Quantum Fourier Transform (QFT) Matrix for a 1-Qubit System

The Quantum Fourier Transform (QFT) matrix for a 1-qubit system is straightforward, as the system operates in a $2^1 = 2$-dimensional Hilbert space. The 2×2 QFT matrix is defined as:

$$QFT = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{2\pi i/2} \end{bmatrix}.$$

Since $e^{2\pi i/2} = -1$, the matrix simplifies to:

$$QFT = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Explanation

Input Basis States: $\{|0\rangle, |1\rangle\}$.

Output States: The QFT maps the computational basis states to the Fourier basis:

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

$$|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Unitary Matrix: The QFT matrix is unitary, meaning:

$$QFT \cdot QFT^\dagger = I.$$

This matrix is identical to the Hadamard gate, which is used to create superpositions in quantum circuits.

Quantum Fourier Transform (QFT) Matrix for a 2-Qubit System

The Quantum Fourier Transform (QFT) for a 2-qubit system operates in a $2^2 = 4$-dimensional Hilbert space. The 4×4 QFT matrix is:

$$QFT = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{bmatrix},$$

where ω is the primitive 4th root of unity:

$$\omega = e^{2\pi i/4} = i.$$

The entries of the matrix $QFT_{j,k}$ are computed as:

$$QFT_{j,k} = \frac{1}{2}\omega^{(j-1)(k-1)}.$$

Substituting the values of ω, the explicit form of the matrix becomes:

$$QFT = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

Transformation of Basis States

This matrix transforms the computational basis states $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ into their corresponding Fourier basis states.

Quantum Fourier Transform (QFT) Matrix for a 3-Qubit System

The Quantum Fourier Transform (QFT) for a 3-qubit system operates in a $2^3 = 8$-dimensional Hilbert space. Its matrix representation is an 8×8 unitary matrix, defined as:

$$QFT = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix},$$

where ω is the primitive 8th root of unity, defined as:

$$\omega = e^{2\pi i/8} = \cos\left(\frac{2\pi}{8}\right) + i \sin\left(\frac{2\pi}{8}\right).$$

Each entry in the matrix $QFT_{j,k}$ is given by:

$$QFT_{j,k} = \frac{1}{\sqrt{8}} \omega^{(j-1)(k-1)},$$

where $j, k = 1, 2, \ldots, 8$.

Transformation of Basis States

This matrix transforms the computational basis states $\{|0\rangle, |1\rangle, \ldots, |7\rangle\}$ into their corresponding Fourier basis states.

Quantum Fourier Transform (QFT) Matrix for a 4-Qubit System

The Quantum Fourier Transform (QFT) matrix for a 4-qubit system operates in a $2^4 = 16$-dimensional Hilbert space. The matrix is a 16×16 unitary matrix, with entries defined as:

$$QFT_{j,k} = \frac{1}{\sqrt{16}} \omega^{(j-1)(k-1)},$$

where:

$$\omega = e^{\frac{2\pi i}{16}}$$

is the 16th root of unity, and $j, k \in \{1, 2, \ldots, 16\}$.

Matrix Structure

The general form of the QFT matrix for 4 qubits is:

$$QFT = \frac{1}{\sqrt{16}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{15} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{30} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{45} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{15} & \omega^{30} & \omega^{45} & \cdots & \omega^{225} \end{bmatrix}.$$

Simplified Example for ω

The powers of $\omega = e^{\frac{2\pi i}{16}}$ (the 16th root of unity) are computed as:

$$\omega^0 = 1, \quad \omega^1 = e^{\frac{2\pi i}{16}}, \quad \omega^2 = e^{\frac{4\pi i}{16}}, \quad \ldots, \quad \omega^{15} = e^{\frac{30\pi i}{16}}.$$

Substituting these values into the matrix provides the full 16×16 structure.

Properties

Unitary: The matrix satisfies:

$$QFT \cdot QFT^\dagger = I,$$

where $QFT^\dagger$ is the conjugate transpose of QFT, and I is the identity matrix.

Efficient Representation: While the matrix can be explicitly constructed, in practice, the QFT is implemented using a sequence of gates (Hadamard and controlled phase gates), which scale efficiently with the number of qubits.

Quantum Fourier Transform (QFT) Matrix for a 5-Qubit System

The Quantum Fourier Transform (QFT) for a 5-qubit system operates in a $2^5 = 32$-dimensional Hilbert space. Its matrix is a 32×32 unitary matrix. Each element is defined as:

$$QFT_{j,k} = \frac{1}{\sqrt{32}} \omega^{(j-1)(k-1)},$$

where ω is the 32nd root of unity:

$$\omega = e^{\frac{2\pi i}{32}} = \cos\left(\frac{2\pi}{32}\right) + i \sin\left(\frac{2\pi}{32}\right).$$

Matrix Representation

The general form of the QFT matrix for 5 qubits is:

$$QFT = \frac{1}{\sqrt{32}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{31} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{62} \\ 1 & \omega^3 & \omega^6 & \cdots & \omega^{93} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{31} & \omega^{62} & \cdots & \omega^{961} \end{bmatrix}.$$

Explanation of Entries

Each entry in the matrix is computed as:

$$QFT_{j,k} = \frac{1}{\sqrt{32}} \omega^{(j-1)(k-1)},$$

where $(j-1)(k-1)$ is calculated using modular arithmetic, and $j, k = 1, 2, \ldots, 32$.

Properties

Unitary: The QFT matrix satisfies:

$$QFT \cdot QFT^\dagger = I,$$

where $QFT^\dagger$ is the conjugate transpose of QFT, and I is the identity matrix.

Fourier Basis Transformation: The QFT maps the computational basis states $\{|0\rangle, |1\rangle, \ldots, |31\rangle\}$ to their Fourier-transformed counterparts.

Quantum Fourier Transform (QFT) Matrix for an 8-Qubit System

The Quantum Fourier Transform (QFT) for an 8-qubit system operates in a $2^8 = 256$-dimensional Hilbert space. The matrix representation is a 256×256 unitary matrix, with elements defined as:

$$QFT_{j,k} = \frac{1}{\sqrt{256}} \omega^{(j-1)(k-1)},$$

where:

$$\omega = e^{\frac{2\pi i}{256}}$$

is the 256th root of unity.

General Matrix Structure

The QFT matrix is:

$$QFT = \frac{1}{\sqrt{256}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{255} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{510} \\ 1 & \omega^3 & \omega^6 & \cdots & \omega^{765} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{255} & \omega^{510} & \cdots & \omega^{65025} \end{bmatrix}.$$

Key Elements

Each element of the matrix $QFT_{j,k}$ is calculated as:

$$QFT_{j,k} = \frac{1}{\sqrt{256}} e^{\frac{2\pi i (j-1)(k-1)}{256}},$$

where:

$$j, k \in \{1, 2, \ldots, 256\}.$$

The term ω^a can be expressed using cosine and sine functions:

$$\omega^a = e^{\frac{2\pi i a}{256}} = \cos\left(\frac{2\pi a}{256}\right) + i \sin\left(\frac{2\pi a}{256}\right).$$

Properties

Unitary: The QFT matrix satisfies:

$$QFT \cdot QFT^\dagger = I,$$

where $QFT^\dagger$ is the conjugate transpose of QFT, and I is the 256×256 identity matrix.

Efficient Representation: Although conceptually large, the QFT matrix is rarely constructed explicitly in quantum computing. Instead, the QFT is implemented as a series of controlled phase gates and Hadamard gates.