§9.3 线性系统状态空间表示 output

一、物理机理

胡克定律：$F = ky$ ↗弹性系数 ↘阻尼系数

阻尼器：$F = f\frac{dy}{dt}$

牛二律：$F_合 = m\frac{d^2y}{dt^2}$

Input：作用于m的 $F(t)$

取 $x_1 = y$，$x_2 = \dot{y} = v$

二、差分/微分方程

步骤 1. 建立微分方程 → 与初值相关；与储能环节相关（电感电流，电容电压，物块的速度）

2. $\boxed{选择状态变量}$ 建立状态方程

3. 建立输出方程

类型1. 线性差分方程不含输入u的微分

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

若输入/输出的值在t≥0时已知，可确定系统状态

$y^{(n)}, y^{(n-1)}, \cdots, \dot{y}, y$ —— Derivatives of output

u —— Input

Select state variables: $x_1 = y$, $x_2 = \dot{y}$, ..., $x_{n-1} = y^{(n-2)}$, $x_n = y^{(n-1)}$ (Output equation)

then: $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, ..., $\dot{x}_{n-1} = x_n$, $\dot{x}_n = -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + bu$ (State equations)

$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$

State space:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b \end{bmatrix}$$

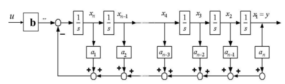
$C = [1\ 0\ 0\ \ldots\ 0]$

draw the following block diagram (state variable diagram) among the state variables.

✓The output of each integrator corresponds to each state variable.

✓The state equations are decided by the relationship of I/O.

✓The output equation is decided obviously from the output part.



$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$

类型2 含输入u的微分

nth-order linear differential equation representation:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

Reference Case 1:

$\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, ..., $\dot{x}_{n-1} = x_n$

$\dot{x}_n = -a_n y - a_{n-1}\dot{y} - \cdots - a_1 y^{(n-1)} + b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$

$= -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$

⇐ $x_1 = y$, $x_2 = \dot{y}$, ..., $x_{n-1} = y^{(n-2)}$, $x_n = y^{(n-1)}$

若u为有限阶跃信号，$\dot{u}$为冲激，$u^{(n)}$为高阶冲激 t=0处无穷大

无法连续输出，故其微分为系统变量

无法描述系统状态

However, the derivative of input u is still contained in 'state equation' which is INCONSEQUENCE. 非因果

The principle of state variable selection:

No derivative of the input/operation function should exist in any system state equation.

Select the state variables:

$x_1 = y - \beta_0 u$

$x_2 = \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$

$x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$

$x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u = \dot{x}_{n-1} - \beta_{n-1}u$

n-1 state equations

The input is contained in the state variables

代入

$\dot{x}_1 = x_2 + \beta_1 u$, $\dot{x}_2 = x_3 + \beta_2 u$, ..., $\dot{x}_{n-1} = x_n + \beta_{n-1}u$

$\dot{x}_n = ?\ ?\ ?$

then

↓对x_n微分

$x_n = y^{(n)} - \beta_0 u^{(n)} \cdots - \beta_{n-1}\dot{u}$

$\dot{x}_n = -a_1 y^{(n-1)} - a_2 y^{(n-2)} - \cdots - a_n y + (b_0 - \beta_0)u^{(n)} + (b_1 - \beta_1)u^{(n-1)} + \cdots + (b_{n-1} - \beta_{n-1})\dot{u} + b_n u$

代入

$\dot{x}_n = -a_n x_1 - a_{n-1}x_2 - \cdots - a_2 x_{n-1} - a_1 x_n + (b_0 - \beta_0)u^{(n)} + (b_1 - \beta_1 - a_1\beta_0)u^{(n-1)} + (b_2 - \beta_2 - a_1\beta_1 - a_2\beta_0)u^{(n-2)} + \cdots + (b_n - a_n\beta_{n-1} - a_{n-2}\beta_{n-2} - \cdots - a_1\beta_{n-1} - a_n\beta_0)u$

Principle: No derivative of input $u(t)$ contained in state equations.

thus: $\beta_0 = b_0$, $\beta_1 = b_1 - a_1\beta_0$, $\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$, ..., $\beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n\beta_0$

state equations:

$x_1 = y - \beta_0 u$

$x_2 = \dot{y} - \beta_0\dot{u} - \beta_1 u$

$x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u$

$x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u$

derivatives of y:

$y = x_1 + \beta_0 u$

$\dot{y} = x_2 + \beta_0\dot{u} + \beta_1 u$

$\ddot{y} = x_3 + \beta_0\ddot{u} + \beta_1\dot{u} + \beta_2 u$

$y^{(n-1)} = x_n + \beta_0 u^{(n-1)} + \beta_1 u^{(n-2)} + \cdots + \beta_{n-2}\dot{u} + \beta_{n-1}u$

bring output y and it derivatives: $y', \ldots, y^{(n-1)}$ in x_n

State-space of the system is:

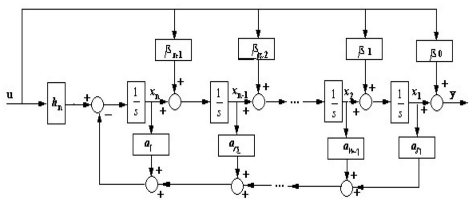
$\dot{x}_1 = x_2 + \beta_1 u$, $\dot{x}_2 = x_3 + \beta_2 u$, ..., $\dot{x}_{n-1} = x_n + \beta_{n-1}u$, $\dot{x}_n = -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + \beta_n u$

Rewrite the system to the matrix representation: $\dot{x} = Ax + Bu$, $y = Cx + Du$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix},\ B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$C = [1\ 0\ 0\ \cdots\ 0]$, $D = \beta_0 = b_0$

$y = x + \beta_0 u$



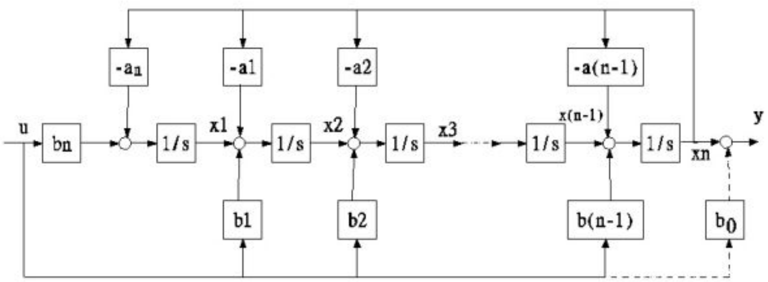
可观测标准型 CP24 重新选择状态变量

Matrix description:

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

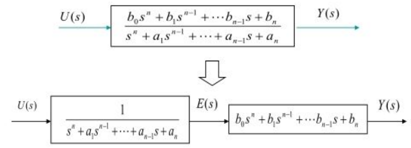
$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_1 \end{bmatrix}\quad b = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix}$$

$C = [0\ \ 0\ \ \cdots\ \ 1]$　　$d = 0$



三、通过系统传递函数

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = b_0 + \frac{b_1' s^{n-1} + \cdots + b_{n-1}' s + b_n'}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = b_0 + G'(s)$$



$$Y(s) = (b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1}s + b_n)E(s)$$
$$U(s) = (s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n)E(s)$$

Select state variables: $x_1 = e(t)$, $x_2 = \dot{e}(t)$, ..., $x_n = e^{(n-1)}(t)$

$$y = b_0\dot{x}_n + b_1 x_n + b_2 x_{n-1} + \cdots + b_{n-1}x_2 + b_n x_1$$
$$u = \dot{x}_n + a_1 x_n + a_2 x_{n-1} + \cdots + a_{n-1}x_2 + a_n x_1$$

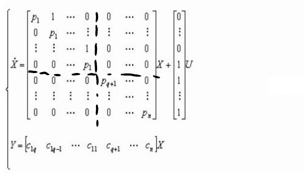
$b_0=0$时可控标准型

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}u$$

$$y = b_0(-a_n\ \ -a_{n-1}\ \cdots\ -a_1)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{b_0 u}_{直接传输矩阵} + (b_n\ \ b_{n-1}\ \cdots b_1)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

If b_0=0, the output equation will be simplified.

※传函串行/并行化的结构图见PPT



System matrix A is Jordan Standard Form(约当标准型).

※利用状态空间求传函

For SISO system: State-space to Transfer Function

The State-space representation of a SISO system:

$\dot{x} = Ax + Bu$, $y = Cx + Du$

$x, \dot{x} \in R^{n\times 1}$　$A \in R^{n\times n}$　$B \in R^{n\times 1}$　$C \in R^{1\times n}$　D is a scalar quantity.

Assume zero-initial condition, and use Laplace Transform:

$sX(s) = AX(s) + BU(s)$

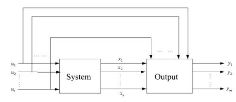
$Y(s) = CX(s) + Du(s)$

$X(s) = (sI - A)^{-1}BU(s)$

The Transfer Function is:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

For MIMO system: State-space to Transfer Function



$u_1, u_2, \cdots, u_r$ —— input signals of the system

$y_1, y_2, \cdots, y_m$ —— output signals of the system

$x_1, x_2, \cdots, x_n$ —— state variables of the system

The dynamic equations: $\dot{x} = Ax + Bu$, $y = Cx + Du$

Equations have the same formation with SISO system, however, the matrices B, C, D have different dimension.

$x, \dot{x} \in R^{n\times 1}$, $U \in R^{r\times 1}$, $Y \in R^{m\times 1}$, $A \in R^{n\times n}$, $B \in R^{n\times r}$, $C \in R^{m\times n}$, $D \in R^{m\times r}$

Laplace transformation:

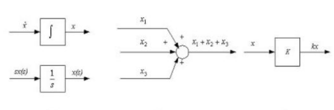
$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$G \in R^{m\times r}$ —— Transfer Function Matrix

四、通过状态变量图

- State variable diagram: the diagram description of the relationship of state variables, which is composed by the integral items, proportion items and sum symbols.

◆ The output of each integral item is chosen as a state variables of the system.


(a)　(b)　(c)