UNIT-III

Multiple Integrals

① Double Integrals

let $I = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy$.

Case I: when y_1, y_2 are func of x and x_1, x_2 are constant. $f(x,y)$ is first Integrated w.r.to 'y' keeping x fixed b/w limits y_1, y_2 and then the resulting expression is Integrated w.r.to 'x' with in the limits x_1, x_2

$$I = \int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx.$$

Case II :- when x_1, x_2 are func of y and y_1, y_2 are constant. $f(x,y)$ is first Integrated w.r.t x. keeping y fixed b/w limits x_1, x_2 and then the resulting expression is Integrated w.r.to 'y' with in the limits y_1, y_2

$$I = \int_{y_1}^{y_2}\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy.$$

Case III when both pairs of limits are constants the region of integration is rectangle.

$$I = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy \Rightarrow \int_{a}^{b}\int_{c}^{d} f(x,y)\,dx\,dy.$$

① Evaluate $\int_0^2\int_0^x y\,dy\,dx$

Sol The given integral is

$$\int_{x=0}^{2}\int_{y=0}^{x} y\,dy\,dx.$$

$$= \int_{x=0}^{2}\left[\int_{y=0}^{x} y\,dy\right]dx$$

$$= \int_{x=0}^{2}\left[\frac{y^2}{2}\right]_0^x dx$$

$$= \frac{1}{2}\int_{x=0}^{2}\left[x^2-0^2\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{2} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$$

② $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy.$

Sol Given the integral is

$$\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} \left[x^2+y^2\right]dy\,dx.$$

$$\int_{x=0}^{1}\left[\int_{y=x}^{\sqrt{x}}(x^2+y^2)\,dy\right]dx = \int_{x=0}^{1}\left[x^2\left[y\right]_x^{\sqrt{x}} + \left[\frac{y^3}{3}\right]_x^{\sqrt{x}}\right]dx.$$

$$= \int_{x=0}^{1}\left[x^2\left[\sqrt{x}-x\right] + \frac{1}{3}\left[(\sqrt{x})^3 - x^3\right]\right]dx.$$

$$= \int_{x=0}^{1}\left[\sqrt{x}\cdot x^2 - x^3 + \frac{1}{3}x^{3/2} - \frac{x^3}{3}\right]dx.$$

$$= \int_{x=0}^{1} x^{5/2} + \frac{1}{3}x^{3/2} + \left[-x^3 - x^3/3\right]dx.$$

$$= \int_0^1 \left[x^{3/2}[1+1/3] - x^3[1+1/3]\right] dx$$

$$= \int_0^1 \left(\frac{4}{3}x^{3/2} - 4/3\, x^3\right) dx$$

$$= \frac{4}{3}\left[\frac{x^{3/2+1}}{3/2+1}\right]_0^1 - 4/3\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{4}{3}\left[\frac{x^{5/2}}{5/2}\right]_0^1 - \frac{1}{3}[x^4]_0^1$$

$$= \frac{8}{15}[$$

wrong.

$$= \int_{x=0}^{1} \left[x^{5/2} + \frac{1}{3}x^{3/2} - 4/3\, x^3\right] dx.$$

$$= \left[\frac{x^{7/2}}{7/2} + \frac{1}{3}\cdot\frac{x^{5/2}}{5/2} - \frac{4}{3}\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{7}[x^{7/2}]_0^1 + \frac{2}{15}[x^{5/2}]_0^1 - \frac{1}{3}[x^4]_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{30+14-35}{105} = \frac{9}{105} = \frac{3}{35}$$

$$= 3/35 \;//$$

① $\int_{x=0}^{1}\int_{0}^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol The given Integral

$$I = \int_{x=0}^{1}\int_{y=0}^{y=\sqrt{1+x^2}} \frac{dy\,dx}{(1+x^2)+y^2}$$

$$I = \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\,dy\right]dx. \quad \text{whe}$$

$$I = \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{p^2+y^2}\,dy\right]dx \quad \text{where } p=\sqrt{1+x^2}$$

$$\int \frac{1}{a^2+x^2}\,dx = \operatorname{Tan}^{-1}\left(\frac{x}{a}\right)\cdot\frac{1}{a}.$$

$$= \int_{x=0}^{1}\left[\frac{1}{p}\operatorname{Tan}^{-1}\left(\frac{y}{p}\right)\right]_0^p dx.$$

$$= \int_{x=0}^{1}\frac{1}{p}\left[\operatorname{Tan}^{-1}\left(\frac{p}{p}\right)-\operatorname{Tan}^{-1}\left(\frac{0}{p}\right)\right]dx$$

$$= \int_{x=0}^{1}\frac{1}{p}\left[\operatorname{Tan}^{-1}(1)-\operatorname{Tan}^{-1}(0)\right]dx.$$

$$= \int_{x=0}^{1}\frac{1}{p}\cdot\left[\pi/4-0\right]dx.$$

$$= \frac{\pi}{4}\int_{x=0}^{1}\frac{dx}{\sqrt{1+x^2}} \qquad (\text{or}) \qquad = \frac{\pi}{4}\int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}\,dx$$

$$= \frac{\pi}{4}\left[\log\left[x+\sqrt{x^2+1}\right]\right]_{x=0}^{1} \qquad = \pi/4\left[\sinh^{-1}x\right]_0^1$$

$$= \pi/4\log\left[1+\sqrt{1+1}\right] \qquad = \pi/4\left[\sinh^{-1}(1)-\sinh^{-1}(0)\right]$$

$$= \pi/4\log\left[1+\sqrt{2}\right] \qquad = \pi/4\sinh^{-1}(1)$$

$\Rightarrow$ 2

① $\int_0^2 \int_0^x e^{x+y}\, dy\, dx$

Sol $\int_0^2 \int_0^x e^x . e^y \, dy\, dx$

$= \int_0^2 e^x \left[\int_0^x e^y \, dy \right] dx$

$= \int_0^2 e^x \left[e^y \right]_0^x dx$

$\int_0^2 e^x [e^x - e^0]\, dx$

$\int_0^2 e^x [e^x - 1]\, dx$

$\int_0^2 [e^{2x} - e^x]\, dx$

$\left[\frac{e^{2x}}{2} - e^x \right]_0^2$

$\left[\frac{e^{4x}}{2} - e^2 \right] \Rightarrow e^2\left[\frac{e^2}{2} - 1\right] + \frac{1}{2}$

$- \left[\frac{1}{2} - 1 \right]$

① $\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dx\, dy$

Sol $\int_0^5 \int_0^{x^2} [x^3 + xy^2]\, dx\, dy$

$= \int_0^5 \left[x^3 \int_0^{x^2} 1 . dy + x \int_0^{x^2} y^2 \, dy \right] dx$

$= \int_0^5 \left[x^3 [y]_0^{x^2} + x \left[\frac{y^3}{3} \right]_0^{x^2} \right] dx.$

$= \int_0^5 \left[x^3 (x^2 - 0) + \frac{x}{3} (y^3)_0^{x^2} \right] dx$

$= \int_0^5 \left[x^5 + \frac{x}{3} ((x^2)^3) \right] dx$

$= \int_0^5 \left[x^5 + \frac{x}{3} (x^6) \right] dx.$

$= \int_0^5 \left[x^5 + \frac{x^7}{3} \right] dx \quad = \left[\frac{x^6}{6} + \frac{1}{3} \cdot \frac{x^8}{8} \right]_0^5$

$= \left(\frac{x^6}{6} + \frac{x^8}{24} \right)_0^5$

$= \frac{5^6}{6} + \frac{5^8}{24} \quad = 5^6 \left[\frac{1}{6} + \frac{25}{24} \right]$

$= \frac{29(5^6)}{24}$ //

① Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$

Sol $\int_0^4 \left[\left[\int_0^{x^2} e^{y/x} \right] dy \right] dx$.

$\int_0^4 \left[\frac{e^{y/x}}{1/x} \right]_0^{x^2} dx$

$\int_0^4 x \left[e^{y/x} \right]_0^{x^2} dx$

$\int_0^4 x \left[e^{x^2/x} - e^{0/x} \right] dx$

$\int_0^4 x \left[e^{x} - 1 \right] dx$

$\int_0^4 \left[xe^{x} - x \right] dx$.

$\left[xe^{x} - e^{x} - x^2/2 \right]_0^4 = \left(4e^4 - e^4 - \frac{4^2}{2} \right) - \left(0(1) - 1 - 0/2 \right)$

$= 4e^4 - e^4 - \frac{\cancel{16}^{8}}{\cancel{2}} + 1$

$= 3e^4 - 7$

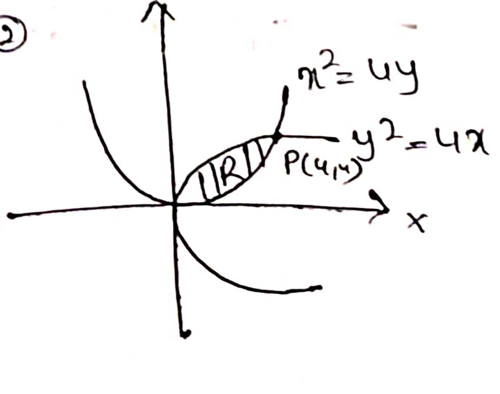
② $\iint (x+y)\, dx\, dy$, over the region in the positive quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol

① Evaluate $\iint_R y^2\, dx\, dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Sol Given parabolas $y^2 = 4x \longrightarrow ①$

and $x^2 = 4y \longrightarrow ②$



To find their points of Intersection solve eq ① & ②.

eq ① $y^2 = 4x \Rightarrow \left(\frac{x^2}{4} \right)^2 = 4x$.

$$\frac{x^4}{4^2} = 4x$$

$$x^4 = 4^3 x$$

$$x^4 - 4^3 x = 0$$

$$x(x^3 - 4^3) = 0$$

$$x = 0, \; x = 4.$$

when $x = 0$, $y = 0$
when $x = 4$, $y = 4$;
Thus the point two parabolas dutersect at the point $O(0,0)$ and $P(4,4)$

$$\iint_R y^2 \,dx\,dy = \int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} y^2 \,dx\,dy$$

$$= \int_0^4 \left[\frac{y^3}{3} \right]_{x^2/4}^{2\sqrt{x}} dx.$$

$$= \frac{1}{3} \int_0^4 \left[y^3 \right]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{3} \left[\int_0^4 \left[(2\sqrt{x})^3 - (x^2/4)^3 \right] dx \right.$$

$$\frac{x^6}{64}$$

$$= \frac{1}{3} \int_0^4 [8x^{3/2} - x^6/4^3] dx$$

$$= \frac{1}{3}\left[8 \int_0^4 x^{3/2} dx - \frac{1}{64} \left[\int_0^4 x^6 dx \right] \right]$$

$$= \frac{1}{3}\left[8 \left[\frac{(x)^{5/2}}{5/2} \right]_0^4 - \frac{1}{64} \left[\frac{x^7}{7} \right]_0^4 \right.$$

$$= \frac{1}{3}\left[\frac{16}{5} \left[x^{5/2} \right]_0^4 - \frac{1}{64 \times 7} \left[x^7 \right]_0^4 \right]$$

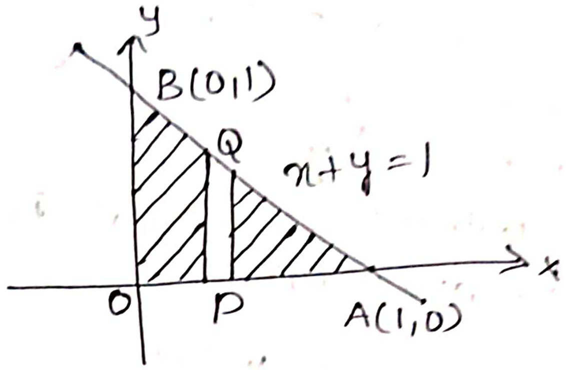
$$= \frac{1}{3}\left[\frac{16}{5} \left[(4)^{5/2} - \frac{1}{64 \times 7} \left[4^7 \right] \right] \right]$$

$$= \frac{1}{3}\left[\frac{16}{5} (2)^5 - \frac{1}{448} (4^7) \right] = \frac{768}{35}$$

① Evaluate $\iint_R (x^2+y^2)\,dx\,dy$, where R is the region in the positive quadrant for which $x+y \le 1$

Sol

$$\iint_R (x^2+y^2)\,dx\,dy.$$



$$= \int_{x=0}^{1}\int_{y=0}^{y=1-x} (x^2+y^2)\,dy\,dx$$

$$= \int_{x=0}^{1}\left[\int_0^{1-x} (x^2+y^2)\,dy\right] dx$$

$$= \int_0^1 \left[x^2 [y]_0^{1-x} + \left[\frac{y^3}{3}\right]_0^{1-x}\right] dx$$

$$= \int_0^1 (x^2)[(1-x)-0] + \frac{1}{3}[(1-x)^3-0]\,dx$$

$$= \int_0^1 [x^2(1-x)+\frac{1}{3}(1-x)^3]\,dx$$

$$= \int_0^1 [x^2-x^3+\frac{1}{3}(1-x)^3]\,dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{1}{3}\frac{(1-x)^4}{4(-1)}\right]_0^1$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{1}{12}(1-x)^4\right]_0^1$$

$$= \left[\frac{1}{3}-\frac{1}{4}-\frac{1}{12}(1-1)^4-\left(0-0-\frac{1}{12}(1-0)^4\right)\right]$$

$$= \frac{1}{3}-\frac{1}{4}+\frac{1}{12}$$

$$= \frac{4-3+1}{12}$$

$$= 2/12 = 1/6.$$

① Evaluate $\iint (x+y)\,dx\,dy$, over the region in the positive quadrant bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol Given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

Hence the region of Integration can be expressed as

$0 \le x \le a,\ 0 \le y \le \frac{b}{a}\sqrt{a^2-x^2}$

$$\iint (x+y)\,dx\,dy = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)\,dx\,dy.$$

$$= \int_0^a \left[xy+\frac{y^2}{2}\right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x\left[\frac{b}{a}\sqrt{a^2-x^2}\right] + \frac{\left(b/a\sqrt{a^2-x^2}\right)^2}{2}\,dx$$

$$= \int_0^a \left[x\frac{b}{a}\sqrt{a^2-x^2} + \frac{b^2}{2a^2}(a^2-x^2)\right]dx$$

$$= \int_0^a \left[\frac{-2x}{-2}\frac{b}{a}\sqrt{a^2-x^2} + \frac{b^2}{2a^2}(a^2-x^2)\right]dx$$

$$= \int_0^a \left[-\frac{b}{2a}\sqrt{a^2-x^2}\,(-2x) + \frac{b^2}{2a^2}(a^2-x^2)\right]dx \qquad \left\{\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}\right\}$$

$$= \left[-\frac{b}{2a}\cdot\frac{2}{3}(a^2-x^2)^{3/2} + \frac{b^2}{2a^2}\left(a^2x-\frac{x^3}{3}\right)\right]_0^a$$

$$= \frac{-2b}{6a}(0) + \frac{b^2}{2a^2}$$

$$= \left[-\frac{2b}{6a}(a^2-x^2)^{3/2} + \frac{b^2}{2a^2}\left(a^2x - x^3/3\right) \right]_0^a$$

$$= -\frac{2b}{6a}\left[(a^2-a^2)^{3/2} - (a^2-0)^{3/2} \right] + \frac{b^2}{2a^2}\left[(a^3 - a^3/3) - (0-0) \right]$$

$$= -\frac{2b}{6a}\left[0 - a^3 \right] + \frac{b^2}{2a^2}\left[\frac{2a^3}{3} \right]$$

$$= \frac{a^3 b}{3a} + \frac{b^2 a^3}{a^2 3}$$

$$= \frac{a^2 b}{3} + \frac{ab^2}{3}$$

$$= \frac{ab}{3}[a+b]$$

Double Integrals in polar co-ordinates.

① Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

$$\int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$$

$$\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta}$$

$$\frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta \Rightarrow \frac{a^2}{2}\int_0^{\pi}\frac{(1-\cos 2\theta}{2}\,d\theta$$

$$\frac{a^2}{4}\left[\int_0^{\pi} 1.\,d\theta - \int_0^{\pi}\cos 2\theta\,d\theta\right]$$

$$\frac{a^2}{4}\left[[\theta]_0^{\pi} - \frac{1}{2}(\sin 2\theta]_0^{\pi} \right]$$

$$= \frac{a^2}{4}\left[\pi - \frac{1}{2}[\sin 2\pi - \sin 0]\right]$$

$$= \frac{a^2}{4}[\pi - 0] = \frac{\pi a^2}{4}.$$

① Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$

Sol $\{ \int f(x)^n \cdot f'(x)\,dx = \frac{f(x)^{n+1}}{n+1}$

$= \int_0^{\pi/2} \left[\left[\frac{1}{2} \int_0^{\infty} \frac{2r}{(r^2+a^2)^2} \right] dr \right] d\theta$

$\frac{1}{2} \int_0^{\pi/2} \left[\int_0^{\infty} \frac{1}{(r^2+a^2)^2}\, 2r\,dr \right] d\theta$

$-\frac{1}{2} \int_0^{\pi/2} \left[\frac{1}{(r^2+a^2)} \right]_0^{\infty} d\theta$

$= -\frac{1}{2} \int_0^{\pi/2} \left[\frac{+1}{\infty} - \frac{1}{a^2} \right] d\theta$

$= -\frac{1}{2} \int_0^{\pi/2} \left[0 - \frac{1}{a^2} \right] d\theta$

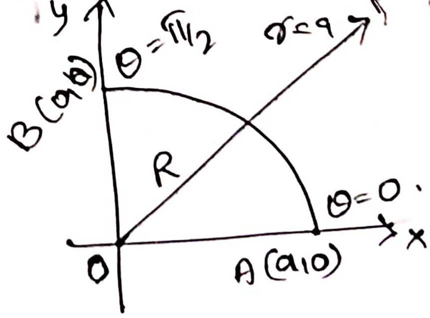
$= \frac{1}{2a^2} \int_0^{\pi/2} d\theta = \frac{1}{2a^2} [\theta]_0^{\pi/2} = \frac{\pi}{4a^2}$ //.

Method next page.

② **Change of variables in Double Integral**

① Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$ by Changing into polar co-ordinates.

Sol let $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$



The Region R. of Integration is specified by the inequalities.

$0 \le x \le \sqrt{a^2-y^2},\ 0 \le y \le a$

$\therefore$ R is the Region bounded by the circle. $x^2+y^2=a^2$ in the first quadrant.

eq ① into the polar form by putting $x = r\cos\theta$, $y = r\sin\theta$ and replace $dx\,dy$ by $r\,dr\,d\theta$ in the cartesian integral.

$$I = \int_0^{\pi/2} \int_{r=0}^{a} r^2 \cdot r \cdot dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{a^4}{4} \int_0^{\pi/2} \theta$$

$$= \frac{\pi a^4}{8}.$$

Change of Variables from cartesian to polar co-ordinates

In this case, we have $u = r$, $v = \theta$ and

$x = r\cos\theta$, $y = r\sin\theta$.

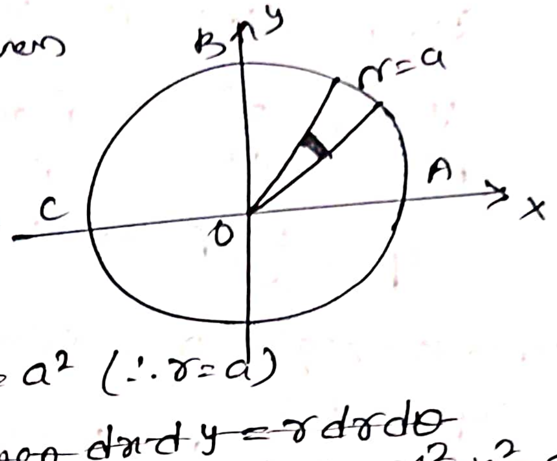
$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

① Transform the following to Cartesian form and hence evaluate $\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr\, d\theta$

Sol The region of integration is given by $r=0, r=a, \theta=0$ and $\theta=\pi$.



In cartesian co-ordinates the same region is given by

$x=0, y=0$ ($\because r=0$) and $x^2+y^2=a^2$ ($\because r=a$)

Since θ varies from 0 to π, ~~Then $dx\,dy = r\,dr\,d\theta$~~

The region of integration is the semi-circle $x^2+y^2=a^2$

The region of integration then $dx\,dy = r\,dr\,d\theta$

let $x = r\cos\theta$, $y = r\sin\theta$

$$\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta \, dr\, d\theta = \int_0^{\pi}\int_0^{a} (r\sin\theta)(r\cos\theta)\, r\, dr\, d\theta$$

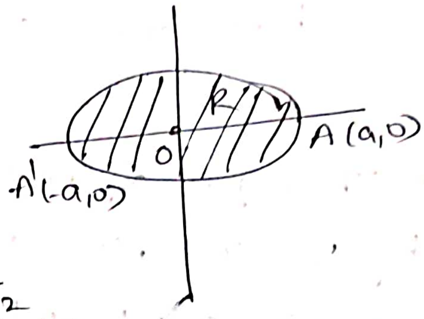
$$= \int_{x=-a}^{a}\int_0^{\sqrt{a^2-x^2}} xy \, dy\, dx$$

$$= \int_{-a}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{-a}^{a} x\left[a^2-x^2-0\right] dx$$

$$= \frac{1}{2}\int_{-a}^{a} x(a^2-x^2)\, dx = 0 \quad \{\because \text{Integrand is odd fun}\}$$

(1) Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Sol Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$

$y^2 = \frac{b^2}{a^2}(a^2-x^2) \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2-x^2}$

Hence the region of integration R can be expressed as

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy$$

$\left\{\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx\right\}$

$$= 2\int_{-a}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy.$$

$$= 2\int_{-a}^{a}\left[x^2[y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} + \left[\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\right]dx$$

$$= 2\int_{-a}^{a} x^2\left[\frac{b}{a}\sqrt{a^2-x^2}\right] + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\,]\,dx.$$

$$= 4\int_0^a \left[\frac{b}{a}x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

$\boxed{\cancel{x = a\sin\theta} \\ \cancel{\theta = \ldots} \\ \cancel{x = a\sin\theta}}$

{putting $x = a\sin\theta$.
$dx = a\cos\theta\,d\theta$
$x = a\sin\theta \Rightarrow 0 = a\sin\theta$
$\theta = 0$
$x = a\sin\theta \Rightarrow a = a\sin\theta$
$\theta = \pi/2$

$$= 4\int_0^{\pi/2}\left(\frac{b}{a}a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta} + \frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right]a\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/2}\frac{b}{a}(a^3)\sin^2\theta\cos\theta + \frac{b^3}{3a^3}(a^2)^{3/2}(1-\sin^2\theta)^{3/2}\,a\cos\theta\,d\theta$$

wrong.

$= 4$

$$= 4\int_0^a \left[\frac{b}{a} x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right] dx.$$

put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta.$

$x = 0 \Rightarrow 0 = a\sin\theta \Rightarrow \sin\theta = \sin 0 = \theta = 0$

$x = a \Rightarrow \cancel{a} = \cancel{a}\sin\theta \Rightarrow \sin\theta = \sin\pi/2 \Rightarrow \theta = \pi/2$

$$= 4\int_0^{\pi/2} \left[\frac{b}{a}(a\sin\theta)^2\sqrt{a^2-a^2\sin^2\theta} + \frac{b^3}{3a^3}(a^2 - a^2\sin^2\theta)^{3/2}\right] a\cos\theta\, d\theta.$$

$$= 4\int_0^{\pi/2} \left[\frac{b}{a} a^2\sin^2\theta\, a\cos\theta + \frac{b^3}{3a^3}(a^2)^{3/2}(1-\sin^2\theta)^{3/2}\right] a\cos\theta\, d\theta.$$

$$= 4\int_0^{\pi/2} \left[b a^2\sin^2\theta\cos\theta + \frac{b^3}{3\cancel{a^3}}\cdot \cancel{a^3}(\cos^2\theta)^{3/2}\right] a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2} \left[b a^2\sin^2\theta\cos\theta + \frac{b^3}{3}\cos^3\theta\right] a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2} \left[b a^3\sin^2\theta\cos^2\theta + \frac{b^3}{3} a\cos^4\theta\right] d\theta.$$

$$= 4\left[a^3 b\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta + \frac{ab^3}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta\right]$$

formulas.

$$\int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{1}{2}, \frac{3}{4}, \frac{5}{6} \cdots = \frac{n-1}{n}\cdot\frac{\pi}{2}, & n \to \text{even} \\ \frac{2\cdot 4\cdot 6 \cdots (n-1)}{1\cdot 3\cdot 5 \cdots n}, & n \to \text{odd.}\end{cases}$$

$$\int_0^{\pi/2} \sin^m x\cos^n x\, dx = \begin{cases} \frac{(m-1)(m-3)\cdots 1\,(n-1)(n-3)}{(m+n)(m+n-2)\cdots 2}\cdot\frac{\pi}{2} & (\&) \text{ both even} \\ \text{otherwise } (0) \end{cases}$$

$$= 4\left[a^3 b\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{ab^3}{\cancel{3}}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{4\pi}{16}\left[a^3b + ab^3\right] \Rightarrow \frac{4\pi ab}{16}\left[a^2+b^2\right] //$$

① Evaluate $\displaystyle\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol

$$\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$

$$= \int_0^{\pi/4}\left[\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\right]d\theta \qquad \left\{ \int f^n(x)\cdot f'(x)\,dx = \frac{f^{n+1}(x)}{n+1}+c \right\}$$

$$= \int_0^{\pi/4}\left[\frac{1}{-2}\int_0^{a\sin\theta} (a^2-r^2)^{-1/2}(-2r)\,dr\right]d\theta$$

$$= \int_0^{\pi/4} \frac{1}{-2}\left[\frac{(a^2-r^2)^{-1/2+1}}{-1/2+1}\right]_0^{a\sin\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{-2}\times\frac{2}{1}\left[(a^2-r^2)^{1/2}\right]_0^{a\sin\theta} d\theta$$

$$= -\int_0^{\pi/4}\left[(a^2-r^2)^{1/2}\right]_0^{a\sin\theta} d\theta$$

$$= -\int_0^{\pi/4}\left\{[a^2-a^2\sin^2\theta]^{1/2}-[a^2-0]^{1/2}\right\} d\theta$$

$$= -\int_0^{\pi/4}\left[(a^2)^{1/2}(1-\sin^2\theta)^{1/2} - a\right] d\theta$$

$$= -a\int_0^{\pi/4}\left[(\cos^2\theta)^{1/2}-1\right]d\theta$$

$$= -a\int_0^{\pi/4}[\cos\theta-1]\,d\theta$$

$$= -a\left[[\sin\theta]_0^{\pi/4}-(\theta)_0^{\pi/4}\right]$$

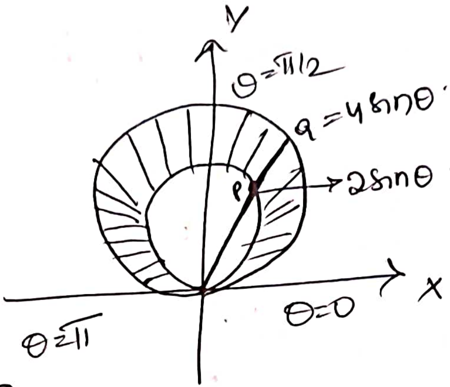
$$= -a\left[\sin\pi/4-\pi/4\right]$$

$$= -a\left[\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right]$$

$$= a\left[\pi/4 - 1/\sqrt{2}\right]$$

(2) Evaluate $\iint r^3 dr d\theta$ over the area included b/w the circle $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol The Region of integration R is shown shaded.
Here r varies from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π



$$\iint r^3 dr d\theta = \int_{\theta=0}^{\pi} \left[\int_{r=2\sin\theta}^{4\sin\theta} r^3 dr\right] d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi} [4^4\sin^4\theta - 2^4\sin^4\theta]\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi} [256\sin^4\theta - 16\sin^4\theta]\, d\theta$$

$$\begin{array}{r} 256 \\ 16 \\ \hline 240 \end{array}$$

$$= \frac{1}{4} \times 240 \int_0^{\pi} \sin^4\theta\, d\theta$$

60

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta \qquad \left\{ \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ if } f(2a-x) = f(x) \right\}$$

$$= 60\times 2\int_0^{\pi/2} \sin^4\theta\, d\theta \qquad \left\{ \int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} \right\}$$

$$= 120\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 120\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

30, 15

$$= \frac{45\pi}{2}$$

Change of variables in double Integral

Transformation of co-ordinates

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations b/wn the old variable (x,y) with the new variables (u,v) of the new co-ordinates system

the $\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \cancel{\dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}} \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

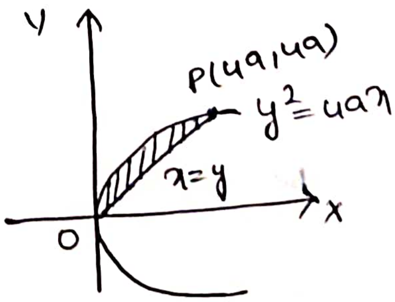
which is called the Jacobian of the co-ordinate Changes transformation.

① Show that $\displaystyle\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = 8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$.

Sol The given Integral $\displaystyle\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$.

The region of integration is given by $x = \dfrac{y^2}{4a}$, $x = y$ and $y = 0$, $y = 4a$.

∴ The region is bounded by the parabola $y^2 = 4ax$ and the straight line $x = y$.



Let $x = r\cos\theta$, $y = r\sin\theta$ then $dx\,dy = r\,dr\,d\theta$.

The limits for r are $r = 0$ at O and for P on the parabola $y^2 = 4ax$.

$$r^2\sin^2\theta = 4a(r\cos\theta)$$

$$r = \frac{4a\cos\theta}{\sin^2\theta}$$

for the line $y = x$, a slope $m = 1$ ie $\tan\theta = 1$, $\theta = \pi/4$.

The limits for $\theta : \pi/4 \rightarrow \pi/2$

Also $x^2-y^2 = r^2(\cos^2\theta - \sin^2\theta)$ and $x^2+y^2 = r^2$

$$\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{(\cos^2\theta-\sin^2\theta)\cancel{r^2}}{\cancel{r^2}}\, r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[[\cos^2\theta-\sin^2\theta]\int_{r=0}^{4a\cos\theta/\sin^2\theta} r\,dr\right]d\theta$$

$$= \int_{\pi/4}^{\pi/2}[\cos^2\theta-\sin^2\theta]\left[\frac{r^2}{2}\right]_0^{4a\cos\theta/\sin^2\theta} d\theta.$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)\left[\frac{4^2a^2\cos^2\theta}{\sin^4\theta} - 0\right]d\theta.$$

$$= \frac{\overset{8}{\cancel{16}}a^2}{\cancel{2}}\int_{\pi/4}^{\pi/2}\left[\frac{\cos^4\theta}{\sin^4\theta} - \frac{\cancel{\sin^2\theta}\cos^2\theta}{\sin^{\cancel{4}}\theta_{\,2}}\right]d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}[\cot^4\theta - \cot^2\theta]\,d\theta.$$

$$= 8a^2\left[\frac{3\pi-8}{12} + \pi/4 - 1\right]$$

$$= 8a^2\left[\pi/2 - 5/3\right]$$ { direct formula way }

Change of order of integration :-

working Rule:

Step 1. First identify the variables for the limits.

Step 2. Draw a rough sketch of the given region of integration.

3. If we are evaluating the integral w.r.to y first then take a vertical strip. (i.e) a strip parallel to y-axis otherwise. take a horizontal strip (i.e) a strip parallel to x-axis

4. Now rotate the strip by an angle 90° in the anti-clock wise direction and identify the starting & ending points of the strip which will be the lower and upper limits of that variable.

Step 5: Identify the limits for other variables for the region of considaration.

Step 6. Evaluate the double integral with new order of integration.

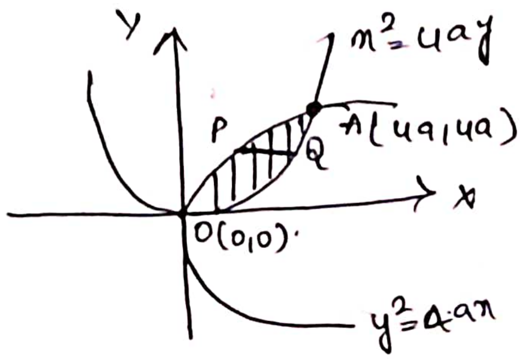
① Change the order of integration and evaluate

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$$

Sol The given Integral $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$

y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from 0 to 4a.

$y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$. (i.e) $x^2 = 4ay$ & $y^2 = 4ax$.



The region of integration is the shaded region in figure. The given integral is

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$ Changing the order of integration, we must fix y first

x varies from $x = \frac{y^2}{4a}$ to $\sqrt{4ax}$ and the y varies from 0 to 4a.

$$\int_0^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}}$$

$$= \int_0^{4a} [2\sqrt{ay} - y^2/4a]\,dy$$

$$= 2\sqrt{a}\int_0^{4a} y^{1/2}dy - 1/4a\int_0^{4a} y^2 dy$$

$$= 2\sqrt{a}\left[\frac{y^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a}\times\frac{2}{3}\left[y^{3/2}\right]_0^{4a} - \frac{1}{12a}\left[y^3\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\left[(4a)^{3/2} - 0\right] - \frac{1}{12a}\left[(4a)^3\right]$$

$$= \frac{\cancel{2}4\sqrt{a}}{3}\cdot(2^{\cancel{2}})^{3/2}a^{3/2} - \frac{1}{\cancel{12}\cancel{a}_{\cancel{6}3}}\left[\cancel{64}^{\cancel{32}\,16}\cancel{a^3}^{2}\right]$$

$$= \frac{32\cdot a^{1/2+3/2}}{3} - \frac{16a^2}{3}$$

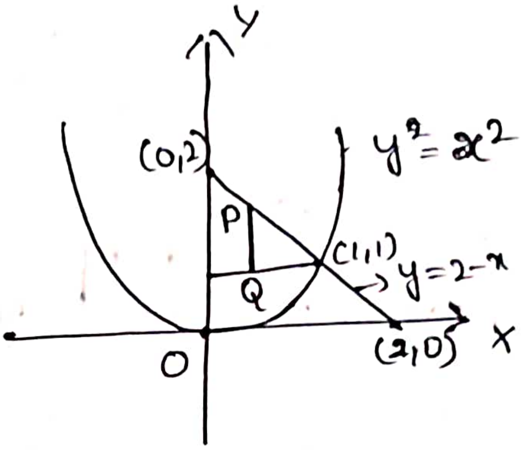
$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2 \quad //.$$

① Change of order of integration $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ evaluate. double Integral.

Sol The given Integral is

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx.$$



The region of Integration is given from $y=x^2$ to $y=2-x$ and $x=0$ to $x=1$.

Hence we shall draw curve $y=x^2$ on line $y=2-x$

The line $y=2-x$ passes through $(2,0)$ & $(0,2)$ the point of Intersection of curve $y=x^2$ and $y=2-x$ are obtain by solving these eqn

$y=2-x$

$x^2=2-x$

$x^2+x-2=0$

$x^2+2x-x-2=0$

$x(x+2)-1(x+2)=0$

$(x+2)(x-1)=0$

$x=-2,\ x=1.$

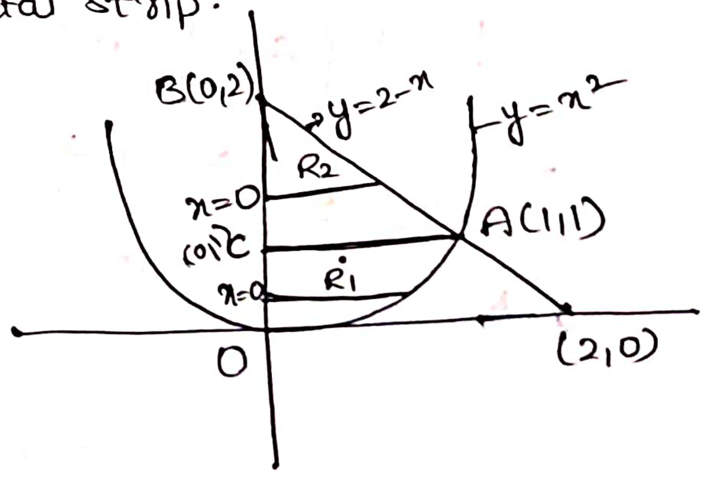
when $x=-2,\ y=(-2)^2=4$

when $x=1\ \ y=1$

Here point of Intersection of curve are $(-2,4),(1,1)$

Suppose we change of order of Integration in the changed order we have to take two horizontal strip.

we take region as follows.



Area OAB = Area OAC + Area CAB.

limits varies from for the 1st Region of Integration is given by $y=0$ to $y=1$ &

$x=0$ to $x=\sqrt{y}$

and 2nd Region of Integration is given by $y=1$ to $y=2$

$x=0$ to $x=2-y$.

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx - \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dy\,dx$$

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{0}^{2-y} xy\,dy\,dx.$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y\left[(\sqrt{y})^2 - 0^2\right] dy + \frac{1}{2}\int_1^2 y\left[x^2\right]_0^{2-y} dy.$$

$$= \frac{1}{2}\int_0^1 y[y]\,dy + \frac{1}{2}\int_1^2 y\left((2-y)^2 - 0^2\right) dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y\left[4+y^2-4y\right] dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[\int_1^2 \left[4y+y^3-4y^2\right] dy\right]$$

$$= \frac{1}{6}\left[1^3 - 0^3\right] + \frac{1}{2}\left[4\left[\frac{y^2}{2}\right]_1^2 + \left[\frac{y^4}{4}\right]_1^2 - 4\left[\frac{y^3}{3}\right]_1^2\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[2\left[2^2-1^2\right] + \frac{1}{4}\left[2^4-1^4\right] - \frac{4}{3}\left(2^3-1^3\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[6 + \frac{1}{4}[15] - \frac{4}{3}[7]\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{3\times 24 + 15\times 3 - 16\times 7}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{24}\left[72+45-112\right] = \frac{1}{6} + \frac{5}{24} = \frac{4+5}{24} = \frac{\cancel{9}^{3}}{\cancel{24}_{8}}$$

$$= 3/8$$

① By Changing the order of integration, evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy dx$.

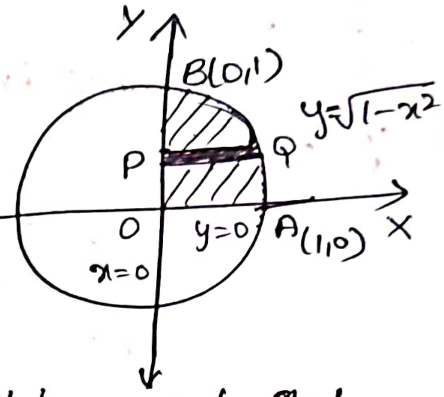
Sol The area of integration lies b/w $y=0$ to $y=\sqrt{1-x^2}$

$x^2+y^2=1$.

The limits of x are 0 to 1.

Hence the region of integration is OAB and is divided into vertical strips.

For changing the order of integration, we shall divide the region of integration into horizontal strips.



The new limits of integration become.

$x=0$ to $x=\sqrt{1-y^2}$ and those for y will be $y=0$ to $y=1$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy dx = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2 dy dx.$$

$$= \int_{y=0}^{1} y^2 \int_{x=0}^{\sqrt{1-y^2}} dx\, dy.$$

$$= \int_{y=0}^{1} y^2 [x]_0^{\sqrt{1-y^2}} dy.$$

$$= \int_0^1 y^2 [\sqrt{1-y^2}-0] dy.$$

To change into polar co-ordinates, put $y=\sin\theta$. Then

$dy = \cos\theta\, d\theta$

Also $y=0 \Rightarrow \theta=0$ and $y=1 \Rightarrow \theta=\pi/2$

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta = \int_{\theta=0}^{\pi/2} \sin^2\theta\, (\cos\theta\, d\theta) \cos\theta.$$

$$= \int_{\theta=0}^{\pi/2} \sin^2\theta \cdot \cos^2\theta\, d\theta \quad \S$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{16}$$ //.

$\S$ $\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{(m-1)(n-1)}{(m+n)(m+n-2)} \frac{\pi}{2}$.

1) Change the order of integration and solve $\int_0^a \int_{x^2}^{2a-x} xy^2 \, dy \, dx$.

<u>Triple Integral</u>

The triple Integral is

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z) \, dx\,dy\,dz = \int_{x_1}^{x_2} \Big[\int_{y_1(x)}^{y_2(x)} \Big[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) dz \Big] dy \Big] dx.$$

① Evaluate $I = \int_0^1 \int_1^2 \int_2^3 xyz \, dx\,dy\,dz$.

$$= \int_0^1 \int_1^2 \Big[\frac{x^2}{2}\Big]_2^3 yz \, dy\,dz.$$

$$= \frac{1}{2} \int_0^1 \int_1^2 [3^2 - 2^2] y\, z \, dy\,dz$$

$$= \frac{5}{2} \int_0^1 \Big[\frac{y^2}{2}\Big]_1^2 z \, dz$$

$$= \frac{5}{4} \int_0^1 z [2^2 - 1^2] dz$$

$$= \frac{15}{4} \int_0^1 z \, dz$$

$$= \frac{15}{4} \Big[\frac{z^2}{2}\Big]_0^1 = \frac{15}{8} \,//.$$

② $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dx\,dy\,dz$.

Sol $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y} (e^z dz) \, dy\,dx$.

$$\int_0^a \int_0^x [e^z]_0^{x+y} e^{x+y} \, dy\,dx$$

$$\int_0^a \int_0^x e^{x+y} [e^{x+y} - e^0] \, dy\,dx.$$

$$\int_0^a \int_0^x [e^{2x+2y} - e^{x+y}]\, dy\, dx.$$

$$\int_0^a [e^{2x}[\int_0^x e^{2y} dy] - e^x \int_0^x e^y dy]\, dx$$

$$= \int_0^a [e^{2x}[\frac{e^{2y}}{2}]_0^x - e^x [e^y]_0^x]\, dx$$

$$= \int_0^a \frac{e^{2x}}{2}[e^{2x} - e^0] - e^x[e^x - e^0]\, dx$$

$$= \int_0^a [\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x]\, dx.$$

$$= \int_0^a [\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x]\, dx.$$

$$= [\frac{e^{4x}}{8}]_0^a - \frac{3}{2}[\frac{e^{2x}}{2}]_0^a + [e^x]_0^a$$

$$= \frac{e^{4a}}{8} - \frac{1}{8} - \frac{3}{4}[e^{2a} - 1] + [e^a - 1]$$ $-1+6-8$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{1}{8} + \frac{3}{4} - 1$$

$$I = \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$$

*

① Evaluate $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx.$

Sol Given Integral $= \int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y} \cdot e^z \cdot dz\, dy\, dx$

$$= \int_0^{\log 2} \int_0^x e^{x+y} [\int_0^{x+\log y} e^z dz]\, dy\, dx$$

$$= \int_0^{\log 2} \int_0^x e^{x+y} [e^z]_0^{x+\log y}\, dy\, dx$$

$$= \int_0^{\log 2}\int_0^{x} [e^{x+y}][e^{x+\log y} - e^0]\,dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x+y}[e^{x}\cdot e^{\log y} - 1]\,dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x+y}[e^{x}\cdot y - 1]\,dy\,dx.$$

$$= \int_0^{\log 2}\int_0^{x} e^{x}\cdot e^{y}[e^{x}\cdot y - 1]\,dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x}[e^{x}\cdot ye^{y} - e^{y}]\,dy\,dx$$

$$= \int_0^{\log 2} e^{x}\left[e^{x}\int_0^{x} ye^{y}\,dy - \int_0^{x} e^{y}\,dy\right]dx$$

$$= \int_0^{\log 2} e^{x}\left[e^{x}[ye^{y} - e^{y}]_0^{x} - [e^{y}]_0^{x}\right]dx$$

$$= \int_0^{\log 2} e^{x}\left[e^{x}[xe^{x} - e^{x} - (0 - e^0)] - (e^{x} - e^0)\right]dx$$

$$= \int_0^{\log 2} e^{x}[e^{x}[xe^{x} - e^{x} + 1] - e^{x} + 1]\,dx$$

$$= \int_0^{\log 2} e^{x}[xe^{2x} - e^{2x} + \cancel{e^{x}} - \cancel{e^{x}} + 1]\,dx$$

$$= \int_0^{\log 2} [xe^{3x} - e^{3x} + e^{x}]\,dx$$

$$= \left[x\frac{e^{3x}}{3} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^{x}\right]_0^{\log 2}$$

$$= \log 2\cdot\frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{\log 2} - \left[0 - \frac{e^0}{9} - \frac{e^0}{3} + e^0\right]$$

$$= \log 2\cdot\frac{8}{3} - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} \cancel{+1} = \frac{8}{3}\log 2 - \frac{19}{9}$$

$-8 - 24 + 18$
$+1 + 3 -$
-14

② Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$.

Sol

$$\int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\int_0^{\sqrt{1-x^2-y^2}} z\,dz\right] dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy[1-x^2-y^2-0]\,dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\,dy\,dx.$$

$$= \frac{1}{2} \int_0^1 \left[x \int_0^{\sqrt{1-x^2}} y\,dy - x^3 \int_0^{\sqrt{1-x^2}} y\,dy - x \int_0^{\sqrt{1-x^2}} y^3\,dy\right] dx$$

$$= \frac{1}{2} \int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} - x^3\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} - \frac{x}{4}\left[y^4\right]_0^{\sqrt{1-x^2}} \Big] dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{x}{2}(1-x^2) - \frac{x^3}{2}[1-x^2] - \frac{x}{4}[1-x^2]^2\right] dx.$$

$$= \frac{1}{2} \int_0^1 \frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4}[x^4+1-2x^2]\Big] dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{x-2x^3+x^5}{2}\right) - \frac{x^5}{4} - \frac{x}{4} + \frac{2x^3}{4}\Big] dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{x-2x^3+x^5}{2} + \frac{-x^5-x+2x^3}{4}\right] dx$$

$$= \cancel{\frac{1}{4}\left[\int_0^1 2(x-2x^3+x^5)\right.} = \frac{1}{8} \int_0^1 [2x - 4x^3 + 2x^5 - x^5 - x + 2x^3]\,dx$$

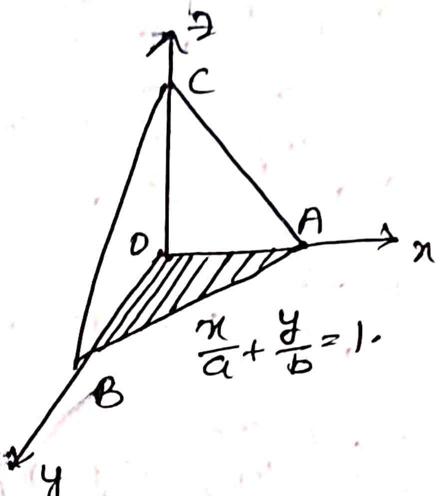
$$= \frac{1}{8} \left[\int_0^1 x - 2x^3 + x^5\right] dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2} - 2\frac{x^4}{4} + \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6} - 0\right] = \frac{1}{48}$$

Volume as a Double Integral

① Using double integration, find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.



Sol

Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \longrightarrow ①$

$$z=c\left[1-\frac{x}{a}-\frac{y}{b}\right] \longrightarrow ②$$

In the xy-plane, $z=0$.

Substituting $z=0$ in ①, we get

$$\frac{x}{a}+\frac{y}{b}=1.$$

The region R in the xy-plane is a triangle OAB bounded by $x=0, y=0$ and the line $\frac{x}{a}+\frac{y}{b}=1$.

In the region x varies from 0 to a, y varies from 0 to $b(1-\frac{x}{a})$

Hence the required volume $= \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} z\,dy\,dx$.

$$=\int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} c\left[1-\frac{x}{a}-\frac{y}{b}\right]dy\,dx.$$

$$=c\int_0^a\left[\frac{(1-\frac{x}{a}-\frac{y}{b})^2}{(-2/b)}\right]_0^{b(1-x/a)}dx \qquad \left\{\int x^n dx = \frac{-x^{n+1}}{-(n+1)}\right.$$

$$\boxed{\cancel{c\int_0^a\left[\frac{(1-x/a)}{-2/b}-\frac{1}{b}[y\right.}}$$

$$=c\int_0^a -\frac{b}{2}\left[\left[1-\frac{x}{a}-\frac{y}{b}\right]^2\right]_0^{b(1-\frac{x}{a})}dx.$$

$$=\frac{-cb}{2}\int_0^a\left[1-\frac{x}{a}-\frac{1}{b}\left[b\left(1-\frac{x}{a}\right)-0\right]\right]^2 dx$$

$$= \boxed{\cancel{-\frac{cb}{2}\int_0^a \left[1-\frac{x}{a}-\left(1-\frac{x}{a}\right)\right]^2 dx}}.$$

$$= -\frac{cb}{2}\int_0^a \left[\left[1-\frac{x}{a}-\frac{1}{b}\left(b\left(1-x/a\right)\right)\right]-\left[1-\frac{x}{a}-\frac{0}{b}\right]\right]^2 dx$$

$$= -\frac{cb}{2}\int_0^a \left[\left[\cancel{1}-\cancel{\frac{x}{a}}\cancel{-1}+\cancel{\frac{x}{a}}\right]^2-\left[\left(1-\frac{x}{a}\right)\right]^2\right] dx$$

$$= \frac{cb}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{cb}{2}\left[\frac{(1+x/a)^3}{-3/a}\right]_0^a$$

$$= -\frac{abc}{6}\left[\left(1-x/a\right)^3\right]_0^a$$

$$= -\frac{abc}{6}\left[0-1\right] = \frac{abc}{6}$$

(2) Find by double integral the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$

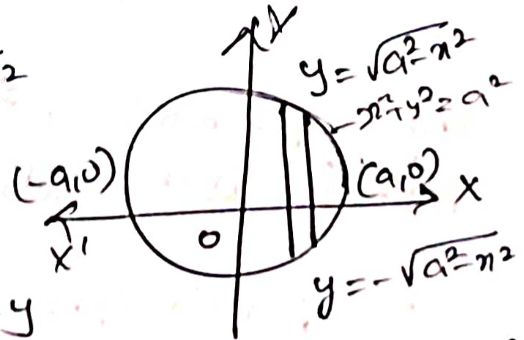
Sol Given cylinders are $x^2+y^2=a^2 \longrightarrow (1)$

$x^2+z^2=a^2 \longrightarrow (2)$

with xy plane $z=0$.

The section of the cylinder $x^2+y^2=a^2$ is the circle $x^2+y^2=a^2$

from eq (2) we have $z=\pm\sqrt{a^2-x^2}$



∴ Each of the surface is symmetrical about the xy-plane.

Hence the required volume $= 2\iint_R z\,dx\,dy$

where $z=\sqrt{a^2-x^2}$ and R is the region bounded by the circle given by eq (1) on the xy-plane i.e

$-a \le x \le a$ and $-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

$$\text{Required volume} = 2\int_{-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z\,dx\,dy.$$

$$= 2\int_{-a}^{a}\left[\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\right]dx.$$

$$= 2\int_{-a}^{a}\sqrt{a^2-x^2}\,[y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\,dx.$$

$$= 2\int_{-a}^{a}\sqrt{a^2-x^2}\left[\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right]dx$$

$$= 4\int_{-a}^{a}\left[\sqrt{a^2-x^2}\cdot\sqrt{a^2-x^2}\right]dx$$

$$= 4\int_{-a}^{a}(a^2-x^2)\,dx$$

$$= 8\int_{0}^{a}(a^2-x^2\,dx$$

$$= 8\left[a^2[x]_0^a-\left(\frac{x^3}{3}\right)_0^a\right]$$

$$= 8\left[a^3-\frac{a^3}{3}\right]$$

$$= 8\left[\frac{2a^3}{3}\right] = \frac{16a^3}{3}$$

(2) Find the area of the circle using double integral.

(1) using triple integral, find the volume of the sphere whose radius is a units.

Sol The eqn of the sphere is $x^2+y^2+z^2=a^2$

In cartesian form, the region of integration is as follows.

z varies from $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$

y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

x varies from $-a$ to a.

Required Volume $= \iiint dx\,dy\,dz$

$$= \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz$$

$$= \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [z]_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dy$$

$$= 2\int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\, dy.$$

$$= 2\int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\, dy \Rightarrow 2\int_{-a}^{a} dx \cdot 2\int_{0}^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\, dy$$

$$= 4\int_{-a}^{a} dx \left[\frac{y}{2}\sqrt{(\sqrt{a^2-x^2})^2-y^2} + \frac{(\sqrt{a^2-x^2})^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_{0}^{\sqrt{a^2-x^2}}$$

$$= 4\int_{-a}^{a} dx \left[0 + \frac{a^2-x^2}{2}\sin^{-1}(1) - \{0+0\}\right]$$

$$= 4\int_{-a}^{a} dx \left(\frac{a^2-x^2}{2}\right)\pi/2$$

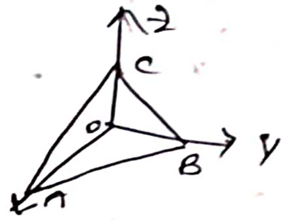
$$= \frac{4\pi}{4}\int_{-a}^{a} (a^2-x^2)\, dx$$

$$= 2\pi\int_{0}^{a} (a^2-x^2)\, dx \Rightarrow 2\pi\left[a^2(x)_0^a - \left(\frac{x^3}{3}\right)_0^a\right] = 2\pi\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{4\pi a^3}{3}$$

① Find the volume of the tetrahedron bounded by the planes $x=0, y=0, z=0$, and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol The Required volume $= \iiint_V dx\,dy\,dz$



on the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,\ z=c\left[1-\frac{x}{a}-\frac{y}{b}\right]$$

∴ The Required volume of the tetrahedron.

Hence for a fixed (x,y) on the xy plane within $\triangle OAB$, z

$$z \to 0 \text{ to } c\left[1-\frac{x}{a}-\frac{y}{b}\right]$$

$$y \to 0 \text{ to } b\left[1-\frac{x}{a}\right]$$

$$x \to 0 \text{ to } a.$$

The Required volume of the tetrahedron.

$$= \int_0^a dx \int_{y=0}^{b[1-\frac{x}{a}]} dy \int_{z=0}^{c[1-\frac{x}{a}-\frac{y}{b}]} dz$$

$$= \int_0^a dx \int_0^{b[1-\frac{x}{a}]} \left[c(1-x/a-y/b)-0\right] dy$$

$$= \int_0^a dx \int_0^{b[1-\frac{x}{a}]} \left[c(1-x/a)-\frac{c}{b}y\right] dy$$

$$= \int_0^a dx \left[\left(c-\frac{c}{a}x\right)y-\frac{c}{b}\frac{y^2}{2}\right]_0^{b(1-x/a)}$$

$$= \int_0^a dx \left[\left(c-\frac{c}{a}x\right)\left(b(1-x/a)-0\right)-\frac{c}{2b}\left[b^2(1-x/a)^2-0\right]\right]$$

$$= \int_0^a dx \left[cb\left(1-\frac{x}{a}\right)^2-\frac{cb}{2}\left(1-\frac{x}{a}\right)^2\right]$$

$$= \frac{cb}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx.$$

$$= \frac{cb}{2}\left[\frac{(1-x/a)^3}{-1/a\cdot 3}\right]_0^a = \left[-\frac{cb}{2}\cdot\frac{a}{3}\left[\left(1-\frac{a}{a}\right)^3-\left(1-\frac{0}{a}\right)^3\right]\right]$$

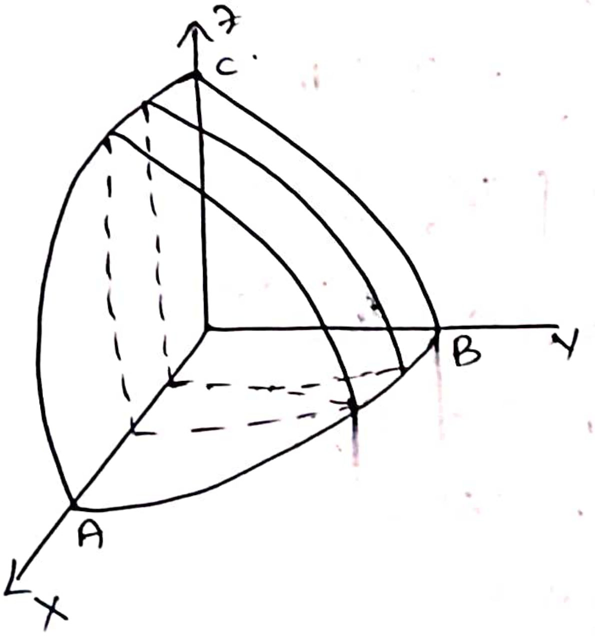
$$= -\frac{abc}{6}[0-1] = \frac{abc}{6} \text{ c.units.}$$

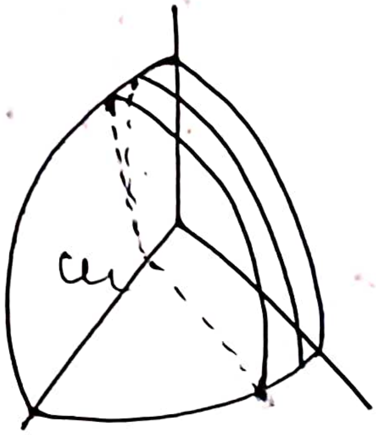
① Find the volume of the greatest rectangular parallelopiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Sol The Solid figure $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is cut into 8 equal pieces by the three co-ordinate planes.

Hence the volume of the Solid is equal to 8 times the volume of the Solid bounded by $x=0,\ y=0,\ z=0$ and surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

z varies from $z=0$ to $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$





Hence the required volume $=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx.$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx.$$

write $1-\frac{x^2}{a^2}=\frac{p^2}{b^2}$

$\therefore$ The required volume $=8\int_0^a\int_0^p \frac{c}{b}\sqrt{p^2-y^2}\,dy\,dx.$

$$=\frac{8c}{b}\int_0^a\int_0^p\left[\sqrt{p^2-y^2}\,dy\right]dx.$$

$$= \frac{8c}{b}\int_0^a\int_0^p \left(\sqrt{p^2-y^2}\,dy\right)dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{p^2-y^2}+\frac{p^2}{2}\sin^{-1}\left(\frac{y}{p}\right)\right]_0^p dx$$

$$= \frac{8c}{b}\int_0^a 0+\left[\frac{p^2}{2}\sin^{-1}\left(\frac{p}{p}\right)-[0+0]\right] dx$$

$$= \frac{8c}{b}\int_0^a \frac{p^2}{2}\sin^{-1}(1)\,dx$$

$$= \frac{8c}{b}\int_0^a \frac{p^2}{2}\cdot\frac{\pi}{2}\,dx.$$

$$= \frac{2c}{b}\int_0^a p^2\,dx. = \frac{2c\pi}{b}\int_0^a b^2\left[1-\frac{x^2}{a^2}\right]dx$$

$$= 2bc\pi\int_0^a \left[1-\frac{x^2}{a^2}\right]dx$$

$$= 2\pi bc\left[[x]_0^a-\frac{1}{a^2}\left[\frac{x^3}{3}\right]_0^a\right]$$

$$= 2\pi bc\left[(a-0)-\frac{1}{3a^2}[a^3-0]\right]$$

$$= 2\pi bc\left[a-\frac{a}{3}\right]$$

$$= 2\pi bc\left[\frac{2a}{3}\right]$$

$$= \frac{4abc\pi}{3} \quad //.$$

$$= \frac{4\pi}{3}abc \text{ cu.units.}$$