$$y = y_c + y_p$$

$$y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x + \frac{1}{16} \cos 3x.$$

(2) $(D^2+5D-6)y = \sin 4x \sin x.$

Sol $f(D) = D^2+5D-6$

The A.E is $f(m)=0$

$$m^2+5m-6=0$$

$$m^2+6m-1m-6=0$$

$$(m+6)(m-1)=0$$

$$m=-6,1$$

The roots are real & different

$$y_c = c_1 e^{-6x} + c_2 e^{x}$$

Now $P.I = \dfrac{1}{D^2+5D-6} \sin 4x \sin x$

$$= \frac{1}{2(D^2+5D-6)}[2\sin 4x \sin x] \quad \{2\sin A \sin B = \cos(A-B) - \cos(A+B)\}$$

$$= \frac{1}{2(D^2+5D-6)}[\cos 3x - \cos 5x]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+5D-6}\cos 3x - \frac{1}{D^2+5D-6}\cos 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-9+5D-6}\cos 3x - \frac{1}{-25+5D-6}\cos 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{5D-15}\cos 3x - \frac{1}{5D-31}\cos 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{5}\left[\frac{D+3}{D^2-3^2}\cos 3x - \frac{5D+31}{(5D)^2-(31)^2}\cos 5x\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{5}\left[\frac{1}{-18}(D+3)\cos 3x\right] - \frac{5D+31}{25D^2-961}\cos 5x\right]$$

$$= \frac{1}{-180}[\sin 3x(-3) + 3\cos 3x] + \frac{1}{2}\left[\frac{1}{-3172} - 25\sin 3x + 31\cos 5x\right]$$

① P.I of $f(D)y = \phi(x)$ when $\phi(x) = x^K$ where K is a positive Integer.

$$P.I = \frac{1}{f(D)} x^K.$$

① $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

2) $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$

3) $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

4) $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$

5) $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots$

6) $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots$

① Solve $(D^3 + 2D^2 + D)y = x^3$

Sol $(D^3 + 2D^2 + D)y = x^3$

A.E is $f(m) = 0$

$$m^3 + 2m^2 + m = 0$$

$$m(m^2 + 2m + 1) = 0$$

$$m = 0,\ m = -1, -1$$

Hence C.F $= C_1 e^{0x} + (C_2 + C_3 x) e^{-x}$

Now P.I $= y_p = \frac{x^3}{D^3 + 2D^2 + D} = \frac{x^3}{D(D+1)^2}$

$$= \frac{1}{D} \cdot \frac{1}{(D+1)^2} x^3$$

$$= \frac{1}{(1+D)^2} \int x^3 dx$$

$$= (1+D)^{-2} \frac{x^4}{4}$$

$$P.I = \frac{1}{4}(1+D)^{-2} \cdot x^4$$

$$P.I = \frac{1}{4}\left[1-2D+3D^2-4D^3+5D^4 - -\right]x^4$$

$$= \frac{1}{4}\left[x^4-2Dx^4+3D^2x^4-4D^3x^4+5D^4x^4\right]$$

$$= \frac{1}{4}\left[x^4-8x^3+36x^2-96x+120\right]$$

$$P.I = \frac{x^4}{4}-2x^3+9x^2-24x+30.$$

General solution is

$$y = C.F + P.I$$

$$y = c_1 + (c_2+c_3x)e^{-x} + \frac{x^4}{4}-2x^3+9x^2-24x+30$$

① Solve $D^2(D^2+4)y = 320(x^3+2x^2+e^x)$

Sol Given eqn is $D^2(D^2+4)y = 320(x^3+2x^2+e^x)$

let $f(D) = D^2(D^2+4)$

A.E is $f(m) = 0$

$$m^2(m^2+4) = 0$$

$$m = 0, m = \pm 2i$$

The roots are real and repeated and two roots are complex conjugate numbers.

Thus C.F is $y_c = (c_1+c_2x)e^{0.x} + (c_3\cos 2x + c_4\sin 2x)$

$$y_c = (c_1+c_2x) + (c_3\cos 2x + c_4\sin 2x)$$

Now $P.I = \frac{1}{D^2(D^2+4)} 320(x^3+2x^2+e^x)$

$$= \frac{1}{D^2(D^2+4)}320(x^3+2x^2) + \frac{1}{D^2(D^2+4)}320e^x.$$

$$= \frac{1}{D^2 4\left(1+\frac{D^2}{4}\right)}320(x^3+2x^2) + \frac{1}{D^2(D^2+4)}320e^x$$

$$P.I = y_{P_1} + y_{P_2}$$

$$y_{P_1} = \frac{\cancel{320}^{\,80}}{\cancel{4}} \frac{1}{D^2}\left(1+\frac{D^2}{4}\right)^{-1}(x^3+2x^2)$$

$$= 80\left[\frac{1}{D^2}\right]\left[1-\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2-\left(\frac{D^2}{4}\right)^3 - - -\right](x^3+2x^2)$$

$3x^2$
$6x$
6
(0)

$$= \frac{80}{D^2}\left[1-\frac{D^2}{4}+\frac{D^4}{16}\right](x^3+2x^2)$$

$$= \frac{80}{D^2}\left[x^3+2x^2 - \frac{1}{\cancel{4}_{2}}[\cancel{6}^{3}x] + \left(\frac{-4}{4}\right) + \frac{1}{16}[0]\right]$$

$$= \frac{80}{D^2}\left[x^3+2x^2-\frac{3}{2}x+1\right]$$

$$= \frac{80}{D}\left[\frac{1}{D}\left(x^3+2x^2-\frac{3}{2}x-1\right)\right]$$

$$= \frac{80}{D}\left[\int (x^3+2x^2-3/2\,x-1)\,dx\right]$$

$$= \frac{80}{D}\left[\frac{x^4}{4}+2\frac{x^3}{3}-\frac{3}{2}\frac{x^2}{2}-x\right]$$

$$= 80\left[\int\left[\frac{x^4}{4}+\frac{2}{3}x^3-\frac{3}{4}x^2-x\right]dx\right]$$

$$= 80\left[\frac{1}{4}\left[\frac{x^5}{5}\right]+\frac{\cancel{2}}{3}\left[\frac{x^4}{\cancel{4}_{2}}\right]-\frac{3}{4}\left[\frac{x^3}{3}\right]-\frac{x^2}{2}\right]$$

$$= \frac{\cancel{80}^{\,4}}{\cancel{20}}x^5 + \frac{\cancel{80}^{\,40}}{\cancel{6}_{3}}x^4 - \frac{\cancel{80\times 3}^{\,20}}{\cancel{4\times 3}}x^3 - \frac{\cancel{80}^{\,40}}{\cancel{2}}x^2$$

$$y_{P_1} = 4x^5+\frac{40}{3}x^4-20x^3-40x^2$$

and $$y_{P_2} = \frac{1}{D^2(D^2+4)}\,320e^x$$

$$= \frac{320}{(1)^2(1^2+4)}e^x = \frac{\cancel{320}^{\,64}}{\cancel{5}}e^x$$

$$y_{P_2} = 64e^x$$

$$y = y_c + y_{P_1} + y_{P_2}$$

$$y = c_1 + c_2x + c_3\cos 2x + c_4\sin 2x + 4x^5 + \frac{40}{3}x^4 - 20x^3 - 40x^2 + 64e^x$$

① $(D^2-4D+4)y = 8(e^{2x}+x^2+\sin 2x)$

Sol The given eqn is $(D^2-4D+4)y = 8e^{2x}+8x^2+8\sin 2x$

$f(D) = D^2-4D+4$

A.E is $f(m)=0$

$m^2-4m+4=0$

$m = 2,2$

$\therefore$ C.F is $y_c = e^{2x}(c_1+c_2x)$

Now $P.I = \frac{1}{D^2-4D+4}(8x^2+8e^{2x}+8\sin 2x)$

$= \frac{1}{(D-2)^2}8x^2 + \frac{1}{(D-2)^2}e^{2x} + \frac{1}{D^2-4D+4}8\sin 2x$

$P.I = y_{P_1}+y_{P_2}+y_{P_3}$.

$y_{P_1} = \frac{1}{(D-2)^2}8x^2$

$= \cancel{8}^{2}\left[\frac{1}{\cancel{(-2)^2}[1-\frac{D}{2}]^2}x^2\right]$

$= +2[1-\frac{D}{2}]^{-2}x^2$

$= +2[1+2(\frac{D}{2})+3(\frac{D}{2})^2 - - -]x^2$

$= +2[1+D+\frac{3}{4}D^2]x^2$

$= 2[x^2+2x+\frac{3}{4}(2)]$

$y_{P_1} = 2x^2+4x+3$

and $y_{P_2} = \frac{8}{(D-2)^2}e^{2x} = \frac{8x^2}{2!}e^{2x} = \cancel{8}^{4}\frac{x^2}{\cancel{2}}e^{2x} = 4x^2e^{2x}$

$y_{P_3} = \frac{1}{D^2-4D+4}8\sin 2x = 8\cdot\frac{1}{-2^2-4D+4}\sin 2x$

$= \cancel{8}^{2}\cdot\frac{1}{-\cancel{4}D}\sin 2x$

$= -2\left\{\frac{1}{D}\sin 2x\right\}$

$= -2\left[\int \sin 2x\, dx\right]$

$-2\cdot\left[-\frac{\cos 2x}{2}\right]$

$y_{P_3} = \cos 2x$

$y = y_{C.F} + y_P$

$y = (c_1+c_2x)e^{2x} + 4x^2e^{2x} + 2x^2+4x+3 + \cos 2x$

① P.I of $f(D)y = \phi(x)$ when $\phi(x) = e^{ax}V$ where 'a' is a constant and V is a func of x.

$$P.I = \frac{1}{f(D)}[e^{ax}V]$$

$$P.I = e^{ax}\frac{1}{f(D+a)}(V)$$

(or) we will use this method To (ii) find P.I when V is $\sin bx$ (or) $\cos bx$ (or) x^K or a polynomial of degree K. In this case.

① Solve (i) $(D^4-1)y = e^x\cos x$.

Sol $f(D) = D^4-1$

A.E is $f(m)=0$

$m^4-1=0$

$(m^2-1)(m^2+1)=0$

$m^2-1=0 \quad m^2+1=0$

$m^2=1 \quad m^2=-1$

$m^2=1 \quad m^2=i^2$

$m=\pm 1 \quad m=\pm i$

Two roots are real and unequal and two roots are complex conjugate numbers.

$$y_c = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x.$$

Now $P.I = \frac{e^x\cos x}{D^4-1} = e^x\frac{\cos x}{(D+1)^4-1}$

$$= e^x.\frac{\cos x}{(D+1)^2(D+1)^2-1}$$

$$= e^x\frac{\cos x}{(D^2+2D+1)(D^2+2D+1)-1}$$

$$= e^x\frac{\cos x}{D^4+2D^3+D^2+2D^3+4D^2+2D+D^2+2D+\cancel{1}-\cancel{1}}$$

$$= e^x\frac{\cos x}{D^4+4D^3+6D^2+4D}$$

$$= e^{x} \frac{\cos x}{(-1)^2 + 4D(-1) + 6(-1) + 4D}$$

$$= e^{x} \frac{\cos x}{1 - 4D - 6 + 4D}$$

$$= e^{x} \frac{\cos x}{-5}$$

$$y_p = -\frac{e^{x}\cos x}{5}$$

The general solution is $y = y_c + y_p$.

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{e^{x}}{5}\cos x.$$

=*=

② Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x$.

Sol Given eqn in operator form is

$$(D^2 - 6D + 13)y = 8e^{3x}\sin 2x.$$

Let $f(D) = D^2 - 6D + 13$

A.E is $f(m) = 0$

$$m^2 - 6m + 13 = 0$$

$$m = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$

$$m = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$$\begin{array}{r} 52 \\ 36 \\ \hline 16 \end{array}$$

The roots are complex & conjugate.

C.F is $y_c = e^{3x}[c_1 \cos 2x + c_2 \sin 2x]$

Now P.I $= y_p = \frac{8e^{3x}\sin 2x}{D^2 - 6D + 13}$

$$y_p = 8e^{3x} \frac{1}{(D+3)^2 - 6(D+3)+13} \sin 2x$$

$$y_p = 8e^{3x} \frac{1}{D^2+\cancel{6D}+9-\cancel{6D}-18+13} \sin 2x.$$

$$y_p = 8e^{3x} \frac{\sin 2x}{D^2+4}$$

$$y_p = \overset{2}{\cancel{8}}e^{3x} \frac{-x}{2(\cancel{2})} \cos 2x \quad \left\{ f(-a^2)=0,\ \frac{\sin ax}{D^2+a^2} = \frac{-x}{2a}\cos ax \right]$$

$$y_p = -2xe^{3x}\cos 2x.$$

General Solution $y = y_c + y_p$

$$y = e^{3x}[c_1\cos 2x + c_2 \sin 2x] - 2xe^{3x}\cos 2x.$$

② solve $(D^3 - 4D^2 - D + 4)y = e^{3x}\cos 2x.$

Sol Given eqn $(D^3-4D^2-D+4)y = e^{3x}\cos 2x$

Let $f(D) = D^3 - 4D^2 - D + 4$

A.E is $f(m) = 0$

$$m^3 - 4m^2 - m + 4$$

$$\begin{array}{c|cccc} m=1 & 1 & -4 & -1 & 4 \\ & 0 & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array}$$

$$(m-1)(m^2-3m-4)=0$$

$$(m-1)(m^2-4m+m-4)=0$$

$$(m-1)(m+1)(m-4)=0$$

$$m = 1, -1, 4$$

Thus C.F $y_c = c_1e^{x} + c_2e^{-x} + c_3e^{4x}.$

Now P.I $y_p = \frac{1}{f(D)} Q(x)$

$$y_p = \frac{1}{D^3 - 4D^2 - D + 4} e^{3x}\cos 2x.$$

$$P.I = e^{3x} \frac{1}{(D+3)^3 - 4(D+3)^2 - (D+3) + 4} \cos 2x$$

$$P.I = e^{3x} \frac{1}{D^3 + 27 + 9D^2 + 27D - 4(D^2 + 9 + 6D) - D - 3 + 4}$$

$$P.I = e^{3x} \frac{\cos 2x}{D^3 + 9D^2 + 27D + 27 - 4D^2 - 36 - 24D - D + 1}$$

$$P.I = e^{3x} \frac{\cos 2x}{D^3 + 5D^2 + 2D - 8}$$

$$= e^{3x} \frac{\cos 2x}{D(-4) + 5(-4) + 2D - 8}$$

$$= e^{3x} \frac{\cos 2x}{-4D - 20 + 2D - 8}$$

$$= e^{3x} \frac{\cos 2x}{-2D - 28} = \frac{e^{3x}}{-2} \frac{1}{(D+14)} \cos 2x$$

$$= \frac{-1}{2} e^{3x} \frac{D-14}{D^2 - (14)^2} \cos 2x$$

$$= \frac{-1}{2} e^{3x} \frac{D-14}{-4-196} \cos 2x$$

$$= \frac{1}{2(200)} e^{3x} (D-14) \cos 2x$$

$$= \frac{1}{400} e^{3x} [D \cos 2x - 14 \cos 2x]$$

$$= \frac{e^{3x}}{400} [-2 \sin 2x - 14 \cos 2x]$$

$$= -\frac{\cancel{2}}{\cancel{400}_{200}} e^{3x} [\sin 2x + 7 \cos 2x]$$

$$\boxed{y_p = -\frac{1}{200} e^{3x} [\sin 2x + 7 \cos 2x]}$$

$$y = y_c + y_p$$

① P.I of $f(D)y = \phi(x)$ when $\phi(x) = x^m v$, m being a positive integer and v is any func of x.

Here v is either sinax or cosax only. It should not be of the form x^m (or) e^{ax}.

① Working Rule for finding P.I of $f(D)y = x^m \sin ax$ or $x^m \cos ax$

(i) $P.I = \frac{1}{f(D)} x^m \sin ax$

$= \text{Imaginary part (I.P) of } \frac{1}{f(D)} x^m (\cos ax + i \sin ax)$

$= \text{I.P of } \frac{1}{f(D)} x^m e^{iax}.$

(ii) $P.I = \frac{1}{f(D)} x^m \cos ax$

$= \text{Real part (R.P) of } \frac{1}{f(D)} x^m e^{iax}.$

② Alternative Method for finding P.I of $f(D)y = \phi(x)$ when $\phi(x) = x.V$ (when m = 1) where V is a func of x.

$P.I = \frac{1}{f(D)}(xV) = \left[x - \frac{1}{f(D)} f'(D)\right] \frac{1}{f(D)} V.$

① Find the solution solve $(D^2-2D+1)y = xe^x \sin x$.

Sol Given $(D^2-2D+1)y = xe^x \sin x$.

$f(D) = D^2-2D+1$

A.E is $f(m)=0$

$m^2-2m+1=0$

$m^2-m-m+1=0$

$m(m-1)-(m-1)=0$

$m=1,1$

The roots are real & equal

$y_c = (C_1 \cancel{+} C_2x)e^x$.

Now P.I $= \frac{1}{f(D)}\phi(x)$

$$P.I = \frac{1}{D^2-2D+1} xe^x \sin x.$$

$$= \frac{1}{D^2-2D+1} e^x x \sin x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+1} x\sin x.$$

$$= e^x \frac{1}{D^2+\cancel{2D}+\cancel{1}-\cancel{2D}-\cancel{2}+\cancel{1}} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x.$$

$$= e^x . \frac{1}{D}\left[\frac{1}{D} x \sin x\right]$$

$$= e^x \frac{1}{D}\left[\int x \sin x \, dx\right]$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - 1.(-\sin x)\right]$$

$$= e^x \frac{1}{D}\left[-x\cos x + \sin x\right]$$

$$= e^x \left[-\int x\cos x\,dx + \int \sin x\,dx\right]$$

$$= e^x \left[-[x(\sin x) - 1.(-\cos x)] + (-\cos x)\right]$$

$$= e^x \left[-x\sin x - \cos x - \cos x\right]$$

$y_p = -e^x[x\sin x + 2\cos x]$

= x =

(2) Solve the differential eqn $(D^2+4)y = x\sin x$.

Sol

$m^2+4=0 \Rightarrow m^2 = (2i)^2 \Rightarrow m = \pm 2i$

$y_c = c_1\cos 2x + c_2 \sin 2x$.

$$P.I = \frac{1}{D^2+4} x\sin x.$$

$$= \left[x - \frac{1}{D^2+4}(2D)\right]\frac{1}{D^2+4}\sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right]\frac{\sin x}{-1^2+4} \qquad \left\{ \frac{1}{f(D)} xV = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}V \right\}$$

$$= \frac{1}{3}\left[x - \frac{2D}{D^2+4}\right]\sin x.$$

$$= \frac{1}{3}\left[x\sin x - \frac{2D}{D^2+4}\sin x\right]$$

$$= \frac{1}{3}\left[x\sin x - \frac{2D}{-1^2+4}\sin x\right]$$

$$= \frac{1}{3}\left[x\sin x + \frac{2}{3}D\sin x\right]$$

$$= \frac{1}{3}\left[x\sin x - \frac{2}{3}[\cos x]\right]$$

$$y_p = \frac{1}{3}\left[x\sin x - \frac{2}{3}\cos x\right]$$

$y = y_c + y_p$

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\left[x\sin x - \frac{2}{3}\cos x\right]$$

Ass Solve $\frac{d^2y}{dx^2} + 9y = x\sin 2x.$

① Solve $(D^2+2D+1)y = x\cos x.$

② $(D^2+3D+2)y = xe^x \sin x.$

④ Solve the differential eqn

⑧ $(D^4+2D^2+1)y = x^2\cos^2 x.$

Sol $f(D) = D^4+2D^2+1$

A.E is $f(m)=0.$

$m^4+2m^2+1=0$

$(m^2+1)^2=0$

$m^2+1=0 \quad m^2+1=0$

$m^2=i^2 \quad m^2=i^2$

The roots are complex number and Repeated.

$y_c = (C_1+C_2x)\cos x + (C_3+C_4x)\sin x.$

$$P.I = \frac{1}{f(D)}\phi(x)$$

$$= \frac{x^2\cos^2 x}{D^4+2D^2+1} = \frac{x^2(\cos 2x+1)}{2(D^4+2D^2+1)}$$

$$y_p = \frac{x^2}{2(D^4+2D^2+1)} + \frac{x^2\cos 2x}{2(D^4+2D^2+1)} = y_{p_1}+y_{p_2}$$

where $y_{p_1} = \frac{x^2}{2(D^2+1)^2} = \frac{x^2}{2}(D^2+1)^{-2}$

$$= \frac{1}{2}(1+D^2)^{-2}x^2$$

$$= \frac{1}{2}[1-2D^2+3D^4-\dots]x^2$$

$$= \frac{1}{2}[x^2-2D^2x^2] = \frac{x^2}{2} - D^2(x^2)$$

$$\boxed{y_{p_1} = \frac{x^2}{2} - 2.}$$

$$y_{P_2} = \frac{x^2\cos 2x}{2(D^2+1)^2} = x^2 \cdot \frac{R.P \text{ of } e^{i2x}}{2(D^2+1)^2}$$

$$= \frac{1}{2} \cdot R.P \text{ of } e^{2ix} \quad = R.P \text{ of } e^{i2x} \cdot \frac{1}{2} \cdot \frac{1}{((D+2i)^2+1)^2} x^2$$

$$= R.P \text{ of } \frac{e^{2ix}}{2}\left[\frac{x^2}{[D^2+4i^2+4Di+1]^2}\right]$$

$$= R.P \text{ of } \frac{e^{2ix}}{2}\left[\frac{x^2}{(D^2+4Di-3)^2}\right]$$

$$= R.P \text{ of } \frac{e^{i2x}}{2}\left[\frac{x^2}{(-3)^2\left[1-\frac{D^2+4Di}{3}\right]^2}\right]$$

$$= R.P \text{ of } \frac{e^{i2x}}{2\times 9}\left[\left[1-\frac{D^2+4Di}{3}\right]^{-2} x^2\right]$$

$$= R.P \text{ of } \frac{e^{2ix}}{18}\left[1+2\left[\frac{D^2+4Di}{3}\right]+3\left[\frac{D^2+4Di}{3}\right]^2 - - -\right]x^2$$

$$= R.P \text{ of } \frac{e^{2ix}}{18}\left[1+\frac{2}{3}D^2+\frac{8i}{3}D+\frac{\cancel{3}}{\cancel{9}_3}\left[D^4-16D^2+8iD^3\right]\right]x^2$$

$$= R.P \text{ of } \frac{e^{2ix}}{18}\left[1+\frac{2}{3}D^2+\frac{8i}{3}D-\frac{16}{3}D^2\right]x^2$$

$$= R.P \text{ of } \frac{e^{2ix}}{18}\left[x^2+\frac{2}{3}(2)+\frac{8}{3}i(2x)-\frac{16}{3}(2)\right]$$

$$= R.P \text{ of } \left[\frac{\cos 2x + i\sin 2x}{18}\right]\left[x^2+\frac{4}{3}+\frac{16}{3}ix-\frac{32}{3}\right]$$

$$= \frac{1}{18}\left[x^2\cos 2x+\frac{4}{3}\cos 2x-\frac{32}{3}\cos 2x-\frac{16}{3}x\sin 2x\right]$$

$$y_{P_2} = \frac{1}{18}\left[x^2\cos 2x-\frac{28}{3}\cos 2x-\frac{16}{3}x\sin 2x\right]$$

Hence General solution is

$$y = y_c + y_{P_1} + y_{P_2}$$

$$y = [(c_1+c_2x)\cos x + [c_3 + c_4x)\sin x] + \frac{x^2}{2} - 2 + \frac{1}{18}[x^2\cos 2x - \frac{28}{3}\cos 2x - \frac{16}{3}x\sin 2x]$$

① Solve $(D^2-4D+4)y = x^2\sin x + e^{2x} + 3$.

Sol Given $(D^2-4D+4)y = x^2\sin x + e^{2x} + 3$.

The A.E is $f(m)=0$

$$m^2-4m+4=0$$

$$m = 2,2$$

The roots are real & equal

$$y_c = (c_1+c_2x)e^{2x}.$$

Now $P.I = \dfrac{x^2\sin x + e^{2x} + 3}{(D-2)^2} = \dfrac{x^2\sin x}{(D-2)^2} + \dfrac{e^{2x}}{(D-2)^2} + \dfrac{3}{(D-2)^2}$

$$P.I = \frac{1}{(D-2)^2}x^2(I.P \text{ of } e^{ix})$$

$$= I.P \text{ of } \frac{1}{(D-2)^2}x^2e^{ix}$$

$$= I.P \text{ of } e^{ix}\frac{1}{(D+i-2)^2}x^2 \qquad \left\{ P.I = \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V \right\}$$

$$= I.P \text{ of } e^{ix}\left[\frac{x^2}{(-2)^2[1-\frac{D+i}{2}]^2}\right]$$

$$= I.P \text{ of } e^{ix}.\frac{1}{4}.\frac{x^2}{[1-\frac{D+i}{2}]^2}$$

$$= I.P \text{ of } e^{ix}\frac{1}{4}[1-\frac{D+i}{2}]^{-2}x^2$$

$$= I.P \text{ of } e^{ix}(\tfrac{1}{4})[1+2(\tfrac{D+i}{2}) + 3(\tfrac{D+i}{2})^2 + 4(\tfrac{D+i}{2})^3 + \ldots]x^2$$

$$= I.P \text{ of } \frac{e^{ix}}{4}\left[1+D+i+\frac{3}{4}(D^2+i^2+2Di)\right]x^2$$

$$= I.P \text{ of } \frac{e^{ix}}{4}\left[1+D+i+\frac{3}{4}D^2-\frac{3}{4}+\frac{3}{2}Di\right]x^2$$

$$= I.P \text{ of } \frac{e^{ix}}{4}\left[x^2+2x+ix^2+\frac{3}{4_2}(2)-\frac{3}{4}x^2+\frac{3}{2}(2x)i\right]$$

$$= I.P \text{ of } \frac{e^{ix}}{4}\left[\frac{1}{4}x^2+ix^2+2x+3xi+3/2\right]$$

$$= I.P \text{ of } \left(\frac{\cos 2x+i\sin 2x}{4}\right)\left[\frac{1}{4}x^2+ix^2+2x+3xi+3/2\right]$$

$$= \frac{1}{4}\left[x^2\cos 2x+3x\cos 2x+\frac{1}{4}x^2\sin 2x+2x\sin 2x+\frac{3}{2}\sin 2x\right]$$

$$= \frac{1}{8}\left[4x^2\cos 2x+12x\cos 2x+x^2\sin 2x+8x\sin 2x+6\sin 2x\right]$$

$$= \frac{1}{8}\left[x^2[4\cos 2x+\sin 2x]+x[12\cos 2x+8\sin 2x]+6\sin 2x\right]$$

$$y_{P_2} = \frac{e^{2x}}{(D-2)^2} = \frac{x^2}{2!}e^{2x} \quad \left\{ f(D)=0,\ \frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2!} \right\}$$

$$y_{P_3} = \frac{3.e^{0.x}}{(D-2)^2} = 3/4.$$

$$y = y_c + y_p$$

$$y = (C_1+C_2x)e^{2x} + 3/4 + \frac{x^2}{2}e^{2x} + y_{P_1}.$$

① General Solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ by the method of Variation of parameters.

<u>Sol</u> <u>working Rule</u>:–

(1) Reduce the given eqn to the standard form, if necessary.

2. Find the solution of $\frac{d^2y}{dx^2} + P\cdot\frac{dy}{dx} + Qy = 0$ and let the solution of

$$C\cdot F = y_c = c_1 u(x) + c_2 v(x)$$

3. Take $P.I = y_p = Au + B\cdot v$ where A and B are func of x.

4. Find $W(u,v) = u\cdot\frac{dv}{dx} - v\cdot\frac{du}{dx}$

5. Find A and B using.

$$A = -\int\frac{vR\,dx}{W(u,v)} = -\int\frac{vR}{u\cdot\frac{dv}{dx} - v\frac{du}{dx}}dx.$$

$$B = \int\frac{uR}{W(u,v)}dx = \int\frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$$

6. Write the general solution of the given eqn as.

$$y = y_c + y_p$$

$$y = c_1 u(x) + c_2 v(x) + A(x)u(x) + B(x)v(x)$$

where c_1 & c_2 are constant

① Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$$

<u>Sol</u> Given eqn in the operator form is $(D^2+1)y = \operatorname{cosec} x$.

Here $P = 0$, $Q = 1$ and $R = \operatorname{cosec} x$.

A.E is $f(m) = 0$, $m^2+1 = 0 \Rightarrow m = \pm i$

C.F is $\boxed{y_c = C_1 \cos x + C_2 \sin x.}$

Let $y_p = A\cos x + B\sin x = A\,u(x) + B\,v(x)$

Here $u = \cos x$, $v = \sin x$.

$$\frac{du}{dx} = -\sin x, \quad \frac{dv}{dx} = \cos x.$$

$$W(u,v) = u\frac{dv}{dx} - v\frac{du}{dx} = \cos x(\cos x) - \sin x(-\sin x)$$

$$W(u,v) = \cos^2 x + \sin^2 x = 1.$$

A and B are given by.

$$A = -\int \frac{v.R}{W(u,v)}\,dx = -\int \frac{\sin x \cdot \operatorname{cosec} x}{1}\,dx$$

$$= -\int \sin x \cdot \frac{1}{\sin x}\,dx$$

$$\boxed{A = -x.}$$

$$B = \int \frac{u.R}{W(u,v)}\,dx = \int \frac{\cos x \cdot \operatorname{cosec} x}{1}\,dx$$

$$= \int \cot x\,dx$$

$$\boxed{B = \log|\sin x|}.$$

$$y_p = A\,u(x) + B\,v(x)$$

$$\boxed{y_p = -x\cos x + \log|\sin x|\sin x.}$$

Hence G.S is $y = y_c + y_p$

② Solve $(D^2-2D+2)y = e^x \tan x$ by the method of variation of parameters.

Sol $(D^2-2D+2)y = e^x \tan x$.

A.E is $f(m)=0$

$m^2-2m+2=0$

$m = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm\sqrt{4i^2}}{2} = \frac{2\pm 2i}{2} = 1\pm i$

We have $y_c = e^x(c_1\cos x + c_2\sin x)$

$y_c = c_1 e^x\cos x + c_2 e^x \sin x$.

$= c_1 u(x) + c_2 v(x)$ where

$u(x) = e^x\cos x,\ v(x) = e^x \sin x$.

$\frac{du}{dx} = e^x(-\sin x) + \cos x\, e^x,\ \frac{dv}{dx} = e^x\cos x + e^x\sin x$.

Now $W(u,v) = u\frac{dv}{dx} - v\frac{du}{dx}$.

$W(u,v) = [e^x\cos x(e^x\cos x + e^x\sin x) - \overset{e^x\sin x}{\cancel{x}}(-e^x\sin x + e^x\cos x)]$

$= e^{2x}\cos^2 x + \cancel{e^{2x}\sin x\cos x} + e^{2x}\sin^2 x - \cancel{e^{2x}\sin x\cos x}$

$= e^{2x}(\cos^2 x + \sin^2 x)$

$W(u,v) = e^{2x}$

using variation of parameters.

$A = -\int \frac{vR}{W(u,v)}dx = -\int \frac{\cancel{e^x}\sin x}{\cancel{e^{2x}}} \cdot \cancel{e^x}\tan x\, dx$

$= -\int \tan x \sin x\, dx$

$= -\int \frac{\sin^2 x}{\cos x}dx$

$= -\int \frac{1-\cos^2 x}{(\cos x)} = -\int\left(\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}\right)dx$

$= -\int(\sec x - \cos x)dx$

$= -\log|\sec x + \tan x) + \sin x$

$B = \int \frac{UR}{W(UN)} dx$

$= \int \frac{e^{x}\cos x \cdot e^{x}\tan x}{e^{2x}} dx$

$B = \int \cos x \cdot \frac{\sin x}{\cos x} dx \quad = -\cos x.$

The general solution is given by

$y = y_c + AU + BV$

$y = e^{x}(c_1\cos x + c_2\sin x) + (\log|\sec x + \tan x| + \sin x) e^{x}\cos x$
$+ (-\cos x) e^{x}\sin x.$

$y = c_1 e^{x}\cos x + c_2 e^{x}\sin x - \cos x\sin x e^{x} + [\sin x + \log|\sec x + \tan x|] e^{x}\cos x$

= * =

① Solve by the method of variation of parameters

$y'' + y = \sec x.$

② $(D^2 - 2D)y = e^{x}\sin x.$

③ $(D^2 + 1)y = \cos x.$

Differential Equation of first order and first degree

Electrical circuits

LR circuit :-

L - Inductance.

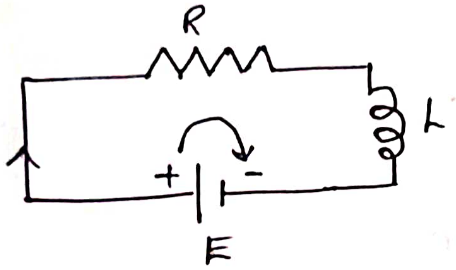
E - Electromotive force.

R - Resistance

L → $\overset{L}{+\text{mmm}-}$ Henry.

E → $\overset{E}{+|\vdash-}$ Volt.

R → $\overset{R}{+\text{MM}-}$ ohm



$i = \frac{dq}{dt}$ (or) $q = \int i dt$ ohm's Law. where 'i' is the current and 'q' is quantity of electricity.

Voltage drop a cross resistance $R = R \cdot i$

Voltage drop a cross Inductance $L = L \cdot \frac{di}{dt}$

Voltage drop a cross capacitance $C = q/c$.

Kirchoff's Law

The algebraic sum of voltage drop resistance and voltage drop in inductance is equal to the electro motic force in the circuit

$$\boxed{R \cdot i + L \cdot \frac{di}{dt} = E}$$

dividing with L on both sides

$$\frac{R}{L} i + \frac{\cancel{L}}{\cancel{L}} \frac{di}{dt} = \frac{E}{L}$$

$$\frac{di}{dt} + \frac{R}{L} \cdot i = \frac{E}{L}$$

This is of the form Linear differential eqn

Now $P = \frac{R}{L}$, $Q = \frac{E}{L}$

$$I.F = e^{\int P dx} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} \cdot t}$$

Solution is

$$i \times I.F = \int Q \times (I.F) dt + c.$$

$$i \cdot e^{\frac{R}{L} \cdot t} = \int \frac{E}{L} \cdot e^{\frac{Rt}{L}} dt + c.$$

$$i \cdot e^{Rt/L} = \frac{E}{L} \int e^{Rt/L} dt + c$$

$$= \frac{E}{\cancel{L}} \cdot \frac{e^{Rt/L}}{R/\cancel{L}} + c$$

$$i \cdot e^{Rt/L} = \frac{E}{R} \cdot e^{Rt/L} + c \longrightarrow ①$$

Initial $t = 0 \rightarrow i = 0$

$$0 \cdot e^0 = \frac{E}{R} \cdot e^0 + c$$

$$0 = \frac{E}{R} + c$$

$$\boxed{c = -\frac{E}{R}}$$

eq ① becomes

$$i \cdot e^{\frac{Rt}{L}} = \frac{E}{R} \cdot e^{\frac{Rt}{L}} - \frac{E}{R}$$

$$i \cdot e^{\frac{Rt}{L}} = \frac{E}{R} [e^{\frac{Rt}{L}} - 1]$$

$$i = \frac{E}{R} \left[\frac{e^{Rt/L} - 1}{e^{Rt/L}} \right]$$

$$\boxed{i = \frac{E}{R} [1 - e^{-Rt/L}]}$$

① An R–L circuit has an emf given (in volts) by 10 sint, a resistance of 90 ohms, an inductance of 4 henries find the current at any time t by assuming zero initial current.

Sol Given $E = 10\sin t$

$R = 90$ ohms

$L = 4$ henries.

By Kirchoff's Law, we get.

$$R \cdot i + L \cdot \frac{di}{dt} = E$$

$$\frac{R}{L} i + \frac{di}{dt} = \frac{E}{L}$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

$$\frac{di}{dt} + \frac{90}{4} i = \frac{10\sin t}{4}$$

$$\frac{di}{dt} + \frac{45}{2} i = \frac{5\sin t}{2} \longrightarrow ①$$

eq ① is a linear diff eqn in 'i'

$$\frac{di}{dt} + P(t)\, i = Q(t)$$

where $I.F = e^{\int P(t)dt} = e^{\int \frac{45}{2} dt} = e^{\frac{45t}{2}}$

General solution of eq ① is given by

$$i.(I.F) = \int Q(t)(I.F)\,dt + c.$$

$$\text{i.e } i.e^{45t/2} = \int \frac{5\sin t}{2} \cdot e^{\frac{45}{2}t} dt + c.$$

$$i.e^{\frac{45t}{2}} = \frac{5}{2}\int e^{\frac{45}{2}t} \sin t\, dt + c.$$

we have $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$

$$i \cdot e^{\frac{45}{2}t} = \frac{5}{2} \cdot \frac{e^{45/2\, t}}{\left(\frac{45}{2}\right)^2 + (1)^2} \left[\frac{45}{2} \sin t - \cos t \right] + c.$$

$$= \frac{5}{2} \cdot \frac{\cancel{4}^{2} \cdot e^{\frac{45}{2}t}}{2025 + 4} \left[\frac{45}{2} \sin t - \cos t \right] + c.$$

$$i \cdot e^{\frac{45}{2}t} = 10 \cdot \frac{e^{45/2\, t}}{2029} \left[\frac{45}{2} \sin t - \cos t \right] + c. \longrightarrow ②$$

$$i = \frac{10 \cdot \cancel{e^{45/2\, t}}}{2029 \cdot \cancel{e^{45/2\, t}}} \left[\frac{45}{2} \sin t - \cos t \right] + c \cdot e^{-\frac{45}{2}t} \longrightarrow ③$$

With initial $i = 0$, $t = 0$. in eq ②

$$0 = \frac{10}{2029} \left[\frac{45}{2} \sin 0 - \cos 0 \right] + c.$$

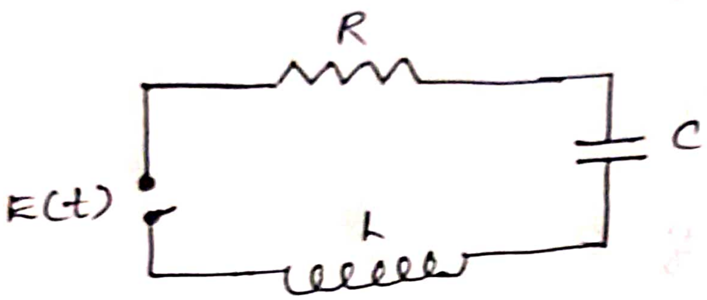
$$\boxed{c = \frac{10}{2029}}$$

eq ③ becomes.

$$\boxed{i = \frac{10}{2029} \left[\frac{45}{2} \sin t - \cos t \right] + \frac{10}{2029} \cdot e^{-\frac{45}{2}t}}$$

Differential Equation of second order Electrical circuit

L-C-R circuit



consider the discharge of a condenser 'C' through an induction L and the resistance R.

The voltage drop a cross L.

$$V_L = L.\frac{di}{dt} \quad \left\{ i = \frac{dq}{dt} \right\}$$

$$V_L = L.\frac{d}{dt}\left[\frac{dq}{dt}\right]$$

$$\boxed{V_L = L.\frac{d^2q}{dt^2}}$$

The voltage drop across R.

$$V_R = i.R$$

$$\boxed{V_R = R.\frac{dq}{dt}}$$

The voltage drop across C.

$$V_C = q/c.$$

Kirchoff's Law :- The algebraic sum of all voltage drops around a closed circuit is Zero.

$$V_L + V_R + V_C = E$$

$$V_L + V_R + V_C = 0$$

$$L.\frac{d^2q}{dt^2} + R.\frac{dq}{dt} + \frac{q}{c} = 0.$$

Dividing with L

$$\frac{K}{K} \cdot \frac{d^2q}{dt^2} + \frac{R}{L} \cdot \frac{dq}{dt} + \frac{q}{CL} = 0$$

$$\frac{d^2q}{dt^2} + \frac{R}{L} \cdot \frac{dq}{dt} + \frac{q}{CL} = 0.$$

Let $\frac{R}{L} = 2d$, $\frac{1}{LC} = \mu^2$

$$\frac{d^2q}{dt^2} + 2d \frac{dq}{dt} + \mu^2 q = 0.$$

$$D^2q + 2d\,Dq + \mu^2 q = 0$$

$$(D^2 + 2d\,D + \mu^2)\,q = 0 \longrightarrow (1)$$

The A.E is $f(m) = 0$

$$m^2 + 2dm + \mu^2 = 0$$

$$a = 1,\ b = 2d,\ c = \mu^2$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-2d \pm \sqrt{4d^2 - 4\mu^2}}{2} = \frac{-2d \pm 2\sqrt{d^2 - \mu^2}}{2}$$

$$\boxed{m = -d \pm \sqrt{d^2 - \mu^2}}$$

Case I $d > \mu$

Roots are real & different

say m_1, m_2

$$q = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

Case-II $d = \mu$

Roots are real & equal

say $m = -d, -d$

$$q = (c_1 + c_2 x)\, e^{-dt}$$

Case-III $d < \mu$

Roots are complex and conjugate $-d \pm i\alpha$

$$q = e^{-dt}[c_1 \cos \alpha t + c_2 \sin \alpha t]$$

① A condensor of capacity 'c' discharged through an inductance L and Resistance R in series and the charge q at time t. Satisfies the eqn $L \cdot \frac{d^2q}{dt^2} + R \cdot \frac{dq}{dt} + \frac{q}{C} = 0$

Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2\times10^{-6}$ farads and that when $t=0$ charge q is 0.002 coulombs. and the current $\frac{dq}{dt} = 0$ obtain the value of q in terms of t.

Sol Given D.E is

$$L \cdot \frac{d^2q}{dt^2} + R \cdot \frac{dq}{dt} + \frac{q}{C} = 0.$$

$$\frac{d^2q}{dt^2} + \frac{R}{L} \cdot \frac{dq}{dt} + \frac{q}{CL} = 0 \longrightarrow ①$$

Given Inductance $L = 0.25$ henries

Resistance $R = 250\,\Omega$

capacity of condenser $c = 2\times10^{-6}$

when $t=0$, charge $q = 0.002$ & current $\frac{dq}{dt} = 0$.

Substituting the above values in eq①, we get

$$\frac{d^2q}{dt^2} + \frac{250}{0.25}\frac{dq}{dt} + \frac{q}{0.25\times2\times10^{-6}} = 0$$

$$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + 2\times10^6 q = 0.$$

$$[D^2 + 1000D + 2\times10^6]q = 0.$$

The A.E is $f(m) = 0$

$$m^2 + 10^3 m + 2\times10^6 = 0$$

$$m = \frac{-10^3 \pm \sqrt{10^6 - 8\times10^6}}{2}$$

$$m = \frac{\overset{500}{10^3}[-1\pm\sqrt{-7}]}{\cancel{2}}$$

$$m = 500[-1\pm\sqrt{-7}]$$

$$m = -500 \pm 500\sqrt{7}\, i$$

$$m = -500 \pm 1323\, i$$

The roots are complex conjugate roots.

The solution is

$$q = e^{-500t}\left[C_1 \cos 1323t + C_2 \sin 1323t \right] \longrightarrow (2)$$

when $t = 0$, $q = 0.002$

$$0.002 = e^{-0.500}\left[C_1 \cos 0 + C_2 \sin 0 \right]$$

$$\boxed{C_1 = 0.002}$$

diff w.r.to t on both sides of (2)

$$\frac{dq}{dt} = -500 e^{-500t}\left[C_1 \cos 1323t + C_2 \sin 1323t \right]$$
$$+ e^{-500t}\left[-C_1 \sin 1323t \cdot (1323) + C_2 \cos(1323t)(1323) \right]$$

when $t = 0$, $\frac{dq}{dt} = 0$.

$$0 = -500\left[C_1 \cos 1323(0) + C_2 \sin 1323(0) \right] + e^{-0500}\left[-C_1 (1323) \sin 1323(0) + 1323\, C_2 \cos(132t)(0) \right]$$

$$0 = -500\left[C_1 + 0 \right] + \left[0 + 1323 C_2 \right]$$

$$1323 C_2 = 500 C_1$$

$$C_2 = \frac{500 \times 0.002}{1323}$$

$$\boxed{C_2 = 0.0008}$$

The general solution is

$$q = e^{-500t}\left[0.002 \cos 1323t + 0.0008 \sin 1323t \right]$$