UNIT - IV

Vector Differentiation and Vector operators

Vector :— A physical quantity which has both magnitude and direction is called a vector.

Ex: Velocity, force, electric force, magnitic force, accetalation

Scalar:

A physical quantity which has only magnitude but no direction is called Scalar.

Ex: time, temperature, distance, speed, mass.

Scalar point func:—

consider a region in 3 dimensional space to each point $p(x,y,z)$ to associate a unique real number (scalar) $\phi(x,y,z)$ this $\phi(x,y,z)$ is called scalar point func.

Vector point func:—

To each point $p(x,y,z)$ to associate a unique vector $\bar{f}(x,y,z)$ when $\bar{f}$ is called vector point func.

Derivative of vector point func:—

Let $\bar{f}$ be a vector point func its derivative is given by $\bar{f}(x,y,z)$.

$$\frac{d\bar{f}}{dt} = \lim_{x \to a} \frac{\bar{f}(t) - \bar{f}(a)}{t-a}$$

Vector Differential operator :-

The vector differential operator ∇. is defined as.

$$\boxed{\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}}$$

Gradient of a scalar point function :-

let $\phi(x,y,z)$ be a scalar point func then the gradient of file defined as.

$$\boxed{\text{grad}\,\phi = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}}$$

properties of Gradient :-

1) If f and g two scalar functions when

$$\text{grad}[f \pm g] = \text{grad}\, f \pm \text{grad}\, g.$$

2) The necessary and sufficient condition for a scalar point func to be constant is that

$$\nabla f = \bar{0} \quad (\text{null vector}).$$

3) $\text{grad}(fg) = f(\text{grad}\, g) + g(\text{grad}\, f)$

4) $\text{grad}\left[\frac{f}{g}\right] = \frac{g(\text{grad}\, f) - f(\text{grad}\, g)}{g^2} \quad (g \neq 0)$

5) $\text{grad}(c.\bar{f}) = c(\text{grad}\, \bar{f})$

6) let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

If ϕ is any scalar point func, then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left[\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right]\left[\bar{i}dx + \bar{j}dy + \bar{k}dz\right]$$

$$= \nabla\phi \cdot d\bar{r}$$

① Find gradϕ where $\phi(x,y,z) = \log(x^2+y^2+z^2)$ at $(1,1,1)$

Sol $\phi(x,y,z) = \log(x^2+y^2+z^2)$

Then $\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\log(x^2+y^2+z^2) = \frac{1}{x^2+y^2+z^2}(2x) = \frac{2x}{x^2+y^2+z^2}$

$\frac{\partial\phi}{\partial y} = \frac{2y}{x^2+y^2+z^2}$, $\frac{\partial\phi}{\partial z} = \frac{2z}{x^2+y^2+z^2}$

$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$= \frac{2x}{x^2+y^2+z^2}\bar{i} + \frac{2y}{x^2+y^2+z^2}\bar{j} + \frac{2z}{x^2+y^2+z^2}\bar{k}$

$\nabla\phi = \frac{2}{x^2+y^2+z^2}[x\bar{i} + y\bar{j} + z\bar{k}]$

Hence gradϕ at $(1,1,1)$ $= \frac{2}{3}[\bar{i}+\bar{j}+\bar{k}]$

① Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $\bar{i}+2\bar{j}+2\bar{k}$ at the point $(1,2,0)$.

Sol Given $f = xy + yz + zx$.

grad $f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

then $\frac{\partial f}{\partial x} = y+z$, $\frac{\partial f}{\partial y} = x+z$, $\frac{\partial f}{\partial z} = x+y$.

grad $f = \bar{i}(y+z) + \bar{j}(x+z) + \bar{k}(x+y)$.

→ If $\bar{e}$ is the unit vector in the direction of vector $\bar{i}+2\bar{j}+2\bar{k}$,

If $\bar{e} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})$

∴ Directional derivative of f along the given direction

$= \bar{e}\cdot\nabla f$

$= \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})\,[(y+z)\bar{i} + (x+z)\bar{k} + (x$

$= \frac{1}{3}[(y+z) + 2(x+z) + 2(x+y)]$

at the point $(1,2,0)$ $= \frac{1}{3}(2+2+6) = \frac{10}{3}$

Directional Derivative. Theorem:-

The directional derivative of a scalar point func ϕ at a point $p(x,y,z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e}\cdot \text{grad}\,\phi$ ie $\bar{e}\cdot\nabla\phi$

$$\bar{e} = \frac{x\bar{i}+y\bar{j}+z\bar{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\bar{e} = \frac{\bar{e}}{|\bar{e}|}$$

① find $\nabla(x^2-yz+z^2)$

Let $f(x,y,z) = x^2-yz+z^2$

Then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = -z$, $\frac{\partial f}{\partial z} = -y+2z$

$$\nabla(x^2-yz+z^2) = \nabla\phi = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

$$\nabla f = 2x\bar{i} + (-z)\bar{j} + (-y+2z)\bar{k}$$

② $\nabla(x^2+y^2z)$ ② Find grad f where

$f = x^3+y^3+3xyz$, $f = x^2y+y^2x+z^2$

$f = x^3-y^3+x^2z$

③ P.T

$\nabla r^n = nr^{n-2}\bar{r}$

Sol

$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and let $r = |\bar{r}|$

Then we have $r^2 = x^2+y^2+z^2$

diff w.r.to x partially, we have.

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla(r^n) = \frac{\partial}{\partial x}(r^n)\bar{i} + \frac{\partial}{\partial y}(r^n)\bar{j} + \frac{\partial}{\partial z}(r^n)\bar{k}$$

$$= nr^{n-1}\frac{\partial r}{\partial x}\bar{i} + nr^{n-1}\frac{\partial r}{\partial y}\bar{j} + nr^{n-1}\frac{\partial r}{\partial z}\bar{k}$$

$$= n r^{n-1}\left[\frac{\partial r}{\partial x}\bar{i}+\frac{\partial r}{\partial y}\bar{j}+\frac{\partial r}{\partial z}\bar{k}\right]$$

$$= n r^{n-1}\left[\frac{x}{r}\bar{i}+\frac{y}{r}\bar{j}+\frac{z}{r}\bar{k}\right]$$

$$= n r^{n-1}\cdot r^{-1}\left[x\bar{i}+y\bar{j}+z\bar{k}\right]$$

$$= n r^{n-2}\left[x\bar{i}+y\bar{j}+z\bar{k}\right]$$

$$\boxed{\nabla r^{n} = n r^{n-2}\bar{r}}$$

(2) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\bar{i}-\bar{j}-2\bar{k}$.

Sol Given

$$\phi = x^2yz+4xz^2$$

$$\therefore \frac{\partial\phi}{\partial x} = 2xyz+4z^2,\quad \frac{\partial\phi}{\partial y} = x^2z,\quad \frac{\partial\phi}{\partial z} = x^2y+8xz.$$

Hence $\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i}+\frac{\partial\phi}{\partial y}\bar{j}+\frac{\partial\phi}{\partial z}\bar{k}$

$$\nabla\phi = (2xyz+4z^2)\bar{i}+(x^2z)\bar{j}+(x^2y+8xz)\bar{k}$$

$$\therefore \nabla\phi \text{ at } (1,-2,-1) = 8\bar{i}-\bar{j}-10\bar{k}$$

The unit vector in the direction $2\bar{i}-\bar{j}-2\bar{k}$ is $\bar{e} = \dfrac{2\bar{i}-\bar{j}-2\bar{k}}{\sqrt{2^2+(-1)^2+(-2)^2}} =$

$$\bar{e} = \frac{1}{\sqrt{9}}(2\bar{i}-\bar{j}-2\bar{k}) = \frac{1}{3}(2\bar{i}-\bar{j}-2\bar{k})$$

$\therefore$ Directional derivative of ϕ in the direction of $\bar{e}$ is $= \bar{e}\cdot\nabla\phi$

$$= \frac{1}{3}(2\bar{i}-\bar{j}-2\bar{k})(8\bar{i}-\bar{j}-10\bar{k})$$

$$= \frac{1}{3}(16+1+20) = \frac{37}{3}.//$$

Ass I Find the directional derivative of $f(x,y,z) = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$.

(2) $f(x,y,z) = 2xy + z^2$ at $(1,-1,3)$ in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.

(3) $f(x,y,z) = 2x^2 + z^2$ at $(1,-1,3)$ " " "

(4) $\phi(x,y,z) = xyz$ along the direction of the normal to the surface at $(1,1,1)$ in the direction of the vector $\bar{i} + \bar{j} + \bar{k}$.

(5) $\phi(x,y,z) = xy^2 + yz^2$ at the point $(2,-1,1)$ in the direction $\bar{i} + 2\bar{j} + 2\bar{k}$.

(2) Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the normal to the surface $f(x,y,z) = x\log z - y^2$ at $(-1,2,1)$.

Sol Given $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,-1)$.

and $f(x,y,z) = x\log z - y^2$ at $(-1,2,1)$

Now $\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2yz + 4xz^2)\bar{i} + \frac{\partial}{\partial y}(x^2yz + 4xz^2)\bar{j} + \frac{\partial}{\partial z}(x^2yz + 4xz^2)\bar{k}$$

$$\nabla\phi = (2xyz + 4z^2)\bar{i} + (x^2z)\bar{j} + (x^2y + 8xz)\bar{k}$$

$\nabla\phi$ at $(1,-2,-1)$

$$\nabla\phi = [2(1)(-2)(-1) + 4(-1)^2]\bar{i} + [(1)^2(-1)]\bar{j} + [1(-2) + 8(1)(-1)]\bar{k}$$

$$\nabla\phi = 8\bar{i} - \bar{j} - 10\bar{k}$$

unit normal to the surface $f(x,y,z) = x\log z - y^2$ is

$$\frac{\nabla f}{|\nabla f|}$$

$$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x\log z - y^2] = \log z$$

$$\frac{\partial f}{\partial y} = [-2y], \quad \frac{\partial f}{\partial z} = \frac{x}{z}$$

$$\nabla f = \log z\,\bar{i} + (-2y)\bar{j} + \left(\frac{x}{z}\right)\bar{k} \quad \text{at } (-1,2,1)$$

$$= \log(1)\bar{i} + (-2(2))\bar{j} + \left(\frac{-1}{1}\right)\bar{k}$$

$$= 0 - 4\bar{j} - \bar{k}$$

$$\therefore \frac{\nabla f}{|\nabla f|} = \frac{-4\bar{j} - \bar{k}}{\sqrt{0^2 + (-4)^2 + (-1)^2}} = \frac{-4\bar{j} - \bar{k}}{\sqrt{17}} = \frac{1}{\sqrt{17}}(-4\bar{j} - \bar{k})$$

$$\therefore \text{Directional derivative} = \nabla\phi \cdot \frac{\nabla f}{|\nabla f|} = (8\bar{i} - \bar{j} + 10\bar{k})\frac{1}{\sqrt{17}}(-4\bar{j} - \bar{k})$$

$$= \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}} \quad //$$

(2) find the directional derivative of $\phi = xyz$ along the direction of the normal to the surface $x^2z + y^2x + yz^2 = 3$ at the point $(1,1,1)$ Ans $\sqrt{3}$

(3) Find the directional derivative of $xyz^2 + xz$ at $(1,1,1)$ in a direction of the normal to the surface $3xy^2 + y = z$ at $(0,1,1)$

Ans $\frac{4}{\sqrt{11}}$

(4) Find the directional derivative of the func. $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(1,1,1)$

Sol

$$f = xy^2 + yz^2 + zx^2$$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = (y^2 + 2xz)\bar{i} + (z^2 + 2xy)\bar{j} + (x^2 + 2yz)\bar{k}$$

at $(1,1,1)$ $\quad \nabla f = 3\bar{i} + 3\bar{j} + 3\bar{k}$

Let $\bar{r}$ be the position vector of any point on the curve $x=t, y=t^2, z=t^3$. Then

$$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} = t\bar{i}+t^2\bar{j}+t^3\bar{k}$$

$$\frac{d\bar{r}}{dt} = \bar{i}+2t\bar{j}+3t^2\bar{k} \text{ at } (1,1,1)$$

$$\frac{d\bar{r}}{dt} = \bar{i}+2\bar{j}+3\bar{k}.$$

We know that $\frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve

$\therefore$ unit vector along the tangent $\bar{t} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}(\bar{i}+2\bar{j}+3\bar{k})$

$\therefore$ Directional derivative along the tangent $= \bar{t}\cdot\nabla f$

$$= \frac{1}{\sqrt{14}}(\bar{i}+2\bar{j}+3\bar{k})(3\bar{i}+3\bar{j}+3\bar{k}) = \frac{1}{\sqrt{14}}[3+6+9] = \frac{18}{\sqrt{14}} \,//$$

(2) Find the unit normal vector to the given surface $x^2y+2xz=4$ at the point $(2,-2,3)$

Sol: Given $f = x^2y+2xz-4$.

$$\frac{\partial f}{\partial x} = 2xy+2z, \quad \frac{\partial f}{\partial y} = x^2, \quad \frac{\partial f}{\partial z} = 2x.$$

$$\text{grad } f = \nabla f = (2xy+2z)\bar{i}+(x^2)\bar{j}+(2x)\bar{k}$$

grad f at $(2,-2,3)$.

$$= (-8+6)\bar{i}+(4)\bar{j}+4\bar{k} = -2\bar{i}+4\bar{j}+4\bar{k}.$$

grad f is the normal vector to the given surface at the given point

Hence the required unit normal vector $= \frac{\nabla f}{|\nabla f|} = \frac{\cancel{2}(-\bar{i}+2\bar{j}+2\bar{k})}{\cancel{2}\sqrt{1^2+2^2+2^2}}$

$$= \frac{-\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{9}} = \frac{1}{3}(-\bar{i}+2\bar{j}+2\bar{k})$$

① find unit normal vector to the surface $z = x^2+y^2$ at $(-1,-2,5)$ Ans $\frac{1}{\sqrt{21}}(2i+4j+k)$

② $x^2y+2xz^2=8$ at $(1,0,2)$

③ $x^2+y^2+2z^2=26$ at $(2,2,3)$

③ $x^3+y^3+3xyz=3$ at $(1,2,-1)$

④ $xy+yz+zx$ at $(1,1,1)$

⑤ $z^2=x^2-y^2$ at $(2,1,\sqrt{3})$

④ find the directional derivative of the func $f = x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where $Q=(5,0,4)$.

Sol The position vector of P and Q with respect to the origin are $\overline{OP} = \bar{i}+2\bar{j}+3\bar{k}$ and $\overline{OQ} = 5\bar{i}+4\bar{k}$

$$\therefore \overline{PQ} = \overline{OQ}-\overline{OP} = 5\bar{i}+4\bar{k}-\bar{i}-2\bar{j}-3\bar{k} = 4\bar{i}-2\bar{j}+\bar{k}$$

let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$. Then

$$\bar{e} = \frac{4\bar{i}-2\bar{j}+\bar{k}}{\sqrt{16+4+1}} = \frac{1}{\sqrt{21}}(4\bar{i}-2\bar{j}+\bar{k})$$

and $\text{grad} f = \frac{\partial f}{\partial x}\bar{i}+\frac{\partial f}{\partial y}\bar{j}+\frac{\partial f}{\partial z}\bar{k}$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2y \quad \frac{\partial f}{\partial z} = 4z$$

$$\nabla f = 2x\bar{i}-2y\bar{j}+4z\bar{k}$$

The direction derivative of f at $(1,2,3)$ in the direction of $\overline{PQ}$

$$= \bar{e}\cdot\nabla f$$

$$= \frac{1}{\sqrt{21}}[4\bar{i}-2\bar{j}+\bar{k}][2x\bar{i}-2y\bar{j}+4z\bar{k}]$$

$$= \frac{1}{\sqrt{21}}[8x+4y+4z] \text{ at } (1,2,3)$$

$$= \frac{1}{\sqrt{21}}[8+8+12] = \frac{28}{\sqrt{21}} \text{ //.}$$

① The scalar point func $\phi(x,y,z) = 4xz^2+2x^2yz$ at the point $A(1,2,3)$ in the direction of the line AB where $B=(5,0,4)$. Ans $\frac{120}{\sqrt{21}}$

① find the angle b/wn the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Sol let $\phi_1 = x^2+y^2+z^2-9$ and $\phi_2 = x^2+y^2-z-3$. be the given surfaces.

$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 2z$, $\frac{\partial g}{\partial x} = 2x, \frac{\partial g}{\partial y} = 2y, \frac{\partial g}{\partial z} = -1$.

$\nabla f = \text{grad} f = 2x\bar{i}+2y\bar{j}+2z\bar{k}$ and

$\nabla g = \text{grad}\, g = 2x\bar{i}+2y\bar{j}-\bar{k}$

The angle b/wn two surfaces at a point is the angle b/wn the normal to the surfaces at that point.

let $\bar{n}_1 = \text{grad} f$ at $(2,-1,2) = 4\bar{i}-2\bar{j}+4\bar{k}$

and $\bar{n}_2 = \text{grad}\, g$ at $(2,-1,2) = 4\bar{i}-2\bar{j}-\cancel{2}\bar{k}$

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at $(2,-1,2)$ let θ be the angle b/wn the two surfaces, Then

$$\cos\theta = \frac{\bar{n}_1}{|\bar{n}_1|} \cdot \frac{\bar{n}_2}{|\bar{n}_2|} = \frac{(4\bar{i}-2\bar{j}+4\bar{k})}{\sqrt{16+4+16}} \cdot \frac{(4\bar{i}-2\bar{j}-\bar{k})}{\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{\sqrt{36}\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}} \,//.$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Ass 1) find the angle b/wn the surfaces $xy^2z = 3x+z^2$ and $3x^2-y^2+2z=1$ at $(1,-2,1)$ $\quad \cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$

② $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at $(4,-3,2)$ $\quad \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$

③ Evaluate the angle b/wn the normals to the surface $xy=z^2$ at the points $(4,1,2)$ & $(3,3,-3)$

② find the angle b/wn the normals to the surface $x^2 = yz$ at the points $(1,1,1)$ & $(2,4,1)$

Sol Given surface is $f(x,y,z) = x^2 - yz$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at the points $(1,1,1)$ & $(2,4,1)$ respectively

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -z, \quad \frac{\partial f}{\partial z} = -y$$

$$\text{grad } f = 2x\bar{i} - z\bar{j} - y\bar{k}$$

$$\bar{n}_1 = (\text{grad } f)_{(1,1,1)} = 2\bar{i} - \bar{j} - \bar{k}$$

$$\bar{n}_2 = (\text{grad})_{(2,4,1)} = 4\bar{i} - \bar{j} - 4\bar{k}$$

Let θ be the angle b/wn the two normals, Then

$$\cos\theta = \frac{\bar{n}_1}{|\bar{n}_1|} \cdot \frac{\bar{n}_2}{|\bar{n}_2|}$$

$$= \frac{2\bar{i} - \bar{j} - \bar{k}}{\sqrt{4+1+1}} \cdot \frac{4\bar{i} - \bar{j} - 4\bar{k}}{\sqrt{16+1+16}} = \frac{8+1+4}{\sqrt{6}.\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}.$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right)$$

① Divergence of a vector :—

Let $\bar{f}$ be any continuously differentiable vector point func Then $\bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is defined as

$$\text{div}.\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z} = \left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right].\bar{f}$$

$$\text{div}.\bar{f} = \nabla.\bar{f}$$

Solenoidal vector :—

A vector point func $\bar{f}$ is said to be solenoidal if $\text{div}\bar{f} = 0$.

Theorem :— If the vector func $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then

$$\text{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

① find div.$\bar{f}$ where $\bar{f}$ = grad $(x^3+y^3+z^3-3xyz)$.

let $\phi = x^3+y^3+z^3-3xyz$.

then $\frac{\partial \phi}{\partial x} = 3x^2-3yz$, $\frac{\partial \phi}{\partial y} = 3y^2-3zx$, $\frac{\partial \phi}{\partial z} = 3z^2-3xy$.

$\bar{f} = \text{grad}\,\phi = \bar{i}\frac{\partial \phi}{\partial x}+\bar{j}\frac{\partial \phi}{\partial y}+\bar{k}\cdot\frac{\partial \phi}{\partial z}$

$= \bar{i}(3x^2-3yz)+\bar{j}(3y^2-3xz)+\bar{k}(3z^2-3xy)$

$\bar{f} = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

Hence $\text{div}\,\bar{f} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$.

$= \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3zx)+\frac{\partial}{\partial z}(3z^2-3xy)$

$= \frac{\partial}{\partial x}\ 3(2x)+3(2y)+3(2z)$

$\text{div}\cdot\bar{f} = 6(x+y+z)$

① Find div $\bar{f}$ where $\bar{f} = r^n\bar{r}$, find n if it is Solenoidal?

② P.T $r^n\bar{r}$ is solenoidal if $n=-3$.

(or) P.T $\text{div}(r^n\bar{r}) = (n+3)r^n$ Hence S.T $\bar{r}/r^3$ is solenoidal?

Sol Given $\bar{f} = r^n\bar{r}$

where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$.

and $r = |\bar{r}| \Rightarrow r^2 = x^2+y^2+z^2$

diff w.r.to x partially, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

& $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \bar{f} = r^n\bar{r} = \bar{r}^n(x\bar{i}+y\bar{j}+z\bar{k})$

Hence $\text{div}\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$.

$= nr^{n-1}\frac{\partial r}{\partial x}\cdot x + r^n + nr^{n-1}\frac{\partial r}{\partial y}\cdot y + r^n + nr^{n-1}\frac{\partial r}{\partial z}\cdot z + r^n$

$= nr^{n-1}\left[\frac{\partial r}{\partial x}x + \frac{\partial r}{\partial y}y + \frac{\partial r}{\partial z}\cdot z\right] + 3r^n$.

$= nr^{n-1}\left[\frac{x}{r}\cdot x + \frac{y}{r}\cdot y + \frac{z}{r}\cdot z\right] + 3r^n$

$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n$

$= nr^{n-1}\cdot\frac{1}{r}[x^2+y^2+z] + 3r^n$

$= nr^{n-1}\cdot\frac{r^{\cancel{2}}}{\cancel{r}} + 3r^n = nr^{n-\cancel{1}+\cancel{1}} + 3r^n = nr^n + 3r^n = r^n(n+3)$

Let $\bar{f} = r^n\bar{r}$ be solenoidal. Then $\text{div}\bar{f} = 0$

$r^n(n+3) = 0 \Rightarrow n = -3$,

$\therefore r^n\bar{r} = \frac{\bar{r}}{r^3}$ is solenoidal.

(2) P.T $\text{div}\left(\frac{\bar{r}}{r}\right) = \frac{2}{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

(2) Curl of A vector :-

$\bar{f}$ be any continuously differentiable vector point func. Then the vector func. defined by $\bar{i}\times\frac{\partial f}{\partial x} + \bar{j}\times\frac{\partial f}{\partial y} + \bar{k}\times\frac{\partial f}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$. (or)

$$\nabla\times\bar{f} = \text{curl}\,\bar{f} = \bar{i}\times\frac{\partial f}{\partial x} + \bar{j}\times\frac{\partial f}{\partial y} + \bar{k}\times\frac{\partial f}{\partial z} = \sum\left(\bar{i}\times\frac{\partial f}{\partial x}\right)$$

Theorem :- If $\bar{f}$ is a differentiable vector point func. given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{curl}\,\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & f & f \end{vmatrix} = \nabla\times\bar{f}$$

① Theorem: -

Def: A vector $\bar{f}$ is said to be irrotational if curl $\bar{f} = \bar{0}$.

Scalar potential func:—

If $\bar{f}$ is irrotational, then there will always exist a scalar func $\phi(x, y, z)$ such that $\bar{f} = \nabla\phi$

This func ϕ is called scalar potential of $\bar{f}$ and $\bar{f}$ is called conservation.

Theorem: If $\bar{f}$ is a conservative vector field then curl $\bar{f} = \bar{0}$.

① P.T $\bar{F} = yz\bar{i} + zx\bar{j} + yx\bar{k}$ is irrotational.

Sol

$$\bar{F} = yz\bar{i} + zx\bar{j} + yx\bar{k}$$

Now $\text{curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$

$$= \bar{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \bar{j}\left[\frac{\partial}{\partial x}xy - \frac{\partial}{\partial z}(yz)\right] + \bar{k}\left[\frac{\partial}{\partial x}zx - \frac{\partial}{\partial y}yz\right]$$

$$= \bar{i}[x - x] - \bar{j}[y - y] + \bar{k}[z - z]$$

$$= 0.\bar{i} + 0.\bar{j} + 0.\bar{k}$$

$$= \bar{0}$$

Hence F is irrotational.

① Find curl $\bar{f}$ where $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol Let $\phi = x^3+y^3+z^3-3xyz$. Then

$$\bar{f} = \text{grad}\,\phi = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 3x^2-3yz, \quad \frac{\partial\phi}{\partial y} = 3y^3-3xz, \quad \frac{\partial\phi}{\partial z} = 3z^2-3xy.$$

$$\bar{f} = 3(x^2-yz)\bar{i}+3(y^2-zx)\bar{j}+3(z^2-xy)\bar{k}$$

$$\bar{f} = 3[(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}]$$

$$\text{curl}\,\bar{f} = \text{curl grad}\,\phi = \nabla\times\text{grad}\,\phi = 3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= 3\left[\bar{i}\left(\frac{\partial}{\partial y}(z^2-xy)-\frac{\partial}{\partial z}(y^2-zx)\right]-\bar{j}\left[\frac{\partial}{\partial x}(z^2-xy)-\frac{\partial}{\partial z}(x^2-yz)\right]+\bar{k}\left[\frac{\partial}{\partial x}(y^2-zx)-\frac{\partial}{\partial y}(x^2-yz)\right]\right]$$

$$= 3[\bar{i}[-x-(-x)]-\bar{j}[-y-(-y)]+\bar{k}[-z-(-z)]]$$

$$= 3[\bar{i}(0)-\bar{j}(0)+\bar{k}(0)]$$

$$= \bar{0}$$

$$\text{curl}\,\bar{f} = \bar{0}$$

Find curl $\bar{F}$ for (i) $\bar{F} = z\bar{i}+x\bar{j}+y\bar{k}$

(ii) $\bar{F} = 2xz^2\bar{i}-yz\bar{j}+3xz^3\bar{k}$

(iii) $\bar{f} = xyz^2\bar{i}+xy^2z\bar{j}+x^2yz\bar{k}$

(iv) $\bar{F} = x\bar{i}-y^2\bar{j}+z^3\bar{k}$

(v) if $\bar{F} = xy^2\bar{i}+2x^2yz\bar{j}-3yz^2\bar{k}$ find curl $\bar{F}$ at the point $(1,-1,1)$

(vi) If $\bar{F} = (x+y+1)\bar{i}+\bar{j}-(x+y)\bar{k}$ then S.T curl $\bar{F}=0$.

② Prove that if $\bar{r}$ is the position vector of any point in space, then $r^n\bar{r}$ is irrotational (or) Show that $\text{curl}(r^n\bar{r}) = 0$.

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

and $r = |\bar{r}|$ $\therefore r^2 = x^2+y^2+z^2$

Differentiating w.r.t. x partially, we get

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\text{Ily} \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

We have $r^n\bar{r} = r^n[x\bar{i}+y\bar{j}+z\bar{k}]$

$$= r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}$$

$$\nabla\times(r^n\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}[r^n z] - \frac{\partial}{\partial z}[yr^n]\right] - \bar{j}\left[\frac{\partial}{\partial x}(zr^n) - \frac{\partial}{\partial z}(xr^n)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(yr^n) - \frac{\partial}{\partial y}(xr^n)\right]$$

$$= \bar{i}\left[nr^{n-1}z\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right] - \bar{j}\left[znr^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$+ \bar{k}\left[ynr^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial y}\right]$$

$$= \bar{i}\left[nr^{n-1}z\left(\frac{y}{r}\right) - ynr^{n-1}\left(\frac{z}{r}\right)\right] - \bar{j}\left[znr^{n-1}\left(\frac{x}{r}\right) - xnr^{n-1}\left(\frac{z}{r}\right)\right]$$

$$+ \bar{k}\left[ynr^{n-1}\left(\frac{x}{r}\right) - xnr^{n-1}\left(\frac{y}{r}\right)\right]$$

$$= \bar{i}\left[nr^{n-2}(\cancel{yz} - \cancel{zy})\right] - \bar{j}\left[nr^{n-2}(\cancel{xz} - \cancel{zx})\right] + \bar{k}\left[nr^{n-2}(\cancel{xy} - \cancel{yx})\right]$$

$$= [\bar{i}0 + \bar{j}0 + \bar{k}0]nr^{n-2} = nr^{n-2}(0) = \bar{0}$$

Hence $r^n\bar{r}$ is irrotational.

① Show that the vector $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

Sol

Let $\bar{f}=(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$, Then.

$$\text{curl}\,\bar{f}=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$=\bar{i}\left[\frac{\partial}{\partial y}(z^2-xy)-\frac{\partial}{\partial z}(y^2-zx)\right]-\bar{j}\left[\frac{\partial}{\partial x}(z^2-xy)-\frac{\partial}{\partial z}(x^2-yz)\right]+\bar{k}\left[\frac{\partial}{\partial x}(y^2-zx)-\frac{\partial}{\partial y}(x^2-yz)\right]$$

$$=\bar{i}[-x+x]-\bar{j}[-y+y]+\bar{k}[-z+z]$$

$=0$

curl $\bar{f}=0$.

$\therefore$ $\bar{f}$ is irrotational. Then there exists ϕ such that $\bar{f}=\nabla\phi$

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}=(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x}=x^2-yz,\quad \frac{\partial\phi}{\partial y}=y^2-zx,\quad \frac{\partial\phi}{\partial z}=z^2-xy.$$

$$\partial\phi=(x^2-yz)\,dx;\quad \partial\phi=(y^2-zx)\,dy,\quad \partial\phi=(z^2-xy)\,dz$$

Integrating

$$\int\partial\phi=\int(x^2-yz)\,dx \Rightarrow \phi=\frac{x^3}{3}-xyz+f_1(y,z)\longrightarrow ①$$

$$\int\partial\phi=\int(y^2-zx)\,dy=\phi=\frac{y^3}{3}-xyz+f_2(z,x)\longrightarrow ②$$

$$\int d\phi=\int(z^2-xy)\,dz=\phi=\frac{z^3}{3}-xyz+f_3(x,y)\longrightarrow ③.$$

from eq ①, ② & ③ $\quad \phi=\dfrac{x^3+y^3+z^3}{3}-xyz.$

$$\phi=\frac{1}{3}(x^3+y^3+z^3)-xyz+\text{constant}$$

which is the required scalar potential.

① Show that the vector field $\bar{f} = 2xyz^2\,\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\,\bar{k}$ is irrotational find the potential func.

Sol Given $= 2xyz^2\,\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\,\bar{k}$

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \bar{i}\left[2x^2\cancel{z} + \cos yz - y\sin y z\cdot z\right] - \bar{j}\left[4x\cancel{y}z - 4\cancel{x}yz\right] + \bar{k}\left[2xz^2 - 2xz^2\right]$$
$$- \left[2x^2z + \cos yz - z\sin yz\cdot y\right]$$

$= \bar{i}(0) + \bar{j}(0) + \bar{k}(0) = \bar{0}$

$\therefore$ The func is irrotational.

There exists ϕ such that $\text{grad}\,\phi = \bar{f}$.

Then ϕ is called the scalar potential func.

$$\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k} = 2xyz^2\,\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\,\bar{k}$$

comparing components.

$$\frac{\partial\phi}{\partial x} = 2xyz^2 \Rightarrow \partial\phi = (2xyz^2)\,dx$$
$$\int d\phi = \int 2xyz^2\,dx \Rightarrow \phi = x^2yz^2 + c_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = x^2z^2 + z\cos yz \Rightarrow \partial\phi = (x^2z^2 + z\cos yz)\,dy$$
$$\int d\phi = \int (x^2z^2 + z\cos yz)\,dy$$
$$\phi = x^2z^2y + \cancel{z}\frac{\sin yz}{\cancel{z}} + c_2(x,z)$$
$$\phi = x^2yz^2 + \sin yz.$$

$$\frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz \Rightarrow \partial\phi = (2x^2yz + y\cos yz)\,dz$$
$$\phi = x^2yz^2 + \cancel{y}\frac{\sin yz}{\cancel{y}} + c_3(x,y)$$
$$\phi = x^2yz^2 + \sin yz + c_3(x,y)$$

$\therefore \phi = x^2yz^2 + \sin yz +$ constant is the potential func.

① Find constants a, b, c so that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$.

Sol $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k} \longrightarrow ①$

Vector $\bar{A}$ is irrotational $\Rightarrow$ curl $\bar{A} = \bar{0}$.

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$\bar{i}[c+1] + \bar{j}[4-a] + \bar{k}[b-2] = 0$

$\bar{i}[c+1] + \bar{j}[a-4] + \bar{k}(b-2] = 0\bar{i} + 0\bar{j} + 0\bar{k}$

Comparing both sides.

$c+1=0 \Rightarrow c=-1$, $a-4=0 \Rightarrow a=4$, $b-2=0 \Rightarrow b=2$

$\therefore a=4, b=2, c=-1$ substituting these values in eq ①

Now $\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$.

we have $\bar{A} = \nabla\phi$

$(x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

comparing both sides, we get

$\frac{\partial\phi}{\partial x} = (x^2+2y+4z)$, $\frac{\partial\phi}{\partial y} = 2x-3y-z$, $\frac{\partial\phi}{\partial z} = 4x-y+2z$

$\partial\phi = (x+2y+4z)\partial x$, $\partial\phi = (2x-3y-z)\partial y$, $\partial\phi = (4x-y+2z)\partial z$

$\int d\phi = \int(x+2y+4z)dx$, $\int d\phi = \int(2x-3y-z)dy$, $\int d\phi = \int(4x-y+2z)dz$

$\phi = \frac{x^2}{2} + 2xy + 4xz + f_1(y,z)$, $\phi = 2xy - 3\frac{y^2}{2} - zy + f_2(z,x)$, $\phi = 4xz - yz + 2\frac{z^2}{2} + f_3(x,y)$

$\phi = \frac{x^2}{2} - \frac{3}{2}y^2 + z^2 + 2xy - yz + 4xz + c$.

① find constants a, b & c if the vector.

$\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$ is irrotational.

② Find whether the func. $F = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational and hence find scalar potential func corresponding to it.

③ Show that $\bar{f} = (y^2-z^2+3yz-2x)\bar{i} + (3xz+2xy)\bar{j} + (3xy-2xz+2z)\bar{k}$ is both Solenoidal and irrotational.

① Vector differential operator ∇:

The operator $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ is defined such that

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$ where ϕ is a scalar point func.

② Scalar differential operator $\bar{a}\cdot\nabla$

The operator $\bar{a}\cdot\nabla = (\bar{a}\cdot\bar{i})\frac{\partial}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial}{\partial z}$ is defined such that

$$(\bar{a}\cdot\nabla)\phi = (\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\cdot\bar{j})\frac{\partial\phi}{\partial z}.$$

and $$(\bar{a}\cdot\nabla)\bar{f} = (\bar{a}\cdot\bar{i})\frac{\partial\bar{f}}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial\bar{f}}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial\bar{f}}{\partial z}.$$

3) Vector differential operator $\bar{a}\times\nabla$

The operator $\bar{a}\times\nabla = (\bar{a}\times\bar{i})\frac{\partial}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial}{\partial z}$ is defined s.t

(i) $(\bar{a}\times\nabla)\phi = (\bar{a}\times\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\phi}{\partial z}$

(ii) $(\bar{a}\times\nabla)\bar{f} = (\bar{a}\times\bar{i})\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\bar{f}}{\partial z}$

(iii) $(\bar{a}\times\nabla)\times\bar{f} = (\bar{a}\times\bar{i})\times\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\times\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\times\frac{\partial\bar{f}}{\partial z}$

4) Scalar differential operator $\nabla\cdot$

The operator $\nabla\cdot = \bar{i}\cdot\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}$ is defined such that

$$\nabla\cdot\bar{f} = \bar{i}\cdot\frac{\partial\bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial\bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial\bar{f}}{\partial z}$$

Note: ~~div.~~ $\nabla\cdot\bar{f}$ is defined as div. $\bar{f}$. it is a scalar point func.

Vector differential operator $\nabla \times$

The operator $\nabla \times = \bar{i} \times \frac{\partial}{\partial x} + \bar{j} \times \frac{\partial}{\partial y} + \bar{k} \times \frac{\partial}{\partial z}$ is defined such that

$$\nabla \times \bar{f} = \bar{i} \times \frac{\partial f}{\partial x} + \bar{j} \times \frac{\partial f}{\partial y} + \bar{k} \times \frac{\partial f}{\partial z}.$$

Note: $\nabla \times \bar{f}$ is defined as curl $\bar{f}$. it is a vector point func.

6) grad div $\phi = \nabla \cdot \nabla\phi = \nabla^2\phi$

① P.T $\operatorname{div}(\operatorname{grad} \phi)^m = (m)(m+1)r^{m-2}$ (or) $\nabla^2 r^m = m(m+1)r^{m-2}$

(or) $\nabla^2 r^n = n(n+1)r^{n-2}$.

Sol $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$. Then $r^2 = x^2 + y^2 + z^2$

Differentiating w.r.t x partially, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \ \frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\operatorname{grad} r^m = \bar{i}\frac{\partial}{\partial x}r^m + \bar{j}\frac{\partial}{\partial y}r^m + \bar{k}\frac{\partial}{\partial z}r^m$

$= \bar{i}\, m r^{m-1}\frac{\partial r}{\partial x} + \bar{j}\, m r^{m-1}\frac{\partial r}{\partial y} + \bar{k}\cdot m r^{m-1}\frac{\partial r}{\partial z}$.

$= \bar{i}\, m r^{m-1}\left(\frac{x}{r}\right) + \bar{j}\, m r^{m-1}\left(\frac{y}{r}\right) + \bar{k}\cdot m r^{m-1}\left(\frac{z}{r}\right)$

$\operatorname{grad} r^m = \bar{i}\, m r^{m-2}x + \bar{j}\, m r^{m-2}y + \bar{k}\, m r^{m-2}z$.

$\operatorname{grad} r^m = \sum \bar{i}\, m r^{m-2}x$.

$\therefore \operatorname{div}(\operatorname{grad} r^m) = \sum \left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right]\left[m r^{m-2}x\,\bar{i} + m r^{m-2}y\cdot\bar{j} + m r^{m-2}z\cdot\bar{k}\right]$

$= \frac{\partial}{\partial x}(m r^{m-2}x) + \frac{\partial}{\partial y}(m r^{m-2}\cdot y) + \frac{\partial}{\partial z}(m r^{m-2}\cdot z)$

$= \sum m\left[r^{m-2} + (m-2)r^{m-3}\cdot x\frac{\partial r}{\partial x}\right]$

$= m\sum\left[r^{m-2} + (m-2)r^{m-3}\cdot x\left(\frac{x}{r}\right)\right]$

$= m\sum\left(r^{m-2} + (m-2)x^2 r^{m-4}\right]$

$= m\left[(m-2)r^{m-4}\sum x^2 + r^{m-2}\sum 1\right]$

$= m\left[(m-2)r^{m-4}\left[x^2+y^2+z^2\right] + r^{m-2}\left[1+1+1\right]\right]$

$$= m[(m-2)r^{m-4}(r^2)+3r^{m-2}]$$
$$= m[(m-2)r^{m-2}+3r^{m-2}]$$
$$= mr^{m-2}[m-2+3]$$
$$= m(m+1)r^{m-2}$$
$$\nabla^2 r^m = m(m+1)r^{m-2}$$

③ prove that $\mathrm{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\mathrm{curl}\,\bar{a} - \bar{a}\cdot\mathrm{curl}\,\bar{b}$ (or)

$$\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot\mathrm{curl}\,\bar{a} - \bar{a}\cdot\mathrm{curl}\,\bar{b} \text{ (or)}$$
$$\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot(\nabla\times\bar{a}) - \bar{a}\cdot(\nabla\times\bar{b})$$

Sol
$$\mathrm{div}(\bar{a}\times\bar{b}) = \sum \bar{i}\cdot\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$$
$$= \sum \bar{i}\cdot\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$A\cdot(B\times C) = (A\times B)\cdot C$ & $\bar{A}\cdot(B\times C) = -(\bar{A}\times C)\cdot\bar{B}$

$$= \sum \bar{i}\cdot\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \sum \bar{i}\cdot\left[\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$
$$= \sum\left[\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right]\cdot\bar{b} - \sum\left[\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right]\cdot\bar{a}$$
$$= (\nabla\times\bar{a})\cdot\bar{b} - (\nabla\times\bar{b})\cdot\bar{a} \Rightarrow \bar{b}\cdot\mathrm{curl}\,\bar{a} - \bar{a}\cdot\mathrm{curl}\,\bar{b}.$$

Hence the theorem.

Theorem:- Prove that

$$\mathrm{grad}(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times\mathrm{curl}\,\bar{a} + \bar{a}\times\mathrm{curl}\,\bar{b}$$

Sol (or) $\nabla(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times(\nabla\times\bar{a}) + \bar{a}\times(\nabla\times\bar{b})$

Sol consider

$$\bar{a}\times(\mathrm{curl}\,\bar{b}) = \bar{a}\times(\nabla\times\bar{b})$$
$$= \bar{a}\times\sum\bar{i}\times\frac{\partial\bar{b}}{\partial x}$$

$\bar{A}\times(B\times C) = (A\cdot C)\bar{B} - (A\cdot\bar{B})\bar{C}$

$$= \sum\bar{a}\times\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right)$$
$$= \sum\left[\left(a\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{i} - (\bar{a}\cdot\bar{i})\frac{\partial\bar{b}}{\partial x}\right]$$

$$\sum \hat{i}\left[\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right]-\left[\bar{a}\cdot\sum \hat{i}\frac{\partial}{\partial x}\right]\bar{b}$$

$$\bar{a}\times \operatorname{curl}\bar{b} = \sum \bar{i}\left[\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right]-(\bar{a}\cdot\nabla)\bar{b} \longrightarrow ①$$

lly $$\bar{b}\times \operatorname{curl}\bar{a} = \sum \bar{i}\left[\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right]-(\bar{b}\cdot\nabla)\bar{a} \longrightarrow ②$$

eq ① + ② gives.

$$\bar{a}\times\operatorname{curl}\bar{b}+\bar{b}\times\operatorname{curl}\bar{a} = \sum \bar{i}\left[\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right]-(\bar{a}\cdot\nabla)\bar{b}+\sum \bar{i}\left[\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right]-(\bar{b}\cdot\nabla)\bar{a}.$$

$$\sum \bar{i}\left[\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right]+\sum \bar{i}\left[b\cdot\frac{\partial \bar{a}}{\partial x}\right] = \bar{a}\times\operatorname{curl}\bar{b}+\bar{b}\times\operatorname{curl}\bar{a}+(\bar{a}\cdot\nabla)\bar{b}+(\bar{b}\cdot\nabla)\bar{a}$$

$$\sum \bar{i}\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}+\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) = \bar{a}\times\operatorname{curl}\bar{b}+\bar{b}\times\operatorname{curl}\bar{a}+(\bar{a}\cdot\nabla)\bar{b}+(\bar{b}\cdot\nabla)\bar{a}$$

$$\sum \bar{i}\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b}) = \bar{a}\times\operatorname{curl}\bar{b}+\bar{b}\times\operatorname{curl}\bar{a}+(\bar{a}\cdot\nabla)\bar{b}+(\bar{b}\cdot\nabla)\bar{a}$$

$$\operatorname{grad}(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a}+(\bar{a}\cdot\nabla)\bar{b}+\bar{b}\times(\nabla\times\bar{a})+\bar{a}\times(\nabla\times\bar{b})$$

① $$\operatorname{curl}(\bar{a}\times\bar{b}) = \bar{a}\operatorname{div}\bar{b}-\bar{b}\operatorname{div}\bar{a}+(\bar{b}\cdot\nabla)\bar{a}-(\bar{a}\cdot\nabla)\bar{b}.$$

Sol (or) $$\nabla\times(\bar{a}\times\bar{b}) = (\nabla\cdot\bar{b})\bar{a}-(\nabla\bar{a})\bar{b}+(\bar{b}\cdot\nabla)a-(\bar{a}\cdot\nabla)\bar{b}.$$

Sol $$\operatorname{curl}(\bar{a}\times\bar{b}) = \sum \bar{i}\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$$

$$= \sum \hat{i}\times\left[\frac{\partial \bar{a}}{\partial x}\times\bar{b}+\bar{a}\times\frac{\partial \bar{b}}{\partial x}\right]$$

$$= \sum \hat{i}\times\left[\left(\frac{\partial \bar{a}}{\partial x}\right)\times\bar{b}\right]+\sum \hat{i}\times\left(\bar{a}\times\frac{\partial \bar{b}}{\partial x}\right)$$

$$= \sum\left[(\hat{i}\cdot\bar{b})\frac{\partial \bar{a}}{\partial x}-\left(\bar{i}\cdot\frac{\partial \bar{a}}{\partial x}\right)\bar{b}\right]+\sum\left[\left(\hat{i}\cdot\frac{\partial \bar{b}}{\partial x}\right)\bar{a}-(\hat{i}\cdot\bar{a})\frac{\partial \bar{b}}{\partial x}\right]$$

$$= \sum(\bar{b}\cdot\hat{i})\frac{\partial \bar{a}}{\partial x}-\sum\left(\bar{i}\cdot\frac{\partial \bar{a}}{\partial x}\right)\bar{b}+\sum\left(\bar{i}\cdot\frac{\partial \bar{b}}{\partial x}\right)\bar{a}-\left[\bar{a}\cdot\sum \hat{i}\frac{\partial}{\partial x}\right]\bar{b}$$

$$= \left[\bar{b}\cdot\sum \hat{i}\frac{\partial}{\partial x}\right]\bar{a}-\sum\left(\bar{i}\cdot\frac{\partial \bar{a}}{\partial x}\right)\bar{b}+\sum \bar{i}\cdot\frac{\partial \bar{b}}{\partial x}\bar{a}-\left[\bar{a}\cdot\sum \hat{i}\frac{\partial}{\partial x}\right]\bar{b}$$

$$= (\bar{b}\cdot\nabla)\bar{a}-(\nabla\cdot\bar{a})\bar{b}+(\nabla\cdot\bar{b})\bar{a}-(\bar{a}\cdot\nabla)\bar{b}$$

$$= \cancel{(\nabla\cdot\bar{b})\bar{a}-(\nabla\cdot\bar{a})\bar{b}}$$

$= (\nabla\cdot\bar{b})\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\,\nabla)\bar{b}$

$= \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a} + (\bar{b}\cdot\nabla)a - (\bar{a}\cdot\nabla)\bar{b}$.

① If f, g are scalar fields P.T $(\nabla f \times \nabla g)$ is solenoidal.

Sol we know that $\text{div}(\bar{a}\times\bar{b}) = \bar{b}\,\text{curl}\,\bar{a} - \bar{a}\,\text{curl}\,\bar{b}$

Take $\bar{a} = \nabla f$, $\bar{b} = \nabla g$

Then $\text{div}(\nabla f \times \nabla g) = \nabla g\,\text{curl}\,\nabla f - \nabla f\cdot \cancel{\text{grad}}\,\nabla g = 0$

$\{\because \text{curl}(\nabla f) = \bar{0} = \text{curl}(\nabla g)\}$

$\therefore \nabla f \times \nabla g$ is solenoidal.

② P.T $\text{div}\,\text{curl}\,\bar{f} = 0$ (or) for any vector A, find $\text{div}\,\text{curl}\,A$.

Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{curl}\,\bar{f} = \nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \bar{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \bar{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$\therefore \text{div}\,\text{curl}\,\bar{f} = \nabla\cdot(\nabla\times\bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$= \cancel{\frac{\partial^2 f_3}{\partial x\partial y}} - \cancel{\frac{\partial^2 f_2}{\partial z\partial x}} - \cancel{\frac{\partial^2 f_3}{\partial x\partial y}} + \cancel{\frac{\partial^2 f_1}{\partial y\partial z}} + \cancel{\frac{\partial^2 f_2}{\partial x\partial z}} - \cancel{\frac{\partial^2 f_1}{\partial y\partial z}}$$

$= 0$

$\therefore \text{div}(\text{curl}\,\bar{f}) = 0$.