可控性：系统输入影响系统所有状态

可观性：系统输出反映系统所有状态

# 一、线性时不变连续系统的可控性

➢ **Definition of Controllability**

State-equations: $\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in T_t$

$x(t) \in R^n,\ u(t) \in R^r,\ A(t) \in R^{n\times n},\ B(t) \in R^{n\times r},\ T_t$: time space

**State controllability**

It is called the state $x_0$ is controllable at $t_0$, if for a non-zero initial state $x(t_0)=x_0$ with the initial time $t_0 \in T$, exists a certain time $t_1 \in T_t$, $t_1 > t_0$ and an unrestricted control $u(t)$, which makes the state transfer from $x(t_0)=x_0$ to $x(t_1)=0$.

**System Controllability**

It is called the system is controllable at time $t_0$, if at time $t_0 \in T_t$, the non-zero initial states in the state space are all controllable.

**Incomplete Controllable**

It is called the system is incomplete controllable, if there are one or some non-zero state variables uncontrollable in the state space.

The controllability of linear time-invariable system has no relation to the initial time $t_0$.

**Cayley- Hamilton Theorem**

Consider a n×n matrix A, whose eigenpolynomial is:

$$f(\lambda) = |\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

Then the matrix A satisfy its eigenpolynomial as well:

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

推论1：矩阵A的k次(k>n)可由A的(n-1)次多项式构造

$$A^k = \sum_{m=0}^{n-1} a_m A^m \quad , k \geq n$$

推论2：矩阵A幂函数可由A的(n-1)次多项式构造

$$e^{At} = \sum_{m=0}^{n-1} a_m(t) A^m$$

**The algebraic criteria of State Controllability :**

**iff (if and only if) the n×n matrix Q is full rank:**

$$rankQ = rank[B \ \ AB \cdots A^{n-1}B] = n$$

**the system is controllable.** n个列向量线性独立

输出可控性

**The n.s. condition of output controllable:**

Iff the $m\times(n+1)r$ dimension output controllability matrix:

$$Q' = [\ CB \ \vdots\ CAB \ \vdots\ CA^2B \ \vdots \cdots \ CA^{n-1}B \ \vdots\ D\ ]$$

Satisfy rank(Q')=m, the system is output controllable.

# 二、线性连续系统可观性

**Completely Observable**

For initial time $t_0 \in T_t$, existing a limitary time $t_1 \in T_t$, $t_1 > t_0$, for all $t \in [t_0, t_1]$, the initial value of the states $x(t_0)$ can be determined uniquely by system output $y(t)$, therefore the system is called Completely Observable in $[t_0, t_1]$.

For the whole time field $[t_0, \infty)$, if the system is observable, the system is observable at $t > t_0$.

If every state $x(t)$ can be observed by $y(t)$ in the period $t_0 \leq t \leq t_1$, the system is Completely Observable.

**Incompletely Observable**

For initial time $t_0 \in T_t$, existing a limitary time $t_1 \in T_t$, $t_1 > t_0$, for all $t \in [t_0, t_1]$, if the initial value of all states $x_i(t_0)$, i=1,2,…,n, cannot be determined by the system output $y(t)$ totally.

In other word, at least one state cannot be determined by y(t), therefore the system is called Incompletely Observable in $[t_0, t_1]$, or Unobservable.

n.s. condition of system Observability

For the linear time-invariable system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

If and only if the rank of the $nm\times n$ dimension observability matrix R is $n$, system is observable.

namely $R = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ $\quad rankR = n$

or $\quad R^T = [\,C^T \ A^TC^T \cdots (A^T)^{n-1}C^T\,] \quad rankR^T = n$

# 三、可控可观对偶性

Consider the following system state space $S_1$ and $S_2$:

$$S_1: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad x \in R^n, u \in R^r, y \in R^m, A \in R^{n\times n}, B \in R^{n\times r}, C \in R^{m\times n}$$

$$S_2: \begin{cases} \dot{z} = A^T z + C^T v \\ n = B^T z \end{cases} \quad z \in R^n, v \in R^m, n \in R^r, A^T \in R^{n\times n}, C^T \in R^{n\times m}, B^T \in R^{r\times n}$$

System $S_1$ and $S_2$ are called dual system.

**Duality Principle:**

if and only if the system $S_1$ is state observable / state controllable, system $S_2$ will be state controllable / state observable.

# 四、线性连续可控可观判据汇总 对离散系仅适用判据1

## 1、可控判据

可控范式

☞Controllable Canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

if the state equation is: $\dot{x} = Ax + Bu$

$x(t) \in R^n, u(t) \in R^r, A \in R^{n\times n}, B \in R^{n\times r}$

判据1

Criterion 1: The linear time-invariable continuous system is state complete controllable if and only if (iff) the controllability matrix $Q_c$ is full rank.

$Q_C = [B \ \ AB \ \ A^2B \ \cdots \ A^{n-1}B]$ $\quad rankQ_c = n$

判据2

Criterion 2: if linear time-invariable system has unequal eigenvalues, the n.s. condition of system controllable is:

the diagonal canonical equation from nonsingular transformation satisfies that there is no zero row in the input matrix $\bar{B}$

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \bar{x} + \bar{B}u$$

以为输入矩阵，B无0行保证输入可控制系统状态，

判据3

Criterion 3: For the Jordan Canonical form $\dot{\bar{x}} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix} \bar{x} + \bar{B}u$

the **rows** of $\bar{B}$ which are corresponding to the **last rows** of Jordan blocks $J_i(i=1,2,\ldots,k)$ are not completely zero.

(If two of Jordan Blocks have the same eigenvalue, the result does not hold.)

## 2、可观判据

可观范式

☞Observable canonical form:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \quad C = [0 \ \cdots \ 0 \ \ 1]$$

判据1

Criterion 1: The linear time-invariable continuous system is state complete observable if and only if the observability matrix $Q_o$ is full rank.

$$Q_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \ (= Q_c^T|_{B\to C}) \qquad \text{Namely, } rankQ_o = n$$

判据2

Criterion 2: if the linear time-invariable continuous system has unequal eigenvalue, the n.s. condition of state observability is that:

the diagonal canonical form of the system has no zero-column in the output matrix $\bar{C}$

$$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \bar{x} \\ y = \overline{cx} \end{cases}$$

判据3

Criterion 3: In the Jordan Canonical Form:

$$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix} \bar{x} \\ y = \overline{cx} \end{cases}$$

the **columns** in $\bar{C}$ corresponding to the **first rows** of Jordan blocks $J_i(i=1,2,\ldots,k)$ are not completely zero.

(If two of Jordan Blocks have the same eigenvalue, the result does not hold.)