Unit – III

Algebraic Structures

Binary operation

It combines 2 elements of the set to produce another element which also belongs to the same set.

Binary operators :- $+, -, \times, \div$

Ex :- $N = \{1, 2, 3, \dots\}$. $1+2=3 \in N$ then $+$ is binary oper

Properties of Binary operations

1) closure property
2) Associative "
3) Identity "
4) Inverse "
5) commutative "

→ closure → Algebraic structure

→ closure + Associative → Semi groups

→ closure + Associative + Identity
↓
Monoids

→ closure + Associative + Identity + Inverse → Groups

→ closure + Associative + Identity + Inverse + commutative
↓
Abelian Groups

(or)

1) closure
2) Associative
3) Identity
4) Inverse
5) commutative

Semi group

Monoid

Group

↓
Abelian Group.

Algebraic structures

(or) closure property

A non-empty set 'S' with one or more binary operations is called as algebraic structure.

let if $\forall\, a, b \in S$ then $a * b \in S$.

$\boxed{* \rightarrow +, -, \times, \div}$

If '+' is a binary operation on N, then the algebraic structure is written as $(N, +)$.

Ex:- 1) If $N = \{1, 2, 3, \ldots\}$

$1 + 2 = 3 \in N \therefore (N, +)$ ✓

2) $1 - 2 = -3 \notin N \therefore (N, -)$ ✗

3) $1 \times 3 = 3 \in N \therefore (N, \times)$ ✓

4) If $Z \rightarrow$ Integers. $(Z, +)$ ✓

2) Associative property

let a, b, c ∈ S.

then $a * (b * c) = (a * b) * c$

$\forall a, b, c \in S.$

Semi Group:-

An algebraic structure $(S, *)$ is called a semigroup if it satisfies 2 properties

(i) closure property

(ii) Associative property.

1) S.T $(N, +)$ is a semigroup

Sol: let $N = \{1, 2, 3, \ldots\}$

must satisfy 2 properties

(i) closure

$\forall a, b \in N$ then $a * b \in N$

Here $* = \oplus$ [question]

means: $\forall a, b \in N$ then $a + b \in N$

let $a = 1, b = 2$

$1 + 2 = 3 \in N$

$\therefore$ closure property satisfies

(ii) Associative

let $a, b, c \in N$

then $a * (b * c) = (a * b) * c \in N$

let $a = 1, b = 2, c = 3$

$\Rightarrow 1 + (2 + 3) = (1 + 2) + 3$

$\Rightarrow 1 + 5 = 3 + 3$

$\Rightarrow 6 = 6 \in N$

$\therefore$ Associative satisfies

$\therefore (N, +)$ is a semigroup.

2) S.T $(N, -)$ is not a semi group

Sol:- closure

Let $a, b \in N$ then $a-b \in N$

$N = \{1, 2, 3, \ldots\}$

let $a = 1, b = 2$

$1-2 = -1 \notin N$

So closure doesn't satisfy.

Associative

$a * (b * c) = (a * b) * c$

$[* = -]$

$a-(b-c) = (a-b)-c$

Let $a = 1, b = 2, c = 3$

$\Rightarrow 1-(2-3) = (1-2)-3$

$\Rightarrow 1-(-1) = (-1)-3$

$\Rightarrow 2 = -4 \notin N$

$\therefore (N, -)$ is not a semi-group.

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3) P.T $(E, *)$ is a semi-group where E is the set of even positive numbers. $[* = \times]$

Sol: let $E = \{2, 4, 6, 8, 10, \ldots\}$.

(i) closure:

let $a, b \in E$ then $a * b \in E$

let $a = 2, b = 4$

$2 * 4 = 8 \in E$

so closure satisfies.

(ii) Associative

$a * (b * c) = (a * b) * c$

let $a = 2, b = 4, c = 6$

$\Rightarrow 2 * (4 * 6) = (2 * 4) * 6$

$\Rightarrow 2 * 24 = 8 * 6$

$\Rightarrow 48 = 48 \in E$

$\therefore$ Associative satisfies

$\therefore (E, *)$ is a semigroup.

4) S.T that the binary operation $*$ defined on $(R, *)$ where $x * y = x^y$ is not a semigroup

Sol:. real no's includes Natural, whole, Integers, rational & Irrational

Given:- $(R, *)$ on $x * y = x^y$

(i) closure :-

Let $x, y \in R \Rightarrow x * y = x^y \in R$

if $x = 1, y = 2$

$1 * 2 = 1^2 \in R$

(ii) Associative

$a * (b * c) = (a * b) * c$

$\Rightarrow x * (y * z) = (x * y) * z$

$\because [x * y = x^y]$

$\Rightarrow$ |||ly $y * z = y^z$

$\Rightarrow x * y^z = x^y * z$

$\Rightarrow x^{(y^z)} \neq (x^y)^z = x^{yz}$

$\therefore$ not Associative

$\therefore$ L.H.S $\neq$ R.H.S

$\therefore$ Not a semigroup.

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5) S.T the operation $*$ defined by $a*b = a+b-ab \ \forall a,b \in Z$ is a semigroup

Sol: Given:

$$a*b = a+b-ab \quad \forall a,b \in Z.$$

(i) closure:

let $a,b \in Z$ then

$$a*b = a+b-ab \in Z.$$

$$1*2 = 1+2-(1)(2)$$
$$= 3-2 = 1 \in Z.$$

$\therefore$ closure satisfies.

(ii) Associative

$$a*(b*c) = (a*b)*c$$

R.H.S: $(a*b)*c$

$$(a+b-ab)*c$$

(a) (b)

$$\Rightarrow a+b-ab+c-(a+b-ab)c$$

$$\Rightarrow a+b-ab+c-ac-bc+abc$$

L.H.S

$a * (b * c)$

$\left[\begin{array}{ll} \because & a*b = a+b-ab \\ \text{similarly} & b*c = b+c-bc \end{array}\right]$

$\Rightarrow a * (b * c)$

$\Rightarrow a * (b+c-bc)$

ⓐ ⓑ

$\Rightarrow a + (b+c-bc) - a(b+c-bc)$

$\Rightarrow a+b+c-bc-ab-ac+abc$

$\because$ L.H.S = R.H.S

$\therefore$ Associative satisfies.

$\therefore$ It is a semigroup.

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6) S.T Q with the operation $*$ defined by $a*b = a-b+ab$ $\forall a,b \in Q$ is not a semigroup

Sol:- Given:

$a*b = a-b+ab \quad \forall a,b \in Q$

(i) closure:

$1 * 2 = 1-2+(1)(2)$
$= -1+2 = 1 \in Q.$

let $a, b \in Q$ then $a * b = a - b + ab$
$\in Q$

$\therefore$ closure satisfies.

Associative:

$a * (b * c) = (a * b) * c$

$\because [a * b = a - b + ab]$ $b * c = b - c + bc$

R.H.S : $(a * b) * c$

$\Rightarrow$ $(a - b + ab) * c$
ⓐ ⓑ

$\Rightarrow$ $a - b + ab - c + (a - b + ab) c$

$\Rightarrow$ $a - b + ab - c + ac - bc + abc.$

L.H.S

$a * (b * c)$

$\Rightarrow a * (b - c + bc)$
ⓐ ⓑ

$\Rightarrow$ $a - (b - c + bc) + a(b - c + bc)$

$\Rightarrow$ $a - b + c - bc + ab - ac + abc$

$\therefore$ L.H.S $\neq$ R.H.S Associative doesn't satisfy.

$\therefore$ $(Q, *)$ is not a semi-Group.

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3) Identity Element [0 & 1]

If there exists an element

$e \in G$ such that

$$e * a = a = a * e \quad \forall a \in G$$

then e is called Identity element.

Ex:- $a * e = a$

[$* = +$]	[$* = \times$]
$1 + \boxed{0} = 1$	$2 \times \boxed{1} = 2$
$-3 + \boxed{0} = -3$	$-3 \times \boxed{1} = -3$
$\therefore e = 0$	$\therefore e = 1$
0 is Identity element	$\therefore$ 1 is Identity element

Monoid: An algebraic structure $(S, *)$ is called a Monoid if it satisfies the following properties.

(i) closure

(ii) Associative

(iii) Identity.

Ex:- 1) $(\mathbb{Z},+)$

$\mathbb{Z} \rightarrow$ set of Integers.

$\{\ldots -3,-2,-1,0,1,2,3\ldots\}$.

(i) closure:-

If $a,b \in \mathbb{Z}$ then $a+b \in \mathbb{Z}$

$a=1, b=2 \Rightarrow 1+2=3 \in \mathbb{Z}$.

(ii) Associative

$a+(b+c) = (a+b)+c$

$a=1, b=2, c=3$

$\Rightarrow 1+(2+3) = (1+2)+3$

$6 = 6$

(iii) Identity

$a * e = a$ $[* = +]$

$a + e = a$

$-2 + \boxed{} = -2$ 0

0 is Identity

$\therefore (\mathbb{Z},+)$ is a Monoid.

(ii) $(\mathbb{Z},*)$ is also a Monoid.

Identity $\Rightarrow$ $a * e = a$

$-2 \times \boxed{} = -2$ 1

1 is Identity

(iii) $(N,+)$ is not a Monoid.

(iv) $(N,*)$ is a monoid.

4) Inverse Element

If $a * b = e = b * a$ for some $b \in G$, then b is called Inverse of $a \in G$.

[$a^{-1} \to$ Inverse of a] not $\frac{1}{a}$

$\boxed{\begin{array}{l}\text{Identity elements are} \\ \quad e = 0 \ \& \ 1 \\ e = 0 \ \text{for addition} \\ e = 1 \ \text{for multiplication}\end{array}}$

Ex:

$a * b = e$ [$* = +$]

$a + \square = 0$

$\square \to -a$

$\therefore -a$ is Inverse of a.

$a * b = e$ [$* = \times$]

$a \times \square = 1$

$\square \to 1/a$

$\therefore 1/a$ is Inverse of a.

$AB = I \Rightarrow B = A^{-1}$

$ab = e \Rightarrow b = a^{-1}$

Group:

An Algebraic structure $(G,*)$ is said to be Group if it satisfies the following properties:

(i) closure (ii) Associative

(iii) Identity (iv) Inverse.

Ex:- S.T $(R,+)$ is a group.

Sol:- (i) closure

if $a,b \in R$ then $a*b \in R$

$a+b \in R$

Let $a=1, b=2$

$a+b = 1+2 = 3 \in R$

$\therefore$ closure satisfies.

(ii) Associative:-

let $a=1, b=2, c=3$

$a+(b+c) = (a+b)+c$

$1+(2+3) = (1+2)+3$

$1+5 = 3+3 \Rightarrow 6=6$

Associative satisfies.

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(iii) Identity

$a * e = a$ $\quad [* = +]$

$a + e = a$ $\quad$ let $a = 1$

$1 + \boxed{} = 1$

$\rightarrow 0$

0 is Identity.

$\therefore$ Identity satisfies.

(iv) Inverse:

$a * b = e$ $\quad [* = +]$

so $a + b = e$

let $a = 1$

$1 + \boxed{} = 0$ $\quad [e = 0]$

-1

$\therefore$ -1 is inverse of 1

$\therefore$ Inverse satisfies.

$\therefore$ $(R, +)$ is a group.

P.T $(G, *)$ where $G = \{1, -1, i, -i\}$ is a group (or) Abelian

$[i^2 = -1]$

Sol:

$*$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

(i) closure:-

$a, b \in G$ then $a * b \in G$

$1 \times 1 = 1 \in G$

$-1 \times -1 = 1 \in G$

∴ closure satisfies

(ii) Associative:-

$a * (b * c) = (a * b) * c$

$a = 1, b = -1, c = i$

$\Rightarrow 1 \times (-1 \times i) = (1 \times -1) \times i$

$\Rightarrow 1 \times (-i) = (-1) \times i$

$\Rightarrow -i = -i$ $[L.H.S = R.H.S]$

∴ Associative satisfies.

(iii) Identity

$a * e = a$

let $a = 1$

$1 \times \boxed{1} = 1$

let $a = i$

$i \times \boxed{1} = i$

$\therefore$ 1 is Identity from the table.

(iv) Inverse

$a * b = e$ $[i^2 = -1]$

$a * b = 1$

let $a = i$ $\Rightarrow i * \boxed{-i} = 1$

$-i^2 \Rightarrow -(-1) = 1$

$\{1, -1, i, -i\}$

$-i$ is inverse of i

let $a = 1$

$a * b = e$

$1 * \boxed{1} = 1$

$\therefore$ 1 is Inverse of 1

Inverse of $-1 = -1$

Inverse of $-i = +i$

$a * b = e$

$-i * \boxed{+i} = 1$

P.T $\{1, \omega, \omega^2\}$ is a group w.r.t multiplication where $1, \omega, \omega^2$ are cube roots of unity.

Sol:

$[\omega^3 = 1]$

$*$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3 = 1$
ω^2	ω^2	$\omega^3 = 1$	$\omega^4 = \omega$

$\omega^4 = \omega^3 \cdot \omega$
$= 1 \cdot \omega$
$= \omega$

(i) closure:-

From the table it is clear that $(G, *)$ satisfies closure.

(ii) Associative

Let $a = 1, b = \omega, c = \omega^2$

$a \times (b \times c) = (a \times b) \times c$

$\Rightarrow 1 \times (\omega \times \omega^2) = (1 \times \omega) \times \omega^2$

$\Rightarrow \omega^3 = \omega^3$

Associative satisfies.

(iii) Identity

$a * e = a$

'1' is Identity element.

$1 \times \square = 1$
$\omega \times \square = \omega$
$\omega^2 \times \square = \omega^2$

(iv) Inverse

$a * b = e$

$a * b = 1$

$1, \omega, \omega^2$

$a \times b = e$

$1 \times \square = \boxed{1}$

$\omega \times \square = 1$ (ω^2)

$\omega^2 \times \square = 1$ (ω)

→ ω^2 is inverse of ω

→ ω is inverse of ω^2

→ 1 is its own inverse.

∴ It is a group. It holds commutative also.

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5) Commutative

$a * b = b * a \quad \forall\ a, b \in G.$

Abelian Group

An algebraic structure $(S, *)$ is called Abelian group if it satisfies following 5 properties.

(i) closure (ii) Associative

(iii) Identity (iv) Inverse.

(v) ~~A~~Commutative.

Commutative

$[* = +]$

$a + b = b + a$

$3 + 2 = 2 + 3$

$[* = -]$

$2 - 3 = 3 - 2$ ✗

$[* = \times]$

$2 \times 3 = 3 \times 2$

P.T the set 'Z' of all integers with binary operation '$*$' defined by $a*b = a+b+1$ is an abelian group.

Sol:- (i) closure:-

If $a, b \in Z$ then $a*b \in Z$

$1 \times 1 \in Z$

closure satisfies

(ii) Associative

$a*(b*c) = (a*b)*c$

R.H.S:- $(a*b)*c$ $\because [a*b = a+b+1]$

$\downarrow$

$(a+b+1)*c$

a b

$\Rightarrow a+b+1+c+1 \Rightarrow a+b+c+2.$

L.H.S:- $a*(b*c)$

$\Rightarrow a*(b+c+1)$

a b

$\Rightarrow a+(b+c+1)+1 \Rightarrow a+b+c+2$

$\therefore$ L.H.S = R.H.S.

$\therefore$ Associative satisfies.

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(iii) Identity

$a * e = a$

$[a * b = a + b + 1]$

$a * e = a + e + 1$

$\because a * e = a$

$a + e + 1 = a \Rightarrow e + 1 = 0 \Rightarrow e = -1$

$\therefore e = -1$ is the identity element in $\mathbb{Z}$.

(iv) Inverse

$a * b = e$

$a * b = e = -1$

$\Rightarrow a * b = -1$

$a + b + 1 = -1 \Rightarrow a + b = -2$

$\Rightarrow b = -2 - a$

$\therefore b = -2 - a$ is inverse of $\mathbb{Z}$

(v) commutative

$a * b = b * a.$

$\Rightarrow a + b + 1 = b + a + 1 = b * a$

$\therefore$ commutative satisfied.

$\therefore (\mathbb{Z}, *)$ is an abelian group.

Q) S.T the set Q_+ of all positive rational numbers forms an abelian group under the composition defined by $*$ such that $a*b = \frac{ab}{3}$ for $a, b \in Q_+$

Sol:- (i) closure :-

let $a, b \in Q_+$ then $a*b \in Q_+$

$$\frac{ab}{3} = \frac{1 \times \cancel{3}}{3} = \frac{1}{3} \in Q_+$$

closure satisfies.

(ii). Associative:-

$a*(b*c) = (a*b)*c$ $\quad\left[a*b = \frac{ab}{3}\right]$

R.H.S :- $(a*b)*c$

$$\left(\frac{ab}{3}\right) * c \Rightarrow \frac{\frac{abc}{3}}{3} = \frac{abc}{9}$$

(a) (b)

L.H.S :- $a*(b*c)$

$$\Rightarrow a * \left(\frac{bc}{3}\right) \Rightarrow \frac{\frac{abc}{3}}{3} \Rightarrow \frac{abc}{9}$$

(a) (b)

$\therefore$ L.H.S = R.H.S.

$\therefore$ Associative satisfies.

(iii) Identity $\left[a*b=\frac{ab}{3}\right]$

$a*e = a$ $\qquad a*e=\frac{ae}{3}$

$\frac{ae}{3}=a \Rightarrow ae = 3a$

$\Rightarrow [e=3]$

$\therefore e=3$ is the Identity element in $Q+$

(iv) Inverse

$a*b = e$

$\frac{ab}{3}=e \Rightarrow \frac{ab}{3}=3 \; [e=3]$

$\Rightarrow ab = 9 \Rightarrow b=\frac{9}{a}$

$\therefore b=\frac{9}{a}$ is the inverse of a

(v) Commutative

$a*b = b*a$

$\Rightarrow \frac{ab}{3} = \frac{ba}{3}$

$a*b = \frac{ab}{3} = \frac{ba}{3} = b*a$

$\therefore$ commutative exists

$\therefore (Q+,*)$ is an Abelian group.//

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Addition modulo 'm':

Let $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$ then addition modulo 'm' is given by

$a +_m b = r$, $0 \le r < m$

Ex:- 1) $20 +_6 5 = 1$

$20 + 5 = 25$

$25 \div 6$

$6)25(4$
24
1

$1 +_5 2 = 3.$

$1 +_5 0 = 1$

2) $16 +_2 3 = 1$

$16 + 3 = 19$

$2)19(9$
18
1

3) $15 +_5 3 = 3.$

$15 + 3 = 18$

$5)18(3$
15
3

4) $5 +_6 4 = 3$ [$9 > 6$]

5) $1 +_5 1 = 2$

$r \to$ no decimals

Multiplication modulo p:-

Let $a, b \in \mathbb{Z}$ and $p \in \mathbb{N}$ then multiplication modulo 'm' is given by $a \times_p b = r$, $0 \le r \le p$

Ex:- $5 \times_3 4 = 2.$

$3)20(6$
18
2

S.T the set $G = \{0, 1, 2, 3, 4\}$ is an abelian group w.r.t addition modulo 5.

Sol: composition table for $+_5$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$4 +_5 4 = 3$

$5)\,8\,(1$
$\underline{5}$
3

$1 +_5 4 =$

$5)\,5\,(1$
$\underline{5}$
0

$2 +_5 3 = 0$

$2 +_5 4 = 5)\,6\,(1$
$\underline{5}$
3

(i) closure

Let $a, b \in G$, then

$a +_5 b \in G$

$a = 1, b = 2$ $\quad 1 +_5 2 = 3 \in G$

$\therefore$ closure satisfies.

(ii) Associative

$a +_5 (b +_5 c) = (a +_5 b) +_5 c$

let = $a = 1, b = 2, c = 3$

$\Rightarrow 1 +_5 (2 +_5 3) = (1 +_5 2) +_5 3$

$\Rightarrow 1 +_5 (0) = 3 +_5 3$

$\Rightarrow 1 = 1$ [L.H.S = R.H.S]

∴ Associative satisfies

(iii) Identity

$a +_5 e = a$

$1 +_5 \boxed{0} = 1$

$2 +_5 \boxed{0} = 2$

0 is the Identity element.

$[e = 0]$

(iv) Inverse:-

$a +_5 b = e \Rightarrow a +_5 b = 0$

let $a = 1$, ~~b=~~ $\Rightarrow 1 +_5 \boxed{4} = 0$

4 is inverse of 1 $\quad 1 +_5 \boxed{4} = 0.$

from table.

let $a = 2$

$2 +_5 \boxed{} = 0$ (e)

from table $\rightarrow$ 3

3 is inverse of 2.

check:- $2 +_5 3 \Rightarrow 0.$

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let $a = 3$

$a +_5 b = 0$

$3 +_5 \boxed{2} = 0$ [from table]

2 is inverse of 3

let $a = 4$

$4 +_5 \boxed{1} = 0$

1 is inverse of 4

0 is its own inverse

(v) commutative

$a +_5 b = b +_5 a$

$a = 1, b = 2$

$\Rightarrow 1 +_5 2 = 2 +_5 1$

$\Rightarrow 3 = 3$ //

$\therefore (G, +_5)$ is abelian group //.

Q) S.T the set $G = \{1, 2, 3, 4\}$ is an abelian group w.r.t multiplication modulo 5.

sol:- composition table

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(i) closure :-

If $a, b \in G$ then $a \times_5 b \in G$
closure satisfies.

(ii) Associative

$$a \times_5 (b \times_5 c) = (a \times_5 b) \times_5 c$$

let $a = 1, b = 2, c = 3$

$\Rightarrow 1 \times_5 (2 \times_5 3) = (1 \times_5 2) \times_5 3$

$\Rightarrow 1 \times_5 (1) = 2 \times_5 3$

$\Rightarrow 1 = 1$ [L.H.S = R.H.S]

$\therefore$ Associative satisfies

(iii) Identity

$1 \times_5 \square = 1$

$$a \times_5 e = a$$

1 is Identity $[e = 1]$

(iv) Inverse

$$a \times_5 b = e$$

$$\Rightarrow a \times_5 b = 1$$

let $a = 1 \Rightarrow 1 \times_5 \square = 1$, $\square = 1$

1 is inverse of 1

let

$a = 2 \Rightarrow 2 \times_5 \square = 1$, $\square = 3$

$\therefore$ 3 is inverse of 2

$a = 3 \Rightarrow 3 \times_5 \square = 1$, $\square = 2$

2 is inverse of 3

$a = 4 \quad 4 \times_5 \square = 1$, $\square = 4$

4 is inverse of 4.

commutative

$$a \times_6 b = b \times_6 a$$

$$1 \times_6 2 = 2 \times_6 1$$

$$2 = 2$$

$\therefore (G, \times_5)$ is abelian group.

G_2:- cancellation law holds in G.

$\forall\ a, b, c \in G$ then $a * b = a * c \Rightarrow b = c$

$b * a = c * a \Rightarrow b = c$

Sol:-

consider

$(a * b) = (a * c)$

multiply with a^{-1} on bs

$\Rightarrow (a * b) * a^{-1} = (a * c) * a^{-1}$

$\Rightarrow (a * a^{-1}) * b = (a * a^{-1}) * c$

$\Rightarrow e * b = e * c$

$\Rightarrow b = c$ $[\because a a^{-1} = e]$

$\begin{bmatrix} e \times b = b \\ e \times c = c \end{bmatrix}$

consider

$(b * a) = (c * a)$

$\Rightarrow (b * a) * a^{-1} = (c * a) * a^{-1}$

$\Rightarrow b * (a * a^{-1}) = c * (a * a^{-1})$

$\Rightarrow b * e = c * e$

$\Rightarrow b = c$

$a * b = e$

$b = a^{-1}$

$\Rightarrow a * a^{-1} = e$

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Ex:- P.T If $a^2 = a$ then $a = e$, $\forall\, a \in G$.

Sol:- Given $a^2 = a$

$$a.a = a$$

multiply with a^{-1} on b.s

$$\Rightarrow (aa)a^{-1} = a.a^{-1}$$ $[2 \times \frac{1}{2} = 1]$

$$\Rightarrow a(aa^{-1}) = a.a^{-1}$$ $[aa^{-1} = e]$

$$\Rightarrow a.e = e$$ $[a.e = a]$

$1.e = 1$

$$\Rightarrow a = e$$ $2.e = 2$

Ex:- If $(G, *)$ is a group, then $(a*b)^{-1} = b^{-1} * a^{-1}$ $\forall\, a, b \in G$

Sol:- concept

If $\left.\begin{array}{l} x*y = e \\ y*x = e \end{array}\right\}$ then $y = x^{-1}$

Here let $x = (a*b)$, $y = b^{-1} * a^{-1}$

$x*y$

$$\Rightarrow (a*b) * (b^{-1} * a^{-1})$$

$$\Rightarrow a * (b * b^{-1}) * a^{-1}$$ [Associative]

$$\Rightarrow a * (e) * a^{-1}$$ [Inverse]

$$\Rightarrow a * a^{-1} = e$$

$y * x$

$(b^{-1} * a^{-1}) * (a * b)$ $[a^{-1} * a = e]$

$\Rightarrow b^{-1} * (a^{-1} * a) * b$

$\Rightarrow b^{-1} * (e) * b$

$\Rightarrow b^{-1} * b = e.$

$\therefore x * y = y * x = e.$ $x * y = e$

$y = x^{-1}$

$(b^{-1} * a^{-1}) = (a * b)^{-1}$

$\therefore (a * b)^{-1} = b^{-1} * a^{-1}$

$x^{-1} = y.$

Ex:- If every element of a group G is its own inverse S.T G is an abelian group

Sol. Let $a, b \in G$ $\because$ own inverse

$\Rightarrow a^{-1} = a, b^{-1} = b$

In the same way if $ab \in G$ then $(ab)^{-1} = ab$

$\because (ab)^{-1} = ab$

$\Rightarrow b^{-1} a^{-1} = ab$

$\Rightarrow ba = ab$ [$\because$ commutative satisfied]

$\therefore$ G is abelian.

Ex:- S.T in a group G, for $a, b \in G$, $(ab)^2 = a^2b^2$

$\Leftrightarrow$ G is an abelian

Sol:- $\because (ab)^2 = a^2b^2$

$\Leftrightarrow (ab)(ab) = (a \cdot a)(b \cdot b)$

$\Leftrightarrow \cancel{a}(ba)\cancel{b} = \cancel{a}(ab)\cancel{b}$ [by Association]

$\Leftrightarrow ba = ab$ [by cancellation law]

$\because ab = ba$ [is commutative]

$\therefore G$ is abelian.

Ex: Show that $G = \{x / x = 2^a 3^b$ for $a, b \in Z\}$ is a gro
under multiplication.

Sol: Let $x, y, z \in G$, we can take

$x = 2^p 3^q$, $y = 2^r 3^s$, $z = 2^l 3^m$, p, q, r, s, l, m

i) closure law :- For $x, y \in G \Rightarrow x \cdot y \in G$

since $x \cdot y = (2^p \cdot 3^q) \cdot (2^r \cdot 3^s)$

$= 2^{p+r} \cdot 3^{q+s} \in G.$

ii) Associative Law :- For $x, y, z \in G$

$x.(y.z) = x.(yz)$:—

$$x.(y.z) = 2^p 3^q (2^r . 3^s . 2^l 3^m)$$

$$= 2^p 3^q (2^{r+l} 3^{s+m})$$

$$= 2^{p+r+l} . 3^{q+s+m}$$

$$= (2^{p+r} . 3^{q+s}) 2^l . 3^m$$

$$= (2^p . 3^q . 2^r . 3^s) 2^l . 3^m = (x.y).z$$

$\therefore$ Associative is satisfied.

iii) Identity Law : Let $x \in G$

we know that $e = 2^0 3^0 \in G$, since $0 \in Z$

$$\therefore x . e = 2^p 3^q . 2^0 . 3^0 = 2^{p+0} 3^{q+0} = 2^p 3^q = x$$

$$\therefore x . e = e . x = x$$

iv) Inverse Law :- Let $x \in G$

Now $y = 2^{-p} 3^{-q} \in G$, since $-p, -q \in Z$

$$\therefore x . y = 2^p . 3^q . 2^{-p} . 3^{-q} = 2^{p-p} 3^{q-q} = 2^0 . 3^0 = e$$

$$\therefore x . y = y . x = e$$

Hence $(G, .)$ is a group.

Order of an element :– Let $(G, *)$ be a group.
$a \in G$ then the least positive integer 'n' if it
such that $a^n = e$ is called the order of $a \in G$.
order of an element $a \in G$ is be denoted by $o(a)$

Ex: If $G = \{1, -1, i, -i\}$

$1^1 = 1^2 = 1^3 = \dots = 1 \Rightarrow o(1) = 1$

$(-1)^2 = (-1)^4 = (-1)^6 = \dots = 1 \Rightarrow o(-1) = 2$

$i^4 = i^8 = \dots = 1 \Rightarrow o(i) = 4$

$(-i)^4 = (-i)^8 = \dots = 1 \Rightarrow o(-i) = 4.$

Sub groups :– Let $(G, *)$ be a group and 'H' be a non empty subset of G. If $(H, *)$ is itself is a group then $(H, *)$ is called sub-group of $(G, *)$

Ex: i) $(Z, +)$ is a sub group of $(Q, +)$

ii) $(N, +)$ is not a sub group of the group $(Z, +)$

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