# UNIT-4

## Elementary combinatorics (permutation & combinations)

**combinatorics :-** combinatorics is an important part of discreate mathematics that solves counting problems without actually enumerating all possible cases.

combinatorics deals with counting the number of ways of arranging or choosing objects from a finite set according to certain specified rules.

In other words, combinatorics is concerned with problems of permutations and combinations

**permutations :-** (arrangement purpose)

An ordered arrangement of $r$ elements of a set containing '$n$' distinct elements is an $r$-permutation of $n$ elements and is denoted by $p(n,r)$ or $^nP_r$ where $r \leq n$

$$p(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

$$\boxed{\begin{aligned} p(n,r) &= \frac{n!}{(n-r)!} \\ p(n,n) &= n! \end{aligned}}$$

combinatorys: An ordered selection of r elements of a set containing n distinct elements is called an r-combination of n elements and is denoted by $C(n,r)$ (or) $nC_r$ (or) $\binom{n}{r}$

$$\boxed{\begin{aligned} \therefore\ C(n,r) &= \frac{n!}{r!(n-r)!} \\ C(n,n) &= 1 \end{aligned}}$$

permutation: An arrangement in a sequence of elements of a set of is called permutation of elements.

$$nP_r = \frac{n!}{(n-r)!}$$

problems:-

① How many 4 digit numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digit allowed.

sol:- Since repetition is allowed, each of the 4 places in a 4 digit numbers can be filled up in 4 ways by

The given 4 digits

| 4 ways | 4 ways | 4 ways | 4 ways |
|---|---|---|---|
| Thousan | Hundreds | Tens | unit |

The required no. of 4 digit numbers

$= n^r \Rightarrow 4^4 \Rightarrow 4 \times 4 \times 4 \times 4$

$= 256$

## Example problems on permutations

① <u>Number</u> of <u>permutations</u> of <u>n distinct objects</u> (without <u>duplication</u>):

The number of different arrangements (permutations) of n distinct objects, taken all at a time is

$$p(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

$$\boxed{p(n,n) = n!}$$

② <u>number</u> of <u>permutations</u> of <u>r objects</u> <u>among</u> <u>n distinct</u> <u>objects</u>:

Suppose we are given n distincts and wish to arrange r of the objects denoted by

$$p(n,r) = \frac{n!}{(n-r)!} \qquad {}^nP_r = P(n,r)$$

(3) Number of permutations ~~of~~ n ~~distincts~~ objects (with duplication) :-

It is required to find the no of permutations that can be formed from a collection of n objects of which $n_1$ are of one type, $n_2$ are of a second type, ....., $n_k$ are of $k^{th}$ type with $n_1 + n_2 + \cdots + n_k = n$ objects then the no of permutations of n objects are

$$\frac{n!}{n_1! * n_2! * \cdots * n_k!}$$

(4) circular permutation :- permutations in a circle are ~~asked~~ called circular permutation.

the tot no. of ways of arranging the n persons in a circle $= (n-1)!$

Here $\because$ n is the no. of permutations problems

(1) How many ways are there to sit 10 boys and 10 girls around a circle circular table?

sol :- Here 10 Boys & 10 girls are sit around a ~~circle~~ circular table.

Tot no. of persons $= 10+10 = 20$

$n = 20$

the tot no. of ways of circular permutations are $(n-1)!$

$= (20-1)! \Rightarrow 19! \Rightarrow 19!$

(2) How many ways are there 3 persons sit around a round table?

sol :- No. of persons $n = 3$

The tot no. of ways of arranging 3 persons around a round table is

$(n-1)! \Rightarrow (3-1)! \Rightarrow 2! \Rightarrow 2$

(3) How many different arrangements of letters in the word BOUGHT?

sol :- The given word ~~conta~~ BOUGHT contains 6 letters that are distinct (without repitition)

The tot no. of arrangements of letters in the word BOUGHT $= p(n,n) = n!$

$= 6!$

$= 6\times5\times4\times3\times2\times1$

$= 720$

(4) How many different strings of length 4 can be formed using the letters of the word PROBLEM?

sol :- the given word PROBLEM has 7 letters $\therefore n=7$

The no. of different strings of length 4 can be formed by using the letters of the world PROBLEM =

$p(n,r) = \frac{n!}{(n-r)!}$ $p(7,4) = \frac{7!}{(7-4)!} = \frac{7!}{3!}$

$= \frac{7\times6\times5\times4\times3!}{3!}$

$= 840$

(5) How many words of three distinct letters can be formed from the letters of the word PASCAL.

sol :- the given word PASCAL contains 6 letters

$\therefore n = 6$

$p(n,r) = p(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!}$

$= \frac{6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}} = 120$ ways.

∴ the no. of ways 3 distinct letters can be formed by using the letters of the word ~~pas~~ PASCAL = 120 ways.

(6) Find the no. of permutations of the letters of the word ENGINEERING.

Sol :- The given word ENGINEERING has 11 letters out of which

$E = 3 \ (n_1)$
$N = 3 \ (n_2)$
$I = 2 \ (n_3)$
$G = 2 \ (n_4)$
$R = 1 \ (n_5)$

∴ total no. of permutations of the letters of the word ENGINEERING =

$$= \frac{n!}{n_1! * n_2! * n_3! * n_4! * n_5!}$$

$$= \frac{11!}{3! \times 3! \times 2! \times 2! \times 1!}$$

$$= \frac{11!}{6 \times 6 \times 4} \Rightarrow \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 6 \times 4}$$

$= 2,70,200$

H.W

① Find permutations of

(a) SUCCESS $= \frac{7!}{3! \times 2! \times 1! \times 1!} = 420$

(b) STRUCTURES $= \frac{10!}{2! \times 2! \times 2! \times 2! \times 1!} = 2,26,8$

(c) MATHEMATICS $= \frac{11!}{2! \times 2! \times 2! \times 4! \times 1!}$

$= 49,89,600$

① prove that $(2n)! = 2^n n! \{1 \cdot 3 \cdot 5 \cdots (2n-1)\}$

sol :- $(2n)! = 2n(2n-1)(2n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= [2n(2n-2) \cdots 4 \cdot 2]\ (2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1)$

$= [2n(2n-2) \cdots 2 \cdot 1)]$

$= [2n\, 2(n-1) \cdots 2 \cdot 1)]$

$= [2n(n-1) 2(2n-2) \cdots 2(2 \cdot 1)]\ [1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)]$

$= 2^n (n)! [1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)]$

② If $nP_2 = 72$ find the value of $n$.

sol :- Here $nP_2 = 72 \Rightarrow \frac{n!}{(n-2)!} = 72$

$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 72$

$n(n-1) = 72$

$n^2 - n = 72 \Rightarrow n^2 - n - 72 = 0$

$n^2 - 9n + 8n - 72 = 0$

$n(n-9)+8(n-9)=0$

$(n+8)(n-9)=0$

$n=-8,\ n=9$

But $n \neq -8$ ( n cannot be negative)

$\therefore \boxed{n=9}$

(8) If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$ find $n$ ;

Sol: ${}^{2n+1}P_{n-1} = \frac{(2n+1)!}{(n+2)!}$ and ${}^{2n-1}P_n = \frac{(2n-1)!}{(n-2)!}$

$\therefore {}^{2n+1}P_{n-1} \div {}^{2n-1}P_n = \frac{(2n+1)!}{(n+2)!} \cdot \frac{(n-2)!}{(2n-1)!} = \frac{3}{5}$

$$\frac{\frac{(2n+1)!}{(2n+1-n+1)!}}{\frac{(2n-1)!}{(2n-1-n)!}} = 3/5$$

$$\frac{(2n+1)!}{(n+2)} \cdot \frac{(n-1)!}{(2n-1)!} = 3/5$$

$$\frac{(2n+1)(2n+1-1)(2n+1-2)!}{(n+2)(n+2-1)(n+2-2)(n+2-3)!} \cdot \frac{(n-1)!}{(2n-1)!} = 3/5$$

$$\frac{(2n+1)(2\cancel{n})\cancel{(2n-1)!}}{(n+2)(n+1)(\cancel{n})\cancel{(n-1)!}} \cdot \frac{\cancel{(n-1)!}}{\cancel{(2n-1)!}} = 3/5$$

$$\frac{(4n+2)}{(n^2+3n+2)} = 3/5 \quad \Rightarrow 20n+10 = 3n^2+9n+6$$

$\Rightarrow 3n^2+9n-20n+6-10=0$

$\Rightarrow 3n^2-11n-4=0$

$\Rightarrow$

$(n-4)(3n+1) = 0$,

$\therefore n = 4$, or $-1/3$

~~$3 \cdot 4 \cdot 4$~~

But $n \neq -1/3$ $\therefore n = 4$

(3) In how many ways can the letters of the word DISCRETE so that the vowels may come together?

Sol:- no. of letters in the given word $n = 8$

vowels in the word $= 3$

$\therefore$ The required number $= 3! \times (8-3)!$

$= 3! \times 5!$

$= 720$

(4) In how many ways can 7 boys and 5 girls be seated in a row so that no two girls may sit together.

Sol:- Since there is no restriction on boys, first of all we fix the position of 7 boys.

Their positions are indicated as

$\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times B_7 \times$

7 boys can be arranged in $7!$ ways.

Now if 5 girls sit at places including the two ends. indicated by X, then no two of the 5 girls will sit together.

clearly, 5 girls can be seated in 8 places in $^8P_5$ ways.

Hence the required no. of ways of seating 7 boys and 5 girls under the given condition $= {}^8P_5 \times 7!$

$= \frac{8!}{3!} \times 7!$

$= 8! \times \frac{7 \times 6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}}$

$= 40320 \times 24$

$= 96768$

## Combinations (with repetition)

Suppose we wish to select a combination of r objects with repetition from a set of n distinct objects. The no. of such selections is given by

$$c(n+r-1, r) = c(r+n-1, n-1)$$

→ The following are the other interpretations of this

(i) $c(n+r-1, r) = c(r+n-1, n-1)$ represents the no. of ways in which r identical objects can be distributed among n distinct containers.

(ii) $c(n+r-1, r) = c(r+n-1, n-1)$ represents the no. of non-negative integers sol's of the equation

n - distinct elements
↓
r - identical object w.r
↓
no. of such selections
$c(n+r-1, r) = c(r+n-1, n-1)$

n distinct identical objects — no. of non-negative integers sol^ns

Eg:- In how many ways can 20 similar books be placed on 5 diff shelves?

Sol:- Here $r = 20$, $n = 5$

Req no. of ways $= c(n+r-1, r)$
$= c(5+20-1, 20)$
$= c(24, 20)$
$= 10,626$

Note :- A non-negative integer solution of the equation $x_1 + x_2 + \cdots + x_n = r$ is an n-tuple, where $x_1, x_2, x_3 \cdots x_n$ are non-negative integers whose sum is 'r'

eg:- ① In how many ways can we distribute 10 identical marbles among 6 distinct containers?

sol:- comparing with 1st interpretations

10 identical marbles $r = 10$

6 distinct containers $n = 6$

no. of such selections

$$C(r+n-1, r) = C(10+6-1, 10)$$
$$= C(15, 10)$$
$$= \frac{15!}{10!\,5!}$$
$$= 3003.$$

eg② : Find the no. of non-negative integer solutions of the equation

sol :- sum of non-negative integers

i.e $x_1 + x_2 + x_3 + x_4 + x_5 = 8$ $n = 5$

No. of non-negative integers $r = 8$

$$C(n+r-1, r) = C(5+8-1, 8)$$
$$= C(12, 8)$$
$$= \frac{12!}{8!\,4!} = 495$$

(3) In how many ways can we distribute 12 identical pencils to 5 children so that every child gets atleast one pencil? (8)

sol :- No.of pencils = 12

No. of childrens $n = 15$

every child get atleast one pencil means each child may get $\geq 1$ pencil

First, we distribute one pencil to each children, Then the remaining identical pencils $(12-5=7)$ 7 can be distributed to 5 children.

$\therefore$ no. of ways to distribute 7 identical pencils to 5 children=

$r=7, n=5$

$$c(r+n-1, r) = c(7+5-1, 7)$$

$$= c(11, 7)$$

$$= \frac{11!}{7!\,4!}$$

$$= 330 \text{ ways.}$$

(4) In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each children gets one apple?

sol :- No. of apples = 7

No. of oranges = 6

no. of children = 4

each children gets atleast one apple, i.e every apple may get ≥ 1 apple.

first we distribute one apple to each child. The remaining applys ~~(7-4=3)~~ 3 can be distributed to 4 children.

no. of way the distribute 3 applys to 4 children

$$c(r+(n-1), r)$$

$$= c(3+(4-1), 3) \quad r=3,\ n=4$$

$$= c(6, 3)$$

$$= \frac{6!}{3!\,3!}$$

$$= 20 \text{ ways.}$$

No. of ways to distribute 6 oranges to 4 children $= c(r+n-1, r)$

$$= c(6+4-1, 6) \quad r=6,\ n=4$$

$$= c(9, 6)$$

$$= \frac{9!}{6!\,3!}$$

$$= 84 \text{ ways}$$

∴ Total no. of ways to distribute the fruits under given condition 84 ways

→ find the no. of positive integer solutions of the equation $x_1 + x_2 + x_3 = 17$ where $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 1$

Sol:- Given equation $x_1 + x_2 + x_3 = 17$

where $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 1$

let us consider three non-negative integers $y_1$, $y_2$ and $y_3$

$y_1 = x_1 - 1 \Rightarrow x_1 = y_1 + 1$

$y_2 = x_2 - 1 \Rightarrow x_2 = y_2 + 1$

$y_3 = x_3 - 1 \Rightarrow x_3 = y_3 + 1$

the above eqn can be defined in terms of $y_1, y_2, \& y_3$

$$x_1 + x_2 + x_3 = 17$$

$$(y_1 + 1) + (y_2 + 1) + (y_3 + 1) = 17$$

$$y_1 + y_2 + y_3 + 3 = 17$$

$$y_1 + y_2 + y_3 = 14$$

$\therefore y_1 + y_2 + y_3 = 14$

no. of positive integer solns of the above equation $= C(n + r - 1, r)$ $\quad n = 3$, $r = 14$

$= C(3 + 14 - 1, 14)$

$= C(16, 14)$

$= \frac{16!}{14!\, 2!}$

$= 120$

$\therefore$ no. of positive integer solns of the given eqn $= 120$

H.W

-) Find the no. of integer soln of the eqn $x_1+x_2+x_3+x_4+x_5=30$ where $x_1 \geq 2,\ x_2 \geq 3,\ x_3 \geq 4,\ x_4 \geq 2,\ x_5 \geq 0$

Sol:- Given equation $x_1+x_2+x_3+x_4+x_5=30$

Let us consider the +ve integers

$y_1, y_2, y_3, y_4$ & $y_5$

Now let us set $y_1 = x_1 - 2 \Rightarrow x_1 = y_1+2$

$y_2 = x_2 - 3 \Rightarrow x_2 = y_2+3$

$y_3 = x_3 - 4 \Rightarrow x_3 = y_3+4$

$y_4 = x_4 - 2 \Rightarrow x_4 = y_4+2$

$y_5 = x_5 \Rightarrow x_5 = y_5$

the above eqn can be written as

$$x_1+x_2+x_3+x_4+x_5=30$$

$$(y_1+2)+(y_2+3)+(y_3+4)+(y_4+5)+y_5=30$$

$$y_1+y_2+y_3+y_4+y_5 = 30-11 = 19$$

$$\therefore y_1+y_2+y_3+y_4+y_5=19$$

The no. of integer solns of the above eqn $= C(n+r-1, r)$ , $n=5$, $r=19$

$= C(5+19-1, 19)$

$= C(23, 19)$

$= \dfrac{23!}{19!\,4!} = 8855$

properties

(1) $^nC_r = {^nC_{n-r}} \quad (0 \le r \le n)$

sol:- we have $^nC_{n-r} = \dfrac{n!}{(n-r)!\,(n-n+r)!}$

$$= \frac{n!}{(n-r)!\; r!} = {^nC_r}$$

(2) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r} \quad (0 \le r \le n)$

sol:- $^nC_r + {^nC_{r-1}} = \dfrac{n!}{r!\,(n-r)!} + \dfrac{n!}{(r-1)!\,(n-r+1)!}$

$$= n!\left[\frac{1}{r!(n-r)!} + \frac{1}{(r-1)!\,(n-r+1)!}\right]$$

$$= n!\left[\frac{1}{(n-r)!\; r(r-1)!} + \frac{1}{(n-r+1)\,(n-r+1-1)!\,(r-1)!}\right]$$

$$= n!\left[\frac{1}{r(r-1)!\,(n-r)!} + \frac{1}{(n-r+1)\,(n-r)!\,(r-1)!}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{(n+1)}{r(n-r+1)}\right]$$

$$= \frac{(n+1)\, n!}{r(r-1)!\, (n-r+1)\, (n-r)!}$$

$$= \frac{(n+1)!}{r!\, (n-r+1)!}$$

$$nC_r + nC_{r-1} = {}^{n+1}C_r$$

$\Leftarrow$..

(3) $nC_x = nC_y \Rightarrow x = y$ & $x+y=n$

sol:- we have $nC_x = nC_y$

$\Rightarrow nC_x = nC_{n-y}$ $[\because nC_r = nC_{n-r}]$

$\Rightarrow x = y$ (or) $x = n-y$

$\Rightarrow x = y$ (or) $x+y=n$.

Relation b/w $nC_r$ and ${}^{n+1}C_r$

$$\frac{{}^{n+1}C_r}{nC_r} = \frac{\frac{(n+1)!}{r!\,(n+1-r)!}}{\frac{n!}{r!\,(n-r)!}} = \frac{(n+1)!}{\cancel{r!}\,(n+1-r)!} \frac{\cancel{r!}\,(n-r)!}{n!}$$

$$= \frac{(n+1)\,(n+1-1)!}{(n+1-r)\,(n+\cancel{1}-r-\cancel{1})!} \times \frac{(n-r)!}{n!}$$

$$= \frac{(n+1)\, \cancel{n!}}{(n+1-r)\,\cancel{(n-r)!}} \times \frac{\cancel{(n-r)!}}{\cancel{n!}}$$

$$= \frac{n+1}{(n+1-r)}$$

(1) Evaluate the following

(i) $6C_3 = \frac{6!}{3!(6-3)!} = \frac{6!}{3!\,3!} = \frac{6\times5\times4\times3\times2\times1}{3\times2\times1\times3\times2\times1}$

$= 20$

(ii) $10C_8 = {}^{10}C_{10-8} \Rightarrow 10C_2$

$= \frac{10!}{2!\,8!} = 45$

(2) compute $8P_5$ & $6C_3$

Sol:- $8P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8\times7\times6\times5\times4\times\cancel{3!}}{\cancel{3!}}$

$= 6720$

$6C_3 = \frac{6!}{3!(6-3)!} = \frac{6!}{3!\,3!} = 20.$

(3) $18C_r = 18C_{r+2}$ find the value of $rC_5$

Sol:- $\because nC_r = nC_{n-r} \Rightarrow r+n-r = n$

so $r+r+2 = 18 \Rightarrow 2r+2 = 18$

$\Rightarrow r+1 = 9$

$r = 8$

$\therefore rC_5 = 8C_5 = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\,3!} = \frac{8\times7\times6\times\cancel{5!}}{\cancel{5!}\times3\times2\times1}$

$= 56$

H.W

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→ Find the no. of integer soln of $x_1+x_2+x_3+x_4+x_5 = 30$ where $x_1 \geq 2,\ x_2 \geq 3,\ x_3 \geq 4,\ x_4 \geq 2,\ x_5 \geq 0$

(4) If $^nC_r = 56$ & $^nP_r = 336$ find $n$ & $r$

Sol :- $^nC_r = 56 \Rightarrow \frac{n!}{r!(n-r)!} = 56$

$^nP_r = 336 \Rightarrow \frac{n!}{(n-r)!} = 336.$

$\frac{^nC_r}{^nP_r} = \frac{56}{336} \Rightarrow \frac{\cancel{n!}}{r!\cancel{(n-r)!}} \times \frac{\cancel{(n-r)!}}{\cancel{n!}} = \frac{\cancel{56}}{\cancel{336}_6} = 1/6$

$= \frac{1}{r!} = 1/6$

$r! = 3!$

$\boxed{r = 3}$

Again $^nP_r = 336$

$^nP_3 = 336. \Rightarrow \frac{n!}{(n-3)!} = 336$

$\frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 336$

$n(n-1)(n-2) = 8 \times 7 \times 6$

$\therefore \boxed{n = 8}$

(5) If $^{1000}C_{98} = {}^{999}C_{97} + {}^xC_{901}$ : find $x$

Sol :- Here $^{1000}C_{98} = {}^{999}C_{97} + {}^xC_{901}$

(or) $^{1000}C_{902} = {}^{999}C_{902} + {}^xC_{901}$ $[\because {}^nC_r = {}^nC_{n-r}]$

(or) $^{999+1}C_{902} = {}^{999}C_{902} + {}^x{}_{902-1}$

$$^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$$

$$\therefore n = 999$$

(6) In how many ways can 4 questions be selected from 7 questions?

Sol :- The required no. of ways

$$^7C_4 = \frac{7!}{4!\cdot(7-4)!} = \frac{7!}{4!\cdot3!} = 35$$

(7) In how many ways can you select five cards at least one king, if you choose from a Deck of 52 cards?

Sol :- There are 4 kings in a deck of 52 cards

The no. of ways of choosing 1 king $= {}^4C_1 \times {}^{48}C_4$

" " " " " " 2 " $= {}^4C_2 \times {}^{48}C_3$

" " " " " " 3 " $= {}^4C_3 \times {}^{48}C_2$

" " " " " " 4 " $= {}^4C_4 \times {}^{48}C_1$

$\therefore$ The required number ${}^4C_1 \times {}^{48}C_4 + {}^4C_2 \times {}^{48}C_3 + {}^4C_3 \times {}^{48}C_2 + {}^4C_4 \times {}^{48}C_1$

$= 886,656$

Note: Total no. of ways in which one or more things are taken $2^n - 1$

eg:- ① you have 4 friends, in how many ways can you invite one or more them for dinner?

sol:- you may invite 1, 2, 3, or 4 of your friends to dinner. Hence the required no. of ways $= 2^n - 1$

$= 2^4 - 1$

$= 63$.

② There are 5 questions in a question paper. In how many was can a boy solve one or more questions.

sol:- $2^5 - 1 = 31$

Note:- the selection made of things p and so on, where p, q, & r of Third kind and so on, the tot no. of possible selections is

$[(p+1)(q+1)(r+1) \cdots -1]$

Eg:- In how many ways can a selection be made out of 3 mangoes, 5 oranges and 2 apples?

sol:- Here $p = 3, q = 5, r = 2$

∴ Req no. of combinations $(3+1)(5+1)(2+1) - 1$

$= 4 \cdot 6 \cdot 3 - 1 = 71$

# pigeonhole principle

pigeonhole principle :— If n pigeons are accomodated in m pigeonholes and n>m then at least one pigeonhole will contain two or more pigeons.

Generalized pigeonhole principle :

If n pigeons are accomodated in m pigeonholes and n>m then one of the pigeon holes must contain atleast $\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1$ pigeons.

Eg:- ① P.T in any set of 29 persons atleast 5 persons must have been born on the same day of the week.

Sol:- A week contains 7 days ∴ 7 pigeon holes
a set of 29 persons ∴ 29 pigeons
According to Generalized pigeonhole principle

$n = 29$
$m = 7$

$\left(\frac{n-1}{m}\right) + 1 \Rightarrow \frac{29-1}{7}$

$= \frac{28}{7} = 4$

Among a set of 29 persons, 5 persons must have been hole on the same day of the week

(2) Find the minimum no. of students in a class to be sure that four out of them are born in the same ~~day of a week~~ month.

Sol:- we consider each month as a pigeonhole. then $m = 12$ we have to find the minimum no. of students (pigeons) so that four out of them are born in the same month.

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 4$$

$$\frac{n-1}{m} = \frac{3}{1} \Rightarrow 3m = n-1$$

$$3(12) = n-1$$

$$36+1 = n \Rightarrow n = 37$$

∴ which is req min no. of students

(3) If 9 books are to be kept in 4 shelves, there must be atleast one shelf which contain at least 3 books.

Sol:- $n = 9$, $m = 4$

By pigeon hole principle $\left(\frac{n-1}{m}\right) + 1 = 3$

$$\left(\frac{9-1}{4}\right) + 1 = 3$$

$$\frac{8}{4} + 1 = 3$$

$$2 + 1 = 3$$

~~so at least 3 of the persons have born~~ so atleast one shelf will contain atleast 3 books

# principle of Inclusion - Exclusion

If $A$ & $B$ are any two finite sets

Then $|A \cup B| = |A| + |B| - |A \cap B|$

where $|A|$ denotes the cardinality of $A$.

$A = \{a, b, c\}$
$|A| = 3$
cardinality of $A$

If $A, B, C$ are any 3 sets.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - (B \cap C) - (A \cap C)| + |A \cap B \cap C|$$

If $A$ and $B$ are two disjoint sets

then $|A \cup B| = |A| + |B|$

for generalisation If $A_1, A_2, \ldots, A_n$ are finite sets Then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Ex ① Give a formula for the no. of elements in the union of four sets $A_1, A_2, A_3, A_4$

A:- $|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| -$

$- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| -$
$|A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| +$
$|A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| -$
$|A_1 \cap A_2 \cap A_3 \cap A_4|$

problems

(1) In a sample of 200 logic chips, 46 have a defect $D_1$, 52 have a defect $D_2$, 60 have defect $D_3$, 14 have defects $D_1$ & $D_2$, 16 have defects $D_1$ and $D_3$; 20 have all the defects $D_2$ and $D_3$, and 3 have all the three defects. Find the no. of chips having
(i) at least one defect
(ii) no defect.

sol:- Let $U$ denotes the set of all chips in a given sample, and $A, B, C$ denote the set of chips having defects $D_1, D_2, D_3$ respectively.

Then we have

$|U| = 200$, $|A| = 46$, $|B| = 52$, $|C| = 60$, $|A \cap B| = 14$
$|A \cap C| = 16$, $|B \cap C| = 20$ and $|A \cap B \cap C| = 3$

(i) The set of chips having at least one defect is $A \cup B \cup C$

∴ the no. of chips having at least one defect = $|A \cup B \cup C|$

$$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

$$= 46 + 52 + 60 - 14 - 16 - 20 + 3$$

$$= 161 - 50$$

$$= 111$$

(2) write the principle of Inclusion - Exclusion from a group of 10 professors. How many ways can committees of 5 members are formed so that at least one professor A and professor B will be included.

Sol :- The tot no. of committees $C(10, 5)$

Let A & B be the sets of committees that include professor A and professor B respectively

Then $|A| = C(9,4)$ & $|B| = C(9,4)$ &

$|A \cap B| = C(8,3)$

By the principle of the Inclusion - Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= C(9,4) + C(9,4) - C(8,3)$$

$$= 2C(9,4) - C(8,3)$$

$= 2 \times \frac{9!}{4!\,5!} - \frac{8!}{3!\,5!}$

$= 2 \times \frac{9 \times 8 \times 7 \times 6}{24} - \frac{8 \times 7 \times 6}{6}$

$= 252 - 56$

$= 196.$

(3) How many integers between 1 & 300 (inclusive) are divisible by at least one of 5, 6 and 8.

Sol:- Let $S = \{1, 2, 3 \ldots 300\}$ then

$|S| = 300$

Let $A, B, C$ be the subset of $S$ whose elements are divisible by $5, 6, 8$ respectively. Then we have.

$|A| = \left\lfloor \frac{300}{5} \right\rfloor = 60$, $|B| = \left\lfloor \frac{300}{6} \right\rfloor = 50$

$|C| = \left\lfloor \frac{300}{8} \right\rfloor = 37$, $|A \cap B| = \left\lfloor \frac{300}{30} \right\rfloor = 10$ ($\because$ LCM of 5 & 6 is 30)

$|A \cap C| = \left\lfloor \frac{300}{40} \right\rfloor = 7$, $|B \cap C| = \left\lfloor \frac{300}{24} \right\rfloor = 12$ ($\because$ LCM of 6 & 8 is 24)

and $|A \cap B \cap C| = \left\lfloor \frac{300}{120} \right\rfloor = 2$ ($\because$ LCM of 5, 6, 8 is 120)

By the principle of inclusion exclusion we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

$$= 60 + 50 + 37 - 10 - 7 - 12 + 2$$

$$= 120$$

Thus 120 elements of S are divisible by at least one of 5, 6, 8.

## BINOMIAL THEOREM

A Binomial theorem describes the algebraic expansion of powers of a Binomial with two variables.

$$(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \cdots + {}^nC_{n-1}\, x^1 y^{n-1} + {}^nC_n\, x^0 y^n$$

It can be written as

$$\boxed{(x+y)^n = \sum_{r=0}^{\infty} {}^nC_r\, x^{n-r} y^r}$$

$\rightarrow$ $(1+x)^n = {}^nC_0\, 1^n x^0 + {}^nC_1\, 1^{n-1} x^1 + {}^nC_2\, 1^{n-2} x^2 + {}^nC_3\, 1^{n-3} x^3 + \cdots$

$\Rightarrow$ $nC_0 x^0 + nC_1 x^1 + nC_2 x^2 + nC_3 x^3 + \cdots$

$\Rightarrow$ $1 \cdot x^0 + \frac{n!}{1!(n-1)!} x + \frac{n!}{2!(n-2)!} x^2 + \frac{n!}{3!(n-3)!} x^3 + \cdots$

$\Rightarrow$ $1.1 \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} x + \frac{n(n-1)\cancel{(n-2)!}}{2!\cancel{(n-2)!}} x^2 +$

$\frac{n(n-1)(n-2)\cancel{(n-3)!}}{3!\cancel{(n-3)!}} x^3 + \cdots$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$

problems :

(1) find out the coefficient of $x^9y^3$ in $(x+2y)^{12}$

Sol :- we know that Binomial Theorem

$(x+y)^n = \sum_{r=0}^{\infty} nC_r \cdot x^{n-r} \cdot y^r$

according to the Binomial theorem compare with $(x+2y)^{12}$

$x = x,\ y = 2y,\ n = 12$

$(x+2y)^{12} = \sum_{r=0}^{\infty} 12C_r (x)^{12-r} (2y)^r$

$= \sum_{r=0}^{\infty} 12C_r\, x^{12-r}\, 2^r\, y^r \quad -(1)$

we have to find out $x^9y^3$

$x^9 = x^{12-r}$ | $y^r = y^3$

$9 = 12 - r$ | $r = 3$

$r = 3$

r value substitute in eq (1)

$12C_3 \; 2^3 \; x^{12-3} \; y^3$

$12C_3 \; 2^3 \; x^9 y^3$

$$\frac{12!}{3!\,9!} \times 2^3$$

$$\frac{12 \times 11 \times \cancel{10}^{5} \times \cancel{9!}}{\cancel{6}_{3} \times \cancel{9!}}$$

$= 1760$

$\therefore$ coefficient of $x^9 y^3$ in $(x+2y)^{12}$ is 1760.

(2) Find out the coefficient of $x^5 y^2$ in the expansion of $(2x-3y)^7$

Sol: we know that, from Binomial theorem

$$(x+y)^n = \sum_{r=0}^{\infty} nC_r \, x^{n-r} y^r$$

$x = 2x, \; y = -3y, \; n = 7$

$$(2x-3y)^7 = \sum_{r=0}^{\infty} 7C_r (2x)^{7-r} (-3y)^r$$

$$= \sum_{r=0}^{\infty} 7C_r \, 2^{7-r} x^{7-r} (-3)^r y^r \quad \text{(1)}$$

we have to findout the coefficient of

$x^5 . y^2$

$x^{7-r} = x^5$

$7 - r = 5$

$r = 2$

$r = 2$

$y^r = y^2$

$r = 2$

$\therefore$ r value substitute on eq$^n$ ①

$7C_2 \, (2)^{7-2} \, (-3)^r \, x^{7-2} \, y^2$

$= 7C_2 \, 2^5 \, 9 . \, x^5 y^2$

$\frac{7!}{2! 5!} \, 32 \cdot 9 \, x^5 y^2$

$\frac{7 \times \cancel{6}^{3} \times \cancel{5!}}{2 \times 1 \times \cancel{5!}} \, 32 \cdot 9 \, x^5 y^2$

$= 21 \times 32 \times 9 \cdot x^5 y^2$

$= 6045 \, x^5 . y^2$

$\therefore$ coefficient of $x^5 y^2$ in $(2x - 3y)^7$ is 6045

③ what is the coefficient of $x^2 y^4$ in $(x+y)^6$?

$(x+y)^n = \sum_{r=0}^{\infty} nC_r \, x^{n-r} \, y^r$, $n = 6$.

$(x+y)^6 = \sum_{r=0}^{\infty} 6C_r \, x^{6-r} \, y^r \rightarrow$ ①

we have to find out the coefficient of

$x^2 y^4$

$x^{6-r} = x^2$

$6 - r = 2$

$r = 6 - 2 = 4$

$y^r = y^4$

$r = 4$

substitute these value in eqn (1)

$$^6C_4 \; x^{6-4} \; y^4$$

$$^6C_4 \; x^2 \; y^4$$

$$\frac{6!}{4!\,2!} x^2 y^4 \Rightarrow \frac{6 \times 5 \times 4!}{4! \times 2 \times 1} = 15$$

(4) $x^{101} \; y^{99}$ in the expansion $(2x-3y)^{200}$ ?

$$(x+y)^n = \sum_{r=0}^{\infty} {}^nC_r \; x^{n-r} \; y^r$$

$x = 2x, \quad y = -3y, \quad n = 200$

$$(2x-3y)^{200} = \sum_{r=0}^{\infty} {}^{200}C_r \; (2x)^{n-r} \; (-3y)^r \rightarrow (1)$$

$$= \sum_{r=0}^{\infty} {}^{200}C_r \; 2^{200-r} \; x^{200-r} \; (-3)^r \; y^r$$

$x^{101} = x^{200-r} \qquad y^r = y^{99}$

$200 - r = 101 \qquad r = 99$

$200 - 101 = r$

$r = 99$

$$\Rightarrow {}^{200}C_{99} \; 2^{200-99} \; x^{200-99} \; (-3)^{99} \; (y)^{99}$$

$$\Rightarrow {}^{200}C_{99} \; 2^{101} \; (-3)^{99} \cdot y^{99} \; x^{101}$$

$$-{}^{200}C_{99} \; 2^{101} \; 3^{99} \cdot y^{99} x^{101}$$

coefficient of $x^{101} \; y^{99}$ is $-{}^{200}C_{99} \; 2^{101} \; 3^{99}$

# Multinomial Theorem

Multinomial theorem is a generalization of the Binomial Theorem with more than two variables.

$$(x_1 + x_2 + \cdots + x_n)^n = \sum_{n_1, n_2, \cdots n_k} \frac{n!}{n_1! * n_2! * \cdots * n_k!}$$

where $n_1 + n_2 + \cdots + n_k = n$

Eg:- (1) compute the following

(a) $\binom{7}{2,3,2}$

It is in the form of

$\binom{n}{n_1, n_2, n_3}$ where $n_1 + n_2 + n_3 = n$

By applying the multinomial theorem

$$\frac{n!}{n_1! * n_2! * n_3!} \Rightarrow \frac{7!}{2! \times 3! \times 2!} = 210$$

(b) $\binom{4}{1,1,2}$ this is in the form of $\binom{n}{n_1, n_2\ n_3}$

$$\frac{4!}{1! \times 1! \times 2!} = \frac{24}{2} = 12$$

(c) $\binom{12}{5,3,2,2}$ It is in the form of $\binom{n}{n_1, n_2 n_3 n_4}$

By multinomial theorm $\frac{2!}{5! \times 3! \times 2! \times 2!} = 166320$

(4) what is the coefficent of $x^3y^2z^2$ in $(x+y+z)^9$ ?

sol :- By the multinomial theorem $\left(\frac{n!}{n_1! \times n_2! \times n_3!}\right)$

$$\frac{9!}{3! \times 2! \times 2!} = 15120.$$

(5) Determine the coefficent of $xyz^2$ in the expansion of $(2x-y-z)^4$

sol :- By Applying Multinomial theorem

$$(x_1+x_2+x_3 \cdots + x_k)^n = \frac{n!}{n_1! \times n_2! \times \cdots \times n_k} (x_1)^{n_1} \times (x_2)^{n_2} \times \cdots \times (x_k)^{n_k}$$

where $n_1 + n_2 + \cdots + n_k = n$

given expansion $(2x-y-z)^4$

$n = 4$, $x_1 = 2x$, $x_2 = -y$, $x_3 = -z$

Apply multinomial theorem

$$\frac{n!}{n_1! \times n_2! \times n_3!} \cdot (x_1)^{n_1} (x_2)^{n_2} (x_3)^{n_3} = \frac{4!}{n_1! \times n_2! \times n_3!} (2x)^{n_1} (-y)^{n_2} (-z)^{n_3} \longrightarrow (1)$$

we have to find out the coefficent of

$xyz^2 = x_1^{n_1} \; x_2^{n_2} \; x_3^{n_3}$

$n_1 = 1$

$n_2 = 1$

$n_3 = 2$

substitute $n_1, n_2, n_3$ values in eq (1)

$$\frac{4!}{1! \times 1! \times 2!} (2x)^1 (-y)^1 (-z)^2$$

$$4 \quad 2^1 x^1 (-1)^1 y^1 (-1)^2 z^2$$

$$= \frac{24}{1 \times 1! \times 2}$$

$$= 12 * 2 . x - y . z^2$$

$$= -24 x y z^2$$

$\therefore$ coefficient of $xyz^2$ is $-24$.

(2) Determine the coefficient of $x^3 y^3 z^2$ in the expansion of $(2x - 3y + 5z)^8$

Sol:- By multinomial theorem

$$\frac{n!}{n_1! * n_2! * n_3!} x_1^{n_1} x_2^{n_2} x_3^{n_3}$$

compare $n = 8$, $n_1 = 3$, $n_2 = 3$, $n_3 = 2$

$x_1 = 2x$, $x_2 = -3y$, $x_3 = 5z$

$$\therefore \frac{8!}{3! \times 3! \times 2!} (2x)^3 (-3y)^3 (5z)^2$$

$$2^3 (-3)^3 5^2 \quad x^3 y^3 z^2$$

$$= \frac{8!}{3! \times 3! \times 2!}$$

$$= -3024000 \; x^3 y^3 z^2$$

$\therefore$ coefficient of $x^3 y^3 z^2$ is $-3024000$

# Pascal's Triangle

1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

2

$(a+b)^0$ — 1

$(a+b)^1$ — $1a$ $1b$

$(a+b)^2 \rightarrow$ $1a^2b^0$ $2ab$ $1a^0b^2$

$(a+b)^3$ $1a^3b^0$ $3a^2b^1$ $3ab^2$ $1a^0b^3$

$(a+b)^4$ $1a^4b^0$ $4a^3b^1$ $6a^2b^2$ $4ab^3$ $1a^0b^4$

pascal's Identity for ...

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \qquad \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$\frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$\frac{n!}{r\,(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)\,(n-r+1-1)!}$$

$$\frac{n!}{r\,(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)\,(n-r)!}$$

$$\frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$\frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n-\cancel{r}+1+\cancel{r}}{r(n-r+1)}\right]$$

$$\frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right]$$

$$\frac{n!\,(n+1)}{r\,(r-1)!\,(n-r+1)\,(n-r)!}$$

$$\frac{(n+1)!}{r!\,(n-r+1)!}$$

$$= \binom{n+1}{r}$$