UNIT-II

Linear Differential Equations of Second and Higher Order

Definition:-

An equation of the form

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}(y)}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n x\, y = Q(x)$$ where

$P_1(x), P_2(x), P_3(x) \ldots P_n(x)$ & $Q(x)$ are all continuous and real valued funcs of x is called a linear differential eqn of order n.

Linear Differential Equation with constant coefficients:-

The general linear differential eqn of order n is of the form

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = f(x).$$

where $a_1, a_2, a_3 \ldots a_n$ are real constant

This eqn also be written in operator form as.

$$D^n y + D^{n-1}y\, a_1 + a_2 D^{n-2}y + a_3 D^{n-3}y + \cdots + a_n y = f(x).$$

$$[D^n + D^{n-1}a_1 + a_2 D^{n-2} + a_3 D^{n-3} + \cdots + a_n]y = f(x) \longrightarrow (1)$$

The solution of eq (1) consists of two parts.

1. complementary func.
2. particular Integral,

i.e $\boxed{y = C.F + P.I}$ or $y = y_c + y_p$

where C.F is a complementary func.

P.I is a particular Integral.

To find complementary func^n :-

we have to form the auxiliary eq^n which is obtained by putting

$D = m$ and $f(x) = 0$.

The auxiliary eq^n ① is

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n) = 0 \longrightarrow ②$$

eq ② is an ordinary algebraic eq^n in m of degree n

By solving this eq^n we get n roots (or values) for m.

say $m_1, m_2, m_3 \cdots m_n$

Root of A.E, $f(m) = 0$	C.F (Complementary function)
① $m_1, m_2, m_3 \cdots$ ie all roots are real and distinct $m_1 \neq m_2 \neq m_3 \neq m_4 \cdots$	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
② $m_1, m_1, m_3, m_4 \cdots m_n$ The two roots are real and equal and remaining roots are real and different	$y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
③ $m_1, m_1, m_1, m_4 \cdots m_n$ Three roots are real and equal and remaining roots are real & different	$y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$
④ The roots are of A.E are complex root say $\alpha + i\beta$ and $\alpha - i\beta$ and the remaining roots are real and different	$y_c = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$.
⑤ The pair of conjugate complex roots $\alpha \pm i\beta$ are repeated twice and the remaining roots are real and different	$y_c = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x) \sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$

① Solve $y'' - y' - 2y = 0$

Sol Given D.E can be written in operator form as

$$(D^2 - D - 2)y = 0$$

Auxiliary eqn is $f(m) = 0$

$$m^2 - m - 2 = 0$$

$$m^2 - 2m + m - 2 = 0$$

$$m(m-2) + 1(m-2) = 0$$

$$(m-2)(m+1) = 0$$

$$m = 2, -1$$

The roots are real and different

and hence the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{2x} + c_2 e^{-x}.$$

where c_1 and c_2 are constants

② Solve $(D^2 - 3D + 4)y = 0$.

Auxiliary eqn is $f(m) = 0$

$$m^2 - 3m + 4 = 0$$

Here $a = 1, b = -3, c = 4$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)}$$

$$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm \sqrt{-7}}{2} = \frac{3 \pm i\sqrt{7}}{2}.$$

$$m = \frac{3}{2} \pm i\frac{\sqrt{7}}{2} = \alpha \pm i\beta.$$

The root are complex and conjugate.

and hence the general solution is

$$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$$

$$y = e^{3/2 x}\left[c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right]$$

where c_1 and c_2 are constant

③ Solve $\frac{d^3x}{dt^3} - x = 0$

The given D.E can be written in the operator form as

$(D^3 - 1)x = 0$ where $D = \frac{d}{d\theta}$

The auxiliary eqn is $m^3 - 1 = 0$

$(m-1)(m^2+m+1) = 0$

$m-1=0, \quad m^2+m+1=0$

$m_1 = 1, \quad a=1, b=1, c=1$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1}{2} \pm i\sqrt{3}/2$$

The root is real and the other two roots are complex and conjugate.

The G. Solution is

$x = c_1 e^t + e^{-t/2}\left[c_2 \cos\frac{\sqrt{3}}{2}t + c_3 \sin\frac{\sqrt{3}}{2}t\right]$

where c_1, c_2 & c_3 are constants.

④ Find the complete solution of

$$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$$

Sol Given eqn in the operator form is

$(D^3 - 9D^2 + 23D - 15)y = 0$.

Let $F(D) = D^3 - 9D^2 + 23D - 15$

Auxiliary eqn is $f(m) = 0$

$m^3 - 9m^2 + 23m - 15 = 0$

Synthetic Division

$m=1$	1	-9	23	-15
	0	1	-8	15
	1	-8	15	0

$(m-1)(m^2 - 8m + 15) = 0$

$m-1, \quad m^2 - 3m - 5m + 15 = 0$

$m(m-3) - 5(m-3) = 0$

$(m-3)(m-5) = 0$

$m = 3, \quad m = 5$

The roots are $-1, 3, 5$

The roots are real and different and hence the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{5x}$$ where c_1, c_2 & c_3 are arbitrary constants

② Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

Sol Given eqn is

$(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

let $f(D) = D^4 - 2D^3 - 3D^2 + 4D + 4$.

The A.E is $f(m) = 0$

$m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$.

Synthetic Division

$m=-1$	1	-2	-3	4	4
	0	-1	3	0	-4
$m=-1$	1	-3	0	4	0
	0	-1	4	-4	
	1	-4	4	0	

$(m+1)(m+1)(m^2 - 4m + 4) = 0$

$m = -1, -1$ $\quad$ $m^2 - 2m - 2m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$

The roots are $m = -1, -1, 2, 2$. Hence the general solution of eqn ① is

$$\boxed{y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}}$$

④ Solve $(D^2 + 6D + 9)y = 0$

Given eqn is $(D^2 + 6D + 9)y = 0$

let $f(D) = D^2 + 6D + 9 = 0$

The A.E is, $f(m) = 0$

$m^2 + 6m + 9 = 0$

$m^2 + 3m + 3m + 9 = 0$

$m(m+3) + 3(m+3) = 0$

$(m+3)(m+3) = 0$

$m = -3, -3$

The roots are real & equal

$y = (c_1 + c_2 x)e^{-3x}$.

① $(D^2-3D+4)y=0$

Sol W.T $f(D)=D^2-3D+4$

The A.E is $f(m)=0$

$m^2-3m+4=0$

$m=3/2+i\sqrt{7}/2$

① $(D^4+18D^2+81)y=0$

② $(D^2+4)y=0$.

③ $(D^4+16)y=0$

④ $4y'''+4y''+y'=0$.

⑤ D^3-2D^2-3D

Inverse operator :—

The operator D^{-1} is called inverse of the differential Operator D.

Def: If Q is any func of x then $D^{-1}Q$ or $\frac{1}{D}Q$ is called the integral of Q.

Ex $\frac{1}{D}(\cos 3x)=\int \cos 3x\,dx=\frac{\sin 3x}{3}$

Theorem :— If Q is any function of x and α is a constant then a particular value of $\frac{1}{D-\alpha}Q$ is equal to $e^{\alpha x}\int Q(x)e^{-\alpha x}dx$.

① Find (i) $\frac{1}{D}(x^2)$

$\frac{1}{D}(x^2)=\int x^2dx=\frac{x^3}{3}$

(ii) $\frac{1}{D^3}\cos x=\frac{1}{D^2}\left[\frac{1}{D}\cos x\right]=\frac{1}{D^2}\left[\int \cos x\,dx\right]=\frac{1}{D^2}(\sin x)$

$=\frac{1}{D}\left(\frac{1}{D}\sin x\right)=\frac{1}{D}\left(\int \sin x\,dx\right)$

$=\frac{1}{D}(-\cos x\,dx)$

$=-\int \cos x\,dx$

$\frac{1}{D^3}\cos x.=-\sin x.$

② Find the particular value of $\frac{1}{D+1}(x)$

$\frac{1}{D+1}(x)=e^{-x}\int xe^{x}dx=e^{-x}[xe^{x}-e^{x}]=x-1$

Particular Integral of $f(D)y = Q(x)$

Given eqn is $f(D)y = Q(x)$

Thus particular Integral = P.I $\rightarrow y = \frac{Q(x)}{f(D)}$

① Solve $(D^2+a^2)y = \sec ax$.

Sol Given eqn is $(D^2+a^2)y = \sec ax$.

Let $f(D) = D^2+a^2$

The A.E is $f(m) = 0$

$$m^2+a^2 = 0$$

$$m^2 = -a^2$$

$$m = \pm ai \quad , m = ai, -ai$$

$$\boxed{y_C = c_1 \cos ax + c_2 \sin ax}$$

and $y_P = \frac{1}{D^2+a^2} \sec ax$

$$y_P = \frac{1}{(D-ai)(D+ai)} \sec ax = \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \sec ax$$

Now $\frac{1}{D-ai} \sec ax = e^{iax} \int e^{-iax} \sec ax \, dx$

$$= e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx$$

$$= e^{iax} \int (1 - i \tan ax) dx$$

$$= e^{iax} \left[x + \frac{i}{a} \log \cos ax\right]$$

Similary $\frac{1}{D+ai} \sec ax = e^{-iax}\left[x - \frac{i}{a} \log \cos ax\right]$

$$y_P = \frac{1}{2ai}\left[e^{iax}\{x + \frac{i}{a} \log \cos ax\} - e^{-iax}\{x - \frac{i}{a} \log \cos ax\}\right]$$

$$= \frac{x}{2ai}\left[e^{iax} - e^{-iax}\right] + \frac{1}{2a^2}(\log \cos ax)\left[e^{iax} + e^{-iax}\right]$$

$$= \frac{x}{ai} \sin ax + \frac{1}{a^2} \cos ax \log(\cos ax)$$

① Solve $(D^2+4D+3)y = e^{e^x}$

Sol Given eqn is $(D^2+4D+3)y = e^{e^x}$

A.E is $f(m)=0$ $\quad m^2+4m+3=0$

$$m^2+3m+m+3=0$$

$$m=-3,-1$$

$\therefore$ Roots are real and different

Hence C.F is $y_c = c_1e^{-3x} + c_2e^{-x}$.

Now P.I is $y_p = \frac{1}{f(D)} Q(x) = \frac{1}{(D+3)(D+1)} e^{e^x}$

Now $\frac{1}{(D+3)(D+1)} = \frac{A}{D+3} + \frac{B}{D+1}$

$1 = A(D+1) + B(D+3) \rightarrow$ ①

put $D=-3$ in eq ① $A = -1/2$

put $D=-1$ in eq ① $B = 1/2$

$$\frac{1}{(D+3)(D+1)} = \frac{-1/2}{D+3} + \frac{1/2}{D+1} = \frac{1}{2}\left[\frac{1}{D+1} - \frac{1}{D+3}\right]$$

$$y_p = \frac{1}{(D+3)(D+1)} e^{e^x} = \frac{1}{2}\left[\frac{1}{D+1} - \frac{1}{D+3}\right] e^{e^x}$$

$$= \frac{1}{2}\left[\frac{1}{D+1}e^{e^x} - \frac{1}{D+3}e^{e^x}\right]$$

$$= \frac{1}{2}\left[e^{-x}\int e^{e^x}.e^x dx - e^{-3x}\int e^{e^x}e^{3x}dx\right]$$

$$= \frac{1}{2}\left[P.I_1 - P.I_2\right]$$

$P.I_1 = e^{-x}\int e^{e^x}.e^x dx$ let $e^x = t \Rightarrow e^x dx = dt$

$= e^{-x}\int e^t.dt$

$= e^{-x}.e^t$

$P.I_1 = e^{-x}e^{e^x}$

$P.I_2 = e^{-3x}\int e^{e^x}e^{3x}dx$ let $e^x = t \Rightarrow e^x dx = dt$

$= e^{-3x}\int e^t.e^{2x}.e^x dx = e^{-3x}\int e^t.t^2 dt$ $\quad \{\int u dv = uv - u'v_1 \ldots$

$= e^{-3t}e^t[t^2-2t+2]$

$$P.I_2 = e^{-3x} e^{e^x} [e^{2x} - 2e^x + 2]$$

$$P.I_2 = e^{e^x} [e^{-x} - 2e^{-2x} + 2e^{-3x}]$$

$$y_p = \frac{1}{2} [P.I_1 - P.I_2]$$

$$= \frac{1}{2} [e^{-x} e^{e^x} - e^{e^x} [e^{-x} - 2e^{-2x} + 2e^{-3x}]]$$

$$= \frac{1}{2} e^{e^x} [\cancel{e^{-x}} - \cancel{e^{-x}} + 2e^{-2x} - 2e^{-3x}]$$

$$y_p = e^{e^x} [e^{-2x} - e^{-3x}]$$

$\therefore$ The general solution is $y = y_c + y_p$

$$y = c_1 e^{-3x} + c_2 e^{-x} + e^{e^x} (e^{-2x} - e^{-3x})$$

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Rules for Finding particular Integral in Some Special cases

① P.I of $f(D)y = \phi(x)$ when $\phi(x) = e^{ax}$, where 'a' is constant—

Case I: $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ if $f(a) \neq 0$.

Case II: let $f(a) = 0$ then $f(D) = (D-a)^k$

(i) $\frac{1}{D-a} e^{ax} = x e^{ax}$ if $f(a) = 0$

(ii) $\frac{1}{D+a} e^{-ax} = x e^{-ax}$ if $f(a) = 0$

(iii) $\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}$ if $f(a) = 0$

(iv) $\frac{1}{(D+a)^2} e^{-ax} = \frac{x^2}{2!} e^{-ax}$ if $f(a) = 0$.

(v) $\frac{1}{(D-a)^k} e^{ax} = \frac{x^k}{k!} e^{ax}$ if $f(a) = 0$ & $\frac{1}{(D+a)^k} e^{-ax} = \frac{x^k}{k!} e^{-ax}$

① Solve $(D^2+4D+13)y = 2e^{-x}$

Sol $(D^2+4D+13)y = 2e^{-x}$

A.E is $f(m)=0$

$$m^2+4m+13=0$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16-4(13)}}{2(1)}$$

$$m = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2}$$

$$m = -2 \pm 3i = \alpha \pm i\beta$$

Thus $CF = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x]$

$$\boxed{y_c = e^{-2x}[C_1 \cos 3x + C_2 \sin 3x]}$$

Now $P.I = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{D^2+4D+13} 2e^{-x}$$

$$= \frac{1}{(-1)^2+4(-1)+13} 2e^{-x}$$

$$= 2 \cdot \frac{1}{10} e^{-x}$$

$$\boxed{P.I = \frac{1}{5} e^{-x}}$$

∴ The general solution is

$$y = y_c + y_p = e^{-2x}[C_1 \cos 3x + C_2 \sin 3x] + \frac{1}{5} e^{-x}$$

① $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol The given eqn is

$(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

The A.E is $f(m)=0$

$$(m+2)(m-1)^2 = 0$$

$$m = -2, 1, 1$$

The complementary func

$$y_c = C_1 e^{m_1 x} + (C_2 + C_3 x) e^{m_2 x}$$

The roots are real and one root is repeated twice.

$$y_c = c_1 e^{-2x} + (c_2 + c_3 x) e^{x}$$

Now $P.I = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{(D+2)(D-1)^2} [e^{-2x} + 2\sinh x]$$

$$= \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} 2\left[\frac{e^{x} - e^{-x}}{2}\right]$$

$$= \frac{1}{(-2-1)^2 (D+2)} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^{x} + \frac{1}{(D+2)(D-1)^2} e^{-x}$$

$$= \frac{1}{9(D+2)} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^{x} + \frac{1}{(-1+2)(-1-1)^2} e^{-x}$$

$$= \frac{x}{9} e^{-2x} + \frac{1}{3} \cdot \frac{x^2}{2!} e^{x} + \frac{1}{4} e^{-x}.$$

$$y_p = \frac{x}{9} e^{-2x} + \frac{1}{6} x^2 e^{x} + \frac{1}{4} e^{-x}.$$

The general solution is

$$y = y_c + y_p$$

$$y = c_1 e^{-2x} + (c_2 + c_3 x) e^{x} + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} - \frac{e^{-x}}{4}.$$

① Solve $y'' - 4y' + 3y = 4e^{3x}$, $y(0) = -1$, $y'(0) = 3$.

Sol $(D^2 - 4D + 3)y = 4e^{3x}$

The A.E is $m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0$

$m = 3, 1$

The roots are real and different

Then $C.F = y_c = c_1 e^{3x} + c_2 e^{x}$.

Now P.I $= y_p = \frac{1}{(D^2 - 4D + 3)} 4e^{3x}$

$= \frac{1}{(D-3)(D-1)} 4e^{3x}$

$= \frac{4}{(D-3)(3-1)} e^{3x}$

$= \frac{\cancel{4}}{\cancel{2}(D-3)} e^{3x}$ $\{\because f(3) = 0\}$

$y_p = 2x e^{3x}$.

The general solution is $y = y_c + y_p$

$y = c_1 e^{3x} + c_2 e^{x} + 2x e^{3x} \longrightarrow$ ①

Differentiating w.r.to x on both sides, we get

$y' = c_1 3e^{3x} + c_2 e^{x} + 2[x e^{3x} 3 + e^{3x}]$

$y' = 3c_1 e^{3x} + c_2 e^{x} + 6x e^{3x} + 2e^{3x} \longrightarrow$ ②

Given $y(0) = -1$, $y'(0) = 3$.

$y(0) = -1 \Rightarrow$ eq ① becomes

$-1 = c_1 + c_2 \longrightarrow$ ③

$y'(0) = 3 \Rightarrow$ eq ② becomes

$3 = 3c_1 + c_2 + 2 \Rightarrow 3c_1 + c_2 = 1 \longrightarrow$ ④

eq ④ & ③

$c_1 + \cancel{c_2} = -1$

$3c_1 + \cancel{c_2} = 1$

$-2c_1 = -2$

$\Rightarrow \boxed{c_1 = 1}$

$c_1 + c_2 = -1$

$c_2 = -1 - 1$

$\boxed{c_2 = -2}$

Eq ① ⇒ $\boxed{y = -2e^{x} + (1+2x)e^{3x}}$.

Case – II

P.I of $f(D)\cdot y = \phi(x)$ when $\phi(x) = \sin bx$ (or) $\cos bx$ where b is constant.

① Case: Let $f(D)y = \sin ax$.

$y = \frac{1}{f(D)} \sin ax$ Let $f(D) = f(D^2)$ then

1) $P.I = \frac{\sin ax}{f(D^2)} = \frac{\sin ax}{f(-a^2)}$ provided $f(-a^2) \neq 0$.

2) $P.I = \frac{\cos ax}{f(D^2)} = \frac{\cos ax}{f(-a^2)}$ provided $f(-a^2) \neq 0$.

Case of failure :–

1) $P.I = \frac{\cos ax}{D^2+a^2} = \frac{x}{2a} \sin ax$, if $f(-a^2) = 0$.

$P.I = \frac{\sin ax}{D^2+a^2} = \frac{-x}{2a} \cos ax$, if $f(-a^2) = 0$.

① Solve $(D^2+3D+2)y = \sin 3x$

Sol Here $f(D) = D^2+3D+2$

A.E is $f(m) = 0$

$m^2+3m+2=0$

$(m+2)(m+1)=0$

$m = -1, 2$

The root are real & different.

$y_c = c_1 e^{-x} + c_2 e^{-2x}$.

Now $P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+3D+2} (\sin 3x)$

$$P.I = \frac{1}{D^2+3D+2}\sin 3x$$

$$= \frac{1}{-3^2+3D+2}\sin 3x$$

$$= \frac{1}{3D-7}\sin 3x.$$

$$= \frac{3D+7}{(3D-7)(3D+7)}\sin 3x$$

$$= \frac{3D+7}{3^2D^2-7^2}\sin 3x$$

$$= \frac{3D+7}{9D^2-49}\sin 3x$$

$$= \frac{3D+7}{9(-3^2)-49}\sin 3x$$

$$= \frac{3D+7}{-81-49}\sin 3x$$

$$= \frac{-1}{130}(3D+7)\sin 3x$$

$$= \frac{-1}{130}[3\cdot 3\cos 3x+7\sin 3x]$$

$$y_p = \frac{-1}{130}[9\cos 3x+7\sin 3x]$$

The general solution is

$$y_p = y_c + y_p$$

$$y = c_1e^{-x} + c_2e^{-2x} - \frac{1}{130}[9\cos 3x + 7\sin 3x]$$

Solve $(D^2-4D+3)y = \cos 2x$

② $f(D) = D^2-4D+3$

Auxiliary eqn (A.E) is $f(m)=0$

$$m^2-4m+3=0$$

$$m^2-3m-m+3=0$$

$$(m-3)(m-1)=0$$

$$m = 1, 3$$

The roots are real & different

$$y_c = c_1e^{m_1x}+c_2e^{m_2x}$$

$$\boxed{y_c = c_1e^{x}+c_2e^{3x}}$$

Now $P.I = \frac{1}{f(D)}Q(x) = \frac{1}{D^2-4D+3}\cos 2x$

$$= \frac{1}{-4-4D+3}\cos 2x$$

$$= \cancel{\frac{1}{-2(2D+3)}} \quad \frac{1}{-1-4D}\cos 2x$$

$$y_p = \frac{1}{-(4D+1)}\cos 2x = \frac{4D-1}{-(4D+1)(4D-1)}\cos 2x$$

$$y_p = \frac{4D-1}{-(16D^2-1^2)}\cos 2x$$

$$= \frac{(1-4D)}{16(-4)-1}\cos 2x \Rightarrow \frac{-1}{65}[1-4D]\cos 2x]$$

$$= \frac{-1}{65}[\cos 2x - 4(\sin 2x(-2))]$$

$$y_p = \frac{-1}{65}[\cos 2x + 8\sin 2x] \qquad \left\{ \frac{d}{dx}\cos ax = -\sin ax \cdot a \right.$$

$$y = y_c + y_p$$

$$y = c_1 e^{x} + c_2 e^{3x} - \frac{1}{65}[\cos 2x + 8\sin 2x]$$

ASS 1) $(D^2-3D+2)y = \cos 3x$ ② $(D^2-2D+2)y = \cos 9x$

$1 \pm i$

$\frac{-1}{6565}[18\sin 9x + 79\cos 9x$

2) $(D^2+4)y = \sin 2x$

③ (D^2-a^2) ~ $(D^2+4)y = \cos 2x$

④ $(D^3-1)y = e^{x} + \sin 3x + 2.$

⑤ $(D^3+1)y = \sin(2x+1)$

⑥ $(D^2+4)y = \cos x.$ ② $(D^2+9)y = \cos 3x$

① Solve $(D^2+4)y = e^{x} + \sin 2x + \cos 2x$

Sol Here $f(D) = D^2+4$

A.E is $f(m) = 0 \Rightarrow m^2+4 = 0$

$$m^2 = -4$$

$$m = \pm 2i = \alpha \pm i\beta$$

The root are real & complex roots.

$$y_c = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$$

$$C.F = y_c = c_1\cos 2x + c_2\sin 2x$$

Now $P.I = y_p = \frac{1}{D^2+4}[e^{x} + \sin 2x + \cos 2x]$

$$y_p = \frac{1}{D^2+4} e^x + \frac{1}{D^2+4} \sin 2x + \frac{1}{D^2+4} \cos 2x$$

$$y_p = \frac{1}{(1)^2+4} e^x + \left(-\frac{x}{2(2)} \cos 2x\right) + \frac{x}{2(2)} \sin 2x$$

$$y_p = \frac{1}{5} e^x - \frac{x}{4} \cos 2x + \frac{x}{4} \sin 2x \quad \left\{ \frac{1}{D^2+a} \sin ax = -\frac{x}{2a} \cos ax \right.$$

$$\frac{1}{D^2+a} \cos ax = \frac{x}{2a} \sin ax$$

$$y = y_c + y_p$$

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^x - \frac{x}{4} \cos 2x + \frac{x}{4} \sin 2x$$

— x —

(1) Solve $(D^2+1)y = \sin x \sin 2x$.

Sol Here $f(D) = D^2+1$

A.E is $f(m) = 0$

$$m^2+1=0$$

$$m = \pm i$$

The roots are complex & conjugate root

$$y_c = c_1 \cos x + c_2 \sin x.$$

Now $P.I = \frac{1}{D^2+1}[\sin x \sin 2x]$

$$= \frac{1}{2(D^2+1)}[2 \sin x \sin 2x] \quad \{ \cos(A-B) - \cos(A+B) = 2\sin A \sin B \}$$

$$= \frac{1}{2(D^2+1)}[2 \sin 2x \sin x]$$

$$= \frac{1}{2(D^2+1)}[\cos x - \cos 3x]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1} \cos x - \frac{1}{D^2+1} \cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2(1)} \sin x - \frac{1}{-9+1} \cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2} \sin x + \frac{1}{8} \cos 3x\right]$$

$$y_p = \frac{x}{4} \sin x + \frac{1}{16} \cos 3x.$$