Unit-2

Set Theory

Set: A collection of well-defined objects is called a set.

Elements of a set are usually denoted by lower-case letters while sets are denoted by capital letters

Ex: The vowels

$$V = \{a, e, i, o, u\}$$

Set of Natural numbers

$$N = \{1, 2, 3, 4 \ldots\}$$

Set of whole Numbers

$$W = \{0, 1, 2, 3, \ldots\}$$

There are 2 types of set

Notations

1) Roster form
2) set builder form

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1) Roster form

It is the list of all elements of the set.

2) set-builder form

All the elements of the set must have same property

Ex:- $S = \{1, 3, 5, 7 \ldots\}$

All odd numbers.

$S = \{x / x$ is an odd number$\}$

cardinality of set:-

The no. of elements present in the set. Denoted as $|S|$.

Ex:- 1) Let 'S' be the set of odd +ve numbers less than 10

$S = \{1, 3, 5, 7, 9\}$

then cardinality of set is

$|S| = 5.$

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2) Let 'A' be the set of letters in English alphabets. Then cardinality of A is $|A| = 26$.

Subset [$\subseteq$]

If every element in set A is also the element of set B then A is called subset of B. It is denoted by $A \subseteq B$

Ex:- $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$

$A \subseteq B$

But $B \nsubseteq A$.

Note:- Null set is a subset of every set.

Equality of sets

Two sets A & B are equal if and only if $A \subseteq B$ & $B \subseteq A$.

Ex:- $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$

$\because A \subseteq B$ & $B \subseteq A$

$\therefore$ We can say that $A = B$.

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Types of sets :

Empty set (or) **void set** (or) **Null set**

$\phi = \{ \}$

No. elements in the set.

Finite set: A set with countable number of elements.

Ex: $A = \{1, 2, 3\}$

Infinite set: A set with uncountable number of elements

Ex: $A = \{1, 2, 3, \ldots\}$

Singleton set: A set with only one element. Ex: $A = \{1\}$

Proper subset: [c]

~~$A \subset B$~~

~~If A is~~

Proper subset :

If $A \subseteq B$ and $B \nsubseteq A$ then

$A \subset B$

Ex: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$

$\because A \subseteq B$ & $B \nsubseteq A$

$\therefore A \subset B$

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Power set: It is the set of all subsets of A

Denoted by P(A)

Ex: If $A = \{1, 2\}$ $\rightarrow 2^n \Rightarrow 2^2 = 4$

then $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Universal set:

It is the union of all the sets denoted by U (or) E

Ex: If $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{a, b\}$

then $U = \{1, 2, 3, 4, a, b\}$

Set operations

Union $[A \cup B]$

All elements in A or B.

Intersection $[A \cap B]$

Common elements

Difference $[A - B]$

Elements in A but not in B.

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complement (A^c or A'):

Elements in universal set U but not in A. $[A^c = U - A]$

Symmetric difference:

$$\boxed{A \Delta B = (A-B) \cup (B-A)}$$

Ex:- Let $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$

$A \cup B = \{1,2,3,4,5,6\}$

$A \cap B = \{3,4\}$

$A - B = \{1,2\}$,

$B - A = \{5,6\}$

$A \Delta B = (A-B) \cup (B-A)$

$A \Delta B = \{1,2,5,6\}$.

$A^c =$ If $U = \{1,2,3,4,5\}$, $A = \{1,3,5\}$

$A^c = U - A = \{2,4\}$.

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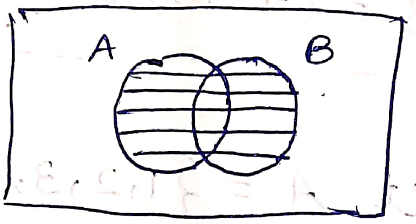
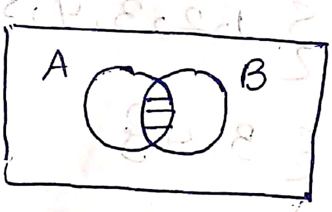
Venn Diagrams

Visual representation of sets using circles inside a rectangle [the universal set]

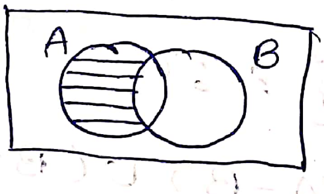
$A \cup B$ =



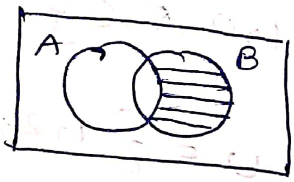
$A \cap B$ =



$A - B$ =

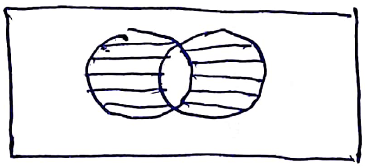


$B - A$ =



$A \Delta B = (A-B) \cup (B-A)$

(or) $A+B$ =



Laws of Set Theory

1) Identity law

$A \cup \phi = A$ $\quad$ $A \cap U = A$

2) Idempotent law

$A \cup A = A$ $\quad$ $A \cap A = A$

3) Commutative law

$A \cup B = B \cup A$ $\quad$ $A \cap B = B \cap A$

4) Associative law

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

5) Distributive law

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$\cancel{(A \cap B) \cap}$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6) Absorption law

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A.$

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7) De-morgan's law

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

8) complement law

$(A')' = A$ (or)

$A \cup A' = U$, $A \cap A' = \phi$

State and prove Distributive laws.

Sol:- Let 'x' is an arbitrary element of $A \cup (B \cap C)$

let $x \in A \cup (B \cap C)$

$\Leftrightarrow$ $x \in A$

$\Leftrightarrow$ $x \in A$ or $x \in (B \cap C)$

$\Leftrightarrow$ $x \in A$ (or) ($x \in B$ and $x \in C$)

$\Leftrightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Leftrightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

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$\Leftrightarrow x \in (A\cup B)\cap(A\cup C)$

$\therefore A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ //

Prove De-morgan's law.

$(A\cup B)' = A'\cap B'$

$\Rightarrow x \in (A\cup B)$

$x\in A$ or $x\in B$

$\Rightarrow x \in (A\cap B)$

$x\in A$ and $x\in B$

$\Rightarrow x\notin A\cup B$

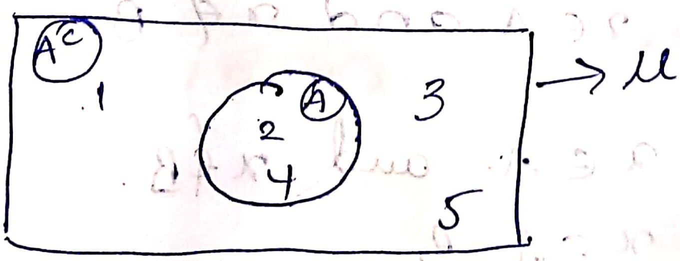
$x\notin A$ and $x\notin B$

$\Rightarrow x\notin A\cap B$

$x\notin A$ or $x\notin B$

Ex: let $U=\{1,2,3,4,5\}$, $A=\{2,4\}$

$A^c = U - A = \{1,3,5\}$



$x\in A^c$ then $x\notin A$

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Prove De-morgan's law

$(A \cup B)' = A' \cap B'$

Sol: L.H.S :- Let $x \in (A \cup B)'$

$\Leftrightarrow \quad x \notin (A \cup B)$

$\Leftrightarrow \quad x \notin A$ and $x \notin B$

$\Leftrightarrow \quad x \in A'$ and $x \in B'$

$\Leftrightarrow \quad x \in (A' \cap B') \Rightarrow$ R.H.S //.

$\therefore (A \cup B)' = (A' \cap B')$ //.

S.T $A - (A \cap B) = A - B$

Sol: Let $x \in A - (A \cap B)$

$\Leftrightarrow \quad x \in A$ and $x \notin A \cap B$

$\Leftrightarrow \quad x \in A$ and $\{x \notin A$ or $x \notin B\}$

$\Leftrightarrow \quad x \in A$ and $x \notin A$ (or)

$x \in A$ and $x \notin B$

$\Leftrightarrow \quad x \in A$ and $x \notin B$

$\Leftrightarrow \quad x \in A - B$

If $A = \{2, 3, 4\}$, $B = \{1, 2\}$, $C = \{4, 5, 6\}$

find $A + B$, $B + C$, $A + B + B + C$.

Sol: $A + B =$

$+$ & Δ are equal

$\boxed{\because A \Delta B = (A - B) \cup (B - A)}$

(i) $[A + B = (A - B) \cup (B - A)]$

$\because A - B = \{3, 4\}$

$B - A = \{1\}$

$A + B = \{3, 4\} \cup \{1\} = \{1, 3, 4\}$

(ii) $B + C$

$\therefore [B + C = (B - C) \cup (C - B)]$

$\Rightarrow B - C = \{1, 2\}$, $C - B = \{4, 5, 6\}$

$\therefore B + C = \{1, 2\} \cup \{4, 5, 6\} = \{1, 2, 4, 5, 6\}$

(iii) $A + B + B + C$

$\Rightarrow (A - B) \cup (B - A) \quad (B - C) \cup (C - B)$

$$\text{(iii)}\quad \overset{Ⓐ}{A+B} + \overset{Ⓑ}{B+C}$$

Sol: $((A+B) - (B+C)) \cup ((B+C) - (A+B))$

$\Rightarrow [\{1,3,4\} - \{1,2,4,5,6\}] \cup [\{1,2,4,5,6\} - \{1,3,4\}]$

$\Rightarrow \{3\} \cup \{2,5,6\}$

$\Rightarrow \{2,3,5,6\}.$

$$\text{(iv)}\quad \overset{Ⓐ}{A+B} + \overset{Ⓑ}{C}$$

Sol

$\Rightarrow ((A+B) - C) \cup (C - (A+B))$

$\Rightarrow [\{1,3,4\} - \{4,5,6\}] \cup [\{4,5,6\} - \{1,3,4\}]$

$\Rightarrow \{1,3\} \cup \{5,6\}$

$\Rightarrow \{1,3,5,6\}.$

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cartesian product

If A and B are 2 sets, the cartesian product is

$A \times B = \{ (a,b) / a \in A, b \in B \}$

Ex:- If $A = \{1,2,3\}$, $B = \{4,5\}$

then $A \times B$

$= \{ (1,4), (1,5), (2,4), (2,5), (3,4), (3,5) \}$

$B \times A = \{ (4,1), (4,2), (4,3), (5,1), (5,2), (5,3) \}$.

$\therefore A \times B \neq B \times A$

If $A = \{2,4,6,8\}$, $B = \{1,3,5,7,9\}$

Find the cartesian product $A \times B$ and $B \times A$

Sol:- $A \times B = \{ (2,1), (2,3), (2,5), (2,7), (2,9), (4,1), (4,3), (4,5), (4,7), (4,9), (6,1), (6,3), (6,5), (6,7), (6,9), (8,1), (8,3), (8,5), (8,7), (8,9) \}$

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$B \times A = \{$

$A = \{2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$

$B \times A = \{(1,2), (1,4), (1,6), (1,8),$
$(3,2), (3,4), (3,6), (3,8), (5,2)$
$(5,4), (5,6), (5,8), (7,2), (7,2),$
$(7,6), (7,8), (9,2), (9,4),$
$(9,6), (9,8)\}$

next sum

Relation

Any subset of $A \times B$ is called a relation

Ex: If $A \times B = \{(a,1), (a,2), (a,3)\}$

$\{(a,1), (a,2)\}$ is a relation because it is subset of $A \times B$.

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Domain and Range of Relations

Domain of $R = \{x / \forall (x, y) \in R\}$

Range of $R = \{y / \forall (x, y) \in R\}$

Ex:- If $R = \{(1, x), (2, y), (2, z)\}$

Domain $= \{1, 2\}$, Range $= \{x, y, z\}$.

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for sets $P = \{a, b, c, d\}$, $Q = \{c, d, e, f, g\}$

find $P \cup Q$, $P - Q$, $Q - P$, $P \cap Q$

Sol:- $P \cup Q = \{a, b, c, d, e, f, g\}$

$P - Q = \{a, b\}$

$Q - P = \{e, f, g\}$

$P \cap Q = \{c, d\}$.

Types of Relations

1) Reflexive Relation
2) Irreflexive "
3) Symmetric "
4) Anti symmetric "
5) Transitive "

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1) Reflexive Relation

R is reflexive if

$(a,a) \in R \ \forall \ a \in A$.

2) Irreflexive:

If $(a,a) \notin R \ \forall a \in A$

3) Symmetric:-

if $(a,b) \in R$ then $(b,a) \in R$.

4) Antisymmetric

if $(a,b) \in R$, $(b,a) \in R$

then $a = b$.

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5) Transitive:

if $(a,b) \in R$, $(b,c) \in R$ then

$(a,c) \in R$.

Ex If $X = \{1,2,3\}$

then ordered pairs

$R = \{(1,1),(1,2),(1,3),(2,1),(2,2),$

$(2,3),(3,1),(3,2),(3,3)\}$.

$\because \left.\begin{matrix}(1,1)\\(2,2)\\(3,3)\end{matrix}\right\} \in R \rightarrow \therefore$ Reflexive.

$\Rightarrow (1,1),(2,2),(3,3) \notin R$. then irreflexive.

$\Rightarrow$ let $X = \{a,b,c,d\}$

$R = \{(a,a),(b,c),(c,b),(d,d)\}$

$\downarrow$

It is symmetric

$\Rightarrow$. $(x,y),(y,x) \in R$ only if $x=y$ $\nRightarrow$

$\{(2,3),(3,2),(2,2)\}$ X

$\{(1,1),(2,2),(3,3)\}$.

$\downarrow$

Antisymmetric

$\{(1,1),(2,2),(1,3)\}$ ✓ $\leftarrow (3,1)$ X

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⇒ Transitive :-

If $A = \{1, 2, 3\}$.

$R = \{(1,2), (2,3), (1,3)\}$

$$\begin{array}{l} 1,2 \\ 2,3 \\ \hline 1,3 \end{array}$$ ✓

If $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

then find if relation is reflexive, symmetric, Antisymmetric and transitive.

Sol:- Given $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$\because \{(1,1), (2,2), (3,3), (4,4)\} \in R$

$\therefore$ It is reflexive.

(ii) Symmetric

$\because \{(1,2), (2,1)\} \in R$

It is symmetric

(iii) Antisymmetric

$\because$ $(2,1) \notin R$.

It is not anti-symmetric

(iv) transitive

$\because$ $\{(1,2),(2,1),(1,1)\} \in R$

$$\begin{array}{l} 1,2 \\ 2,1 \\ \hline 1,1 \end{array}$$

$\therefore$ It is transitive.

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2) Let $A = \{1,2,3,4\}$.

$R = \{(1,1),(1,2),(2,3),(1,3),(4,4)\}$

Sol:- ~~Rot~~ Not reflexive

$\{\cancel{(1,1)},(2,2),(3,3),\cancel{(4,4)}\} \notin R$

Not symmetric $(1,2) \in R$

$\{(1,2),(2,1)\} \notin R$ but $(2,1) \notin R$

$\Rightarrow$ It is Anti-symmetric.

$\Rightarrow$ It is transitive.

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Equivalence Relation

A relation is said to be equivalence if it is

(i) reflexive.

(ii) symmetric

(iii) Transitive

Poset:

A relation is said to be poset if it is

(i) Reflexive

(ii) Antisymmetric

(iii) Transitive.

Ex If $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,4), (2,2), (2,3),$
$(4,1), (4,4), (3,2), (3,3)\}$

P.T R is an equivalence Relation.

Sol: $\because \{(1,1), (2,2), (3,3), (4,4)\} \in R$ $\therefore (2,2) \in R$

(i) $\therefore$ R is reflexive.

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(ii) R is symmetric

for each $(x,y) \in R$ we have $(y,x) \in R$.

$(1,4) \in R \rightarrow (4,1) \in R$.

$(2,3) \in R \rightarrow (3,2) \in R$

$\therefore$ R is symmetric

$\Rightarrow$ R is transitive.

$\forall (x,y) \in R, (y,z) \in R$ we have $(x,z) \in R$.

$$\frac{\begin{matrix}(1,4)\\(4,1)\end{matrix}}{(1,1) \in R} \qquad \frac{\begin{matrix}2,3\\3,2\end{matrix}}{(2,2) \in R.}$$

$\therefore$ R is reflexive, symmetric, transitive.

$\therefore$ R is equivalence relation.

Matrix representation of relation

A relation can be represented by a matrix called Relation matrix and is defined by $M_R = \{1$

$M_R = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

Ex:- Let $A = \{1, 2, 3, 4\}$ and $R = \{(x, y) / x > y\}$ Find the matrix relation and directed Graph of R.

Sol:- Given :- $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,1),$
$(2,2), (2,3), (2,4), (3,1), (3,2),$
$(3,3), (3,4), (4,1), (4,2), (4,3),$
$(4,4)\}$ $\longrightarrow$ ordered pairs.

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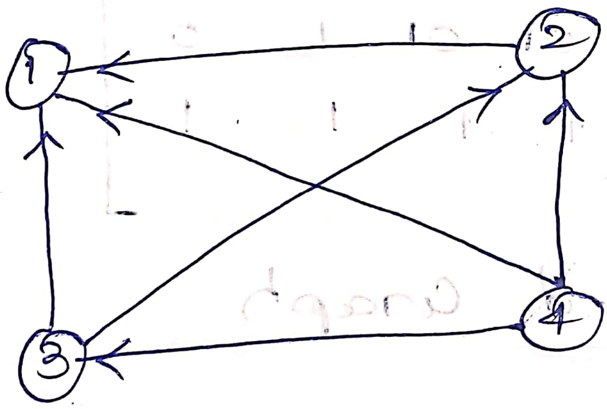
∵ Given $R = \{(x, y) \mid x > y\}$.

$\Rightarrow \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$$\therefore M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Di-graph [Directed Graph]



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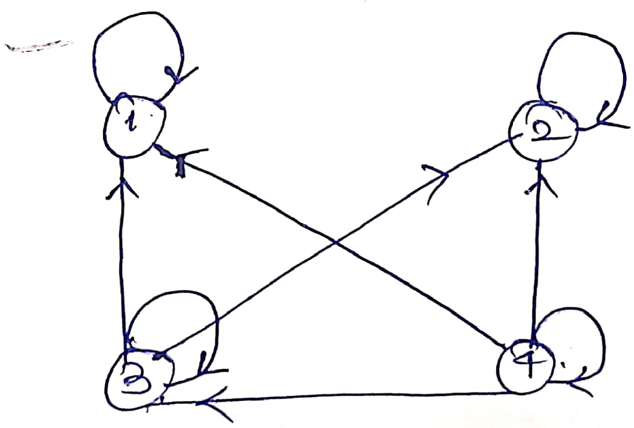
2) If $A = \{1, 2, 3, 4\}$ defined by $R = \{(1,1), (2,2), (3,3), (4,4), (4,3), (4,2), (4,1), (3,2), (3,1)\}$ Find the matrix relation and directed graph.

Sol: Given: $A = \{1, 2, 3, 4\}$

R is given.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Directed Graph



Is 4 divisible by 2?
2 divides 4
$4/2$

2 divides 4 or not ? $\rightarrow 4/2$

Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and

$R = \{(x, y) / x - y$ is divisible by $3\}$

in X. S.T 'R' is an equivalence relation

Sol:. Given:- $X = \{1, 2, 3, 4, 5, 6, 7\}$

$R = \{(x, y) / (x - y)$ is divisible by $3\}$

Reflexive

$\forall\ x \in X$, we have $x - x = 0$

is divisible by 3

$\therefore (x, x) \in R\ \forall\ x \in X$.

$\therefore$ R is reflexive.

Symmetric :-

Let $x, y \in X$ and $(x, y) \in R$

then $x-y$ is divisible by 3

$\Rightarrow$ $-(x-y)$ is also divisible by 3

$\Rightarrow$ $y-x$ is also divisible by 3.

$\therefore$ $(y, x) \in R$

$\therefore$ $(x, y) \in R \rightarrow (y, x) \in R$.

$\therefore$ R is symmetric

Transitive :-

Let $x, y, z \in X$ & $(x, y) \in R$, $(y, z) \in R$ then

$\frac{x-y}{3} = k_1$ $\quad x-y = 3k_1$ & $y-z = 3k_2$

$\Rightarrow x-y+y-z = 3k_1 + 3k_2$

$\Rightarrow$ $x-z = 3(k_1+k_2)$ is divisible by 3

$\Rightarrow$ $(x, z) \in R$

$\therefore$ R is Transitive

$\therefore$ It is equivalence

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composition of Relations

If $A = \{1,2,3,4\}$ and R and S are relations on a set A defined by

$R = \{(1,2),(1,3),(2,4),(4,4)\}$

$S = \{(1,1),(1,2),(1,3),(1,4),(2,3),(2,4)\}$

find ROS, SOR, ROR, SOS, $RO(SOR)$, $(ROS)OR$, $(RORO)OR$, $(SOS)OS$, $R \cup S$, $R \cap S$, S^c, S^{-1}, $Dom(R)$, $Range(R)$, $Dom(S)$, $Range(S)$, $Dom(R \cup S)$, $Range(R \cap S)$

Sol:. ROS

$\Rightarrow \{(1,3),(1,4)\}$

SOR

$\Rightarrow \{(1,2),(1,3),(1,4),(2,4)\}$

$R^2 = ROR$

$\Rightarrow \{(1,4),(2,4)\}$

$S^2 = SOS$

$= \{(1,2),(1,3),(1,4)\}$

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$R \circ (S \circ R) = \{(1,4)\}$

$(R \circ S) \circ R = \{(1,4)\}$

$(R \circ R) \circ R = \{(1,4), (2,4)\}$

$(S \circ S) \circ S = \{(1,3), (1,4)\}$

$R \cup S = \{(1,1), (1,2), (1,3), (1,4),$
$(2,3), (2,4), (4,4)\}$

$R \cap S = \{(1,2), (1,3), (2,4)\}$

~~S'~~ =

$S^{-1} = \{(1,1), (2,1), (3,1), (4,1), (3,2),$
$(4,2)\}$

$D(R) = \{1,2,4\}$

$R(R) = \{2,3,4\}$

$D(S) = \{1,2\}$

$R(S) = \{1,2,3,4\}$

$D(R \cup S) = \{1,2,4\}$

$R(R \cap S) = \{2,3,4\}$

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$A = \{1,2,3,4\}$

$A = \{1,2,3,4\}$

$S^c = A - S$

$\Rightarrow (A \times A) - S$

$\because A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2),$
$(2,3), (2,4), (3,1), (3,2), (3,3), (3,4),$
$(4,1), (4,2), (4,3), (4,4)\}$

$\because (A \times A) - S \Rightarrow$

$S = \{(1,1), (1,2), (1,3), (1,4), (2,3), (2,4)\}$

$\because (A \times A) - S \Rightarrow \{(2,1), (2,2), (3,1), (3,2),$
$(3,4), (3,3), (4,4), (4,1), (4,2), (4,3)\}.$

H.W.: let $A = \{a,b,c,d\}$ defined by.

$R \Rightarrow \{(a,a), (a,c), (c,b), (c,d), (d,b)\}.$

$S = \{(b,a), (c,c), (c,d), (d,a)\}$

Find (i) $R \circ S$

(ii) $S \circ R$

(iii) $R \circ R$

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congruence modulo m:

for any positive integers x, y, m if m divides $x-y$ then x and y are said to be congruent modulo m and is written as

$x \equiv y \pmod{m}$

divides

Ex:- $5 \equiv 3 \pmod 2$ $\quad 2/5-3$

$14 \equiv 2 \pmod 3$ $\quad 3/14-2$

$\boxed{\text{rem } 0}$

P.T the congruence relation is an equivalence relation.

Pf: (i) Reflexive

As $x-x=0$ is divisible by m

i.e, $x \equiv x \pmod m$

divides
$m/x-x$

(ii) Symmetric

$\because x \equiv y \pmod m$

$m/x-y$

similarly $\Rightarrow m/y-x$ also

Ex:- 3 divides 7-1

$3/7-1 \Rightarrow 3/6$

$\therefore y \equiv x \pmod{m}$

$\therefore$ It is Symmetric

(iii) Transitive :-

$\because x \equiv y \pmod{m}$, $y \equiv z \pmod{m}$

$m | x-y$ and $m | y-z$.

$\Rightarrow m | x-y+y-z \Rightarrow m | x-z$

$\Rightarrow x \equiv z \pmod{m}$.

$\therefore$ congruence modulo 'm' is reflexive, symmetric and transitive.

hence it is equivalence relation //.

S.T the relation "greater than or equal to" is a partial ordering on the set of integers.

Sol: let Z^+ be the set of all positive integers

Given $R = \{(x,y) / x \geq y\}$ on Z^+.

(i) Reflexive:

$\forall x \in Z^+$, we have $x \geq x$

$\therefore (x,x) \in R \quad \forall x \in Z^+$

$\therefore$ R is reflexive

(ii) Anti-symmetric

$\forall x,y \in Z^+$, clearly $x \geq y$ and $y \geq x \Rightarrow x = y$

i.e., $(x,y) \in R$ & $(y,x) \in R$

then $x = y$

$\therefore$ R is Anti-symmetric.

(iii) Transitive

$\forall x,y,z \in Z^+$, If $x \geq y$ & $y \geq z$ then $x \geq z$

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i.e., $(x,y) \in R$ & $(y,z) \in R$ then
$(x,z) \in R$

hence $(Z^+, \geq)$ is a POSET

Partial ordering set [POSET]

A set P with a partial ordering "$\leq$" is called a partially ordered set (OR) POSET.
It is denoted by $(P, \leq)$.

Hasse-Diagram

A partial ordering $(\leq)$ on a set 'P' can be represented by means of a diagram known as Hasse diagram of $(P, \leq)$.

→ In Hasse-diagram, each element is represented by a small circle or a dot.

→ we donot put arrows on edges and we donot draw self-loops at vertices

→ There will be no transitive edges

✓ A . B C   A B C ✗

→ Diagram will be from bottom to top.

→ There will be no arrow heads.

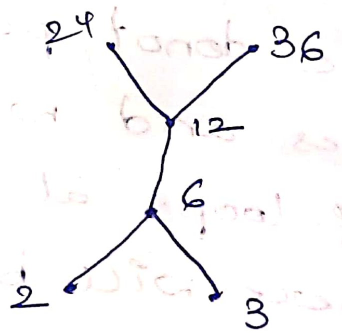
Q) let $X = \{2, 3, 6, 12, 24, 36\}$ and $\leq$ be the relation "divides". Draw the hasse-diagram of $(X, \leq)$.

(or)

Q) Draw the hasse-diagram for the divisibility relation ~~on~~ defined on set $A = \{2, 3, 6, 12, 24, 36\}$

Sol:- $R = \{(2,2), (2,6), (2,12), (2,24), (2,36),$
$\leq$?
$(3,3), (3,6), (3,12), (3,24), (3,36),$
$(6,6), (6,12), (6,24), (6,36), (12,12),$
$(12,24), (12,36), (24,24),$ (~~24~~, $(36,36)\}$

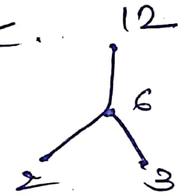
Hasse-diagram



∵ 2 doesn't 3
so 2 & 3 are represented with seperate dots

→ ∵ 2 divides 6
& 3 divides 6 also, so 6
in the middle of 2 & 3

→ ∵ 6 divides 12
so 6 to 12 one edge.

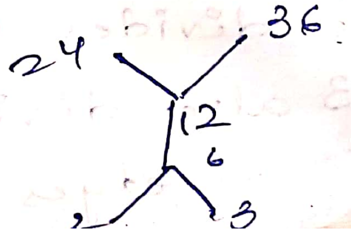


* 2 & 3 also divides 12,
but they are transitive edges

2, 6	3, 6
6, 12	6, 12
2, 12	3, 12

transitive edges

→ ∵ 12 divides 24
12 also divides 36.
so an edge from 12 to 24 & 36

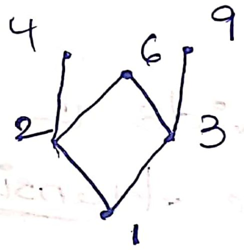


2) Draw hasse-diagram for $\{1, 2, 3, 4, 6, 9\}$ for divisibility

sol: $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,9), (2,2), (2,4), (2,6), (3,3), (3,6), (3,9), (4,4), (6,6), (9,9)\}$

Hasse-diagram

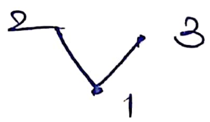
$\{1, 2, 3, 4, 6, 9\}$



$\because$ 1 divides 2

1 divides 3 also.

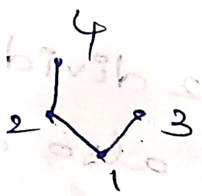
so edges.



---

2 divides 4

so 2 to 4 edge

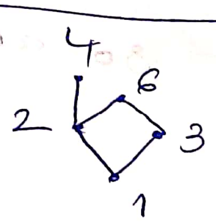


---

2 divides 6.

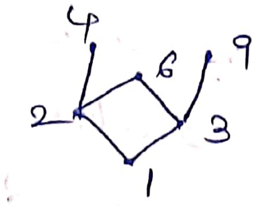
3 also divides 6

so edges from 2 & 3 to 6



3 divides 9

so 3 to 9 edge

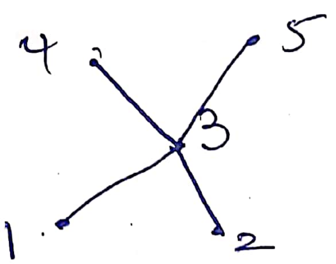


→ Draw the hasse-diagram for the relation R on $A = \{1,2,3,4,5\}$. whose relation matrix is given below:

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Sol: let $R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (5,5)\}$.

Hasse-diagram $\{1,2,3,4,5\}$



∵ $1 \leq 2$ [But not in relation]

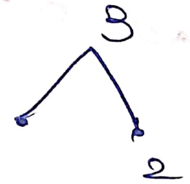
So no edge for 1 to 2

$\because 1 \leq 3$

2 also $2 \leq 3$

so edge from

1,2 to 3.

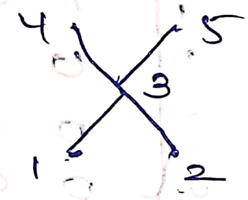


$\because 3 \leq 4$

3 also less than 5

$3 \leq 5$

so edges from 3 to 4 & 5.



Draw Hasse diagram for

$\{1, 3, 5, 9, 15, 45\}$ for divisibility

Sol:. $R = \{(1,1), (1,3), (1,5), (1,9), (1,15), (1,45),$

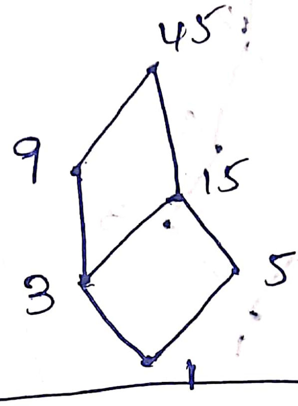
$(3,3), (3,9), (3,15), (3,45), (5,5),$

$(5,15), (5,45), (9,9), (9,45), (15,15)$

$(15,45), (45,45)\}.$

Hasse- Diagram

$\{1, 3, 5, 9, 15, 45\}$



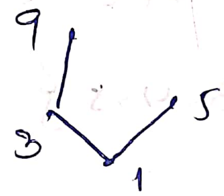
∵ 1 devides 3

1 also divides 5

so edge from 1 to 3 & 5

5 doesn't divides 9

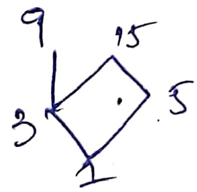
but, 3 divides 9, so 3 to 9 → edge



9 doesn't divide 15

but 3 & 5 divide 15

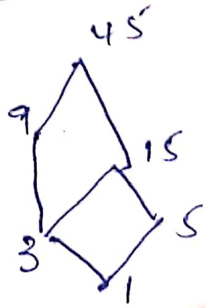
so 3 & 5 to 15 → edge.



Both 15 & 9 divides 45

so edge from 15, 9 to 45

→ no transitive edges should be drawn



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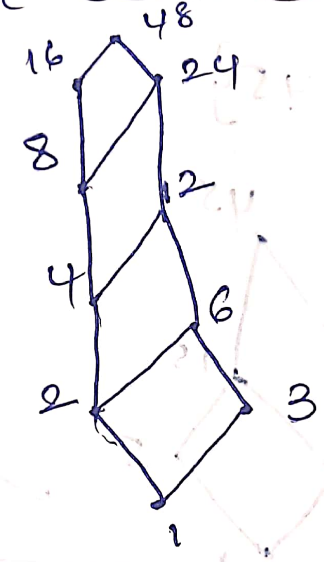
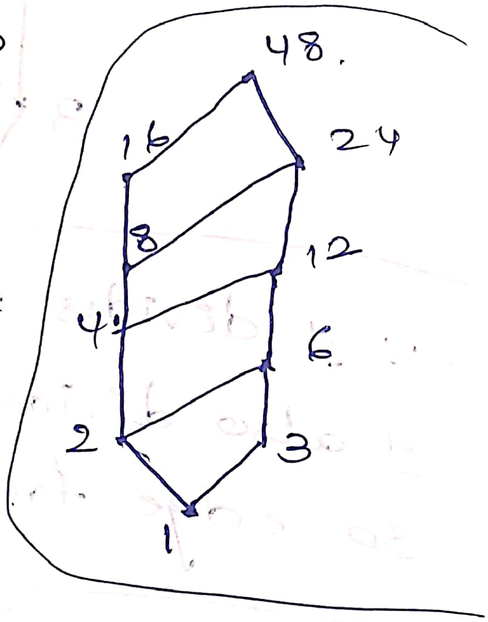
Q) Let $A = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$

Sol :-



(or)

neatly ⇒



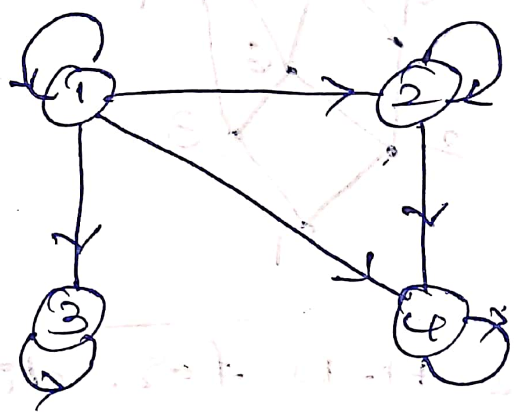
Q) Draw Hasse diagram for $\{1, 2, 3, 4, 5\}$ for $\leq$

∴ Totally ordered set.

→ A partial ordering R on a set $A = \{1, 2, 3, 4\}$ is represented by the following digraph. Draw the hasse-diagram of $(R, \leq)$. → $\leq = /$

Sol :- $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), ~~(2,3)~~, (2,4), (3,3), (4,4)\}$.

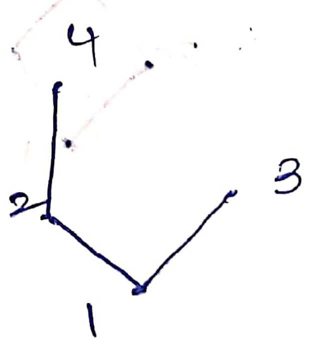
Di-graph



Hasse-Diagram

divides

$\{1, 2, 3, 4\}, /$



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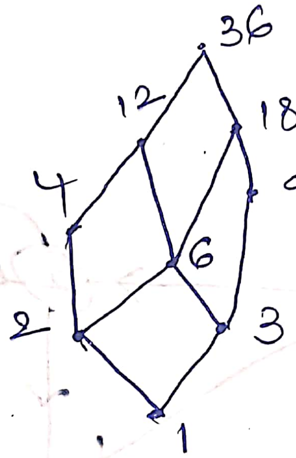
→ Draw hasse-diagram for divisors of 36 i.e, $(D_{36}, |)$

Sol: $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

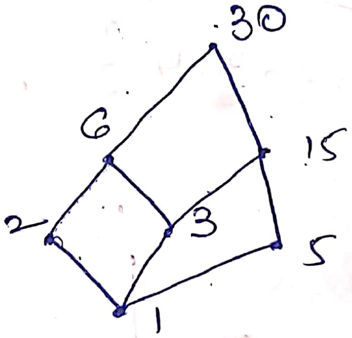
$R = \{(1,1), (1,2),$

$\therefore A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



→ Draw H-D for divisibility for $A = \{1, 2, 3, 5, 6, 15, 30\}$

Sol:-



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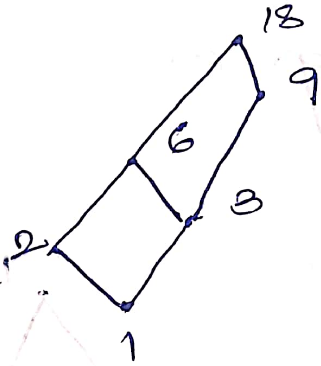
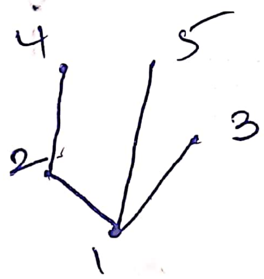
H.D for $[\{1, 2, 3, 6, 9, 18\}, |]$

Sol:- $\{1, 2, 3, 6, 9, 18\}$

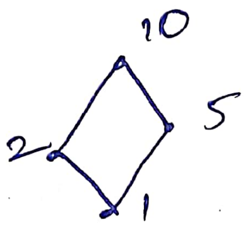


$\{1, 2, 3, 4, 5, |\}$



D(10)

let $A = \{1, 2, 5, 10\}$



D(15), D(32), D(45).

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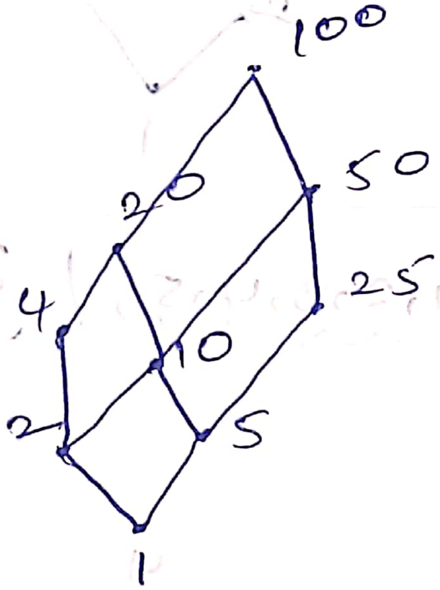
D100

Let $A = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$

Hasse-Diagram



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Lattice

A lattice is a partially ordered set (POSET), denoted as $(L, \leq)$ in which every pair of elements such as $(a, b) \in L$ has a least upper bound (LUB) & greatest lower bound (GLB)

LUB:- It is also called as supremum, join, sum of elements

→ It is denoted by $a \vee b$.

$LUB(a, b) = a \vee b$ (or) a Joins b
(or) $\Rightarrow a \oplus b$

GLB:- It is also called as infimum, meet, product of element

→ It is denoted as $a \wedge b$.

$GLB(a, b) = a \wedge b$, (or) a meets b
(or)
$\Rightarrow a * b$

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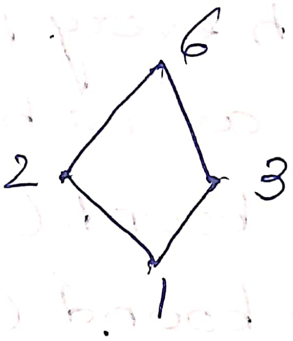
Ex:- Defined whether POSET defined by hasse diagram is lattice or not.

1)



LUB table

∨	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

GLB table

∧	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

∵ Every pair of elements in the given hasse-diagram has LUB & GLB

∴ It is a lattice//.

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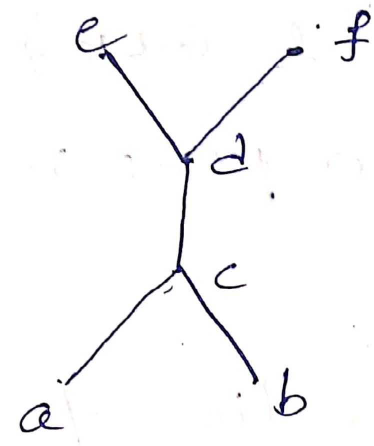
2)



LUB table

∨	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	—
f	f	f	f	f	—	f

GLB table

∧	a	b	c	d	e	f
a	a	—	a	a	a	a
b	—	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	d
f	a	b	c	d	d	f

$\because$ LUB$\{e,f\}$ & GLB$\{a,b\}$ do not exist, the given poset is not a lattice.

Some properties of lattices

If 'L' is any lattice then $\forall a,b,c \in L$

(i) $a * a = a$, $a \oplus a = a$ Idempotent

(ii) $a * b = b * a$, $a \oplus b = b \oplus a$

$\downarrow$

commutative.

(iii) $$\left.\begin{aligned} a*(b*c) &= (a*b)*c \\ a \oplus (b \oplus c) &= (a \oplus b) \oplus c \end{aligned}\right\}$$

$\downarrow$

Associative.

(iv) $$\left.\begin{aligned} a*(b \oplus c) &= (a*b) \oplus (a*c) \\ (a*b) \oplus c &= (a \oplus c) * (b \oplus c) \end{aligned}\right\}$$

$\downarrow$

Distributive.

(v) $$\left.\begin{aligned} a*(a \oplus b) &= a \\ a \oplus (a*b) &= a \end{aligned}\right\}$$ Absorption

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Duality law:

The dual of any statement can be obtained by interchanging $*$ and $\oplus$.

Ex:- 1) The dual of

$a * (b \oplus c)$ is $a \oplus (b * c)$

2) The dual of

$(a \wedge b) \vee c$ is $(a \vee b) \wedge c$.

Types of lattices:

PTO

Distributive lattice

A lattice $(L, \leq)$ is said to be distributive lattice if and only if it satisfies the following properties.

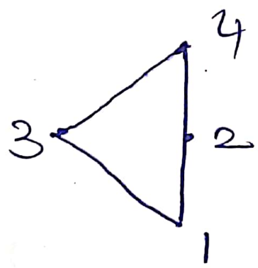
(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

where $a, b, c \in L$.

Ex:- verify the following hasse-diagrams (or) lattices are distributive lattice or not.

i)



Sol:- It is said to be distributive lattice if it satisfies the following properties

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$\Rightarrow$ let $a = 1, b = 2, c = 3$

$\Rightarrow 1 \vee (2 \wedge 3) = (1 \vee 2) \wedge (1 \vee 3)$

$\Rightarrow 1 \vee (1) = 2 \wedge 3$

$\Rightarrow 1 = 1$ $\quad \therefore [L.H.S = R.H.S]$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$\because a = 1, b = 2, c = 3$

$\Rightarrow 1 \wedge (2 \vee 3) = (1 \wedge 2) \vee (1 \wedge 3)$

$\Rightarrow 1 \wedge 4 = 1 \vee 1$

$\Rightarrow 1 = 1$ $\quad \{L.H.S = R.H.S\}$

$\because$ It satisfied the 2 properties

$\therefore$ The Given hasse-diagram is a distributive lattice //.

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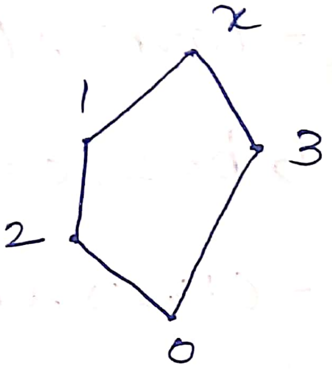
2)



(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Let $a = 1, b = 2, c = 3$

$\Rightarrow 1 \vee (2 \wedge 3) = (1 \vee 2) \wedge (1 \vee 3)$

$\Rightarrow 1 \vee (0) = 1 \wedge x$

$\Rightarrow 1 = 1$ [Satisfied

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a = 1, b = 2, c = 3$

$\Rightarrow 1 \wedge (2 \vee 3) = (1 \wedge 2) \vee (1 \wedge 3)$

$\Rightarrow 1 \wedge (x) = 2 \vee 0$

$\Rightarrow 1 \neq 2$ [L.H.S $\neq$ R.H.S]

$\therefore$ It is not a distributive lattice.

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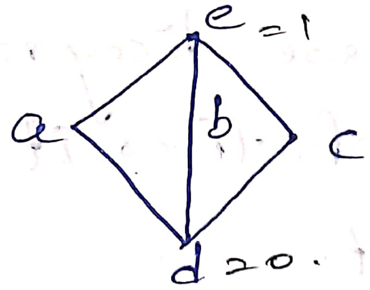
3)



(i) $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$

let $a = a,\ b = b,\ c = c$

$\Rightarrow a\vee(d) = (e)\wedge(e)$

$\Rightarrow a\vee d = e\wedge e$

$\Rightarrow a \neq e$

(ii) $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$

$\Rightarrow a\wedge(e) = d\vee d$

$\Rightarrow a \neq d$

$\boxed{\text{trick}}$

★ must have exactly one complement ~~for~~ each element

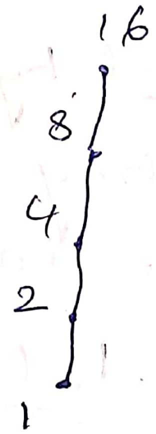
+

then it becomes distributive lattice//.

Verify that $(D_{16}, |)$ is a lattice or not?

Sol: $D_{16} = \{1, 2, 4, 8, 16\}$

Hasse-Diagram



LUB ($\vee$)

$\vee$	1	2	4	8	16
1	1	2	4	8	16
2	2	2	4	8	16
4	4	4	4	8	16
8	8	8	8	8	16
16	16	16	16	16	16

GLB

$\wedge$	1	2	4	8	16
1	1	1	1	1	1
2	1	2	2	2	2
4	1	2	4	4	4
8	1	2	4	8	8
16	1	2	4	8	16

Every pair of elements in D_{16} has LUB & GLB

$\therefore (D_{16}, |)$ is a lattice

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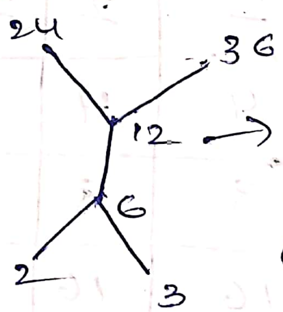
Bounds of a lattice

The greatest element and the least element in a lattice are denoted by 1 & 0 respectively and they are called as bounds of the lattice.

Bounded lattice

A lattice $(L, \leq)$ is bounded if it has the greatest element (1) and the least element (0).

Ex



→ not a bounde lattice

complete lattice

If every subset of L has a join and meet, then the lattice $(L, \leq)$ is called a complete lattice.

complete lattice.

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complement of an element.

Let $a \in L$ be an element in a lattice $(L, *, \oplus)$ with bounds 0 and 1.

If $\exists$ $b \in L \ni a * b = 0$ and $a \oplus b = 1$ then

b is said to be the complement of a.

complemented lattice :- If each element has a complement in a lattice, then the lattice is said to be complemented lattice

→ complement is not unique.

Need to get
$$\begin{bmatrix} a \vee b = 1 \\ a \wedge b = 0 \end{bmatrix}$$

$\oplus \to \vee$

$* \to \wedge$

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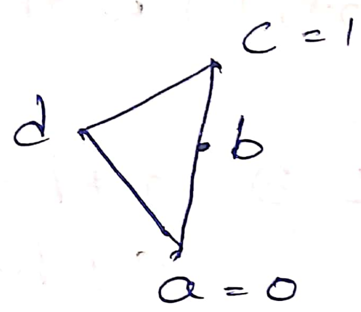
S.T the given lattice is a complemented lattice or not.

1)



Need to get $\begin{bmatrix} a \vee b = 1 \\ a \wedge b = 0 \end{bmatrix}$.

Here $c = 1, a = 0$

Any way $c \vee a = c = 1$

$c \wedge a = a = 0$.

$\therefore$ c is complement of a

& a is complement of c

<u>Next</u> b & d.

$b \vee d = c = 1$

$b \wedge d = a = 0$

$\therefore$ b is complement of d

d is complement of b

$\therefore$ In the given hasse-diagram for each & every element, there is a complement

$\therefore$ It is a complemented lattice.

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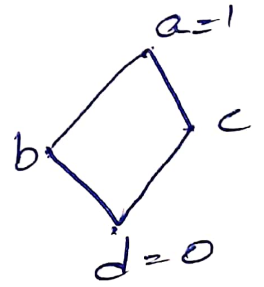
c)



$\because$ $a \vee d = a = 1$

$a \wedge d = d = 0$

So $a^c = d$

$d^c = a$

Next for b & c

$b \vee c = a = 1$

$b \wedge c = d = 0$

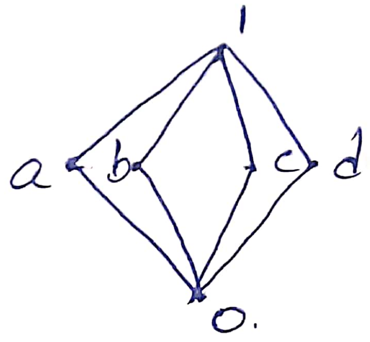
$\therefore b^c = c$

$c^c = b$

$\because$ All the elements have complements.

$\therefore$ It is a complemented lattice.

37)



Sol:- One element may have any no. of complements.

Anyway $1 \vee 0 = 1$

$1 \wedge 0 = 0$

$1^c = 0$

$0^c = 1$

for a & b

$a \vee b = 1$

$a \wedge b = 0$

$a^c = b$

$b^c = a$.

for b & c

$b \vee c = 1$ $\quad \therefore b^c = c$

$b \wedge c = 0$ $\quad c^c = b$.

c & d

$c \vee d = 1$ $\quad c^c = d$

$c \wedge d = 0$ $\quad d^c = c$

for a & d

$a \vee d = 1$ $\quad a^c = d$

$a \wedge d = 0$ $\quad d^c = a$

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So one may have any no. of elements.

$a^c \to d, b, c$

$b^c \to c, d, a$

$c^c \to b, a, d$

$d^c \to a, c, b$

$b, d.$
$b \vee d = 1$
$b \wedge d = 0$
$b^c = d$
$d^c = b$

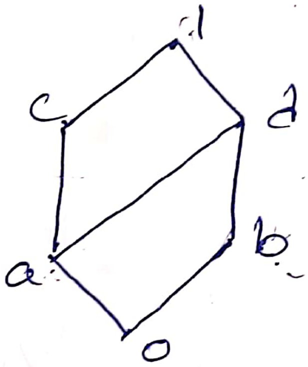
$b \vee a = 1$
$b \wedge a = 0$
$b^c = a$
$a^c = b$

$\because$ All elements are having complements

$\therefore$ It is a complemented lattice.

---

4)



$c^c = b$
$b^c = c$
$c \vee b = 1$
$c \wedge b = 0.$

$a \vee b = d \neq 1$
$a \wedge b = 0$
$a^c \neq b$
$b^c \neq a$

$\because 1^c = 0, \; 0^c = 1$

check for a.

suppose a, d

$a \vee d = d \neq 1$

$a \wedge d = a \neq 0.$

$\therefore a^c \neq d$
$d^c \neq a$

Suppose a, c

$a \vee c = c \neq 1$

$a \wedge c = a \neq 0$

$a^c \neq c$
$c^c \neq a$

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a, doesn't have complement

~~A~~ Even if one element doesn't have a complement

It is not a complemented lattice

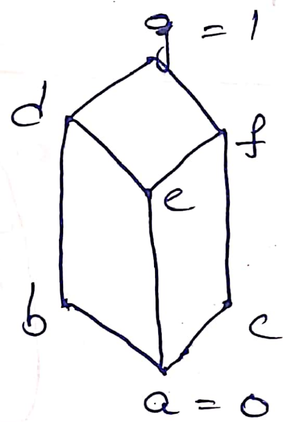
If for b,d

$b \vee d = d \neq 1$

$b \wedge d = d \neq 0$

---

5)



$\therefore a^c = g$, $g^c = a$

because $a \vee g = 1$

$a \wedge g = 0$.

$b^c = c$, $c^c = b$

$b \wedge c = 1$

$b \vee c = 0$

Suppose e, f

$e \vee f = f \neq 1$ $\quad e^c \neq f$

$e \wedge f = e \neq 0$ $\quad f^c \neq e$

Suppose e, d

$e \vee d = d \neq 1$, $\quad e^c \neq d$

$e \wedge d = e \neq 0$ $\quad d^c \neq e$

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suppose e, a

$e \vee a = e \neq 01$

$e \wedge a = a = 0$

$\therefore$ e doesn't have even one complement

$\therefore$ It is not a complemented lattice

Functions

Let X and Y be any 2 sets

A relation f from X to Y is called a function if for every $x \in X$ there is a unique $y \in Y$.
Such that $(x, y) \in f$

Ex: let $X = \{1, 5, P, Jack\}$,

$Y = \{2, 5, 7, 9, Jill\}$.

& $f = \{(1,2), (5,7), (P,9), (Jack,9)\}$

clearly every element has unique image

$\therefore$ f is a function.

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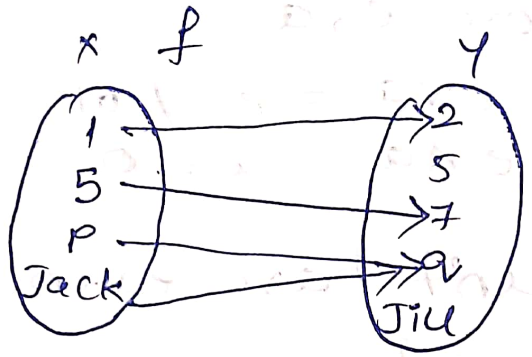

domain of $f = \{1, 5, P, \text{Jack}\}$

Range of $f = \{2, 7, 9\}$

codomain of $f = \{2, 5, 7, 9, \text{Jill}\}$

→ If an element x in X is mapped to y in Y, then y is said to be image of x and x is said to be pre-image of y.

Ex:- If the function 'f' is defined by $f(x) = x^2+1$ on the set $\{-2, -1, 0, 1, 2\}$ find the range of f.

Sol:- Given $f(x) = x^2+1$

$f(-2) = (-2)^2+1 = 5$

$f(-1) = (-1)^2+1 = 2$

$f(0) = 0^2+1 = 1$

$f(1) = 1^2+1 = 2$

$f(2) = 2^2+1 = 5.$

$\therefore$ Range of $f = \{1, 2, 5\}$

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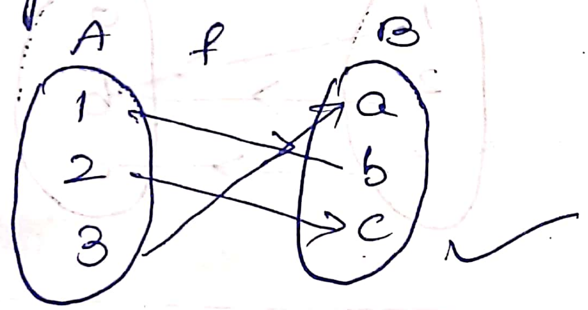
Types of functions

1) one-to-one [Injective function]

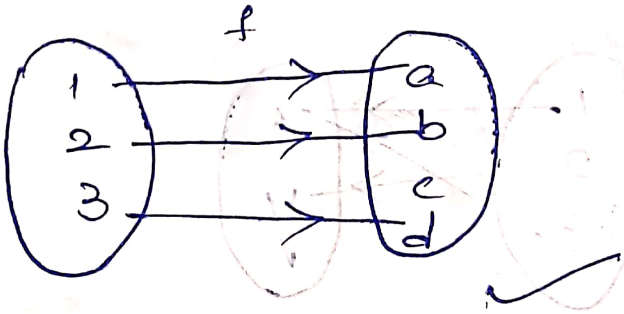
A function $f : A \to B$ is called one-to-one function if distinct elements of set A are mapped to the distinct elements of set B.

★ → Two elements in set A can't be mapped to the same element of set B.

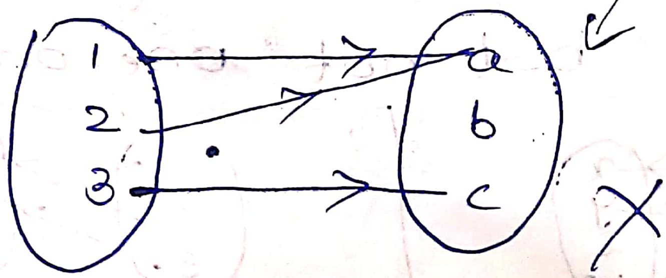
Ex. 1)



2)



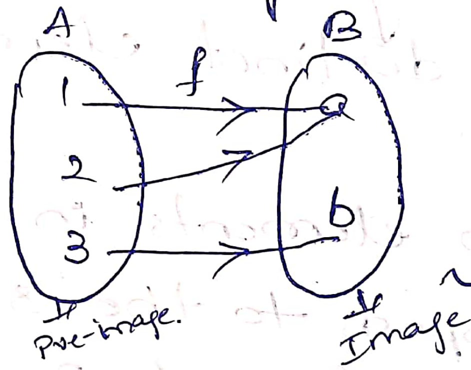
3)



Onto function [surjective]

A function $f: A \to B$ is onto function if every element of set B will be an image of some element of set A.

Ex:- 1)



$f(x) = 2x+3$

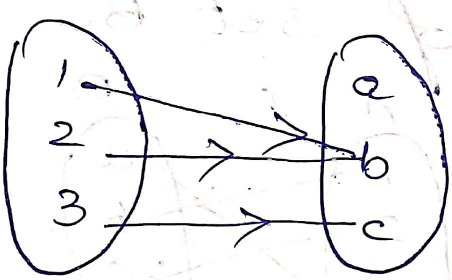
$f(x) = y$

$f(2) = 7$

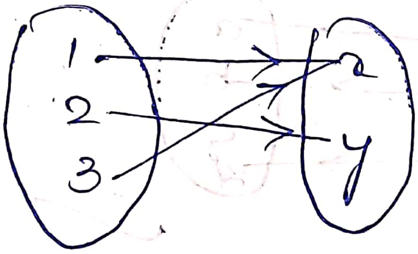
$(2, 7)$

P I

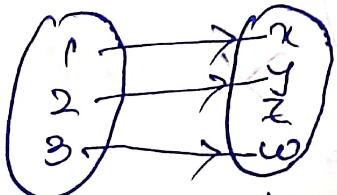
2)



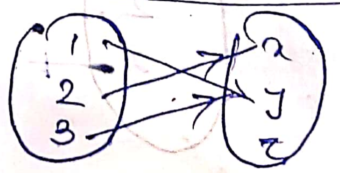
X



It is onto but not one-one.



is injective
but not surjective



not one-one
and onto.

Bijective function : A mapping $f: X \to Y$ is said to be bijective if it is both one-to-one and onto

S.T the function $f: R \to R$ defined by $f(x) = 3x^3 + 5$ $\forall x \in R$ is a bijection

Sol: Let $x, y \in R$. then

$\because$ bijection means both one-one & onto

one-one

$f(x) = f(y)$ $\quad [f(x) = 3x^3+5]$ $\quad [f(y) = 3y^3+5]$

$\Rightarrow 3x^3 + 5 = 3y^3 + 5$

$\Rightarrow 3x^3 = 3y^3$

$\Rightarrow x^3 = y^3 \Rightarrow x = y$

$\therefore f$ is one-one //

onto

Let $f(x) = y$

$\Rightarrow 3x^3 + 5 = y$

$\Rightarrow 3x^3 = y - 5$

$\Rightarrow x^3 = \frac{y-5}{3} \Rightarrow x = \left(\frac{y-5}{3}\right)^{1/3}$

$\in R$

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$\because f(x) = 3x^3+5 \xrightarrow{\in R} ①$

Sub $x = \left(\frac{y-5}{3}\right)^{1/3}$ in ①

$\Rightarrow ① \Rightarrow 3\left[\left(\frac{y-5}{3}\right)^{1/3}\right]^3 + 5$

$\Rightarrow \quad y-5+5 = y$

$\therefore$ we got $f(x) = y$

$\therefore$ f is onto

$\because$ f is one-one and onto

$\therefore$ f is a bijection//

Inverse function :

A function $f : X \to Y$ is invertible if $f : X \to Y$ is bijective.

Ex:- Let $f : R \to R$ be given by $f(x) = x^3-2$. Find f^{-1} if it exists

Sol:- one-one

$f(x) = f(y)$

$\Rightarrow x^3-2 = y^3-2$

$\Rightarrow x^3 = y^3 \Rightarrow x = y$

$\therefore f$ is one-one.

onto

let $f(x) = y$

$\Rightarrow x^3 - 2 = y \Rightarrow x^3 = y + 2$

$\Rightarrow \left[x = (y+2)^{1/3}\right] \rightarrow (1)$

$\because f(x) = y \Rightarrow x^3 - 2 = y \in R$

$\Rightarrow \left[(y+2)^{1/3}\right]^3 - 2$ — not needed

$\Rightarrow y + 2 - 2 = y$

$\therefore$ got $f(x) = y$

$\therefore f$ is onto.

To get Inverse

$\because f(x) = y$

$\Rightarrow x = f^{-1}(y)$

$\Rightarrow (y+2)^{1/3} = f^{-1}(y)$ [from (1)]

$\Rightarrow$ similarly

$f^{-1}(x) = (x+2)^{1/3}$ //.

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composition of functions

let $f: X \to Y$ and $g: Y \to Z$ be 2 functions, then the composition of gof is the function from X to Z defined by

$$(gof)(x) = g(f(x)) \; \forall \; x \in X.$$

Ex. let $f(x) = x+2$, $g(x) = 3x^2+4$ & $h(x) = x^2+1$ for $x \in R$. Find $gof, fog, fof, gog, foh, hog, hof,$ & $fo(goh)$, $(fog)oh$. verify that $fo(goh) = (fog)oh$

Sol:- Given:- $f(x) = x+2$, $g(x) = 3x^2+4$,
$h(x) = x^2+1$

(i) $(gof)(x)$

$\Rightarrow g[f(x)] \Rightarrow g[x+2]$

$\Rightarrow 3(x+2)^2+4$

$\Rightarrow 3[x^2+4+4x]+4 \quad [\because g(x) = 3x^2+4]$

$\Rightarrow 3x^2+12x+12+4$

$\Rightarrow 3x^2+12x+16$ //.

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(ii) $fog(x) = f[g(x)]$

$= f[3x^2+4]$

$= (3x^2+4)+2 \Rightarrow 3x^2+6$ //

(iii) $fof(x) = f[f(x)]$

$= f[x+2]$

$\Rightarrow (x+2)+2 \Rightarrow x+4$ //

(iv) $gog(x) = g[g(x)]$

$= g[3x^2+4]$

$= 3(3x^2+4)^2+4$

$= 3[9x^4+16+24x^2]+4$

$\Rightarrow 27x^4+48+72x^2+4$

$\Rightarrow 27x^4+72x^2+52$ //

(v) $foh(x) = f[h(x)]$

$= f[x^2+1]$

$= (x+2)^2+1$ $\quad$ $[\because f(x) = x+2]$

$\Rightarrow x^2+4+4x+1$ $\quad$ $f(x^2+1) = (x^2+1)+2$

$\Rightarrow x^2+4x+5$ $\quad$ $= x^2+1+2$

(v) $f \circ h(x) = f[h(x)]$

$= f[x^2+1]$

$= \because f(x) = x+2$

$f(x^2+1) = (x^2+1)+2$

$\Rightarrow x^2+3$ //

(vi) $h \circ g(x) = h[g(x)]$

$= h[3x^2+4]$

$= (3x^2+4)^2+1 \quad [\because h(x) = x^2+1]$

$= 9x^4+16+24x^2+1$

$\Rightarrow 9x^4+24x^2+17$ //

(vii) $h \circ f(x) = h[f(x)]$

$= h[x+2]$

$= (x+2)^2+1 \quad [\because h(x) = x^2+1]$

$= x^2+4+4x+1$

$= x^2+4x+5$ //

(viii) $f \circ (g \circ h)(x) = f \circ [g[h(x)]]$

$= f \circ [g[x^2+1]]$

$= f[3(x^2+1)^2+4]$

$= f[3[x^4+1+2x^2]+4]$

$= f[(3x^4+3+6x^2)+4]$

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$\Rightarrow f[3x^4+6x^2+7]$

$\Rightarrow (3x^4+6x^2+7)+2 \quad [\because f(x)=x+2]$

$\Rightarrow \underline{3x^4+6x^2+9}$ //

ii) $(fog)oh(x) \Rightarrow f[g[h(x)]]$

$\Rightarrow$ $(fog)oh$

$\Rightarrow fog(x) = f[g(x)]$

$= f[3x^2+4]$

$= (3x^2+4)+2$

$[fog(x) = 3x^2+6]$

$(fog)oh(x)$

$\Rightarrow (fog)h(x)$

$\Rightarrow (fog)(x^2+1)$

$\Rightarrow 3(x^2+1)^2+6$

$\Rightarrow 3[x^4+1+2x^2]+6$

$\Rightarrow 3x^4+3+6x^2+6$

$\Rightarrow \underline{3x^4+6x^2+9}$ //

$\therefore fo(goh)(x) = (fog)oh(x)$ //

S.T $f(x) = 3-4x$ is bijective, $f: R \to R$

$f(x) = 3-4x$

one-one

let $f(x_1) = f(x_2)$

$3-4x_1 = 3-4x_2$

$\Rightarrow -4x_1 = -4x_2$

$\Rightarrow x_1 = x_2$

f is one-one

onto

let $f(x) = y$

$\Rightarrow 3-4x = y$

$\Rightarrow -4x = y-3$

$\Rightarrow x = \frac{y-3}{-4}$

$x = \frac{-(3-y)}{-4} = \frac{3-y}{4} \in R$

$\therefore$ f is onto.

$\therefore$ f is bijective//

$f(x) = 2x+1 \ \forall x \in R$ is also bijective

onto $\to \left[x = \frac{y-1}{2} \in R\right]$

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