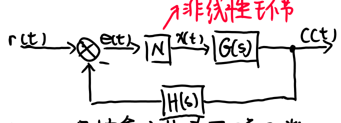
描述函数描述非线性系统

Basic idea



若 $e(t)=A\sin\omega t$，则输入也为正弦函数 假设

$x(t)$ 若为周期非正弦函数可傅立叶展开进行转换，忽略谐波后

$x(t)=x_1\sin(\omega t+\phi_1)$ 输入输出频率同

故在该假设下可对非线性系统进行频率分析 ⇒ 描述函数法 ⇓

Advantage:
- It is not confined by ***the order of the system***. 分析稳定性与自激振荡

Disadvantages:
- It is an ***approximate analysis method***.
- It can only be used to study the system ***frequency characteristics***.

一、描述函数法概念

应用前提 ① 线性部分与非线性部分可分离为两者串联

② 非线性特性部分奇对称，输入输出为静特性 static(无储能环节)

* 此时输入正弦 输出周期函数，无直流分量

③ 线性部分低通滤波特性良好，完全忽略高次谐波仅考虑基波

Definition of the describing function

The describing function $N(A)$ of the nonlinear element is: the *complex ratio* of the fundamental component of the output $y(t)$ and the sinusoidal input $e(t)$, that is:

For $e(t)=A\sin\omega t$,

$$x(t)\approx A_1\cos\omega t+B_1\sin\omega t=x_1\sin(\omega t+\phi_1)\longrightarrow N(A)=\frac{x_1e^{j\phi_1}}{A}$$

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Assume the input of nonlinear is sinusoidal $e(t)=A\sin\omega t$

Normally, the output is periodic, which can be expressed as a *Fourier series*:

$$x(t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n\cos n\omega t+B_n\sin n\omega t)$$

The nonlinearity is *odd-symmetric* ⇨ $A_0=0$

$$A_n=\frac{1}{\pi}\int_0^{2\pi}x(t)\cos n\omega t\ d(\omega t)$$

$$B_n=\frac{1}{\pi}\int_0^{2\pi}x(t)\sin n\omega t\ d(\omega t)$$

For the fundamental component, we have

$$A_1=\frac{1}{\pi}\int_0^{2\pi}x(t)\cos\omega t\ d(\omega t)$$

$$B_1=\frac{1}{\pi}\int_0^{2\pi}x(t)\sin\omega t\ d(\omega t)$$

Thus, the fundamental component is

$$x_1(t)=A_1\cos\omega t+B_1\sin\omega t=x_1\sin(\omega t+\phi_1)$$

where

$$x_1=\sqrt{A_1^2+B_1^2}\qquad \phi_1=arctg\frac{A_1}{B_1}$$

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$$N(A)=\frac{x_1}{A}e^{j\phi_1}=\frac{\sqrt{A_1^2+B_1^2}}{A}e^{j\arctan\frac{A_1}{B_1}}$$

$$\Downarrow\quad=\frac{B_1}{A}+j\frac{A_1}{A}$$

单值奇对称

$\frac{1}{\pi}\int_0^{2\pi}x(t)\cos\omega t\,d\omega t$

$A_1=0$ 时，$N(A)=\frac{B_1}{A}$，A 的实值函数

$x(t)$ 非单值奇对称(偶对称/无对称性)

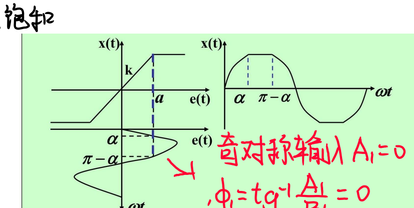
$A_1\neq0$，$B_1\neq0$，$N(A)=\frac{B_1}{A}+j\frac{A_1}{A}$，A 的复变函数

输入信号幅值与频率的函数 $N(A,\omega)$

而非线性环节常无储能环节，输入输出频率独立，通常非线性环节可视为一个仅与幅值相关的函数

二、常见非线性环节的描述函数 $e(t)=A\sin\omega t$

1、饱和



奇对称输入 $A_1=0$，$\phi_1=tg^{-1}\frac{A_1}{B_1}=0$

Fig 8-10 Input and output waveform of a saturation nonlinearity

$$x(t)=\begin{cases}kA\sin\omega t, & 0\le\omega t\le\alpha\\ ka, & \alpha\le\omega t\le\pi-\alpha\\ kA\sin\omega t, & \pi-\alpha\le\omega t\le\pi\end{cases}$$

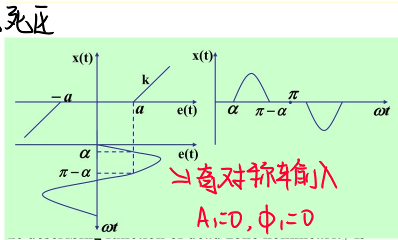
$A\sin\alpha=a\Rightarrow\alpha=\sin^{-1}\frac{a}{A}$

$$B_1=\frac{2}{\pi}\int_0^{\pi}x(t)\sin\omega t\,d\omega t=\frac{2}{\pi}kA\left[\sin^{-1}\frac{a}{A}+\frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

$$N(A)=\frac{B_1}{A}=\frac{2}{\pi}K\left[\sin^{-1}\frac{a}{A}+\frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

⇒ 与 A 相关的增益放大器

2、死区



奇对称输入 $A_1=0$，$\phi_1=0$

$$x(t)=\begin{cases}0, & 0\le\omega t\le\alpha\\ k(A\sin\omega t-a), & \alpha<\omega t\le\pi-\alpha\\ 0, & \pi-\alpha<\omega t\le\pi\end{cases}$$

$A\sin\alpha=a\Rightarrow\alpha=\sin^{-1}\frac{a}{A}$

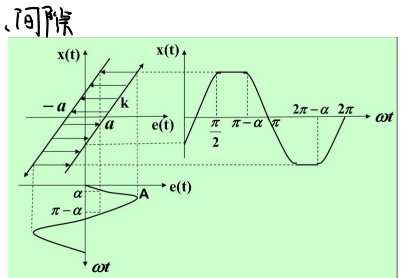
$$B_1=\frac{2}{\pi}kA\left[\frac{\pi}{2}-\sin^{-1}\frac{a}{A}-\frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

$$N(A)=\frac{B_1}{A}=\frac{2}{\pi}K\left[\frac{\pi}{2}-\sin^{-1}\frac{a}{A}-\frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

⇒ $N(A)$ 会下降随 $\frac{a}{A}$ 上升，$\frac{a}{A}=1$ 时 $N(A)=0$

$\frac{a}{A}$ 极小时，死区小，$N(A)$ 近似 K

3、间隙



$$x(t)=\begin{cases}k(A\sin\omega t-a), & 0\le\omega t<\frac{\pi}{2}\\ k(A-a) & \frac{\pi}{2}\le\omega t<\pi-\alpha\\ k(A\sin\omega t+a) & \pi-\alpha\le\omega t\le\pi\end{cases}$$

$A\sin(\pi-\alpha)=A-2a\Rightarrow\alpha=\sin^{-1}\frac{A-2a}{A}$

$$A_1=\frac{2}{\pi}\int_0^{\pi}x(t)\cos\omega t\,d(\omega t)=\frac{4kA}{\pi}\left[\left(\frac{a}{A}\right)^2-\frac{a}{A}\right]$$

$$B_1=\frac{2}{\pi}\int_0^{\pi}x(t)\sin\omega t\,d(\omega t)=\frac{kA}{\pi}\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{A-2a}{A}\right)+\frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right]$$

$$N(A)=\frac{B_1}{A}+j\frac{A_1}{A}$$

$$=\frac{K}{\pi}\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{A-2a}{A}\right)+\frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right]+j\frac{4K}{\pi}\left[\frac{a(a-A)}{A^2}\right]$$

$$=|N(A)|e^{j\phi_1}$$

$$|N(A)|=\sqrt{\left[\frac{4K}{\pi}\left(\frac{a(a-A)}{A^2}\right)\right]^2+\left[\frac{K}{\pi}\left(\frac{\pi}{2}+\sin^{-1}\frac{A-2a}{A}+\frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right)\right]^2}$$

$$\phi_1=tg^{-1}\frac{4\frac{a(a-A)}{A^2}}{\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{A-2a}{A}\right)+\frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right]}$$

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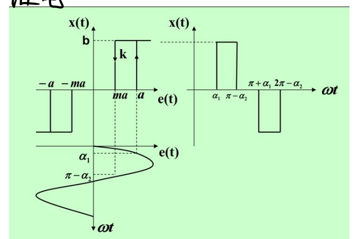
4、继电



$$x(t)=\begin{cases}0, & 0\le\omega t<\alpha_1\\ b, & \alpha_1\le\omega t<\pi-\alpha_2\\ 0, & \pi-\alpha_2\le\omega t\le\pi\end{cases}$$

$A\sin\alpha_1=a\Rightarrow\alpha_1=\sin^{-1}\frac{a}{A}$

$A\sin(\pi-\alpha_2)=ma\Rightarrow\alpha_2=\sin^{-1}\frac{ma}{A}$

$$A_1=\frac{2ab(m-1)}{\pi A}$$

$$B_1=\frac{2b}{\pi}\left[\sqrt{1-\left(\frac{ma}{A}\right)^2}+\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

$$N(A)=|N(A)|e^{j\phi_1}=\sqrt{\left(\frac{A_1}{A}\right)^2+\left(\frac{B_1}{A}\right)^2}e^{jtg^{-1}\frac{A_1}{B_1}}$$

$$|N(A)|=\frac{2b}{\pi A}\sqrt{2\left[1-m\left(\frac{a}{A}\right)^2+\sqrt{1+m^2\left(\frac{a}{A}\right)^4-(m^2+1)\left(\frac{a}{A}\right)^2}\right]}$$

$$\phi_1=tg^{-1}\frac{(m-1)\left(\frac{a}{A}\right)}{\sqrt{1-m^2\left(\frac{a}{A}\right)^2}+\sqrt{1-\left(\frac{a}{A}\right)^2}}$$

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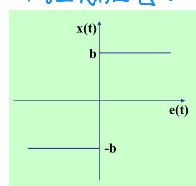
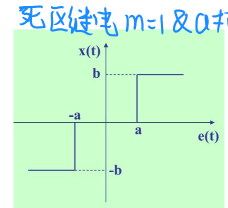
* 理想继电 $a=0$



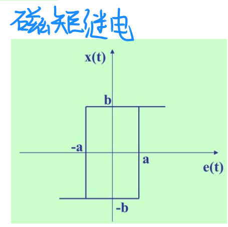
$$N(A)=\frac{4b}{\pi A}$$

死区继电 $m=1$ & $a\neq0$



$$N(A)=\frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$

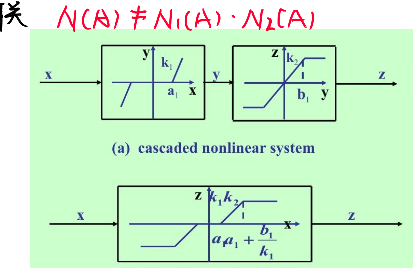
磁滞继电



$$N(A)=\frac{4b}{\pi A}e^{jtg^{-1}\frac{-\left(\frac{a}{A}\right)}{\sqrt{1-\left(\frac{a}{A}\right)^2}}}$$

三、多重非线性系统描述函数

1、级联 $N(A)\neq N_1(A)\cdot N_2(A)$



(a) cascaded nonlinear system

(b) composite nonlinearity

2、并联 $N(A)=N_1(A)+N_2(A)$

四、非线性系统描述函数分析

$N(A)=\frac{\text{输出}x(t)\text{基波}}{\text{正弦输入}e(t)\text{幅值}}$ 可应用描述函数法 部分动态特性

故自激振荡时，输入 $x(t)$ 可被视为正弦信号，因为线性部分近似低通滤波

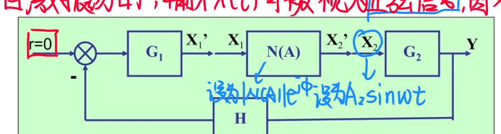


Fig. 8-21 nonlinear system

有 $X_1'=-|G_1(j\omega)G_2(j\omega)H(j\omega)|A_2\sin(\omega t+\theta)$，$\theta=\angle G_1(j\omega)+\angle G_2(j\omega)+\angle H(j\omega)$

$x_2'(t)=-|N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)|A_2\sin(\omega t+\theta+\phi)$

故当 $x_2'(t)=x_2(t)$ 时产生自激振荡，即

$G(s)=G_1(s)G_2(s)H(s)$

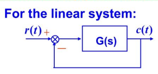
$$\begin{cases}|N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)|=1\\ \theta+\phi=(2n+1)\pi\end{cases}$$

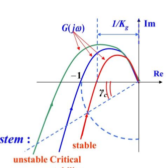
$$\Rightarrow 1+N(A)G(j\omega)=0\ \text{或}\ G(j\omega)=-\frac{1}{N(A)}$$

☆ Nyquist Law

Review of Nyquist criterion

For the linear system:





The characteristic equation of the system:

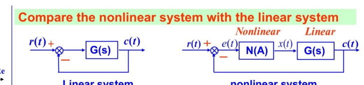
$$1+G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-1+j0$$

If $G(s)$ is a minimum phase transfer function, the necessary and sufficient condition of the stable system is:

$G(j\omega)$ *does not circle the point* $(-1,j\omega)$

Compare the nonlinear system with the linear system



Transfer function of the system:

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}\qquad \phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{N(A)G(j\omega)}{1+N(A)G(j\omega)}$$

Characteristic equation:

$$1+G(j\omega)=0\qquad 1+N(A)G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-1\qquad \Rightarrow G(j\omega)=-\frac{1}{N(A)}$$

In the $G(j\omega)$ plane — A point — A curve

(1) $G(j\omega)$ don't circle the $-\frac{1}{N(A)}$ curve, the nonlinear system is stable

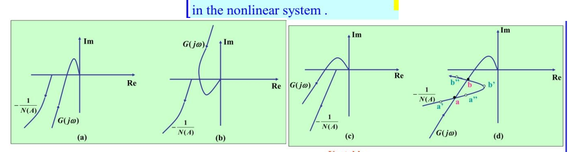
(2) $G(j\omega)$ circle the $-\frac{1}{N(A)}$ curve, the nonlinear system is unstable;

(3) $G(j\omega)$ intersect with the $-\frac{1}{N(A)}$ curve, there will be a self-oscillation in the nonlinear system.

(1) $G(j\omega)$ don' t circle the point $(-1,j\omega)$, the system is stable;

(2) $G(j\omega)$ circle the point $(-1,j\omega)$, the system is unstable;

(3) $G(j\omega)$ intersect with the point $(-1,j\omega)$, the system is in the critical stability.



$G(j\omega)$ do not circle $-1/N(A)$ (Stable)

$G(j\omega)$ circle $-1/N(A)$ (Unstable)

Unstable

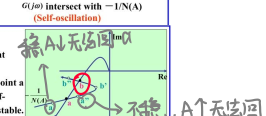
$G(j\omega)$ intersect with $-1/N(A)$ (Self-oscillation)

Analysis:

(1) Self-oscillation occurs at point a and point b.

(2) The self-oscillation at point a is unstable while the self-oscillation at point b is stable.

(3) There is only one stable self-oscillation in an actual physical system.



The *amplitude* and *frequency* of the self-oscillation can be obtained by solving

$$|G(j\omega)N(A)|=1$$

$$\theta+\phi=-\pi$$