Quantum Gates Proof

1. **Hadamard Gate (H)**
   Symbol: H

The Hadamard Gate (H)

The Hadamard gate (denoted as H) is one of the most important single-qubit gates in quantum computing. It creates a superposition state from a computational basis state and is essential for many quantum algorithms. The Hadamard gate is represented by the following matrix:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

When applied to the computational basis states $|0\rangle$ and $|1\rangle$, it creates superpositions. Let's go through the transformations and properties of the Hadamard gate.

1. Hadamard Gate Acting on |0⟩

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the Hadamard gate to $|0\rangle$, we compute:

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

Thus, the Hadamard gate transforms $|0\rangle$ into an equal superposition of $|0\rangle$ and $|1\rangle$:

$$H|0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

2. Hadamard Gate Acting on |1⟩

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the Hadamard gate to $|1\rangle$, we compute:

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

Thus, the Hadamard gate transforms $|1\rangle$ into:

$$H|1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

3. Properties of the Hadamard Gate

The Hadamard gate has several important properties:

a. Self-Inverse Property: $H \cdot H = I$

The Hadamard gate is its own inverse, meaning that if you apply it twice, you return to the original state. To see this, we calculate $H \cdot H$:

$$H \cdot H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus, $H \cdot H = I$, where I is the identity matrix. This means applying the Hadamard gate twice restores the original state of the qubit.

b. Effect on the Computational Basis States

The Hadamard gate maps the computational basis states to the equal superposition states $|+\rangle$ and $|-\rangle$:

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

Conversely, applying the Hadamard gate to $|+\rangle$ and $|-\rangle$ maps them back to the computational basis states:

$$H|+\rangle = |0\rangle$$

$$H\,|-\rangle = |1\rangle$$

c. Eigenvalues and Eigenvectors of the Hadamard Gate

The Hadamard gate has eigenvalues ± 1 with corresponding eigenvectors $|+\rangle$ and $|-\rangle$:

$$H\,|+\rangle = |+\rangle \quad \text{with eigenvalue } +1$$
$$H\,|-\rangle = -\,|-\rangle \quad \text{with eigenvalue } -1$$

Summary

The Hadamard gate creates superposition states from computational basis states and is self-inverse, meaning that applying it twice returns the qubit to its original state. This property, along with its ability to generate superpositions, makes the Hadamard gate a critical component in many quantum algorithms.

2. **Pauli-X Gate (NOT Gate)**
   Symbol: X

The Pauli-X Gate

The Pauli-X gate, also known as the quantum NOT gate, is a fundamental single-qubit gate in quantum computing. It flips the state of a qubit from $|0\rangle$ to $|1\rangle$ and from $|1\rangle$ to $|0\rangle$. This gate is represented by the following matrix:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To prove the properties of the Pauli-X gate, we will apply it to the computational basis states $|0\rangle$ and $|1\rangle$ and show how it transforms these states.

1. Pauli-X Gate Acting on $|0\rangle$

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the Pauli-X gate to $|0\rangle$, we compute:

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

This shows that the Pauli-X gate flips the $|0\rangle$ state to the $|1\rangle$ state.

2. Pauli-X Gate Acting on $|1\rangle$

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the Pauli-X gate to $|1\rangle$, we compute:

$$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

This shows that the Pauli-X gate flips the $|1\rangle$ state to the $|0\rangle$ state.

3. Pauli-X Gate Properties

The Pauli-X gate has some key properties that we can prove:

a. Self-Inverse Property: $X \cdot X = I$

The Pauli-X gate is its own inverse, meaning that if you apply it twice, you return to the original state. Mathematically:

$$X \cdot X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus, $X \cdot X = I$, where I is the identity matrix. This means applying the Pauli-X gate twice restores the original state of the qubit.

b. Eigenvalues and Eigenvectors of Pauli-X

The Pauli-X gate has eigenvalues ± 1 and corresponding eigenvectors:

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

These states $|+\rangle$ and $|-\rangle$ are the eigenstates of the Pauli-X gate:

$$X|+\rangle = |+\rangle$$
$$X|-\rangle = -|-\rangle$$

These properties make the Pauli-X gate one of the essential building blocks in quantum computing, acting as a quantum analog of the classical NOT gate.

3. **Pauli-Y Gate**
   Symbol: Y

The Pauli-Y Gate

The Pauli-Y gate is one of the fundamental single-qubit gates in quantum computing. It combines the properties of both the Pauli-X and Pauli-Z gates, applying both a bit-flip (like the Pauli-X) and a phase-flip (like the Pauli-Z). The Pauli-Y gate is represented by the following matrix:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The Pauli-Y gate acts on the computational basis states $|0\rangle$ and $|1\rangle$ as follows:

1. Pauli-Y Gate Acting on $|0\rangle$

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the Pauli-Y gate to $|0\rangle$, we compute:

$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

This shows that the Pauli-Y gate transforms the $|0\rangle$ state into $i|1\rangle$, introducing a phase of i (imaginary unit).

2. Pauli-Y Gate Acting on $|1\rangle$

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the Pauli-Y gate to $|1\rangle$, we compute:

$$Y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$$

This shows that the Pauli-Y gate transforms the $|1\rangle$ state into $-i|0\rangle$, introducing a phase of $-i$.

3. Pauli-Y Gate Properties

The Pauli-Y gate has several key properties that we can verify:

a. Self-Inverse Property: $Y \cdot Y = I$

The Pauli-Y gate is its own inverse, meaning that if you apply it twice, you return to the original state. Mathematically:

$$Y \cdot Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus, $Y \cdot Y = I$, where I is the identity matrix. This means applying the Pauli-Y gate twice restores the original state of the qubit.

b. Eigenvalues and Eigenvectors of Pauli-Y

The Pauli-Y gate has eigenvalues $\pm i$ and corresponding eigenvectors:

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

These states $|+\rangle$ and $|-\rangle$ are the eigenstates of the Pauli-Y gate:

$$Y|+\rangle = i|+\rangle \quad \text{with eigenvalue } i$$

$$Y|-\rangle = -i|-\rangle \quad \text{with eigenvalue } -i$$

These properties make the Pauli-Y gate an essential component in quantum computing, as it combines both the bit-flip and phase-flip operations, often used in algorithms that require these transformations simultaneously.

4. **Pauli-Z Gate**
   Symbol: Z

The Pauli-Z Gate

The Pauli-Z gate is one of the fundamental single-qubit gates in quantum computing. It acts as a phase-flip gate, flipping the sign of the $|1\rangle$ state while leaving the $|0\rangle$ state unchanged. The Pauli-Z gate is represented by the following matrix:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To prove the properties of the Pauli-Z gate, we will apply it to the computational basis states $|0\rangle$ and $|1\rangle$ and show how it transforms these states.

1. Pauli-Z Gate Acting on $|0\rangle$

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the Pauli-Z gate to $|0\rangle$, we compute:

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

This shows that the Pauli-Z gate leaves the $|0\rangle$ state unchanged.

2. Pauli-Z Gate Acting on $|1\rangle$

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the Pauli-Z gate to $|1\rangle$, we compute:

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

This shows that the Pauli-Z gate flips the phase of the $|1\rangle$ state, changing it to $-|1\rangle$.

3. Pauli-Z Gate Properties

The Pauli-Z gate has some key properties that we can verify:

a. Self-Inverse Property: $Z \cdot Z = I$

The Pauli-Z gate is its own inverse, meaning that if you apply it twice, you return to the original state. Mathematically:

$$Z \cdot Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus, $Z \cdot Z = I$, where I is the identity matrix. This means applying the Pauli-Z gate twice restores the original state of the qubit.

b. Eigenvalues and Eigenvectors of Pauli-Z

The Pauli-Z gate has eigenvalues ± 1 and corresponding eigenvectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These states $|0\rangle$ and $|1\rangle$ are the eigenstates of the Pauli-Z gate:

$$Z|0\rangle = |0\rangle \quad \text{with eigenvalue } +1$$
$$Z|1\rangle = -|1\rangle \quad \text{with eigenvalue } -1$$

These properties make the Pauli-Z gate an essential building block in quantum computing, used to apply a phase flip and distinguish between the $|0\rangle$ and $|1\rangle$ states in algorithms requiring phase changes.

5. **CNOT Gate (Controlled-NOT Gate)**
   Symbol: CX or CNOT

The CNOT Gate (Controlled-NOT Gate)

The CNOT gate (Controlled-NOT gate) is a two-qubit gate in quantum computing. It flips the state of the target qubit if and only if the control qubit is in the $|1\rangle$ state. Otherwise, it leaves the target qubit unchanged. The CNOT gate is essential for creating entanglement between qubits and is commonly used in quantum algorithms.
The matrix representation of the CNOT gate is as follows:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This matrix acts on two-qubit states $|q_1 q_2\rangle$, where q_1 is the control qubit and q_2 is the target qubit. Let's examine how the CNOT gate transforms each of the computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

1. CNOT Gate Acting on $|00\rangle$

The computational basis state $|00\rangle$ is represented as:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the CNOT gate to $|00\rangle$, we compute:

$$\text{CNOT } |00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

So, the CNOT gate leaves $|00\rangle$ unchanged.

2. CNOT Gate Acting on $|01\rangle$

The computational basis state $|01\rangle$ is represented as:

$$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the CNOT gate to $|01\rangle$, we compute:

$$\text{CNOT } |01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

So, the CNOT gate leaves $|01\rangle$ unchanged as well.

3. CNOT Gate Acting on $|10\rangle$

The computational basis state $|10\rangle$ is represented as:

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

When we apply the CNOT gate to $|10\rangle$, we compute:

$$\text{CNOT } |10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$$

This shows that the CNOT gate flips the target qubit, changing $|10\rangle$ to $|11\rangle$.

4. CNOT Gate Acting on $|11\rangle$

The computational basis state $|11\rangle$ is represented as:

$$|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

When we apply the CNOT gate to $|11\rangle$, we compute:

$$\text{CNOT } |11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$$

This shows that the CNOT gate flips the target qubit, changing $|11\rangle$ to $|10\rangle$.

Summary of CNOT Gate Action

The CNOT gate acts as follows on each basis state:

$$\text{CNOT } |00\rangle = |00\rangle$$
$$\text{CNOT } |01\rangle = |01\rangle$$
$$\text{CNOT } |10\rangle = |11\rangle$$
$$\text{CNOT } |11\rangle = |10\rangle$$

This behavior matches the description of the CNOT gate: it flips the target qubit if the control qubit is $|1\rangle$ and leaves the target qubit unchanged if the control qubit is $|0\rangle$.

Properties of the CNOT Gate

- **Unitary Property:** The CNOT matrix is unitary, meaning that $\text{CNOT}^\dagger \cdot \text{CNOT} = I$, where $\text{CNOT}^\dagger$ is the conjugate transpose of the CNOT matrix and I is the identity matrix. This ensures that the CNOT gate is reversible, as required for all quantum gates.
- **Self-Inverse Property:** The CNOT gate is its own inverse, meaning that if you apply it twice, you return to the original state. Mathematically:

$$\text{CNOT} \cdot \text{CNOT} = I$$

- **Entanglement Creation:** The CNOT gate is essential for creating entanglement. When applied to a superposition state, such as after a Hadamard gate, it can produce an entangled state. For example, applying a Hadamard gate to the first qubit and then applying a CNOT gate to both qubits produces the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, which is a maximally entangled state.

Conclusion

The CNOT gate flips the target qubit if the control qubit is in the $|1\rangle$ state, and leaves it unchanged if the control qubit is in the $|0\rangle$ state. It is a unitary, self-inverse gate, and is critical in quantum computing for creating entanglement, which is essential for many quantum algorithms.

6. **T Gate (**$\pi/4$ **Phase Gate)**
   Symbol: T

The T Gate

The T gate is a single-qubit gate in quantum computing that applies a phase shift of $\frac{\pi}{4}$ (or 45 degrees) to the state $|1\rangle$ while leaving $|0\rangle$ unchanged. The T gate is also known as the $\frac{\pi}{8}$ gate because the phase shift is $\frac{\pi}{4}$. It is a crucial gate in quantum computation as it, along with the Hadamard and CNOT gates, forms a set of gates sufficient for universal quantum computation when combined with the Clifford gates.
The T gate is represented by the following matrix:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \end{pmatrix}$$

To understand and prove the properties of the T gate, we will analyze its effect on the computational basis states $|0\rangle$ and $|1\rangle$, and explore its key properties.

1. T Gate Acting on $|0\rangle$

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the T gate to $|0\rangle$, we compute:

$$T|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

This shows that the T gate leaves the $|0\rangle$ state unchanged.

2. T Gate Acting on $|1\rangle$

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the T gate to $|1\rangle$, we compute:

$$T|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\pi/4} \end{pmatrix} = e^{i\pi/4}|1\rangle$$

This shows that the T gate applies a phase shift of $e^{i\pi/4}$ to the $|1\rangle$ state.

3. Properties of the T Gate

The T gate has several important properties that we can verify:

- **Phase Shift by $\frac{\pi}{4}$:** The T gate applies a phase shift of $\frac{\pi}{4}$ to the $|1\rangle$ state, while leaving the $|0\rangle$ state unchanged. This is confirmed by the calculations above, where:
$$T|0\rangle = |0\rangle \quad \text{and} \quad T|1\rangle = e^{i\pi/4}|1\rangle$$
- **Self-Inverse Property of the T Gate's Square:** $T \cdot T = S$
When the T gate is applied twice, it creates a phase shift of $2 \times \frac{\pi}{4} = \frac{\pi}{2}$, which is equivalent to the S gate:
$$T \cdot T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = S$$
where
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$
Thus, applying the T gate twice is equivalent to the S gate.
- **Relationship with the Z Gate:** $T \cdot T \cdot T \cdot T = Z$
When the T gate is applied four times, it produces a phase shift of $4 \times \frac{\pi}{4} = \pi$, which is equivalent to the Z gate:
$$T \cdot T \cdot T \cdot T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$
where
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
Thus, applying the T gate four times is equivalent to the Z gate.

- **Eigenvalues and Eigenvectors of the T Gate:**
  The T gate has eigenvalues 1 and $e^{i\pi/4}$. The corresponding eigenvectors are:
  – $|0\rangle$ with eigenvalue 1,
  – $|1\rangle$ with eigenvalue $e^{i\pi/4}$.

Summary

The T gate applies a phase shift of $\frac{\pi}{4}$ to the $|1\rangle$ state while leaving the $|0\rangle$ state unchanged. When applied twice, the T gate is equivalent to the S gate, and when applied four times, it is equivalent to the Z gate. The T gate, together with other single-qubit gates, plays a crucial role in achieving universal quantum computation due to its ability to introduce a precise phase shift.

7. **S Gate ($\pi/2$ Phase Gate)**
   Symbol: S

The S Gate

The S gate is a single-qubit gate in quantum computing that applies a phase shift of $\frac{\pi}{2}$ (or 90 degrees) to the state $|1\rangle$ while leaving the state $|0\rangle$ unchanged. The S gate is also known as the Phase gate or the $\frac{\pi}{2}$-gate.
The S gate is represented by the following matrix:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

To understand and prove the properties of the S gate, we will analyze its effect on the computational basis states $|0\rangle$ and $|1\rangle$, and explore its key properties.

1. S Gate Acting on $|0\rangle$

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the S gate to $|0\rangle$, we compute:

$$S|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

This shows that the S gate leaves the $|0\rangle$ state unchanged.

2. S Gate Acting on |1⟩

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the S gate to $|1\rangle$, we compute:

$$S|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

This shows that the S gate applies a phase shift of i to the $|1\rangle$ state.

3. Properties of the S Gate

The S gate has several important properties that we can verify:

- **Phase Shift by $\frac{\pi}{2}$:** The S gate applies a phase shift of $\frac{\pi}{2}$ (90 degrees) to the $|1\rangle$ state while leaving the $|0\rangle$ state unchanged. This is confirmed by the calculations above, where:

$$S|0\rangle = |0\rangle \quad \text{and} \quad S|1\rangle = i|1\rangle$$

- **Self-Inverse Property of the S Gate's Square:** $S \cdot S = Z$
  When the S gate is applied twice, it produces a phase shift of $2 \times \frac{\pi}{2} = \pi$, which is equivalent to the Z gate:

$$S \cdot S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

  where

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  Thus, applying the S gate twice is equivalent to the Z gate.
- **Relationship with the T Gate:** $T \cdot T = S$
  The T gate, when applied twice, is equivalent to the S gate:

$$T \cdot T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = S$$

- **Eigenvalues and Eigenvectors of the S Gate:**
  The S gate has eigenvalues 1 and i with corresponding eigenvectors:
  - $|0\rangle$ with eigenvalue 1,
  - $|1\rangle$ with eigenvalue i.

Summary

The S gate applies a phase shift of $\frac{\pi}{2}$ (90 degrees) to the $|1\rangle$ state while leaving the $|0\rangle$ state unchanged. When applied twice, the S gate is equivalent to the Z gate. Additionally, applying the T gate twice is equivalent to the S gate. The S gate is a crucial component in quantum computing, particularly for algorithms that require phase adjustments.

8. **Toffoli Gate (CCNOT or Controlled-Controlled-NOT Gate)**
   Symbol: CCX

The CCNOT Gate (Toffoli Gate)

The CCNOT gate (also known as the Toffoli gate) is a three-qubit gate in quantum computing. It performs a NOT operation on the target qubit if and only if both control qubits are in the $|1\rangle$ state. If either of the control qubits is in the $|0\rangle$ state, the target qubit remains unchanged. The CCNOT gate is commonly used in quantum circuits for reversible computation and arithmetic operations.
The matrix representation of the CCNOT gate is as follows:

$$\text{CCNOT} = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{pmatrix}$$

This matrix acts on three-qubit states $|q_1 q_2 q_3\rangle$, where q_1 and q_2 are the control qubits and q_3 is the target qubit. Let's examine how the CCNOT gate transforms each of the eight computational basis states $|000\rangle$, $|001\rangle$, $|010\rangle$, $|011\rangle$, $|100\rangle$, $|101\rangle$, $|110\rangle$, and $|111\rangle$.

1. CCNOT Gate Acting on $|000\rangle$

The computational basis state $|000\rangle$ is represented as:

$$|000\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the CCNOT gate to $|000\rangle$, we get:

$$\text{CCNOT } |000\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |000\rangle$$

So, the CCNOT gate leaves $|000\rangle$ unchanged.

2. CCNOT Gate Acting on $|001\rangle$

The computational basis state $|001\rangle$ is represented as:

$$|001\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the CCNOT gate to $|001\rangle$, we get:

$$\text{CCNOT } |001\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |001\rangle$$

So, the CCNOT gate leaves $|001\rangle$ unchanged.

3. CCNOT Gate Acting on $|010\rangle$

The computational basis state $|010\rangle$ is represented as:

$$|010\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the CCNOT gate to $|010\rangle$, we get:

$$\text{CCNOT } |010\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |010\rangle$$

So, the CCNOT gate leaves $|010\rangle$ unchanged.

4. CCNOT Gate Acting on $|011\rangle$

The computational basis state $|011\rangle$ is represented as:

$$|011\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the CCNOT gate to $|011\rangle$, we get:

$$\text{CCNOT } |011\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |011\rangle$$

So, the CCNOT gate leaves $|011\rangle$ unchanged.

5. CCNOT Gate Acting on $|100\rangle$

The computational basis state $|100\rangle$ is represented as:

$$|100\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the CCNOT gate to $|100\rangle$, we get:

$$\text{CCNOT } |100\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |100\rangle$$

So, the CCNOT gate leaves $|100\rangle$ unchanged.

6. CCNOT Gate Acting on $|101\rangle$

The computational basis state $|101\rangle$ is represented as:

$$|101\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the CCNOT gate to $|101\rangle$, we get:

$$\text{CCNOT } |101\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |101\rangle$$

So, the CCNOT gate leaves $|101\rangle$ unchanged.

7. CCNOT Gate Acting on $|110\rangle$

The computational basis state $|110\rangle$ is represented as:

$$|110\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

When we apply the CCNOT gate to $|110\rangle$, we get:

$$\text{CCNOT } |110\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |111\rangle$$

This shows that the CCNOT gate flips the target qubit, changing $|110\rangle$ to $|111\rangle$.

8. CCNOT Gate Acting on $|111\rangle$

The computational basis state $|111\rangle$ is represented as:

$$|111\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

When we apply the CCNOT gate to $|111\rangle$, we get:

$$\text{CCNOT } |111\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |110\rangle$$

This shows that the CCNOT gate flips the target qubit, changing $|111\rangle$ to $|110\rangle$.

Summary of CCNOT Gate Action

The CCNOT gate acts as follows on each basis state:

$$\text{CCNOT } |000\rangle = |000\rangle$$

$$\text{CCNOT } |001\rangle = |001\rangle$$

$$\text{CCNOT } |010\rangle = |010\rangle$$

$$\text{CCNOT } |011\rangle = |011\rangle$$

$$\text{CCNOT } |100\rangle = |100\rangle$$

CCNOT

9. **SWAP Gate**
   Symbol: SWAP

The SWAP Gate

The SWAP gate is a two-qubit gate in quantum computing that swaps the states of two qubits. This gate is useful for rearranging qubits within a circuit. The SWAP gate is symmetric and simply exchanges the state of the first qubit with that of the second qubit.
The matrix representation of the SWAP gate is as follows:

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This matrix acts on two-qubit states $|q_1 q_2\rangle$, where q_1 and q_2 represent the states of the two qubits. Let's examine how the SWAP gate transforms each of the computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

1. SWAP Gate Acting on $|00\rangle$

The computational basis state $|00\rangle$ is represented as:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the SWAP gate to $|00\rangle$, we compute:

$$\text{SWAP } |00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

So, the SWAP gate leaves $|00\rangle$ unchanged.

2. SWAP Gate Acting on $|01\rangle$

The computational basis state $|01\rangle$ is represented as:

$$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

When we apply the SWAP gate to $|01\rangle$, we compute:

$$\text{SWAP } |01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$$

This shows that the SWAP gate transforms $|01\rangle$ to $|10\rangle$.

3. SWAP Gate Acting on $|10\rangle$

The computational basis state $|10\rangle$ is represented as:

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

When we apply the SWAP gate to $|10\rangle$, we compute:

$$\text{SWAP } |10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

This shows that the SWAP gate transforms $|10\rangle$ to $|01\rangle$.

4. SWAP Gate Acting on $|11\rangle$

The computational basis state $|11\rangle$ is represented as:

$$|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

When we apply the SWAP gate to $|11\rangle$, we compute:

$$\text{SWAP } |11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$$

So, the SWAP gate leaves $|11\rangle$ unchanged.

Summary of SWAP Gate Action

The SWAP gate acts as follows on each basis state:

$$\text{SWAP } |00\rangle = |00\rangle$$

$$\text{SWAP } |01\rangle = |10\rangle$$

$$\text{SWAP}\,|10\rangle = |01\rangle$$

$$\text{SWAP}\,|11\rangle = |11\rangle$$

This behavior matches the description of the SWAP gate: it exchanges the states of the two qubits, swapping $|01\rangle$ with $|10\rangle$ and leaving $|00\rangle$ and $|11\rangle$ unchanged.

Properties of the SWAP Gate

The SWAP gate has several important properties:

- **Unitary Property:** The SWAP matrix is unitary, meaning that $\text{SWAP}^\dagger \cdot \text{SWAP} = I$, where $\text{SWAP}^\dagger$ is the conjugate transpose of the SWAP matrix and I is the identity matrix. This ensures that the SWAP gate is reversible, as required for all quantum gates.
- **Self-Inverse Property:** The SWAP gate is its own inverse, meaning that if you apply it twice, you return to the original state. Mathematically:

  $$\text{SWAP} \cdot \text{SWAP} = I$$

  This means that applying the SWAP gate twice results in the identity operation, restoring the original state of the qubits.

Conclusion

The SWAP gate exchanges the states of two qubits, flipping $|01\rangle$ to $|10\rangle$ and vice versa, while leaving $|00\rangle$ and $|11\rangle$ unchanged. The SWAP gate is unitary and self-inverse, making it a useful tool for rearranging qubits in quantum circuits.

10. **Rotation Gates (Rx, Ry, Rz)**
Symbols: Rx, Ry, Rz
Function: These gates rotate the qubit state around the specified axis by an angle θ.

The three primary rotation gates are:

- **Rx (Rotation around X-axis)**

The R_x Gate

The R_x gate is a single-qubit rotation gate that rotates the qubit state around the X-axis of the Bloch sphere by an angle θ. The matrix representation of the R_x gate is:

$$R_x(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

This matrix operates on a single qubit and can be used to rotate the state vector by an arbitrary angle around the X-axis.

Action of $R_x(\theta)$ on Basis States

To understand how the R_x gate works, let's apply it to the computational basis states $|0\rangle$ and $|1\rangle$ and see how it transforms them.

1. $R_x(\theta)$ Acting on $|0\rangle$

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the $R_x(\theta)$ gate to $|0\rangle$, we get:

$$R_x(\theta)\,|0\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

So, the state $|0\rangle$ is transformed into:

$$R_x(\theta)\,|0\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle - i\sin\left(\frac{\theta}{2}\right)|1\rangle$$

This represents a superposition of $|0\rangle$ and $|1\rangle$, with amplitudes $\cos\left(\frac{\theta}{2}\right)$ and $-i\sin\left(\frac{\theta}{2}\right)$, respectively.

2. $R_x(\theta)$ Acting on $|1\rangle$

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the $R_x(\theta)$ gate to $|1\rangle$, we get:

$$R_x(\theta)|1\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

So, the state $|1\rangle$ is transformed into:

$$R_x(\theta)|1\rangle = -i\sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle$$

This also represents a superposition of $|0\rangle$ and $|1\rangle$, but with different amplitudes than those of the transformed $|0\rangle$ state.

Properties of the R_x Gate

The R_x gate has several important properties:

a. Unitary Property

The $R_x(\theta)$ gate is a unitary matrix, meaning that $R_x(\theta)^\dagger R_x(\theta) = I$, where $R_x(\theta)^\dagger$ is the conjugate transpose of $R_x(\theta)$ and I is the identity matrix. This ensures that the transformation it performs is reversible.

b. Special Cases

- $R_x(0) = I$: When $\theta = 0$, we have:

$$R_x(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

  This means no rotation is applied, and the gate acts as the identity operator.
- $R_x(\pi) = X$: When $\theta = \pi$, we have:

$$R_x(\pi) = \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & -i\sin\left(\frac{\pi}{2}\right) \\ -i\sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = X$$

  Thus, $R_x(\pi)$ acts as the Pauli-X (NOT) gate, which flips $|0\rangle$ to $|1\rangle$ and vice versa.
- Self-Inverse Property for $R_x(2\pi)$: The R_x gate has a periodicity property, meaning $R_x(2\pi) = I$. When the rotation angle is 2π, the qubit is rotated back to its original state, and the operation acts as the identity.

$$R_x(2\pi) = \begin{pmatrix} \cos(\pi) & -i\sin(\pi) \\ -i\sin(\pi) & \cos(\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Summary

The $R_x(\theta)$ gate performs a rotation of the qubit state around the X-axis by an angle θ. It transforms the computational basis states $|0\rangle$ and $|1\rangle$ into superpositions of $|0\rangle$ and $|1\rangle$, depending on the rotation angle. The gate is unitary, has special cases that correspond to common gates (e.g., $R_x(\pi) = X$), and is periodic, with $R_x(2\pi) = I$.

- **Ry (Rotation around Y-axis)**

The R_y Gate

The R_y gate is a single-qubit rotation gate that rotates the qubit state around the Y-axis of the Bloch sphere by an angle θ. The matrix representation of the R_y gate is:

$$R_y(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

This matrix operates on a single qubit and rotates the state vector by an arbitrary angle θ around the Y-axis.

Action of $R_y(\theta)$ on Basis States

To understand how the R_y gate works, let's apply it to the computational basis states $|0\rangle$ and $|1\rangle$ and see how it transforms them.

1. $R_y(\theta)$ Acting on $|0\rangle$

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the $R_y(\theta)$ gate to $|0\rangle$, we get:

$$R_y(\theta)\,|0\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

So, the state $|0\rangle$ is transformed into:

$$R_y(\theta)\,|0\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle$$

This represents a superposition of $|0\rangle$ and $|1\rangle$, with amplitudes $\cos\left(\frac{\theta}{2}\right)$ and $\sin\left(\frac{\theta}{2}\right)$, respectively.

2. $R_y(\theta)$ Acting on $|1\rangle$

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the $R_y(\theta)$ gate to $|1\rangle$, we get:

$$R_y(\theta)\,|1\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

So, the state $|1\rangle$ is transformed into:

$$R_y(\theta)\,|1\rangle = -\sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle$$

This also represents a superposition of $|0\rangle$ and $|1\rangle$, but with different amplitudes than those of the transformed $|0\rangle$ state.

Properties of the R_y Gate

The R_y gate has several important properties:

a. Unitary Property

The $R_y(\theta)$ gate is a unitary matrix, meaning that $R_y(\theta)^\dagger R_y(\theta) = I$, where $R_y(\theta)^\dagger$ is the conjugate transpose of $R_y(\theta)$ and I is the identity matrix. This ensures that the transformation it performs is reversible.

b. Special Cases

– $R_y(0) = I$: When $\theta = 0$, we have:

$$R_y(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

This means no rotation is applied, and the gate acts as the identity operator.

– $R_y(\pi) = iY$: When $\theta = \pi$, we have:

$$R_y(\pi) = \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = iY$$

Here, $R_y(\pi)$ acts as a rotation that produces a phase equivalent to the Y gate, up to a global phase i.

– Self-Inverse Property for $R_y(2\pi)$: The R_y gate has a periodicity property, meaning $R_y(2\pi) = I$. When the rotation angle is 2π, the qubit is rotated back to its original state, and the operation acts as the identity.

$$R_y(2\pi) = \begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Summary

The $R_y(\theta)$ gate performs a rotation of the qubit state around the Y-axis by an angle θ. It transforms the computational basis states $|0\rangle$ and $|1\rangle$ into superpositions of $|0\rangle$ and $|1\rangle$, depending on the rotation angle. The gate is unitary, has special cases that correspond to common gates (e.g., $R_y(\pi) \approx Y$ up to a phase), and is periodic, with $R_y(2\pi) = I$.

- **Rz (Rotation around Z-axis)**

The R_z Gate

The R_z gate is a single-qubit rotation gate that rotates the qubit state around the Z-axis of the Bloch sphere by an angle θ. The R_z gate is widely used in quantum circuits to adjust the phase of a qubit without affecting its probability amplitudes in the computational basis.
The matrix representation of the R_z gate is:

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

This matrix applies a phase shift of $-\theta/2$ to the $|0\rangle$ state and $+\theta/2$ to the $|1\rangle$ state.

Action of $R_z(\theta)$ on Basis States

To understand how the R_z gate works, let's apply it to the computational basis states $|0\rangle$ and $|1\rangle$ and see how it transforms them.

1. $R_z(\theta)$ Acting on $|0\rangle$

The computational basis state $|0\rangle$ is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we apply the $R_z(\theta)$ gate to $|0\rangle$, we get:

$$R_z(\theta)\,|0\rangle = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\theta/2} \\ 0 \end{pmatrix} = e^{-i\theta/2}\,|0\rangle$$

So, the state $|0\rangle$ is transformed into $e^{-i\theta/2}\,|0\rangle$. This represents a phase shift of $-\theta/2$ applied to $|0\rangle$.

2. $R_z(\theta)$ Acting on $|1\rangle$

The computational basis state $|1\rangle$ is represented as:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When we apply the $R_z(\theta)$ gate to $|1\rangle$, we get:

$$R_z(\theta)\,|1\rangle = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\theta/2} \end{pmatrix} = e^{i\theta/2}\,|1\rangle$$

So, the state $|1\rangle$ is transformed into $e^{i\theta/2}\,|1\rangle$. This represents a phase shift of $+\theta/2$ applied to $|1\rangle$.

Properties of the R_z Gate

The R_z gate has several important properties:

a. Unitary Property

The $R_z(\theta)$ gate is a unitary matrix, meaning that $R_z(\theta)^\dagger R_z(\theta) = I$, where $R_z(\theta)^\dagger$ is the conjugate transpose of $R_z(\theta)$ and I is the identity matrix. This ensures that the transformation it performs is reversible.

b. Special Cases

– $R_z(0) = I$: When $\theta = 0$, we have:

$$R_z(0) = \begin{pmatrix} e^0 & 0 \\ 0 & e^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

This means no rotation is applied, and the gate acts as the identity operator.

– $R_z(2\pi) = I$: When $\theta = 2\pi$, we get:

$$R_z(2\pi) = \begin{pmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

Since $-I$ corresponds to a global phase of π, it effectively acts as the identity operation on the state.

– $R_z(\pi) = Z$: When $\theta = \pi$, we get:

$$R_z(\pi) = \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

This matrix is equivalent to the Pauli-Z gate up to a global phase $-i$, and thus, $R_z(\pi)$ performs a phase flip similar to the Z gate.

c. Self-Inverse Property for $R_z(2\pi)$

The R_z gate has a periodicity property, meaning $R_z(2\pi) = I$ up to a global phase. When the rotation angle is 2π, the qubit is rotated back to its original state, and the operation effectively acts as the identity.

$$R_z(2\pi) = \begin{pmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

This global phase factor of -1 does not affect measurement outcomes and can be ignored in quantum computing.

Summary

The $R_z(\theta)$ gate performs a rotation of the qubit state around the Z-axis by an angle θ. It applies a phase shift of $-\theta/2$ to the $|0\rangle$ state and $+\theta/2$ to the $|1\rangle$ state. The gate is unitary, has special cases that correspond to common gates (e.g., $R_z(\pi) = Z$ up to a global phase), and is periodic, with $R_z(2\pi) = I$ up to a global phase.

Each gate takes an angle (in radians) as a parameter and performs a rotation around the corresponding axis by that angle.