Problem Set 9

Problem Sets due Jul 26, 2022 23:00 PKT Completed

Problems 1-6 correspond to "Linear Projections"

Problem 1

1/1 point (graded)

In $\mathbb{R}^2$, what is the unit vector corresponding to the x_1-direction?

- ( ) $(0, 0)$
- (•) $(1, 0)$
- ( ) $(0, 1)$
- ( ) $(1, 1)$

✔

Submit

Problem 2

1/1 point (graded)

What is the unit vector in the same direction as $(3, 2, 2, 2, 2)$?

- ( ) $(1.5, 1, 1, 1, 1)$
- ( ) $(1, 0.67, 0.67, 0.67, 0.67)$
- (•) $(0.6, 0.4, 0.4, 0.4, 0.4)$
- ( ) $(0.5, 0.33, 0.33, 0.33, 0.33)$

✔

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Problem 3

1/1 point (graded)

What is the projection of the vector $(3, 5, -9)$ onto the direction $(0.6, -0.8, 0)$?

-2.2 ✔

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Problem 4

Problem 4

1/1 point (graded)

What is the (unit) direction along which the projection of $(4,\ -3)$ is largest?

- (●) $(0.8,\ -0.6)$
- ( ) $(-0.6,\ -0.8)$
- ( ) $(-0.8, 0.6)$
- ( ) $(0.8, 0.6)$

✔

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Problem 5

1/1 point (graded)

What is the (unit) direction along which the projection of $(4,\ -3)$ is smallest?

- ( ) $(0.8,\ -0.6)$
- ( ) $(-0.6,\ -0.8)$
- (●) $(-0.8, 0.6)$
- ( ) $(0.8, 0.6)$

✔

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Problem 6

1/1 point (graded)

The projection of vector x onto direction u is exactly zero. Which of the following statements is necessarily true? Select all that apply.

- [x] u is orthogonal to x.
- [ ] u is in the opposite direction to x.
- [x] u is at right angles to x.
- [ ] It is not possible to have a projection of zero.

✔

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Problems 7-8 correspond to "Principal component analysis I: one-dimensional projection"

Problem 7

4/4 points (graded)

A three-dimensional data set has covariance matrix

$$\Sigma = \begin{pmatrix} 4 & 2 & -3 \\ 2 & 9 & 0 \\ -3 & 0 & 9 \end{pmatrix}.$$

a) What is the variance of the data in the x_1-direction?

4

b) What is the correlation between x_1 and x_3?

-0.5

c) What is the variance in the direction $(0, -1, 0)$?

9

d) What is the variance in the direction of $(1, 1, 0)$?

8.5

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Problem 8

1/1 point (graded)

Which of the following covariance matrices has the property that the variance is the same in any direction? Select all that apply.

- [x] The all-zeros matrix.
- [ ] The all-ones matrix.
- [x] The identity matrix.
- [ ] Any diagonal matrix.

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Problems 9-11 correspond to "Principal component analysis II: the top k directions"

Problem 9

8/8 points (graded)

Let $u_1, u_2 \in \mathbb{R}^d$ be two vectors with $\|u_1\| = \|u_2\| = 1$ and $u_1 \cdot u_2 = 0$. Define U to be the matrix whose columns are u_1 and u_2.

What are the dimensions of the following matrices?

a) U

# of Rows =

d

# of Columns =

2

b) U^T

# of Rows =

2

# of Columns

d

c) UU^T

# of Rows =

d

# of Columns =

d

d) $u_1 u_1^T$

# of Rows =

d

# of Columns =

d

Submit

Problem 10

1/1 point (graded)

Continuing from the previous problem, let $u_1, u_2 \in \mathbb{R}^d$ be two vectors with $\|u_1\| = \|u_2\| = 1$ and $u_1 \cdot u_2 = 0$, and define U to be the matrix whose columns are u_1 and u_2.

Which of the following linear transformations sends points $x \in \mathbb{R}^d$ to their (two-dimensional) projections onto directions u_1 and u_2? Select all that apply.

- [x] $x \mapsto (u_1 \cdot x, u_2 \cdot x)$
- [ ] $x \mapsto (u_1 \cdot x)u_1 + (u_2 \cdot x)u_2$
- [x] $x \mapsto U^T x$
- [ ] $x \mapsto UU^T x$

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Problem 11

2/2 points (graded)

For a particular four-dimensional data set, the top two eigenvectors of the covariance matrix are:

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

a) What is the PCA projection of point $(2, 4, 2, 6)$ into two dimensions? Write it in the form (a, b).

- ( ) $(2, 2)$
- ( ) $(2, 3)$
- (•) $(7, 3)$
- ( ) $(4, 6)$

b) What is the reconstruction, from this projection, to a point in the original four-dimensional space? Write it in the form (a, b, c, d)

- (•) $(2, 5, 2, 5)$
- ( ) $(2, 1, 2, 2)$
- ( ) $(4, 2, 2, 2)$
- ( ) $(2, 6, 2, 4)$

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Problems 12-14 correspond to "Linear algebra V: eigenvalues and eigenvectors"

Problem 12

2/2 points (graded)

Consider the 2×2 matrix $M = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$.

a) One of its eigenvectors is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. What is the corresponding eigenvalue?

6

b) Its other eigenvector is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. What is the corresponding eigenvalue?

4

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Problem 13

6/6 points (graded)

A 2×2 matrix M has eigenvalues 10 and 5.

a) What are the eigenvalues of $2M$ (that is, each entry of M is multiplied by 2)?

Larger eigenvalue =

20

Smaller eigenvalue =

10

b) What are the eigenvalues of $M + 3I$, where I is the 2×2 identity matrix?

Larger eigenvalue =

13 ✔

Smaller eigenvalue =

8 ✔

c) What are the eigenvalues of $M^2 = MM$?

Larger eigenvalue =

100 ✔

Smaller eigenvalue =

25 ✔

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Problem 14

7/7 points (graded)

A certain three-dimensional data set has covariance matrix

$$\begin{pmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

.

a) Consider the direction $u = (1, 1, 1)/\sqrt{3}$. What is variance of the projection of the data onto direction u?

8/3 ✔

b) Which of the following are eigenvectors of the covariance matrix? Select all that apply.

- [ ] $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

- [ ] $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- [x] $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- [x] $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
- [ ] $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
- [x] $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

✓

c) Find the eigenvalues of the covariance matrix. List them in decreasing order.

8 ✓

4 ✓

2 ✓

d) Suppose we used principal component analysis (PCA) to project points into *two* dimensions. What would be the resulting two-dimensional projection of the point $x = (\sqrt{2}, -3\sqrt{2}, 2)$?

- ○ $(1, 0)$
- ◉ $(4, 2)$
- ○ $(1, 4)$
- ○ $(4, 1)$

✓

e) Now suppose we use the projection in (d) to reconstruct a point $\hat{x}$ in the original three-dimensional space. What is the Euclidean distance between x and $\hat{x}$, that is, $\|x - \hat{x}\|$?

2

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Problems 15-17 correspond to "Linear algebra VI: spectral decomposition"

Problem 15

1/1 point (graded)
M is a 2×2 real-valued symmetric matrix with eigenvalues $\lambda_1 = 6, \lambda_2 = 1$ and corresponding eigenvectors

$$u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

What is M?

- ( ) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- ( ) $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$
- ( ) $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$
- (•) $\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$

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Problem 16

1/1 point (graded)
For a certain data set in d-dimensional space, the covariance matrix has the following interesting property: there are k positive eigenvalues and the rest are zero (where $k < d$). What can we conclude from this? Select all that apply.

- [ ] Each of the data points has at most k nonzero coordinates.
- [x] The data can be perfectly reconstructed from their PCA projection onto k dimensions.

- [x] Each data point can be expressed as a linear combination of the top k eigenvectors.
- [ ] It is possible to discard $d - k$ of the coordinates without losing any of the variance in the data.

✓

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Problem 17

1/1 point (graded)

A data set in $\mathbb{R}^d$ has a covariance matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Under which of the following conditions is PCA most likely to be effective as a form of dimensionality reduction? Select all that apply.

- [ ] When the λ_i are approximately equal.
- [x] When most of the λ_i are close to zero.
- [ ] When most of the λ_i are close to 1.
- [x] When the sequence $\lambda_1, \lambda_2, \ldots$ is rapidly decreasing.

✓

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