

# Polynomial-Solvability of $\mathcal{NP}$ -class Problems

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## Abstract

Let Hamiltonian complement of the graph  $G = (V(G), E(G))$  be the minimal cardinality set  $H(G) \subset V(G) \times V(G)$  such that graph  $(V(G), E(G) \cup H(G))$  is a Hamiltonian one. Possibility to recognize the cardinality of Hamiltonian complement  $H(G)$  based on reduction to solving the linear programming problem is presented in this paper. Polynomial solvability of  $\mathcal{NP}$ -class follows from the fact that  $\mathcal{NP}$ -complete problem "Hamiltonian circute" is special case of the problem under consideration.

*Keywords:*

computational complexity, graph theory, Hamiltonian path, network programming, P vs NP

## 1. Introduction

One of the top problems in the theory of algorithms is  $\mathcal{P}$  vs  $\mathcal{NP}$  [1], here  $\mathcal{P}$  is the class of problems solvable by algorithms with polynomial computational complexity by deterministic machines,  $\mathcal{NP}$  is the class of recognition problems solvable by nondeterministic machines in polynomial time.

The foundation of  $\mathcal{NP}$ -completeness theory was laid down by S. Cook [2]. He introduced the class  $\mathcal{NP}$  of recognition problems, and the concepts of polynomial reducibility and  $\mathcal{NP}$ -complete problem [3].  $\mathcal{NP}$ -completeness of a wide range of recognition problems was proved. In particular, this also applies to the "Hamiltonian circuit" problem [4]. The most complete guide to the theory of  $\mathcal{NP}$ -completeness is the monograph [5].

Currently, the question "Are  $\mathcal{NP}$ -complete problems difficult to solve?" is considered one of the main issues of modern mathematics [1]. It is known that the proof of the possibility to solve at least one  $\mathcal{NP}$ -complete problem by a polynomial algorithm be a proof of the coincidence of the classes  $\mathcal{P}$  and  $\mathcal{NP}$ .

The polynomial method for reducing the problem of recognizing the cardinality of a Hamiltonian complementation of a graph to solve the linear programming problem is proposed in this preprint. It proves polynomial-solvability of all problems of  $\mathcal{NP}$ -class.

### Hamiltonian covering of the graph

**Definition 1.** Let Hamiltonian complementation of graph  $G = (V(G), E(G))$  be minimal cardinality set  $H(G) \subset V(G) \times V(G)$  such that the graph  $(V(G), E(G) \cup H(G))$  is a Hamiltonian one.

Obviously, the problem to recognize the Hamiltonian complement cardinality is a generalization of the  $\mathcal{NP}$ -complete problem "Hamiltonian circuit" [4]. Consequently, the existence of a polynomial algorithm for the problem of recognizing the Hamiltonian complement cardinality is a proof of the polynomial solvability of  $\mathcal{NP}$ -class problems.

### Chain location problem for graph $G$

Let  $C$  be a chain with the set of vertices  $V(C) = \{c_1, c_2, \dots, c_n\}$  and the set of edges  $E(C) = \{\{c_i, c_{i+1}\} : i = 1, 2, \dots, n\}$ .

The challenge to recognize the existence of a Hamiltonian path in graph  $G$  is equivalent to the problem of recognition of the bijection

$$\varphi : V(C) \leftrightarrow V(G) : \{\varphi(c_i), \varphi(c_{i+1})\} \in E(G), \quad i = 1, 2, \dots, n - 1.$$

existence.

Task of recognizing existence of the bijection can be represented as a Boolean quadratic programming problem. Indeed, let us define

$$x = \{x_v^i : i = 1, 2, \dots, n; v \in V(G)\}$$

as following

$$x_v^i = \delta_v^{\varphi(c_i)} = \begin{cases} 1, & \text{if } \varphi(c_i) = v, \\ 0, & \text{if } \varphi(c_i) \neq v, \end{cases} \quad i = 1, 2, \dots, n, v \in V(G). \quad (1)$$

It is clear that the definition of elements of the set  $x$  by (1) establishes a one-to-one correspondence between the mapping  $\varphi : V(C) \rightarrow V(G)$  and a

point of the unit cube  $\{0, 1\}^{n^2}$ . *Unambiguous* mappings  $\varphi$  correspond to the vertices of the unit cube, belonging to the set

$$D_1 = \left\{ x : \sum_{v \in V(G)} x_v^i = 1, i = 1, 2, \dots, n, x \geq 0 \right\} \quad (2)$$

because restrictions (2) of the problem are the requirement that every element of  $c_i \in V(C)$  receives exactly one destination.

*Surjective* mappings of  $\varphi$  correspond to the vertices of the unit cube belonging to the set

$$D_2 = \left\{ x : \sum_{i=1}^n x_v^i = 1, v \in V(G) \right\} \quad (3)$$

because group of restrictions (3) is the requirement that each element  $v \in V(G)$  is assigned a unique element  $c_i \in V(C)$ . *Bijective* mappings of  $\varphi$  correspond to the vertices of the unit cube

$$D_3 = \left\{ x_v^i \in \{0, 1\} : i = 1, 2, \dots, n - 1, v \in V(G) \right\}$$

belonging to the set  $D_1$ , and the set  $D_2$ , i.e. all elements of the set

$$D = D_1 \cap D_2 \cap D_3.$$

Let us consider the Boolean optimization problem

$$F(x) = \sum_{i=1}^{n-1} \left( \sum_{v,u \in V(G) : \{v,u\} \notin E(G)} x_v^i x_u^{i+1} \right) \rightarrow \min_{x \in D}. \quad (4)$$

The value of the objective function  $F(x)$  is equal to the number of edges in the set  $\overline{E(G)} = [V(G)]^2 \setminus E(G)$  that is image  $\varphi(C)$  of the arranged chain  $C$ .

**Proposition 1.** *Let  $x^*$  be an optimal solution of the problem (4). The graph  $G$  contains a Hamiltonian path if and only if  $F(x^*) = 0$ .*

**Proposition 2.** *Let  $x^*$  be an optimal solution of the problem (4), then  $|H(G)| = F(x^*)$ .*

### Presentation of the problem (4) as the ILP problem

Let us introduce boolean variables

$$y_{(u,v)}^{(i,i+1)} = x_u^i x_v^{i+1}, \quad i = 1, 2, \dots, n-1, \quad u, v \in V(G). \quad (5)$$

It follows from (2) and (5) that

$$\begin{aligned} \sum_{v \in V(G)} y_{(u,v)}^{(i,i+1)} &= x_u^i \sum_{v \in V(G)} x_v^{i+1} = x_u^i, \\ \sum_{u \in V(G)} y_{(u,v)}^{(i,i+1)} &= x_v^{i+1} \sum_{u \in V(G)} x_u^i = x_v^{i+1}. \end{aligned}$$

Let us consider the boolean linear programming problem

$$F_Q(x, y) = \sum_{i=1}^{n-1} \sum_{v,u \in V(G)} y_{(v,u)}^{(i,i+1)} \chi_{\overline{E(G)}}(\{u, v\}) \rightarrow \min_{\substack{x \in D, \\ (x,y) \in M}} \quad (6)$$

where

$$M = \tilde{M} \cap \left[ D_3 \times \left\{ y_{(u,v)}^{(i,i+1)} \in \{0, 1\} : i = 1, 2, \dots, n-1, u, v \in V(G) \right\} \right], \quad (7)$$

$$\tilde{M} = \left\{ (x, y) \geq 0 \mid \begin{aligned} &\sum_{w \in V(G)} y_{(u,w)}^{(i,i+1)} = x_u^i, \\ &\sum_{w \in V(G)} y_{(w,v)}^{(i,i+1)} = x_v^{i+1}, \quad i = 1, 2, \dots, n-1, \quad u, v \in V(G) \end{aligned} \right\}. \quad (8)$$

Note that the system  $\tilde{M}$  of restrictions differs from the system  $M$  of restrictions by the lack of integrality conditions. Later the set  $\tilde{M}$  is used for construction of relaxed problems.

It follows from (6) that

**Proposition 3.** *A necessary and sufficient condition for optimality of the problem (4) solution  $x^*$  is the optimality of the problem (6) solution  $(x^*, y^*)$ .*

*Proof.* Let  $x^{(1)}$  be the optimal solution of problem (4). Let us define  $y^{(1)}$  in accordance with (5). Then  $(x^{(1)}, y^{(1)})$  is a valid problem (6) solution. Conversely, if  $(x^{(2)}, y^{(2)})$  is the problem (6) optimal solution, then (6) – (8) imply that  $x^{(2)}$  is a valid problem (4) solution. In this way,

$$F(x^{(1)}) = F_Q((x^{(1)}, y^{(1)})) \geq F_Q((x^{(2)}, y^{(2)})) = F(x^{(2)}) \geq F(x^{(1)}). \quad (9)$$

Consequently, equalities hold in the chain (9), i.e. all considered in the proof solutions are optimal solutions of the corresponding problems.  $\square$

### Relaxed chain location problem

Let us consider the relaxation of problem (6)

$$F_Q(x, y) = \sum_{i=1}^{n-1} \sum_{v, u \in V(G)} y_{(v, u)}^{(i, i+1)} \chi_{\overline{E(G)}}(\{u, v\}) \rightarrow \min_{\substack{x \in D_1 \cap D_2 \\ (x, y) \in \tilde{M}}}. \quad (10)$$

Constraints of relaxed problem (10) are different from the constraints of source problem (6) by absence of integrity restriction  $x \in D_3$ .

The dual to (10) problem is the following

$$F_Q^*(\xi, \eta, \lambda) = \sum_{i=1}^n \xi_i + \sum_{v \in V(G)} \lambda_v \rightarrow \max_{(\xi, \eta, \lambda) \in \tilde{M}_Q^*} \quad (11)$$

where the feasible set

$$\begin{aligned} \tilde{M}^* = \left\{ (\xi, \lambda, \eta) : \xi_1 - \eta_v^{(1, 2)} \leq \lambda_v, \xi_n - \eta_v^{(n, n-1)} \leq \lambda_v, v \in V(G), \right. \\ \left. \xi_i - \eta_v^{(i, i+1)} - \eta_v^{(i, i-1)} \leq \lambda_v, \quad 2 \leq i \leq n-1, v \in V(G), \right. \\ \left. \eta_v^{(i, i+1)} + \eta_u^{(i+1, i)} \leq \chi_{\overline{E(G)}}(\{u, v\}), 1 \leq i \leq n-1, u, v \in V(G). \right\}, \end{aligned} \quad (12)$$

here variables  $\xi$  correspond to the restrictions of the set  $D_1$ , variables  $\lambda$  correspond to restrictions of the set  $D_2$ , variables  $\eta$  correspond to restrictions of the set  $\tilde{M}$ .

Let us introduce the subset

$$\tilde{L}^* = \{(\xi, \lambda, \eta) \in \tilde{M}^* : \sum_{v \in V} \lambda_v = 0\} \subset \tilde{M}^*$$

and the problem

$$\tilde{F}_{Q_L}^*(\xi, \lambda, \eta) = \sum_{i=1}^n \xi^i \rightarrow \max_{(\xi, \lambda, \eta) \in \tilde{L}^*}. \quad (13)$$

**Proposition 4.** All optimal solutions of problem (13) are optimal solutions of problem (11).

*Proof.* Let us put

$$(\xi^\pi = \xi + \hat{\lambda}, \lambda^\pi = \lambda - \hat{\lambda}, \eta^\pi = \eta), \quad \hat{\lambda} = \left( \frac{\sum_{v \in V} \lambda_v}{|V|} \right) \mathbf{e}.$$

Obviously,

$$(\xi^\pi, \lambda^\pi, \eta^\pi) \in \tilde{M}^*, \quad \sum_{v \in V} \lambda_v^\pi = 0.$$

So  $(\xi^\pi, \lambda^\pi, \eta^\pi) \in \tilde{L}^*$ , and

$$\sum_{i=1}^n (\xi^\pi)^i = \sum_{v \in V} \lambda_v + \sum_{i=1}^n \xi^i.$$

Proposition 4 is proved.  $\square$

**Theorem 1.** The set of optimal solutions of the relaxed problem (10) contains an integer solution.

*Proof.* Let

$$(\xi^*, \lambda^*, \eta^*) = \arg \max_{(\xi, \lambda, \eta) \in \tilde{M}^*} \left( \sum_{v \in V} \lambda_v + \sum_{i=1}^n \xi^i \right)$$

be optimal solutions of problem (11).

It is easy to see that for fixed values of dual variables  $\lambda^*$  the problem (11) turns out to be a dual problem for the problem  $\Theta_W(\lambda^*)$ :

$$\begin{aligned} F_W(\lambda^*)(x, y) = & - \sum_{i \in V(C), v \in V(G)} \lambda_v^* x_v^i + \\ & \sum_{(i, i^+) \in E(C)} \sum_{v, u \in V(G)} y_{(v, u)}^{(i, i^+)} \chi_{\overline{E(G)}}(\{u, v\}) \rightarrow \min_{\substack{x \in D_1 \\ (x, y) \in \tilde{M}}} . \end{aligned} \quad (14)$$

Here, in contrast to problem (4), the surjective condition (3) is absent, and the cost  $\lambda_v^*$  of placing of vertices  $i \in V(C)$  onto vertices  $v \in V(G)$  is added.

**Proposition 5.** All optimal solutions of problem (14) belong to the convex hull of its integer optimal solutions.

*Proof.* Let us introduce graph  $\mathcal{G}$  with vertexes set

$$V(\mathcal{G}) = \{v_0, v_{n+1}\} \cup [V(G) \times \{1, 2, \dots, n\}]$$

and edges set

$$\begin{aligned} E(\mathcal{G}) = & \left[ \bigcup_{v \in V(G)} \{(v_0, (v, 1))\} \right] \cup \left[ \bigcup_{v \in V(G)} \{((v, n), v_{n+1})\} \right] \\ & \cup \left[ \bigcup_{i=1}^{n-1} \left( \bigcup_{u, v \in V(G)} \{((u, i), (v, i+1))\} \right) \right]. \end{aligned}$$

Problem (14) in terms of  $\mathcal{G}$  is the problem of finding the minimum weight path between vertices  $v_0$  and  $v_{n+1}$ , provided that the weights of the vertices  $(v, i) \in V(\mathcal{G})$  equal to  $(-\lambda_v)$ , the weights of the edges  $\{(u, i), (v, i+1)\} \in E(\mathcal{G})$  are equal to  $\chi_{\overline{E(G)}}(\{u, v\})$ , the weights of the vertices  $v_0, v_{n+1}$  and the edges incident to them are zero.

We can find the set

$$S = \left\{ (x^o, y^o)^{(k)} = \arg \min_{(x, y) \in \tilde{M}} F_W(\lambda^*)(x, y), k = 1, 2, \dots, K \right\} \quad (15)$$

of all optimal solutions using the known shortest path algorithms. Obviously, these solutions satisfy the condition  $x^o \in D_1 \cap D_3$ .

The restriction matrix of the problem  $F_W(\lambda^*)$  is completely unimodular. Consequently the set of all optimal solutions of the problem  $F_W(\lambda^*)$  represents the convex hull of  $\text{Conv } S$  of the set of its optimal integer solutions (that is defining by all optimal paths between the vertices  $v_0$  and  $v_{n+1}$ ).  $\square$

Let us show that the chain of relations

$$\begin{aligned} \min_{\substack{x \in D_1 \cap D_2, \\ (x, y) \in \tilde{M}}} F_Q(x, y) &= \max_{(\xi, \lambda, \eta) \in \tilde{M}^*} \tilde{F}_Q^*(\xi, \lambda, \eta) = \\ &= \tilde{F}_Q^*(\xi^*, \lambda^*, \eta^*) = \max_{(\xi, \eta): (\xi, \lambda^*, \eta) \in \tilde{L}^*} \tilde{F}_Q^*(\xi, \lambda^*, \eta) = \\ &= \min_{\substack{x \in D_1, \\ (x, y) \in \tilde{M}}} F_W(\lambda^*)(x, y) = \min_{\substack{x \in D_1 \cap D_3, \\ (x, y) \in \tilde{M}}} F_W(\lambda^*)(x, y) \leq \\ &\leq \min_{\substack{x \in D_1 \cap D_2, \\ (x, y) \in \tilde{M}}} F_W(\lambda^*)(x, y) = \min_{\substack{x \in D_1 \cap D_2, \\ (x, y) \in \tilde{M}}} F_Q(x, y). \quad (16) \end{aligned}$$

is hold.

The first equality is a consequence of the linear programming first duality theorem for the pair of mutually dual problems (10) and (11). The next two equalities follow from the theorem conditions and Proposition 4. The fourth equality is a consequence of the first linear programming duality theorem for a pair of mutually dual problems (14) and (11). The fifth equality is a consequence of the Proposition 5. Inequality is the result of the restriction of an admissible set for a minimum function. The last equality follows from the definition of the set  $\tilde{L}^*$  and the inclusion  $x \in D_2$ .

So, the optimal value of problem (14) for  $\lambda^* \in \tilde{L}^*$  coincides with the optimal value of problems (10) and (11). In accordance with the Proposition 5, the optimal solution  $(x^*, y^*)$  belongs to the set  $D_2 \cap \text{Conv } S$ , but this is possible only if  $S \subset D_1 \cap D_2 \cap D_3$ . Indeed, the inclusion of  $S \subset D_1 \cap D_3$  is a consequence of Proposition 5. The assumption  $S \not\subset D_2 \cap D_3$  contradicts to optimality of  $\lambda^*$ . Theorem now follows.  $\square$

The proof of the theorem establishes the existence of an optimal integer solution of the problem (10), but does not give an algorithm for finding this solution. Nevertheless, the existence of an optimal solution of the problem (10) makes it possible to determine the Hamiltonian complementation cardinality.

**Corollary 1.** *The optimal value of the problem (11) for the graph  $G$  is equal to the cardinality of the Hamiltonian complementation  $|H(G)|$ .*

## Conclusion

Problem (11) represents a linear programming problem to solve which the polynomial algorithms [6] are known. The recognizing problem of the presence of a Hamiltonian circuit in a graph belongs to the class  $\mathcal{NP}$  [4]. Hence, we have proved the theorem

**Theorem 2.** *All problems of  $\mathcal{NP}$  class are polynomial-solvable with deterministic machine.*

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