Divergence of a Binomial Series

Abstract

We show that the series

$$\sum_{n\geq 1} \sqrt[n]{2} - 1$$

diverges. The solution was inspired by perturbation theory.

Proof. The generalized binomial theorem states that

$$(a+b)^x = \sum_{k\geq 0} \binom{x}{k} a^n b^{x-k}$$

where x is any real number, and

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

denotes the generalized binomial coefficient.

Note that $1+1=2$, and so $(1+1)^{1/n} - 1 = \sqrt[n]{2} - 1$.

If $(x, a, b) = (1/n, 1, 1)$, then

$$\binom{\frac{1}{n}}{k} = \frac{1}{k!}\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\cdots\left(\frac{1}{n}-k+1\right)$$

and

$$\sum_{n\geq 1} \sqrt[n]{2} - 1 = \sum_{n\geq 1}\sum_{k\geq 1} \binom{\frac{1}{n}}{k}$$

Note that $\sum_{k\geq 1} \binom{\frac{1}{n}}{k}$ is a convergent alternating series, so we may bound it below using the first two terms.

For example,

$$\sum_{k\geq 1} \binom{\frac{1}{2}}{k} = \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256} - + \cdots \geq \frac{1}{2} - \frac{1}{8}.$$

Thus

$$\sum_{n\geq 1}\sum_{k\geq 1} \binom{\frac{1}{n}}{k} \geq \sum_{n\geq 1}\left(\frac{1}{n} + \frac{1}{2n}\left(\frac{1}{n}-1\right)\right) = \sum_{n\geq 1}\left(\frac{1}{2n} + \frac{1}{2n^2}\right),$$

which diverges by comparision with the harmonic series. $\square$