图示描述二阶微分方程解：由点移动导出系统行为模式

一、相平面法基本概念

二阶系统可描述为 $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{\frac{d\dot{x}}{dx}}{\frac{dt}{dx}} = \frac{\frac{d\dot{x}}{dx}}{\frac{1}{\dot{x}}} = \dot{x}\cdot\frac{d\dot{x}}{dx}$

$\ddot{x} = f(x, \dot{x})$

令 $x_1 = x$，$x_2 = \dot{x}$，有

$$\frac{dx_2}{dx_1} = \frac{f(x_1, x_2)}{x_2}$$

或

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) \end{cases}$$

系统状态 → 微分

对所有t的解 $x(t)$

x_1-x_2 平面即为相平面，轴上轨迹即为相轨迹

1、相平面与相轨迹

Phase plane：

The $x_1 - x_2$ plane is called Phase Plane, where x_1, x_2 are the system state and its derivative .

Phase portrait（相图）：

The locus(轨迹）in the x_1-x_2 plane of the solution x(t) for all t。

The family of phase plane trajectories corresponding to various initial conditions is called ***Phase Portrait*** of the system.

2、相轨迹特征

系统方程 $\ddot{x} + f(x, \dot{x}) = 0$

运动方向 $\begin{cases} \dot{x} > 0 \text{ 上半平面 向右运动} \\ \dot{x} < 0 \text{ 下半平面 向左运动} \end{cases}$ ⇒ 顺时针

*奇点：点斜率不定（平衡点）

$$\alpha = \frac{d\dot{x}}{dx} = \frac{\frac{d\dot{x}}{dt}}{\frac{dx}{dt}} = \frac{-f(x, \dot{x})}{\dot{x}} = \frac{0}{0} \Rightarrow \begin{cases} \ddot{x} = 0 \\ \dot{x} = 0 \end{cases}$$

Except the equilibrium points , there is only one phase trajectory passing through any point in the phase plane.**(相平面上除平衡点外的任意一点只有一根相轨迹通过。）** 微分方程解存在性与唯一性

3、相轨迹构造

① 解析法

对任一二阶非线性系统有 $\ddot{x} + f(x, \dot{x}) = 0$

$x_1 = x$　　$x_2 = \dot{x}_1 = \dot{x}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} = -f(x_1, x_2) \end{cases}$$

$$\Rightarrow \frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = -\frac{f(x_1, x_2)}{x_2}$$

有广义系统方程

$$\frac{dx_2}{dx_1} = \frac{Q(x_1, x_2)}{P(x_1, x_2)}$$

$\dot{x}_2$

$\dot{x}_1$ 在 (x_1, x_2) 点轨迹斜率

If $P(x_1, x_2)$ 、 $Q(x_1, x_2)$ is analytic ，the differential equation can then be solved.

Given a initial condition, the solution can be plotted in the phase plane. This curve is named Phase trajectory（相轨迹）. The family of phase plane trajectories is called Phase plane portrait（相平面图）.

令 $\begin{cases} P(x_1, x_2) = 0 \\ Q(x_1, x_2) = 0 \end{cases}$ 得点 (x_{10}, x_{20}) 为系统奇点、

② 等倾线法

$\alpha = \frac{d\dot{x}}{dx}$ 为相轨迹在某点切线斜率，令α为一常量可确定 x-$\dot{x}$ 平面上一条满足该关系的曲线（即等倾斜）。相轨迹必然以斜率α经过等倾线，给定不同数值可得一簇等倾线，以初始条件起点递推作图可得相轨迹

求 $\frac{d\dot{x}}{x} = \alpha$ 后化简得 $\dot{x} = f(\alpha, x)$，定下α可得等倾线

二、线性系统相轨迹

PPT P23-P35

三、奇点、平衡点与极限环

1、奇点 → 仅能在x轴上

令奇点为 (x_{10}, x_{20}) 有如下方程组

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) = 0 \\ \dot{x}_2 = Q(x_1, x_2) = 0 \end{cases}$$

对 $P(x_1, x_2)$，$Q(x_1, x_2)$ 使用泰勒展开

Ignoring the higher-order terms, without loss of generality we assume that $x_{10} = x_{20} = 0$

then

$$P(x_1, x_2) = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$Q(x_1, x_2) = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

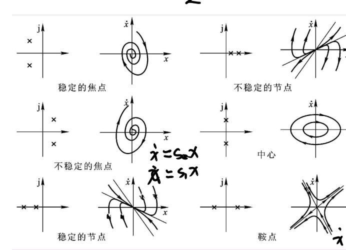
Assume $a = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)}$　$b = \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)}$

$c = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)}$　$d = \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)}$

故 $\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases}$，特征方程 $|\lambda I - A|^2 = \lambda^2 - (a+b)\lambda + (ad-bc) = 0$

则 $\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

依据哈特曼-格罗布曼定理：若一次近似方程的奇点属于节点、焦点和鞍点，则非线性方程的奇点也属于同一类型



$\dot{x} = s_2 x$　$\dot{x} = s_1 x$

不是极限环

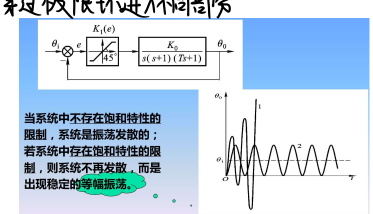
$\dot{x} = s_1 x$　$\dot{x} = s_2 x$

2、极限环 → 非线性系统特有，由非线性特性引起

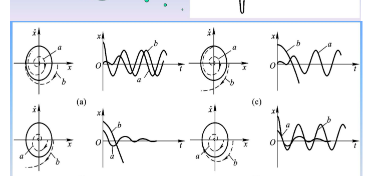
在相平面中一个封闭独立相轨迹被称为极限环

谐振系统中存在极限环，将相平面划分为两部分，任一相轨迹均不能穿过极限环进入不同部分

常见（饱和非线性）



当系统中不存在饱和特性的限制，系统是振荡发散的；若系统中存在饱和特性的限制，则系统不再发散，而是出现稳定的等幅振荡。



3、相平面法分析非线性系统

Algorithm of phase plane analysis(分段线性化)

1. Select ***appropriate coordinate axis*** during the analysis.
2. Divide the phase plane into ***several areas*** according to nonlinear characteristics. Establish ***linear differential equations*** for each area.
3. Establishing equations for the ***switching lines（开关线、切换线）*** in the phase plane according to different nonlinear characteristics.
4. ***Solve the differential equations*** of each area and then draw phase trajectory.
5. The phase trajectory of the whole system can be obtained by ***connecting all the partial trajectories*** in different areas.