一、状态空间状态变量不唯一性

设有状态空间方程 $\dot{x}=Ax+Bu$

将状态变量由 $(x_1,x_2,\cdots,x_n)$ 转换到 $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)$

$$\begin{cases} x_1 = p_{11}\bar{x}_1 + p_{12}\bar{x}_2 + \cdots + p_{1n}\bar{x}_n \\ x_2 = p_{21}\bar{x}_1 + p_{22}\bar{x}_2 + \cdots + p_{2n}\bar{x}_n \\ \cdots\cdots \\ x_n = p_{n1}\bar{x}_1 + p_{n2}\bar{x}_2 + \cdots + p_{nn}\bar{x}_n \end{cases} \qquad x = P\bar{x} \qquad P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

若 P 非奇异 $(|P|\neq 0)$，$\bar{x}$ 也为系统 $\dot{x}=Ax+Bu$ 的一组状态变量

Proof: $\dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu$

Let: $A_1 = P^{-1}AP,\ B_1 = P^{-1}B$

then $\dot{\bar{x}} = A_1\bar{x} + B_1u$

A_1 and A are similar matrices (相似矩阵), the characteristic polynomial: $|sI-A_1| = |sI-A|$

Thus, the state-equations from x and $\bar{x}$ have same eigenvalues

下面讨论几种规范化形式

1、可控型 companion matrix

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$c = [b'_0 \quad b'_1 \quad \cdots \quad b'_{n-1}]$$

2、可观型

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix} \quad b = \begin{bmatrix} b'_0 \\ b'_1 \\ \vdots \\ b'_{n-1} \end{bmatrix}$$

$$c = [0 \quad 0 \quad \cdots \quad 1]$$

3、对角型I

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

4、对角型II

Select state variables: $X_i(s) = \dfrac{c_i}{s-\lambda_i}U(s)$

System Output is: $Y(s) = \sum_{i=1}^{n} X_i(s)$

Inverse Laplace Transformation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} u$$

$$y = [1 \quad 1 \quad \cdots \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

5、约当型 约当块

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & & & \\ & \lambda_1 & 1 & & 0 & \\ & & \lambda_1 & & & \\ & & & \lambda_4 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [c_{11} \quad c_{12} \quad c_{13} \quad c_4 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix}$$

状态空间线性变换目的在于将非规范型转化为规范型

(1) Outlines:

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \xrightarrow[\text{equivalent transformation}]{x=P\bar{x}} \left.\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ \bar{y} &= \bar{C}\bar{x} + \bar{D}u = y \end{aligned}\right\}$$

$(non-canonical)$ $\qquad$ $(canonical)$

$$\because \quad x = P\bar{x}, \quad \dot{x} = P\dot{\bar{x}},$$

P is NxN non-singular constant matrix

Substitute into the equations:

$$\left.\begin{aligned} P\dot{\bar{x}} &= AP\bar{x} + Bu \\ y &= CP\bar{x} + Du \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \dot{\bar{x}} &= P^{-1}AP\bar{x} + P^{-1}Bu \\ y &= CP\bar{x} + Du \end{aligned}\right\}$$

$$\Downarrow$$

$$\bar{A} = P^{-1}AP, \quad \bar{B} = P^{-1}B, \ \bar{C} = CP, \bar{D} = D$$

the constraint satisfied systems {A,B,C,D} and {$\bar{A},\bar{B},\bar{C},\bar{D}$} are similar systems

the related dynamic equations are equivalent euqations

the linear tranformation is called equivalent transformation

二、常用线性变换方法

1、将A变换为对角形式

(a) 设A有n个不同实特征值 $\lambda_1,\cdots,\lambda_n$，满足 $\det(\lambda I-A)=|\lambda I-A|=0$，有

$$\bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

由实特征向量构成的非奇异矩阵 $P=[p_1,\cdots,p_n]$

满足 $Ap_i=\lambda_i p_i$ / $(\lambda_i - A)p_i = 0$

(b) 若A为有n个特征值 $\lambda_1,\cdots,\lambda_n$ 的 campanion matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \\ \lambda_1^{n-1} & & \cdots & \lambda_n^{n-1} \end{bmatrix} \quad \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

P: vandor-mode矩阵

(c) 有n重复实特征根，m-n个不同特征根，若A仍有n个独立特征向量 $p_1,\cdots,p_n$ 满足 $Ap_i=\lambda_i p_i$

构造方式同(a)

2、将矩阵A约当化

(a) 若矩阵A有m个重复实根 $\lambda_1=\cdots=\lambda_m$，有 $(n-m)$ 个不同实根，仅有一个独立特征向量 p_1，

满足 $Ap_i=\lambda_i p_i\ (i=1\sim m)$

则A可转化为唯一约当矩阵

$$J = \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ & \lambda_1 & \ddots & & & 0 & \\ & & \ddots & 1 & & & \\ & & & \lambda_1 & & & \\ & & & & \lambda_{m+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \lambda_n \end{bmatrix}$$

Jordan Block

$P=[p_1,\cdots,p_m \mid \underbrace{p_{m+1},\cdots,p_n}_{n-m\text{个独立特征向量}}]$ 其中 $p_1,p_2,\cdots,p_m$ 满足

$$[p_1,\cdots,p_m]\begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{bmatrix} = A[p_1,\cdots,p_m]$$

(b) 若友矩阵A为可控范式矩阵，有m重特征根 $\lambda_1=\cdots=\lambda_m$，仅有一个独立实特征向量

$$p_1 = [1 \quad \lambda_1 \quad \lambda_1^2 \cdots \lambda_1^{n-1}]^T$$

满足

$$Ap_i = \lambda_i p_i \ (i=1\sim m)$$

有将A约当化的变换矩阵

$$P = \left[p_1 \quad \frac{\partial p_1}{\partial \lambda_1} \quad \frac{\partial^2 p_1}{\partial \lambda_1^2} \cdots \frac{\partial^{m-1} p_1}{\partial \lambda_1^{m-1}} \ \Big| \ p_{m+1} \cdots p_n\right]$$

(c) 设矩阵A有5重特征根

$$\lambda_1 = \cdots = \lambda_5$$

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad (i=1\sim m)$$

and 2 independent real eigenvectors: p_1 and p_2.

Other (n-5) eigenvalues are different.

The matrix A can be transformed to the Jordan form:

$$J = \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & 1 & & & & & \\ & & \lambda_1 & & & & & \\ & & & \lambda_1 & 1 & & & \\ & & & & \lambda_1 & & & \\ & & & & & \lambda_{m+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \end{bmatrix}$$

There are 2 upper Jordan blocks in *J*, in which:

$$P = \left[p_1 \quad \frac{\partial p_1}{\partial \lambda_1} \quad \frac{\partial^2 p_1}{\partial \lambda_1^2} \ \vdots\ p_2 \quad \frac{\partial p_2}{\partial \lambda_1} \ \vdots\ p_6 \quad \cdots \quad p_n\right]$$

3、将可控系统化为可控标准型

- Any controllable system, if its A, b are not controllability canonical form, they can be transformed to the canonical form by appropriate transformation.
- Assume a dynamic system: $\dot{x} = Ax + bu$

Execute the P^{-1} transformation:

$$x = P^{-1}z$$

and we have $\qquad \dot{z} = PAP^{-1}z + Pbu$

Satisfy:

$$PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad Pb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Analyze the transformation matrix: P

Assume P is

$$P = [p_1^T \quad p_2^T \quad \cdots \quad p_n^T]^T$$

Based on the matrix A, P should satisfy:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} A = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}}_{=PAP^{-1}} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}$$

即

$$\begin{cases} p_1A = p_2 \\ p_2A = p_3 = p_1A^2 \\ \vdots \\ p_{n-1}A = p_n = p_1A^{n-1} \\ p_nA = -a_0p_1 - a_1p_2 - \cdots - a_{n-1}p_n \end{cases} \Rightarrow P = \begin{bmatrix} p_1 \\ p_1A \\ \vdots \\ p_1A^{n-1} \end{bmatrix}$$

$$\therefore Pb = p_1\begin{bmatrix} 1 \\ A \\ \cdots \\ A^{n-1} \end{bmatrix} b = p_1\begin{bmatrix} b \\ Ab \\ \cdots \\ A^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore p_1 = [0 \cdots 0 \ 1][b \quad Ab \quad \cdots \quad A^{n-1}b]^{-1}$$

逆可控矩阵的最后一行

The construction methods of transformation matrix $\boxed{P^{-1}}$:

(i) Find the controllability matrix $S = [b \quad Ab \quad \cdots \quad A^{n-1}b]$;

(ii) Find the inverse matrix S^{-1}, which is:

$$S^{-1} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

(iii) Take out the last row of S^{-1}, (the n[th] row) and compose the vector:

$$p_1 = [S_{n1} \quad S_{n2} \quad \cdots \quad S_{nn}]$$

(iv) Construct matrix P

$$P = \begin{bmatrix} p_1 \\ p_1A \\ \vdots \\ p_1A^{n-1} \end{bmatrix}$$

(v) Then, P^{-1} is required transforming matrix from non-canonical form to controllability canonical form.

Obtain the transforming matrix P

If system is completely controllable, there exists nonsingular transformation, which can transform the system to the controllable canonical form.

Define the transforming matrix: **P=QW**

$$Q = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

In which, a_i are coefficients of the eigenpolynomials.

三、SISO系统的范式形式

动态系统状态空间描述

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

设有变换 $x = P\bar{x}$

考虑使用传函下的变换

$$\frac{Y(s)}{U(s)} = \frac{b_0s^n + b_1s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n}$$

有如下范式

① 可控范式 便于设置系统极点

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

分母升幂系数

同幂次系数 $\qquad$ 升幂系数

$$y = [b_n - a_nb_0 \quad b_{n-1} - a_{n-1}b_0 \quad \cdots \quad b_1 - a_1b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0u$$

分子最高次系数

② 可观范式

同幂次系数 $\qquad$ 分子最高次系数

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_nb_0 \\ b_{n-1} - a_{n-1}b_0 \\ \cdots \\ b_1 - a_1b_0 \end{bmatrix} u$$

分母升幂系数顺序 $\qquad$ 升幂系数顺序

$$y = [0 \ 0 \ \cdots \ 0 \ 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0u$$

③ 对角范式

If the roots of denominate polynomial are different, transfer function can be written as follow:

$$\frac{Y(s)}{U(s)} = \frac{b_0s^n + b_1s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)} = b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$

The canonical form is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ 0 & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0u$$

④ 约当范式

If there are repeated eigenvalues in denominator, transform the diagonal form to Jordan form. Assume 3 repeated roots are $-p_1=-p_2=-p_3$ and others are different.

$$\frac{Y(s)}{U(s)} = \frac{b_0s^n + b_1s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)^3(s+p_4)(s+p_5)\cdots(s+p_n)}$$

$$= b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{(s+p_1)} + \frac{c_4}{s+p_4} + \cdots + \frac{c_n}{s+p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & \vdots & & \vdots \\ 0 & 0 & -p_1 & 0 & & 0 \\ 0 & \cdots & 0 & -p_4 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \qquad y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0u$$

四、线性变换的不变性

System: $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \qquad \mathbf{y} = C\mathbf{x} + D\mathbf{u}$

If $\mathbf{x} = P\bar{\mathbf{x}} \qquad \dot{\bar{\mathbf{x}}} = P^{-1}AP\bar{\mathbf{x}} + P^{-1}B\mathbf{u} \qquad \mathbf{y} = CP\bar{\mathbf{x}} + D\mathbf{u}$

- characteristic equation, eigenvalues and eigenvectors

(1). Eigenequation of linear system $\{A, B, C, D\}$

$$f(\lambda) = \det(\lambda I - A) = |\lambda I - A| = 0$$

(2). Eigenvalues λ of matrix A: roots of eigenequation, called characteristic roots, as well.

(3). Eigenvectors: if exist vector ρ and satisfy:

$$\lambda\rho = A\rho$$

ρ is the eigenvector of matrix A according to λ.

(1) Invariance of eigenequation and eigenvalues

Eigenvalues after transformation:

Analysis:

$$\begin{aligned} |\lambda I - P^{-1}AP| &= |\lambda P^{-1}P - P^{-1}AP| = |P^{-1}\lambda P - P^{-1}AP| \\ &= |P^{-1}(\lambda I - A)P| = |P^{-1}||\lambda I - A||P| \\ &= |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\ &= |I||\lambda I - A| = |\lambda I - A| \end{aligned}$$

Obviously, the eigenvalues of the system are same before and after the non-singular linear transformation.

(2) Invariance of system transfer function matrix

The transfer matrix after transformation:

$$\begin{aligned} G'(s) &= CP(sI - P^{-1}AP)^{-1}P^{-1}B + D \\ &= CP(P^{-1}sIP - P^{-1}AP)^{-1}P^{-1}B + D \\ &= CP\left[P^{-1}(sI - A)P\right]^{-1}P^{-1}B + D \\ &= CPP^{-1}(sI - A)^{-1}PP^{-1}B + D \\ &= C(sI - A)^{-1}B + D = G(s) \end{aligned}$$

The transfer matrix of the system is invariant before and after the non-singular linear transformation.