状态空间(模型) ⇔ 动态分析(求解)

保证解存在性与独特性 ⇒ A, B中元素均有界

一、LTI连续系统求解

1、齐次状态方程 $\dot{x}=Ax$ 求解

幂级数法

设上述方程解为t的幂向量

$$x(t)=b_0+b_1t+b_2t^2+\cdots+b_kt^k+\cdots$$

$x, b_0, b_1, \cdots, b_k, \cdots$ 为n维向量

$$\dot{x}=b_1+2b_2t+\cdots+kb_kt^{k-1}+\cdots=A(b_0+b_1t+b_2t^2+\cdots+b_kt^k+\cdots)$$

Assume the coefficients with the same power are uniform.

$$b_1=Ab_0$$
$$b_2=\frac{1}{2}Ab_1=\frac{1}{2}A^2b_0$$
$$b_3=\frac{1}{3}Ab_2=\frac{1}{3\times 2}A^3b_0$$
$$\vdots$$
$$b_k=\frac{1}{k}Ab_{k-1}=\frac{1}{k!}A^kb_0$$
$$\vdots$$

$\because \quad x(0)=b_0$

$\therefore \quad x(t)=(I+At+\frac{1}{2}A^2t^2+\cdots+\frac{1}{k!}A^kt^k+\cdots)x(0)$

Define: $\boxed{e^{At}=I+At+\frac{1}{2}A^2t^2+\cdots+\frac{1}{k!}A^kt^k+\cdots=\sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k}$

$x(t)=e^{At}x(0)$ → 矩阵指数函数，状态转移矩阵$\Phi(t)$

$\dot{x}=Ax$的拉氏变换

$$sX(s)=Ax(s)+x(0)$$
$$(Is-A)x(s)=x(0)$$
$$x(s)=(Is-A)^{-1}x(0)$$

进行拉氏反变换有 $x(t)=\mathcal{L}^{-1}[(sI-A)^{-1}]x(0)$

解齐次状态方程

⇓

故

$\Phi(t)=e^{At}=\mathcal{L}^{-1}[(sI-A)^{-1}]$ 状态转移矩阵闭式解(收敛)

2、非齐次状态方程 $\dot{x}(t)=Ax(t)+Bu(t)$ 求解

(1) 直接积分法 $\dot{x}(t)-Ax(t)=Bu(t)$

left multiply e^{-At} simultaneously: $e^{-At}[\dot{x}(t)-Ax(t)]=e^{-At}Bu(t)$

$\boxed{\frac{d}{dt}[e^{-At}x(t)]}$

$$\frac{d}{dt}[e^{-At}x(t)]=e^{-At}Bu(t)$$

$$e^{-At}x(t)-x(0)=\int_o^t e^{-A\tau}Bu(\tau)d\tau$$

$$x(t)=e^{At}x(0)+\int_o^t e^{A(t-\tau)}Bu(\tau)d\tau \qquad \boldsymbol{x}(t)|_{t=0}=\boldsymbol{x}(0)$$

$$x(t)=\Phi(t)x(0)+\int_o^t\Phi(t-\tau)Bu(\tau)d\tau$$

response of initial condition

Response of input u(t)

(2) 拉氏变换法

$sX(s)-x(0)=AX(s)+Bu(s)$

$(sI-A)X(s)=x(0)+Bu(s)$

$X(s)=(sI-A)^{-1}x(0)+(sI-A)^{-1}Bu(s)$

then $x(t)=L^{-1}[(sI-A)^{-1}x(0)]+L^{-1}[(sI-A)^{-1}Bu(s)]$

from $e^{At}=L^{-1}[(sI-A)^{-1}]$,

we have:

$$x(t)=e^{At}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau=\Phi(t)x(0)+\int_0^t\Phi(t-\tau)Bu(\tau)d\tau$$

二、状态转移矩阵性质

$$e^{At}=I+At+\frac{1}{2}A^2t^2+\cdots+\frac{1}{k!}A^kt^k+\cdots=\sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

① Initial value: $\Phi(0)=I$

② $\dot{\Phi}(t)=A\Phi(t)=\Phi(t)A \qquad \dot{\Phi}(0)=A$

③ Linear relationship: $\Phi(t_1\pm t_2)=\Phi(t_1)\Phi(\pm t_2)=\Phi(\pm t_2)\Phi(t_1)$

④ Reversibility: $\Phi^{-1}(t)=\Phi(-t),\ \Phi^{-1}(-t)=\Phi(t)$

⑤ $x(t_2)=\Phi(t_2-t_1)x(t_1)$

⑥ $\Phi(t_2-t_0)=\Phi(t_2-t_1)\Phi(t_1-t_0)$

⑦ $[\Phi(t)]^k=\Phi(kt)$

⑧ if $AB=BA,\quad e^{(A+B)t}=e^{At}e^{Bt}=e^{Bt}e^{At}$;

if $AB\neq BA,\quad e^{(A+B)t}\neq e^{At}e^{Bt}\neq e^{Bt}e^{At}$

⑨ if $\Phi(t)$ is state transfer matrix of $\dot{x}(t)=Ax(t)$, the newly state tranfer matrix after non-singular transform $x=P\overline{x}$ is:

$$\overline{\Phi}(t)=P^{-1}e^{At}P$$

⑩ Two common state transition matrices

If A is n-order Diagonal Matrix,

$$A=\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix},\qquad \Phi(t)=\begin{bmatrix}e^{\lambda_1t} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & e^{\lambda_nt}\end{bmatrix}$$

If A is m-order Jordan Matrix,

$$A=\begin{bmatrix}\lambda & 1 & \cdots & 0\\ 0 & \lambda & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 0 & \cdots & 0 & \lambda\end{bmatrix}_{m\times m},\qquad \Phi(t)=\begin{bmatrix}e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t}\\ 0 & e^{\lambda t} & \ddots & \vdots\\ \vdots & \ddots & \ddots & te^{\lambda t}\\ 0 & \cdots & 0 & e^{\lambda t}\end{bmatrix}$$

三、状态转移矩阵e^{At}计算

1、直接法(矩阵指数方程)

$$e^{At}=I+At+\frac{1}{2!}A^2t^2+\frac{1}{3!}A^3t^3+\cdots=\sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

对任一常值A与有限t,上述无穷级数一定收敛

2、线性变换法(对角型/约当型)

If the matrix A can be transferred to diagonal form, e^{At} can be given as:

$$e^{At}=Pe^{\Lambda t}P^{-1}=P\begin{bmatrix}e^{\lambda_1t} & & & 0\\ & e^{\lambda_2t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_nt}\end{bmatrix}P^{-1}$$

非奇异线性变换矩阵

对约当型A有 $e^{At}=Se^{Jt}S^{-1}$

3、拉氏变换

$$e^{At}=\mathcal{L}^{-1}[(sI-A)^{-1}]$$

四、线性离散系统状态空间表示

1、离散线性系统状态空间描述

DLTV:

$$\begin{cases}x(k+1)=A(k)x(k)+B(k)u(k)\\ y(k)=C(k)x(k)+D(k)u(k)\end{cases}$$

DLTI

系统矩阵 输入矩阵

$$\begin{cases}x(k+1)=Ax(k)+Bu(k)\\ y(k)=Cx(k)+Du(k)\end{cases}$$

输出矩阵 转移矩阵

2、由差分方程建立离散时间状态空间方程 (PPT 241

$$y(k+n)+a_1y(k+n-1)+\cdots+a_{n-1}y(k+1)+a_ny(k)$$
$$=b_0u(k+n)+b_1u(k+n-1)+\cdots+b_{n-1}u(k+1)+b_nu(k)$$

In which, k is time of kT; T is sampling period; u(k) and y(k) are input and output at time of kT; a_i and b_i are constants decided by system performance; consider the Z-transfer with zero initial condition:

$$Z[y(k)]=y(z),\quad Z[(y(k+i)]=z^iy(z)$$

3、连续系统状态空间表达的离散化

$$x(t)=e^{At}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau=\Phi(t-t_0)x(t_0)+\int_{t_0}^t\Phi(t-\tau)Bu(\tau)d\tau$$

assume $\quad t_0=kT,\quad x(t_0)=x(kT)=x(k)$

$t=(k+1)T,\quad x[(k+1)]T=x(k+1)$

at $t\in[k,k+1]$, u(k)=u(k-1) is constant

$$\boldsymbol{x}(k+1)=\Phi[(k+1)T-kT]\boldsymbol{x}(k)+\boxed{\int_{kT}^{(k+1)T}\Phi[(k+1)T-\tau]Bd\tau}\,\boldsymbol{u}(k)$$

$$G(T)=\int_{kT}^{(k+1)T}\Phi[(k+1)T-\tau)]Bd\tau$$

Variable replacement $(k+1)T-\tau=\tau'$

then $\quad G(T)=\int_0^T\Phi(\tau)Bd\tau$

$$\boldsymbol{x}(k+1)=\Phi[(k+1)T-kT]\boldsymbol{x}(k)+\int_{kT}^{(k+1)T}\Phi[(k+1)T-\tau]Bd\tau\boldsymbol{u}(k)$$

$$G(T)=\int_{kT}^{(k+1)T}\Phi[(k+1)T-\tau)]Bd\tau \qquad G(T)=\int_0^T\Phi(\tau)Bd\tau$$

State equation of discrete system is:

$$\boldsymbol{x}(k+1)=\Phi(T)\boldsymbol{x}(k)+G(T)\boldsymbol{u}(k)$$

The relationship between $\Phi(T)$ and state transition matrix $\Phi(t)$ of continuous system:

$$\Phi(T)=\Phi(t)|_{t=T}$$

The output equation of discrete system is:

$$y(k)=C\boldsymbol{x}(k)+\mathrm{D}u(k)$$

4、离散时不变系统动态方程求解

recursive method and Z-transformation method

$$\boldsymbol{x}(k+1)=\Phi(T)\boldsymbol{x}(k)+G(T)\boldsymbol{u}(k)\qquad \boldsymbol{k=0,1,\cdots,k-1,}$$

The states at time of T, 2T, … ,kT time:

$k=0 \quad \boldsymbol{x}(1)=\Phi(T)\boldsymbol{x}(0)+G(T)\boldsymbol{u}(0)$

$k=1 \quad \boldsymbol{x}(2)=\Phi(T)\boldsymbol{x}(1)+G(T)\boldsymbol{u}(1)$
$\qquad =\Phi^2(T)\boldsymbol{x}(0)+\Phi(T)G(T)u(0)+G(T)\boldsymbol{u}(1)$

$k=2 \quad \boldsymbol{x}(3)=\Phi(T)\boldsymbol{x}(2)+G(T)\boldsymbol{u}(2)$
$\qquad =\Phi^3(T)\boldsymbol{x}(0)+\Phi^2(T)G(T)\boldsymbol{x}(1)+\Phi(T)G(T)u(1)+G(T)\boldsymbol{u}(2)$

$\vdots \qquad \vdots$

$k=k-1$

$$\begin{aligned}\boldsymbol{x}(k)&=\Phi(T)\boldsymbol{x}(k-1)+G(T)\boldsymbol{u}(k-1)\\ &=\Phi^k(T)\boldsymbol{x}(0)+\Phi^{k-1}(T)G(T)u(0)+\Phi^{k-2}(T)G(T)u(1)+\cdots\\ &\quad+\Phi(T)G(T)\boldsymbol{u}(k-2)+G(T)\boldsymbol{u}(k-1)\\ &=\Phi^k(T)\boldsymbol{x}(0)+\sum_{i=0}^{k-1}\Phi^{k-i-1}(T)G(T)\boldsymbol{u}(i)\end{aligned}$$

It is the solution of discrete state equation, which is named as discrete state transition equation.

when $\quad u(i)=0,\quad (i=0,1,\cdots,k-1)$

$$\boldsymbol{x}(k)=\Phi^k\boldsymbol{x}(0)=\Phi(kT)\boldsymbol{x}(0)=\Phi(k)\boldsymbol{x}(0)$$

$\Phi(k)\longrightarrow$ state transition matrix of discrete system

The output equation:

$$\begin{aligned}\boldsymbol{y}(k)&=C\boldsymbol{x}(k)+\mathrm{D}u(k)\\ &=C\Phi^k(T)\boldsymbol{x}(0)+C\sum_{i=0}^{k-1}\Phi^{k-i-1}(T)G(T)\boldsymbol{u}(i)+D\boldsymbol{u}(k)\end{aligned}$$

For the following discrete state equation:

$$\boldsymbol{x}(k+1)=A\boldsymbol{x}(k)+B\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k)=C\boldsymbol{x}(k)+D\boldsymbol{u}(k)$$

Its solution is:

$$\boldsymbol{x}(k)=A^k\boldsymbol{x}(0)+\sum_{i=0}^{k-1}A^{k-i-1}B\boldsymbol{u}(i)$$

$$\boldsymbol{y}(k)=CA^k\boldsymbol{x}(0)+C\sum_{i=0}^{k-1}A^{k-i-1}B\boldsymbol{u}(i)+D\boldsymbol{u}(k)$$