho^{c-1}(1-\rho)^{d-1}\,\beta^{n-\gamma}Z_{i,\tau}(\beta,\rho)}{[b+Z(\beta,\rho)]^{n+a+1}}\,d\rho\,d\beta \tag{40}$$

where C is a normalizing constant.

For $\tau = t$ and $i = n$, $Z_{n,t}(\beta,\rho) = Z(\beta,\rho)$. With the non informative prior for α ($a = b = 0$), it is easy to check that $E[\Lambda_t|n,\mathbf{t_n}] = n$.

4 Simulation results

4.1 Comparison of MLE and Bayesian estimators

In this subsection, we compare the maximum likelihood and Bayesian estimators by means of Monte-Carlo simulations. We simulate 200 samples of size $n \in \{5, 10, 25\}$ of both ARA_1 and ARA_∞ models, with parameters equal to $\alpha = 0.01$, $\beta = 3$ and $\rho = 0.6$. These values are chosen in order to reflect the behaviour of a system with important ageing ($\beta = 3$), good repair efficiency ($\rho = 0.6$) and a mean time of the first failure around 4 ($E[T_1] = 4.14$).

For each sample, we compute the MLE and several Bayesian estimators of each parameter, corresponding to different choices of priors.

The prior for β is non informative ($\gamma = 1, \beta_1 = 0, \beta_2 = +\infty$) or uniform on $[1, 4]$ ($\gamma = 0, \beta_1 = 1, \beta_2 = 4$). This is a rather good prior since the prior mean is 2.5 and the prior standard deviation is 0.87.

Three priors are chosen for ρ : the non informative ($c = d = 1$), a good one ($c = 1.652, d = 0.708$ with a prior mean equal to 0.7) and a bad one ($c = 0.708, d = 1.652$ with a prior mean equal to 0.3). For both informative priors, the standard deviation is 0.25.

For α, we consider several possibilities :

- The non informative prior : $a = b = 0$.
- A good informative prior : $a = 2.25, b = 150$, with a prior mean equal to 0.015 and a prior standard deviation equal to 0.01.
- The estimator of α is computed as a function of the estimators of β and ρ, according to (12).
- The prior for α given β is computed from a prior on $E[T_1]$, according to (30). The prior for $E[T_1]$ is inverse gamma with $A = 8.25$ and $B = 36.25$, leading to a prior mean equal to 5 and a prior standard deviation equal to 2.

The posterior distributions are derived by the MCMC algorithms with $K = 10000$ iterations. In the algorithms, the inital values of the parameters are the MLE. The Bayesian point estimators are the posterior means of the last 1000 obtained values. The empirical means and variances of the 200 samples are computed for each estimator. Figures 1 to 6 give the corresponding mean square errors (MSE) as functions on the sample size.

In these figures, the following notations are adopted.

- MLE : maximum likelihood estimator.
- NI : non informative prior on each parameter : $a = b = 0$, $\gamma = 1$, $\beta_1 = 0, \beta_2 = +\infty$, $c = d = 1$.
- GP : good prior for ρ, prior for α : $a = 2.25, b = 150$, $\gamma = 0$, $\beta_1 = 1, \beta_2 = 4$, $c = 1.652, d = 0.708$.
- GC : good prior for ρ, α computed : $\gamma = 0$, $\beta_1 = 1, \beta_2 = 4$, $c = 1.652, d = 0.708$.
- GNI : good prior for ρ, non informative for α : $a = b = 0$, $\gamma = 0$, $\beta_1 = 1, \beta_2 = 4$, $c = 1.652, d = 0.708$.
- GT1 : good prior for ρ, prior on $E[T_1]$: $\gamma = 0$, $\beta_1 = 1, \beta_2 = 4$, $c = 1.652, d = 0.708$, $A = 8.25, B = 36.25$.
- BP : bad prior for ρ, prior for α : $\gamma = 0$, $\beta_1 = 1, \beta_2 = 4$, $c = 0.708, d = 1.652$.
- BC : bad prior for ρ, α computed : $\gamma = 0$, $\beta_1 = 1, \beta_2 = 4$, $c = 0.708, d = 1.652$.
- BNI : bad prior for ρ, non informative for α : $a = b = 0$, $\gamma = 0$, $\beta_1 = 1, \beta_2 = 4$, $c = 0.708, d = 1.652$.
- BT1 : bad prior for ρ, prior on $E[T_1]$:$\gamma = 0$, $\beta_1 = 1, \beta_2 = 4$, $c = 0.708, d = 1.652$, $A = 8.25, B = 36.25$.

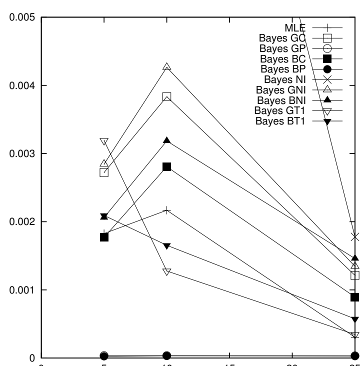

FIGURE 1 – ARA_1 - MSE for α

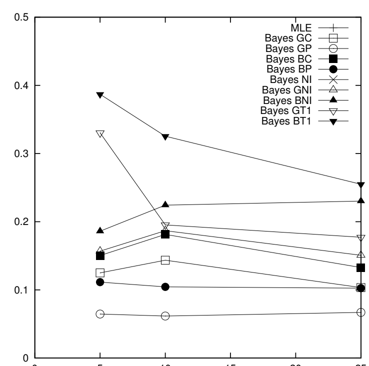

FIGURE 2 – ARA_1 - MSE for β

The behaviour of the estimators is essentially similar for the ARA_1 and ARA_∞ models. The decrease of the MSE is not obvious. The worst estimator is the non informative. The MLE overestimates strongly β, but is correct for α and ρ. For the non informative and ML estimators of β, the MSE is so large

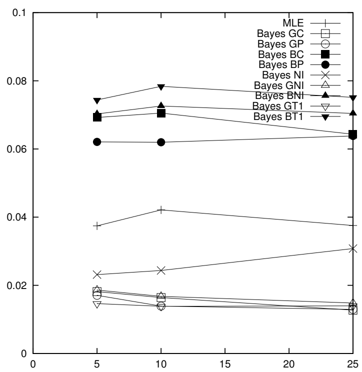

FIGURE 3 – ARA_1 - MSE for ρ

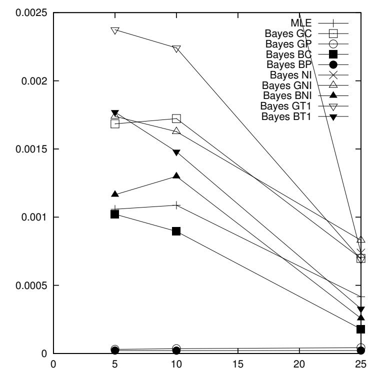

FIGURE 4 – ARA_∞ - MSE for α

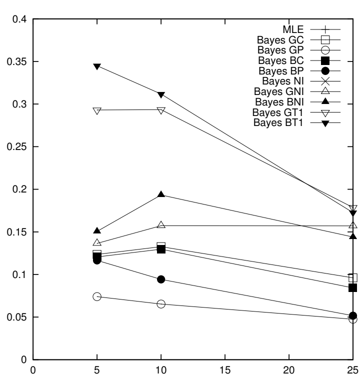

FIGURE 5 – ARA_∞ - MSE for β

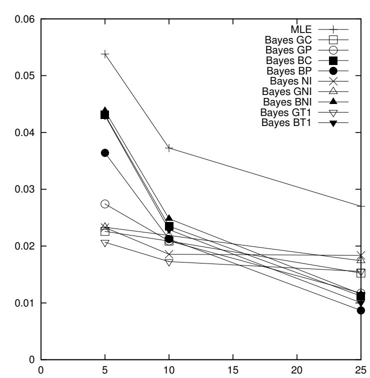

FIGURE 6 – ARA_∞ - MSE for ρ

that it has been removed from figures 2 and 5.

Unsurprisingly, the estimators are better when the prior on ρ is good (G) than when it is bad (B).

The comparison of the ways to estimate α lead to the logic conclusion that the best estimators are obtained for a good informative prior (P). The three other estimators have similar behaviours, which slightly differ depending on the model and parameters. The estimator with a prior on $E[T_1]$ (T1) needs to know such a prior. The estimator with a non informative prior on α (NI) needs the MCMC algorithm on 3 parameters. The estimator with α computed (C) does not need any kind of knowledge on α and is simpler to compute than NI because the MCMC algorithm involves only two parameters. Then, in case that no real knowledge is known on α, the estimator with α computed is a good choice.

For the estimation of ρ in the ARA_1 model, the prior has a strong impact. We can see in figure 3 that the MSE of all the estimators with good prior are much lower than the MSE of all the estimators with bad prior. For small n, this result is still true but much weaker for the ARA_∞ model. Then, the Bayesian estimation of ρ is more accurate in the ARA_∞ model than in the ARA_1 model.

Finally, the Bayesian approach will be particularly profitable for the estimation of β. Of course, the greater is the knowledge on the parameters, the better are the estimations. But Bayesian estimators are still satisfactory even with a bad prior.

4.2 Study of one data set

In practice, one have to analyze one data set made of n successive failure times of a repairable system. The first objective is to estimate the parameters of ARA models and reliability indicators. Thanks to the Bayesian analysis, it is also possible to compute credibility intervals on these quantities. We can also compare prior and posterior distributions. In this subsection, we apply this approach first to data simulated from the ARA_1.In the next section, it will be applied to a real data set on AMC Ambassador cars.

4.2.1 ARA_1 model

Table 1 presents $n = 5$ failure times simulated from the ARA_1 model with parameters $\alpha = 0.01$, $\beta = 3$ and $\rho = 0.6$.

6.17	8.30	9.20	10.47	12.84

TABLE 1 – Simulated data set from the ARA_1 model

Tables 2 to 4 present point estimates of the parameters α, β and ρ. The considered estimators are the MLE and the Bayesian GP, GNI, BNI (*à remplacer par GC et BC*). For the Bayesian estimators, the 80% credibility intervals and posterior standard deviations are also given.

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	0.00003			
GP	0.0123	0.00360	0.0236	0.000074
GNI	0.0277	0.00095	0.0718	0.004199
BNI	0.0122	0.00053	0.0357	0.000728

Table 2 – ARA_1 - Estimation of α

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	5.869			
GP	2.921	2.344	3.572	0.216
GNI	2.954	1.826	3.825	0.570
BNI	3.070	2.030	3.829	0.429

Table 3 – ARA_1 - Estimation of β

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	0.554			
GP	0.605	0.339	0.821	0.0368
GNI	0.538	0.242	0.818	0.0459
BNI	0.317	0.040	0.629	0.0461

Table 4 – ARA_1 - Estimation of ρ

The MLE of α and β are very far from the true values, but the MLE of ρ is quite good. The GP estimates (good priors for α and ρ) are always good, in mean and variance, as well as for the length of credibility intervals. The GNI is a good estimate for β and ρ, but surprisingly not for α. *C'est bizarre que le GNI soit bien moins bon que le BNI pour α.* The BNI is a good estimate for α and β. For ρ, the BNI is highly biased, because it is close to the (bad) prior mean.

Figures 7 and 8 compare the prior density (solid line) and the posterior density (histogram from the MCMC samples) of the GNI and BNI estimators of ρ. In both cases, the posterior is brought closer to the MLE. *On pourrait avoir plus convaincant, avec un a posteriori proche d'un bon a priori et loin d'un mauvais a priori.*

Finally, the Bayesian estimates of the failure intensity and expected number of failures are plotted in figures 9 and 10. The solid blue line represents the GNI Bayesian point estimate, the credibility limits are in dotted blue lines. The true value and maximum likelihood estimates of the functions are respectively plotted in red and green. We can see that for $t = t_n$, the ML and Bayesian estimates of the functions are equal to n. For MLE, this property comes from our choice to estimate α, as a function of the other parameters.

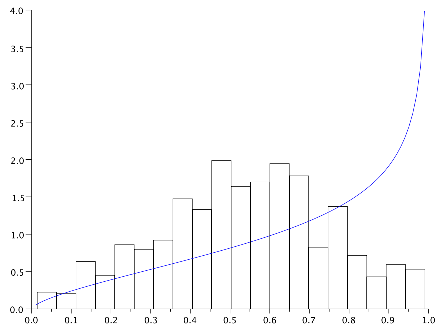

FIGURE 7 – GNI - Comparison of prior and posterior densities

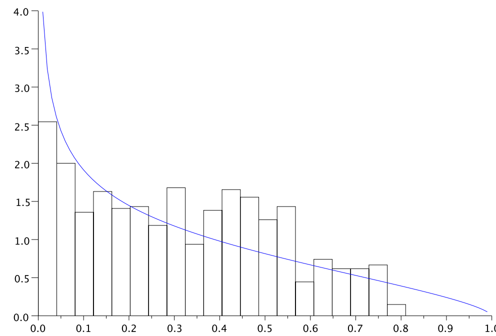

FIGURE 8 – BNI - Comparison of prior and posterior densities

These figures show that the Bayesian estimate provides a better fit than the MLE : the blue line is closer to the red line than the green line. The real intensity and expected number of failures are always within the credibility limits.

5 Application to automobile data

In this section, we present an application to automobile data studied by Ahn et al [1] and Guida and Pulcini [13]. The data given in table 5 are $n = 18$ failure times of an AMC Ambassador car owned by the Ohio state government.

202	265	363	508	571	755	770	818	868
999	1054	1068	1108	1230	1268	1330	1376	1447

TABLE 5 – AMC Ambassador data

The maximum likelihood estimates of the parameters for the ARA_1 and ARA_∞ models are given in table 6. The maximum log-likelihood is also given.

	$\hat{\alpha}_t$	$\hat{\beta}_t$	$\hat{\rho}_t$	max log-likelihood
ARA_1	$1.30\,10^{-7}$	3.102	0.898	-91.996
ARA_∞	$2.12\,10^{-9}$	3.583	0.246	-92.678

TABLE 6 – AMC data - Maximum Likelihood Estimates

The values of the maximum log-likelihoods indicate that ARA_1 may be a better model than ARA_∞ for these data. The estimates of β are close to 3,

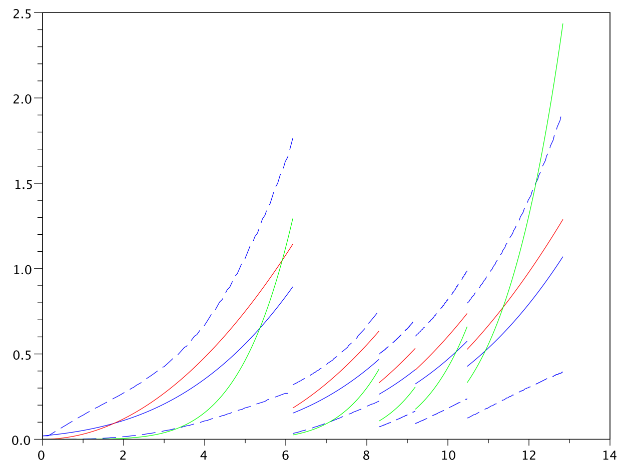

FIGURE 9 – ARA_1 - Intensity

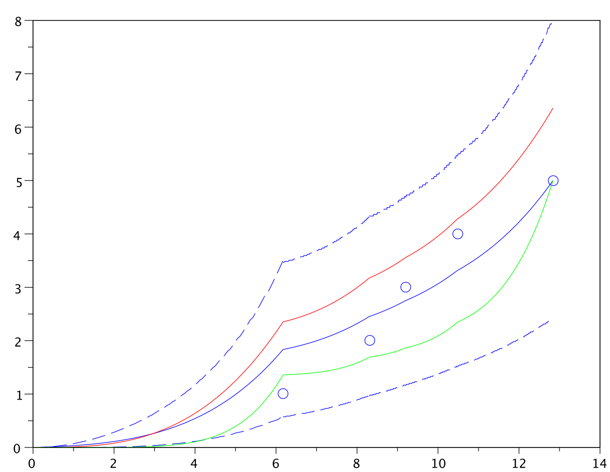

FIGURE 10 – ARA_1 - Expected number of failures

indicating a strong intrinsic wear-out. The estimates of α and ρ in the two models are very different. It is not surprising since the models have different meanings (considerations of that kind are given in [9]). Anyway, the estimates of ρ indicate a good repair efficiency.

Now we will perform the Bayesian analysis of these data.

We keep the same notation for the Bayesian estimators as in the simulation section. For α, we consider several possibilities :

- The non informative prior : $a = b = 0$.
- An informative prior : $a = 0.01$, $b = 100000$, with a prior mean equal to 10^{-7} (close to the ML estimation) and a prior standard deviation equal to 10^{-6}.
- The estimator of α is computed as a function of the estimators of β and ρ, according to (12).

For β and ρ, we choose the same prior distribution as in simulation section. Tables 7 to 12 present point estimates and the 80% credibility intervals and posterior standard deviations of the parameters α, β and ρ for ARA_1 and ARA_∞ models. The GNI estimates correspond to a Beta distribution with prior mean equal to 0.7 and 0.3 respectively for ARA_1 and ARA_∞ models (prior means close to MLE in both cases).

	Point estimate	Lower limit	Upper limit	Standard deviation
GC	5.05310^{-5}	6.23110^{-9}	7.46010^{-5}	2.28910^{-4}
GP	3.90710^{-6}	9.53910^{-7}	7.86810^{-6}	2.96610^{-6}
GNI	3.05010^{-7}	1.20610^{-9}	6.73410^{-7}	2.78010^{-7}

TABLE 7 – ARA_1 - Estimation of α for AMC data

	Point estimate	Lower limit	Upper limit	Standard deviation
GC	2.820	1.910	3.625	0.635
GP	2.442	2.204	2.611	0.134
GNI	3.104	2.718	3.894	0.418

TABLE 8 – ARA_1 - Estimation of β for AMC data

	Point estimate	Lower limit	Upper limit	Standard deviation
GC	0.845	0.694	0.942	0.129
GP	0.813	0.624	0.932	0.130
GNI	0.866	0.800	0.928	0.054

TABLE 9 – ARA_1 - Estimation of ρ for AMC data

	Point estimate	Lower limit	Upper limit	Standard deviation
GC	$5.996 10^{-5}$	$9.344 10^{-10}$	$7.284 10^{-5}$	$3.104 10^{-4}$
GP	$4.454 10^{-7}$	$3.541 10^{-8}$	$1.349 10^{-6}$	$7.988 10^{-7}$
GNI	$4.123 10^{-5}$	$4.828 10^{-6}$	$1.091 10^{-4}$	$4.792 10^{-5}$

TABLE 10 – ARA_∞ - Estimation of α for AMC data

	Point estimate	Lower limit	Upper limit	Standard deviation
GC	2.820	1.899	3.721	0.666
GP	2.874	2.672	3.074	0.149
GNI	2.075	1.824	2.313	0.200

TABLE 11 – ARA_∞ - Estimation of β for AMC data

	Point estimate	Lower limit	Upper limit	Standard deviation
GC	0.3241814	0.1166052	0.5804992	0.1813819
GP	0.3131733	0.1526346	0.5364926	0.1534005
GNI	0.3561041	0.1090298	0.6747647	0.2117258

TABLE 12 – ARA_∞ - Estimation of ρ for AMC data

The Bayesian estimates of ρ are quite similar in all cases and close to MLE.

For ARA_1 model, the Bayesian estimates of β and ρ indicates a wear out of the system with efficient corrective maintenances. For this model, we can note that the GNI estimates of α and β are very close to MLE.

For ARA_∞ model, the Bayesian estimates are quite similar to ML estimates, excepted for GNI ones. The GNI estimation of ρ is close to prior mean, which implies a great difference for the estimation of β, compared to the MLE.

As ARA_1 model seems to be more adequate for these real data, we present in figure 11 and 12 respectively the Bayesian (GNI) intensity and expected number of failures.

In these figures, the Bayesian intensity and expected number of failures are very close to MLE estimates. Moreover, we can see that all failures are within the bounds of credibility interval.

6 Conclusion and future work

Parler de la généralisation à CM+PM avec référence à Pulcini (2000) et Sheu et al (2001) en disant que ces auteurs n'ont traité que le cas CM ABAO.

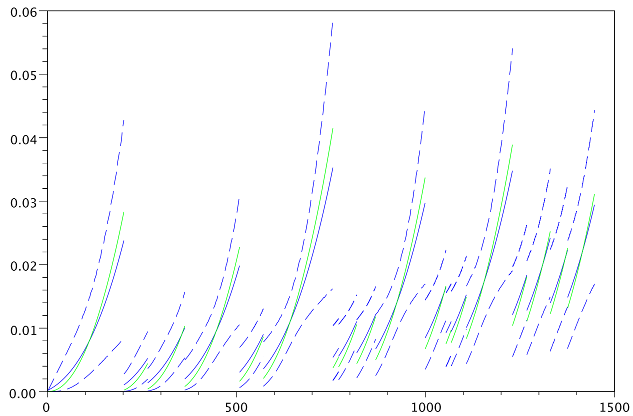

Figure 11 – ARA_1 - Intensity for AMC data

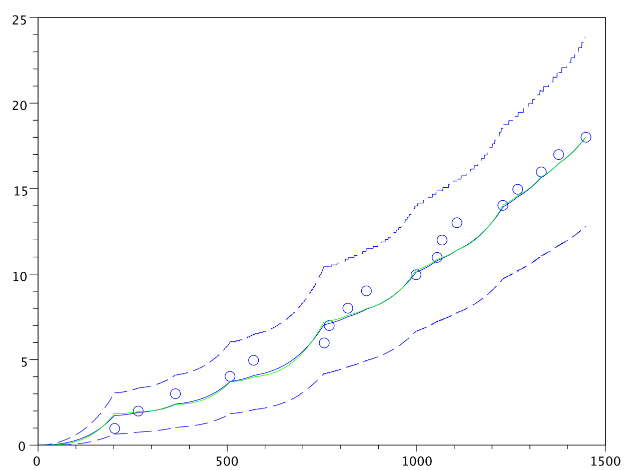

Figure 12 – ARA_1 - Expected number of failures for AMC data

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