UNIT - III

2. Application of Definite Integral

① Volumes of solids of revolution:-

1. Cartesian form :-

(i) Volume of the Solid about X-axis :-

The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$ the x-axis and the lines $x=a, y=b$ is given by $\int_a^b \pi y^2 dx$.

(ii) Volume of the Solid about Y-axis :-

The volume of the solid generated by the revolution of the area bounded by the curve $x = f(y)$ the Y-axis and line $y=a, y=b$ is given by $\int_a^b \pi x^2 dy$.

(iii) Volume bounded by two curves :-

(a) Volume of the solid generated by the revolution of the area bounded by the curves $y = y_1(x), y = y_2(x)$ and the ordinates $x=a, x=b$ about x-axis is

$$V = \int_a^b \pi (y_1^2 - y_2^2)\, dx.$$ where y_1 & y_2 are the ordinates of the upper & lower curves

(b) Volume of the solid generated by the curves $x = x_1(y), x = x_2(y)$ and the ordinates ~~x~~ $y=a, y=b$. about Y-axis is

$$V = \int_a^b \pi (x_1^2 - x_2^2)\, dy.$$

① find the volume of the Solid that result when the Region enclosed by the curve $y = x^3, x = 0, y = 1$ is revolved about the y-axis.

Sol Given curve is $y = x^3$

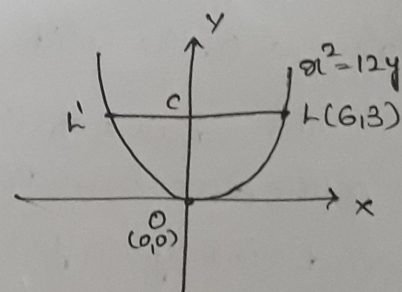
$\therefore$ Required volume $= \int_0^1 \pi x^2 \, dy$.

$= \int_0^1 \pi [y^{1/3}]^2 \, dy$

$= \pi \int_0^1 y^{2/3} dy = \pi \left[\frac{y^{2/3+1}}{2/3+1} \right]_0^1 = \pi \frac{3}{5}(1)$

Volume $= \frac{3\pi}{5}$ units.

② find the volume of the Solid generated by revolving the arc of the parabola $x^2 = 12y$ bounded by its Latus Rectum. about y-axis



Sol Given parabola is

$x^2 = 12y = 4(3)y$.

Let O be the vertex and LL' be the Latus Rectum.

for the arc OL, y varies from 0 to 3.

Required volume $= \int_0^3 \pi x^2 dy$.

$= \int_0^3 \pi \, 12y \, dy$.

$= 12\pi \int_0^3 y \, dy$

$= 12\pi \left[\frac{y^2}{2} \right]_0^3$

$= 6\pi [9 - 0]$

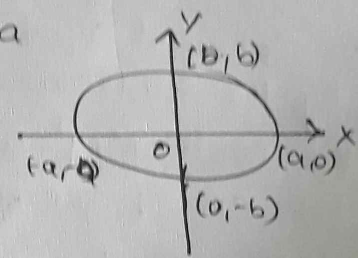
$= 54\pi$ cubic units.

(3) find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(0<b<a)$ about the major axis.

Sol Given eqn of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

when $y=0$, $x = \pm a$

Major axis of the ellipse is $x=-a$ to $x=a$



The volume of the solid generated by the given ellipse revolving about the Major axis $= \int_{-a}^{a} \pi y^2 dx$

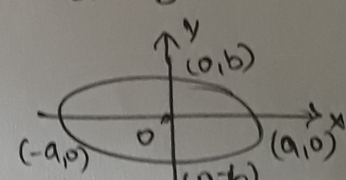
$= 2\pi \int_{0}^{a} y^2 dx$.

$= 2\pi b^2 \int_{0}^{a} [1 - \frac{x^2}{a^2}] dx$

$= 2\pi b^2 [x - \frac{x^3}{3a^2}]_0^a$

$= 2\pi b^2 [a - \frac{a^3}{3a^2}] = 2\pi b^2 [\frac{2a}{3}] = \frac{4}{3}\pi a b^2$

(4) find the volume of the solid when ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(0<b<a)$ rotates about minor axis.



Sol Given ellipse eqn $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

when $x=0$, $y = \pm b$.

minor axis of the ellipse is $y=-b$ to $y=b$.

Required Volume $= \int_{-b}^{b} \pi x^2 dy = 2\pi \int_{0}^{b} x^2 dy$.

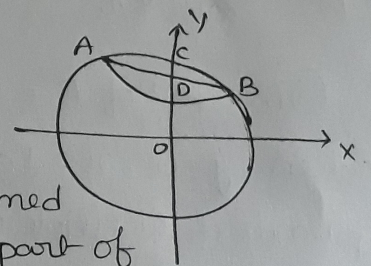
$= 2\pi a^2 \int_{0}^{b} [1 - \frac{y^2}{b^2}] dy$

$= 2\pi a^2 [y - \frac{y^3}{3b^2}]_0^b$

$= 2\pi a^2 [b - \frac{b^3}{3b^2}] = 2\pi a^2 [\frac{2b}{3}]$

$= \frac{4}{3}\pi a^2 b$ cubic units.

(5) Find the volume of spherical cap of height h cut off from a sphere of radius a.



Sol Let $x^2+y^2+z^2=a^2$ be a sphere of radius a.

The spherical cap of a sphere is formed by the revolution of the arc ACD part of the circle $x^2+y^2=a^2$ about Y-axis

Let CD = height of the cap = h.

Then $CD = h$

$OD = OC - CD = a-h$.

The limits of integration are $OD = a-h$ to $OC = a$ for y

∴ Required volume $= \int_{a-h}^{a} x^2 \pi \, dy$

$= \pi \int_{a-h}^{a} ((a^2-y^2)dy)$

$= \pi \int_{a-h}^{a} [a^2-y^2] dy \Rightarrow \pi \left[a^2 (y)_{a-h}^{a} - \left[\frac{y^3}{3}\right]_{a-h}^{a}\right]$

$= \pi \left[a^2[a-(a-h)] - \frac{1}{3}[a^3-(a-h)^3]\right]$

$= \pi \left[a^2h - \frac{a^3}{3} + \frac{1}{3}[a^3 + h^3 - 3a^2h + 3ah^2]\right]$

$= \pi \left[\cancel{a^2h} - \cancel{\frac{a^3}{3}} + \cancel{\frac{a^3}{3}} + \frac{h^3}{3} - \cancel{a^2h} + ah^2\right]$

$= \left[\frac{h^3}{3} + ah^2\right] \pi$

$= \frac{\pi}{3} h^2 [3a-h]$

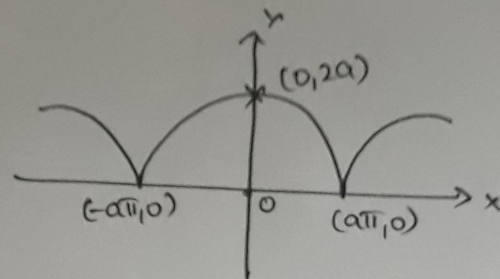
(1) find the volume of solid obtained by revolving one arc of the cycloid $x = a(\theta + \sin\theta), y = a(1+\cos\theta)$ about x-axis

Sol

Replacing θ with $-\theta$.

y is unchanged.

$\therefore$ Curve is symmetrical about y-axis.



$\theta = 0 \Rightarrow x = 0, y = 2a$

$\theta = \pi \Rightarrow x = a\pi, y = 0$

$\theta = -\pi \Rightarrow x = -a\pi, y = 0$.

The curve does not pass through the origin.

Required volume $= \int_{\theta=-\pi}^{\pi} \pi y^2 \frac{dx}{d\theta} . d\theta$

$$= 2\pi \int_0^{\pi} y^2 . \frac{dx}{d\theta} . d\theta .$$

$\{ y = a(1+\cos\theta)$.
$x = a(\theta + \sin\theta)$
$dx = a[d\theta + \cos\theta d\theta]$
$dx = a(1+\cos\theta) d\theta$.
$\frac{dx}{d\theta} = a(1+\cos\theta)$.

$$= 2\pi \int_0^{\pi} a^2(1+\cos\theta)^2 . a(1+\cos\theta) d\theta .$$

$$= 2a^3\pi \int_0^{\pi} [1+\cos\theta]^3 d\theta .$$

$\{ 1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$= 2\pi a^3 \int_0^{\pi} [2\cos^2\tfrac{\theta}{2}]^3 d\theta .$$

$$= 16\pi a^3 \int_0^{\pi} \cos^6(\theta/2) \, d\theta .$$

$\{$ put $\frac{\theta}{2} = \phi \rightarrow d\theta = 2d\phi \}$
when $\frac{\theta}{2} = \phi \Rightarrow \frac{0}{2} = \phi \Rightarrow \phi = 0$
$\pi/2 = \phi \Rightarrow \phi = \pi/2$

$$= 16\pi a^3 \int_0^{\pi/2} \cos^6\phi \, (2d\phi)$$

$$= 32\pi a^3 \left[\int_0^{\pi/2} \cos^6\phi \, d\phi \right]$$

$\{ \int_0^{\pi/2} \cos^n\theta d\theta = \frac{(n-1)(n-3)\cdots 1}{n(n-2)(n-4)\cdots 2} \cdot \frac{\pi}{2}$

$$= \cancel{32}\pi a^3 \left[\frac{5}{\cancel{6}_2} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{\cancel{2}} \cdot \frac{\pi}{\cancel{2}} \right]$$

$$= 5\pi^2 a^3$$

Surface Areas of Revolution:—

1. **Revolution about the x-axis:—**

The surface area of the solid generated by the revolution about the x-axis of the area bounded by the curve $y=f(x)$ the x-axis and the ordinates $x=a, x=b$ is.

$$S=\int_a^b 2\pi y\cdot ds. = \int_a^b 2\pi y\cdot\frac{ds}{dx}\cdot dx. \quad \left\{\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right\}$$

2. **Revolution about y-axis**

Area of the surface of the solid of revolution generated by revolving an arc of the curve $x=g(y)$ from $y=c$ to $y=d$ about y-axis is given by.

$$S=\int_c^d 2\pi x\,ds = \int_c^d 2\pi x\cdot\frac{ds}{dy}\,dy. \quad \left\{\frac{ds}{dy}=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right\}$$

① Find the surface area of a sphere generated by the circle $x^2+y^2=16$ about its diameter.

Sol Given circle is $x^2+y^2=16$ & $y^2=16-x^2 \rightarrow ①$

differentiating eq① w.r.to x, we get

$$\frac{dy}{dx}\cancel{2}y=0-\cancel{2}x$$

$$\frac{dy}{dx}=-\frac{x}{y}.$$

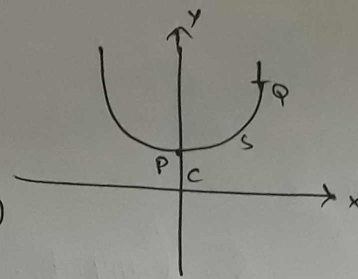
Required surface Area $= 2\pi\int_{-4}^{4} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\pi\int_{-4}^{4} y\sqrt{1+\frac{x^2}{y^2}}\,dx.$$

$$= 2\pi\int_{-4}^{4}\sqrt{x^2+y^2}\,dx = 2\pi\int_{-4}^{4}\sqrt{16}\,dx \text{ using eq①}$$

$$= 8\pi\int_{-4}^{4}dx = 8\pi\left[x\right]_{-4}^{4} = 64\pi.$$

(2) Find the surface area generated by the revolution of an arc of the catenary $y = c\cosh\frac{x}{c}$ about the x-axis from $x = 0$ to $x = c$.



Sol Given that $y = c\cosh\left(\frac{x}{c}\right)$

$$\frac{dy}{dx} = \cancel{c}\sinh\left(\frac{x}{c}\right)\cdot\frac{1}{\cancel{c}} = \sinh\left(\frac{x}{c}\right)$$

Now $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$\frac{ds}{dx} = \sqrt{1 + \sinh^2\left(\frac{x}{c}\right)} = \cosh\left(\frac{x}{c}\right)$$

$\therefore$ Required surface area $= \int_0^c 2\pi y \cdot \frac{ds}{dx} \cdot dx$.

$$= 2\pi \int_0^c c\cdot\cosh\left(\frac{x}{c}\right)\cdot\cosh\left(\frac{x}{c}\right) dx.$$

$$= 2\pi c \int_0^c \cosh^2\left(\frac{x}{c}\right) dx.$$

$\left\{ \text{put } \frac{x}{c} = \theta,\ dx = c\cdot d\theta \right\}$

$$= 2\pi c \int_0^1 \cosh^2\theta\, d\theta\cdot c$$

$$= 2\pi c^2 \int_0^1 \cosh^2\theta\, d\theta.$$

$$= \cancel{2}\pi c^2 \int_0^1 \frac{1+\cosh 2\theta}{\cancel{2}} d\theta \quad \left\{ \cosh 2x = 2\cosh^2 x - 1 \right\}$$

$$= \cancel{2}\pi c^2 \left[\frac{1}{2}[\theta]_0^1 + \left[\frac{\sinh 2\theta}{2\cdot 2}\right]_0^1 \right]$$

$$= \pi c^2 \left[\theta + \frac{\sinh 2\theta}{2} \right]_0^1$$

$$= \pi c^2 \left[1 + \frac{\sinh 2}{2} \right]$$ //

3. Beta and Gamma functions

Definition :-

Improper integrals :-

consider the integral $\int_a^b f(x)dx$ such that an integral for which

① either the interval of integral is not finite that means (i.e) $a=-\infty$ or $b=\infty$ or both.

② The func $f(x)$ is unbounded at one or more points in $[a,b]$ is called an improper integral.

Integral which satisfy both the condition ① & ② are called improper integrals third kind.

Ex:

1) $\int_0^\infty \frac{dx}{1+x^4}$ and $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ are improper integrals of the first kind.

(2) $\int_0^1 \frac{dx}{1-x^2}$ is an improper integral of the second kind.

③ The Gamma func defined by the integral $\int_0^\infty e^{-x}x^{n-1}dx$ when $n>0$ is an improper integral of the third kind.

Beta function definition :-

The definite integral

$\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called Beta func and is denoted by $B(m,n)$ and read as "Beta m,n"

The above integral converges for $m>0, n>0$ thus

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx. \text{ where } m>0, n>0$$

properties of Beta function:—

① Symmetry of Beta function i.e $B(m,n) = B(n,m)$.

proof: By definition, we have.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

put $1-x=y$ so that $-dx=dy$ & $1-0=y \Rightarrow y=1$

$dx=-dy$. $1-1=y \Rightarrow y=0$

$$B(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= -\int_0^1 (1-y)^{m-1} \cdot y^{n-1}(-dy)$$

$$= \int_0^1 (1-y)^{m-1} y^{n-1} dy. \qquad \left\{ \int_a^b f(t)\,dt = \int_a^b f(x)\,dx \right\}$$

$$= \int_0^1 x^{m-1} \cdot (1-x)^{m-1} dy.$$

Hence $B(m,n) = B(n,m)$.

② $\frac{1}{2} B(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

proof By definition of Beta func. we have.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \longrightarrow ①$$

put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$.

Also when $x=0$, $\theta=0$ and when $x=1$, $\theta=\pi/2$

then of ①, we have.

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta$$

$$B(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cdot (\cos^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta.$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \cdot \cos^{2n-2+1}\theta \, d\theta.$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

$$\boxed{\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}B(m,n)}$$

Part-B

2b

(3) $B(m,n) = B(m+1,n) + B(m,n+1)$

5m

proof: By definition of Beta function we have.

$$B(m+1,n) = \int_0^1 x^m (1-x)^{n-1} dx.$$

$$B(m,n+1) = \int_0^1 x^{m-1}(1-x)^n dx.$$

$$\therefore B(m+1,n) + B(m,n+1) = \int_0^1 x^m (1-x)^{n-1} dx + \int_0^1 x^{m-1}(1-x)^n dx.$$

$$= \int_0^1 [x^m(1-x)^{n-1} + x^{m-1}(1-x)^n] dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[\cancel{x} + (1-\cancel{x})] dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1} dx.$$

$$= B(m,n)$$

$$B(m+1,n) + B(m,n+1) = B(m,n)$$

(4) $B(m,n) = \dfrac{(m-1)!\,(n-1)!}{(m+n-1)!}$ if m and n are +ve integer.

Other Forms of Beta functions :—

① $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$.

② $B(m,n) = \int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

③ $B(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$

④ $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx = \frac{B(m+n)}{a^n(1+a)^m}$.

⑤ $\int_a^b (x-b)^{m-1}(a-x)^{n-1} dx = (a-b)^{m+n-1} B(m,n), \; m>0, n>0$.

① prove that $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5} B\left[\frac{2}{5}, \frac{1}{2}\right]$

Sol put $x^5 = y$. i.e $x = y^{1/5}$

so that $dx = \frac{1}{5} y^{1/5 - 1} . dy$

$dx = \frac{1}{5} y^{-4/5} dy$.

Also when $x = 0, y = 0$, when $x = 1, y = 1$.

$\int_0^1 \frac{x}{\sqrt{1-x^5}} dx = \int_0^1 \frac{y^{1/5}}{\sqrt{1-y}} . \frac{1}{5} y^{-4/5} dy$.

$= \frac{1}{5} \int_0^1 y^{\frac{1}{5} - \frac{4}{5}} . (1-y)^{-1/2} dy$

$= \frac{1}{5} \int_0^1 y^{-\frac{3}{5}} (1-y)^{-1/2} dy$

$= \frac{1}{5} \int_0^1 y^{\frac{2}{5}-1} (1-y)^{\frac{1}{2}-1} . dy$

$= \frac{1}{5} B\left[\frac{2}{5}, \frac{1}{2}\right]$

Gamma function :-

The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the Gamma function and is denoted by $\Gamma(n)$ and read as "Gamma n". The integral converges only for $n>0$.

Thus, $\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \text{, where } n>0}$

properties of Gamma function :-

1) $\Gamma(1) = 1$

2) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

3) $\Gamma(n) = (n-1)!$

4) $\Gamma(n+1) = n!$

5) $\Gamma(n+1) = n\Gamma(n)$

(or)

$\Gamma(n) = \frac{\Gamma(n+1)}{n}$ (for -ve value)

6) $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

7) $\Gamma(n) = (n-1)\Gamma(n-1)$ where $n>1$.

1) Find the value of $\Gamma(1/2)$

Sol we know that $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Taking $m=n=1/2$, we have.

$$B\left[\frac{1}{2},\frac{1}{2}\right] = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma\left(1/2+\frac{1}{2}\right)} = \frac{\left[\Gamma(1/2)\right]^2}{\Gamma(1)} = \left(\Gamma(1/2)\right)^2 \longrightarrow (1)$$

But $B\left[\frac{1}{2},\frac{1}{2}\right] = \int_0^1 x^{1/2-1}(1-x)^{1/2-1} dx = \int_0^1 x^{-1/2}(1-x)^{-1/2} dx$

put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$

Also we have $x=0, \theta=0$, when $x=1, \theta=\pi/2$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^{\pi/2} (\sin^2\theta)^{-1/2}(1-\sin^2\theta)^{-1/2}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{-1}\theta\,(\cos^2\theta)^{-1/2}\, 2\sin\theta\cos\theta\, d\theta.$$

① Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$ in terms of Beta func.

Sol $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx = \int_0^1 x^2(1-x^5)^{-1/2}\,dx$.

put $x^5 = t \Rightarrow$ i.e $x = t^{1/5}$

so that $dx = \frac{1}{5}t^{1/5-1}\,dt$

$dx = \frac{1}{5}t^{-4/5}\,dt$

Also when $x=1, t=1$, and when $x=0, t=0$

$$\int_0^1 x^2(1-x^5)^{-1/2}\,dx = \int_0^1 (t^{1/5})^2(1-t)^{-1/2}\,\frac{1}{5}\,t^{-4/5}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{\frac{2}{5}-\frac{4}{5}}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{-2/5}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{\frac{3}{5}-1}(1-t)^{\frac{1}{2}-1}\,dt$$

$$\int_0^1 x^2(1-x^5)^{-1/2}\,dx = \frac{1}{5}B\left[\frac{3}{5},\frac{1}{2}\right]$$

Relation between Beta and Gamma functions:–

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \text{ where } m>0, n>0.$$

proof: The definition of B Gamma function, we have.

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx. \longrightarrow (1)$$

put $x = yt$ So that $dx = y\,dt$

Also when $x=0, t=0$, and $x=\infty, t=\infty$

then eq (1), we have.

$$\Gamma(m) = \int_0^\infty e^{-yt} (yt)^{m-1} \cdot y\,dt$$

$$\Gamma(m) = \int_0^\infty e^{-yt} y^{m-1} \cdot t^{m-1} \cdot y\,dt$$

$$\Gamma(m) = \int_0^\infty e^{-yt} y^{m\cancel{-1}\cancel{+1}} t^{m-1} dt$$

$$\Gamma(m) = y^m \int_0^\infty e^{-yt} t^{m-1} dt$$

$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yt} \cdot t^{m-1} dt \quad \left\{ \int_a^b f(t)\,dt = \int_a^b f(x)\,dx \right\}$$

$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} \cdot x^{m-1} dx \longrightarrow (2)$$

Integrating on both sides w.r.t y from 0 to ∞ and multiplying on both sides $e^{-y} \cdot y^{m+n-1}$ in eq (2), we get—

$$\int_0^\infty \frac{\Gamma(m)}{y^m} (e^{-y}) y^{m+n-1} \cdot dy = \int_0^\infty \left[\int_0^\infty e^{-yx} \cdot x^{m-1} dx \right] e^{-y} y^{m+n-1} dy.$$

$$\Gamma(m)\int_0^\infty e^{-y}\cdot y^{m+n-1}\cdot y^{-m}dy = \int_0^\infty\left[\int_0^\infty e^{-yx}\cdot e^{-y}x^{m-1}y^{m+n-1}dx\right]dy.$$

$$\Gamma(m)\int_0^\infty e^{-y}y^{\cancel{m}+n-1-\cancel{m}}dy = \int_0^\infty\left[\int_0^\infty e^{-y(x+1)}\cdot x^{m-1}y^{m+n-1}dx\right]dy.$$

$$\Gamma(m)\int_0^\infty e^{-y}y^{n-1}dy = \int_0^\infty\left[\int_0^\infty e^{-y(1+x)}\cdot y^{m+n-1}dy\right]x^{m-1}dx.$$

{by interchanging the order of Integrating}.

$$\Gamma(m)\int_0^\infty e^{-x}x^{n-1}dx = \int_0^\infty\left[\frac{\Gamma(m+n)}{(1+x)^{m+n}}\right]x^{m-1}dx$$ { by ~~from~~ eq 2 }

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx.$$ { by eq 1 and by form 1 of Beta function }

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\cdot B(m,n)$$

$$\boxed{B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

① S.T $\Gamma(n) = \int_0^1 (\log 1/x)^{n-1} dx, \quad n > 0$.

Sol

we have $\Gamma(n) = \int_0^\infty e^{-x} . x^{n-1} dx \longrightarrow$ ①

put $x = \log 1/y$

$x = \log y^{-1}$ $\{\log m^n = n \log m\}$

$x = -\log y$.

$-x = \log y$. $\{x = \log a \Rightarrow e^x = a\}$

$\boxed{\begin{matrix} e^{-x} = y. \\ y = e^{-x} \end{matrix}}$ when $x = 0, y = 1$, when $x = \infty$ $y = 0$

$\left\{ \begin{matrix} e^0 = 1 \\ e^{-\infty} = 0 \end{matrix} \right\}$

So that $dy = -e^{-x} dx$. & $dx = -\frac{1}{y} dy$.

Now eq ① becomes.

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$. $\left\{ \int_a^b f(x) dx = -\int_b^a f(x) dx \right\}$

$= \int_1^0 y . [\log 1/y]^{n-1} (-1/y) dy$.

$= -\int_0^1 [\log 1/y]^{n-1} y (-1/y) dy$

$\Gamma(n) = \int_0^1 [\log (1/y)]^{n-1} dy$. $\left\{ \int_a^b f(x) dx = \int_a^b f(t) dt \right\}$

$\Gamma(n) = \int_0^1 [\log 1/x]^{n-1} dx$.

① Evaluate $\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta$ using $\beta \cdot \Gamma$ func:

Sol

we know that

$$\frac{1}{2} B(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \longrightarrow ①$$

put $2m-1 = 6$ and $2n-1 = 7$ so that

$2m = 6+1$ and $2n = 7+1$

$m = 7/2$ $\quad n = 4$

Then eq ① becomes.

$$\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta = \frac{1}{2} B[7/2, 4]$$

$$= \frac{1}{2} \frac{\Gamma(7/2)\Gamma(4)}{\Gamma(7/2+4)}$$

$$= 1/2 \frac{\Gamma(7/2)\Gamma(3+1)}{\Gamma(15/2)} \quad \left\{ \Gamma\left(\frac{15}{2}\right) = \frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \Gamma(7/2) \right.$$

$$= \frac{1}{2} \cdot \frac{\cancel{\Gamma(7/2)} \cdot 3!}{\frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cancel{\Gamma(7/2)}}$$

$$= \frac{1}{2} \cdot \frac{3! \; 2^4}{13 \times 11 \times 9 \times 7}$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{3} \times \cancel{2} \times 16}{13 \times 11 \times \cancel{9}_3 \times 7}$$

$$\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta \cdot = \frac{16}{3003}$$

① $\int_0^1 x^3\sqrt{1-x}\,dx$ using β-Γ func.

Sol $\int_0^1 x^3\sqrt{1-x}\,dx = \int_0^1 x^3(1-x)^{1/2}\,dx$

$= \int_0^1 x^{4-1}(1-x)^{\frac{3}{2}-1}\,dx.$

$= B[4, 3/2]$ { using definition of Beta func

$= \dfrac{\Gamma(4)\Gamma(3/2)}{\Gamma(4+3/2)} = \dfrac{\Gamma(4)\Gamma(3/2)}{\Gamma(11/2)}$

$= \dfrac{\Gamma(3+1)\Gamma(\frac{1}{2}+1)}{} \Rightarrow \dfrac{\Gamma(3+1)\Gamma(3/2)}{(11/2-1)\Gamma(11/2-1)}$

$= \dfrac{\Gamma(3+1)\cancel{\Gamma(3/2)}}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\cancel{\Gamma(3/2)}}$ $\{\Gamma(n) = (n-1)\Gamma(n-1)\}$

$= \dfrac{3!\,2^4}{9.7.5.3}$

$= \dfrac{\cancel{3}\times 2\times 16}{9\times 7\times 5\times \cancel{3}}$

$\int_0^1 x^3\sqrt{1-x}\,dx. = \dfrac{32}{315}$

④ compute $\int_0^\infty e^{-x} x^3 dx$

Sol we know that $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma(n) \longrightarrow ①$

$\therefore \int_0^\infty e^{-x} x^3 dx = \int_0^\infty e^{-x} x^{4-1} dx$

$= \Gamma(4)$ using ①

$= \Gamma(3+1)$ $\{\Gamma(n+1) = n!\}$
if n is +ve Integer.

$= 3!$

$\int_0^\infty e^{-x} x^3 dx = 6$

SM. (1M)
short

⑤ Evaluate $\int_0^1 x^7 (1-x)^5 dx$ by using β-Γ func.

Sol

$\int_0^1 x^7 (1-x)^5 dx = \int_0^1 x^{8-1} (1-x)^{6-1} dx = B(8,6)$

$= \dfrac{\Gamma(8)\Gamma(6)}{\Gamma(8+6)}$

$= \dfrac{\Gamma(8)\Gamma(6)}{\Gamma(14)}$ $\{\Gamma(n+1) = n!\}$

$= \dfrac{\Gamma(7+1)\Gamma(5+1)}{\Gamma(13+1)}$

$= \dfrac{7!\,5!}{13!}$

$= \dfrac{\cancel{7!}\ \cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}}{13\times\underset{4}{\cancel{12}}\times 11\times\cancel{10}\times 9\times\underset{2}{\cancel{8}}\times\cancel{7!}}$

$= \dfrac{1}{13\times 44\times 18}$

$\int_0^1 x^7 (1-x)^5 dx = \dfrac{1}{10296}$

① Show that $4\int_0^\infty \frac{x^2}{1+x^4}\,dx = \sqrt{2}\,\pi$

Sol: $4\int_0^\infty \frac{x^2}{1+x^4}\,dx$

put $x = \sqrt{\tan\theta}$ so that $\left\{ \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}},\ \frac{d}{dx}\tan x = \sec^2 x \right\}$.

$dx = \frac{1}{2}\tan^{1/2-1}\theta \cdot \sec^2\theta\,d\theta$.

$dx = \frac{1}{2}\tan^{-1/2}\theta\,\sec^2\theta\,d\theta$.

Also when $x=0$, $\theta=0$ and when $x\to\infty$, $\theta\to\pi/2$

$$\therefore 4\int_0^\infty \frac{x^2}{1+x^4}\,dx = 4\int_0^{\pi/2} \frac{(\sqrt{\tan\theta})^2}{1+(\sqrt{\tan\theta})^4}\cdot\frac{1}{2}\tan^{-1/2}\theta\,\sec^2\theta\,d\theta.$$

$$= 4\int_0^{\pi/2} \frac{\tan\theta}{1+\tan^{4/2}\theta}\cdot\frac{1}{2}\tan^{-1/2}\theta\,\sec^2\theta\,d\theta$$

$$= \frac{4}{2}\int_0^{\pi/2} \frac{\tan^{1-1/2}\theta}{1+\tan^2\theta}\,\sec^2\theta\,d\theta \quad \{1+\tan^2\theta = \sec^2\theta\}$$

$$= 2\int_0^{\pi/2} \frac{\tan^{1/2}\theta}{\cancel{\sec^2\theta}}\,\cancel{\sec^2\theta}\,d\theta$$

$$= 2\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$$

$$= 2\int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\,d\theta.$$

$$4\int_0^\infty \frac{x^2}{1+x^4}\,dx = 2\int_0^{\pi/2} \sin^{1/2}\theta\cdot\cos^{-1/2}\theta\,d\theta \longrightarrow ①$$

we know that $\frac{1}{2}B(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta.$

put $2m-1 = 1/2$ and $2n-1 = -1/2$

$2m = 1/2+1$ and $2n = -1/2+1$

$2m = 3/2$ & $2n = 1/2$

$m = 3/4$ & $n = 1/4$

Then eq ① becomes.

$$4\int_0^\infty \frac{x^2}{1+x^4}dx = 2\int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta = \cancel{2}\frac{1}{\cancel{2}}B\left[\frac{3}{4}, \frac{1}{4}\right]$$

$$B\left[\frac{3}{4}, \frac{1}{4}\right] = \frac{\Gamma(3/4).\Gamma(1/4)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} \quad \left\{ B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right.$$

$$= \frac{\Gamma(3/4).\Gamma(1/4)}{\Gamma\left(\frac{\cancel{4}}{\cancel{4}}\right)}$$

$$= \frac{\Gamma(3/4).\Gamma(1/4)}{\Gamma(1)} \quad \{\Gamma(1) = 1\}.$$

$$= \Gamma(1/4)\,\Gamma(1-1/4)$$

$$= \frac{\pi}{\sin \pi/4} \quad \left\{ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \right\}.$$

$$= \frac{\pi}{1/\sqrt{2}}$$

$$4\int_0^\infty \frac{x^2}{1+x^4}dx. = \sqrt{2}\pi$$

② Find the value of $\Gamma(5/2)$

Sol $\Gamma(5/2) = (\frac{5}{2}-1)\Gamma(5/2-1)$ $\{\Gamma(n) = (n-1)\Gamma(n-1)\}$

$= \frac{3}{2}\Gamma(3/2)$

$= \frac{3}{2}[3/2-1]\Gamma(3/2-1)$

$= \frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)$

$= \frac{3}{4}\Gamma(1/2)$ $\{\because \Gamma(1/2) = \sqrt{\pi}\}$.

$\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$

③ Evaluate $\int_0^1 x^5(1-x)^3 dx$.

$\int_0^1 x^5(1-x)^3 dx = \int_0^1 x^{6-1}(1-x)^{4-1} dx$

$= B(6,4)$

$= \frac{\Gamma(6)\Gamma(4)}{\Gamma(6+4)}$ $\{\because \Gamma(n) = (n-1)!$

$= \frac{(6-1)!\,(4-1)!}{\Gamma(10)} = \frac{5!\,3!}{(10-1)!}$

$= \frac{5!\,3!}{9!}$

$= \frac{\cancel{5\times4\times3\times2\times1}\times\cancel{3}\times\cancel{2}\times1}{9\times8\times7\times\cancel{6}\times\cancel{5\times4\times3\times2\times1}}$

$= \frac{1}{9\times8\times7}$

$\int_0^1 x^5(1-x)^3 dx. = \frac{1}{504}$

Short - Part-A

1. Find the value of $\Gamma(-5/2)$
2. Evaluate $\int_0^1 x^7 (1-x)^3 dx$.
3. Compute $\beta\left[\frac{9}{2}, \frac{7}{2}\right]$
4. Relation b/w Beta & Gamma function. & {long Answer part B} (proof)
$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (1M)
5. Compute $\int_0^\infty e^{-2x} x^6 dx$.

Part-B

1. prove that $B(m,n) = B(m+1,n) + B(m,n+1)$
2. Evaluate $\int_0^\infty \frac{x}{1+x^6} dx$ in terms of Beta-Gamma func.
3. P.T $\int_0^1 x^m \log x^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is +ve integer $m > -1$,
4. S.T $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx, \; n > 0$
5. S.T $4\int_0^\infty \frac{x^2}{1+x^4} dx = \sqrt{2}\pi$.
6. Evaluate $\int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}$

① Find the value of $\Gamma(-5/2)$

Sol

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-5/2} \qquad \left\{ \Gamma(n) = \frac{\Gamma(n+1)}{n}. \right.$$

$$= -\frac{2}{5}\Gamma(-3/2)$$

$$= -\frac{2}{5}\frac{\Gamma(-3/2+1)}{-3/2} \Rightarrow \frac{-2}{5}\times\frac{-2}{3}\Gamma(-1/2)$$

$$= \frac{4}{15}\frac{\Gamma(-1/2+1)}{-1/2}$$

$$= \frac{-8}{15}\Gamma(1/2)$$

$$= -\frac{8}{15}\Gamma\left(\frac{1}{2}\right) \quad \left\{\Gamma(1/2)=\sqrt{\pi}\right\}$$

$$\Gamma\left(-\frac{5}{2}\right) = \frac{-8\sqrt{\pi}}{15}.$$

② $\int_0^1 x^7(1-x)^3\,dx$.

$$= \int_0^1 x^{8-1}(1-x)^{4-1}\,dx \qquad \left\{ B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \right\}.$$

$$= B(8,4) = \frac{\Gamma(8)\Gamma(4)}{\Gamma(8+4)}$$

$$= \frac{7!\,3!}{\Gamma(12)} = \frac{7!\,3!}{11!}$$

$$= \frac{7!\,3!}{11!} \qquad \left\{ \Gamma(n+1) = n! \right\}$$

$$= \frac{\cancel{7!}\,\cancel{3}\times\cancel{2}\times 1}{11\times 10\times\cancel{8}\times\cancel{9}\times\cancel{7!}} = \frac{1}{1320}$$

$$\begin{array}{r} 120 \\ \times 11 \\ \hline 120 \\ 120 \\ \hline 1320 \end{array}$$

(3) $B\left(\frac{9}{2},\frac{7}{2}\right) = \dfrac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{2})}{\Gamma(\frac{9}{2}+\frac{7}{2})} = \dfrac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{2})}{\Gamma(\frac{\cancel{16}^{8}}{\cancel{2}})}$

$$= \frac{\Gamma(\frac{9}{2})\Gamma(7/2)}{\Gamma(8)}$$

$\{ \Gamma(n) = (n-1)\Gamma(n-1) \}$, $\Gamma(n) = (n-1)!$ & $\Gamma(n+1) = n! \}$

$$= \frac{(\frac{9}{2}-1)\Gamma(\frac{9}{2}-1)\ (\frac{7}{2}-1)\Gamma(7/2-1)}{7!}$$

$$= \frac{\frac{\cancel{7}}{2}\times\frac{5}{2}\times\frac{\cancel{3}}{\cancel{7}}\cdot\frac{1}{2}\Gamma(\frac{1}{2})\cdot\frac{\cancel{5}}{2}\times\frac{\cancel{3}}{2}\times\frac{1}{2}\Gamma(\frac{1}{2})}{\cancel{7}\times\cancel{6}\times\cancel{5}\times4\times\cancel{3}\times2\times1}$$

64
×8
2 3

$$= \frac{5\sqrt{\pi}\times\sqrt{\pi}}{64\times4\times2} = \frac{5}{512}\pi$$

(5) $\int_0^\infty e^{-2x} x^6 dx.$

put $2x = y$ so that $dx = \frac{1}{2}dy.$

$$\int_0^\infty e^{-2x} x^6 dx = \int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y}\cdot\frac{1}{2}dy$$

$$= \frac{1}{2^7}\int_0^\infty y^6 e^{-y} dy$$

$$= \frac{1}{2^7}\int_0^\infty e^{-y}\cdot y^{7-1}\cdot dy. = \frac{1}{2^7}\Gamma(7)$$

$$= \frac{1}{2^7}\times 6! = \frac{45}{8} \quad \{\Gamma(n) = (n-1)!\}.$$

① Evaluate $\int_0^\infty \frac{x}{1+x^6}\,dx$.

Sol put $x^6 = y$ Then $6x^5 dx = dy$.

$$x\,dx = \frac{dy}{6x^4} = \frac{dy}{6(y^{1/6})^{4}} \quad \{ x = y^{1/6} \}.$$

$$x\,dx = \frac{dy}{6y^{2/3}}$$

Also when $x \to 0$, $y \to 0$
when $x \to \infty$, $y \to \infty$

$$\int_0^\infty \frac{x}{1+x^6}\,dx = \frac{1}{6}\int_0^\infty \frac{1}{1+y}\cdot\frac{1}{y^{2/3}}\cdot dy.$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{-2/3}}{1+y}\,dy.$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{1/3-1}}{(1+y)^{1/3+2/3}}\,dy. \quad \left\{ \text{form ① } B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \right\}.$$

$$= \frac{1}{6} B\left(\frac{1}{3},\frac{2}{3}\right) \quad \left\{ B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right.$$

$$= \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(2/3\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}$$

$$= \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}{\Gamma\left(3/3\right)} \quad \left\{ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(\pi/3)} \right\}$$

$$= \frac{1}{6}\,\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-1/3\right)$$

$$= \frac{1}{6}\cdot\frac{\pi}{\sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{1}{\cancel{6}3}\times\frac{\pi}{\frac{\sqrt{3}}{\cancel{2}}} = \frac{\pi}{3\sqrt{3}}.$$

P.T

(3) $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$

Prof

put $\log x = -t$

i.e $x = e^{-t}$

$dx = -e^{-t} dt$

$\left\{ e^{\infty} = \infty \atop e^{-0} = 0 \right\}$

Also when $x=0$, $t=\infty$; when $x=1$, $t=0$.

$$\int_0^1 x^m (\log x)^n dx = \int_\infty^0 (e^{-t})^m (-t)^n (-e^{-t}) dt$$

$$= \int_\infty^0 e^{-tm} (-e^{-t}) \cdot (-1)^n t^n dt$$

$$= -\int_0^\infty -e^{-tm-t} \cdot (-1)^n t^n dt$$

$\left\{ \int_a^b f(x)dx = -\int_b^a f(x)dx \right\}$

$$= (-1)^n \int_0^\infty e^{-t(m+1)} \cdot t^n dt$$

$$= (-1)^n \int_0^\infty e^{-t(m+1)} \cdot t^{(n+1)-1} \cdot dt$$

$\left\{ \int_0^\infty e^{-kx} \cdot x^{n-1} dx = \frac{\Gamma(n)}{k^n}, n>0, k>0 \right\}$.

$$= (-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

$$= \frac{(-1)^n n!}{(m+1)^{n+1}}.$$

$$\therefore \int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

(5) Evaluate $\int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}$

P.T.

Sol

put $x^2 = y$ (i.e) $x = y^{1/2}$ so that

$$dx = \frac{1}{2} y^{1/2 - 1} dy$$

$$dx = \frac{1}{2} y^{-1/2} dy.$$

Also when $x = 0$, $y = 0$, when $x \to \infty$, $y \to \infty$.

$$\int_0^\infty x^4 e^{-x^2} dx = \int_0^\infty y^2 \cdot e^{-y} \cdot \frac{1}{2} y^{-1/2} dy.$$

$$= \frac{1}{2} \int_0^\infty y^{2 - \frac{1}{2}} \cdot e^{-y} dy.$$

$$= \frac{1}{2} \int_0^\infty y^{3/2} e^{-y} dy$$

$$= \frac{1}{2} \int_0^\infty e^{-y} \cdot y^{5/2 - 1} dy.$$

$$\left\{ \int_0^\infty e^{-x} \cdot x^{n-1} dx = \Gamma(n) \right\}.$$

$$= \frac{1}{2} \Gamma\left(\frac{5}{2}\right) \qquad \left\{ \Gamma(n) = (n-1)\Gamma(n-1) \right\}.$$

$$= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{3}{8} \sqrt{\pi}$$