The Complexity Of The NP-Class

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Abstract

This paper presents a novel and straight formulation, and gives a complete insight towards the understanding of the complexity of the problems of the so called NP-Class. In particular, this paper focuses in the Searching of the Optimal Geometrical Structures and the Travelling Salesman Problems. The main results are the polynomial reduction procedure and the solution to the Noted Conjecture of the NP-Class.

Algorithms, NP, Numerical optimization.

1 Introduction

Over the previous century the problem of the complexity of the NP-Class problems has been remain open. The broad types of problems are quite important the science and the technology. Just to mention nano-technologies, design of drugs, and process optimization have driven many researchers to design algorithms, and to publish thousands of papers devoted to improve and solve them. However besides the equivalence of members of the NP-Class no unique efficient algorithm has been designed and moreover the possible determination of complexity has not differentiate the P-Class from the NP-Class.

In order to study in details the complexity of the NP-Class, the following points are presented in this paper.

1. Research Space, Complexity, and Properties of two NP-Problems.

2. Polynomial Reduction Procedure.

The main result shows, that the complexity of the NP-Class is exponential and lacks of a optimization property in their structure for arbitrary and large instances of NP-Problems. Also, It is presented that the special case of the 2D Euclidian Traveller Salesman Problem has polynomial complexity. This modifies the frontier's spectrum of computational complexity [14], in the sense that some problem classified as intractable changes to tractable.

The next proposition is the key of this research.

Proposition 1.1. *The problems of the NP-Class have not an polynomial algorithm for checking their solution.*

To explain its importance, let us to assume the contrary, i.e., a polynomial algorithm for checking the solution exists for any arbitrary and large instances of a NP-Problem. Therefore, it is possibly to use and joining it with a polynomial research algorithm to come out with a joining polynomial algorithm to solve any NP-Problems, furthermore, the properties used into the checking algorithm could help to design and improve the joining algorithm. On the other hand, if it exists the polynomial algorithm able to solve any arbitrary and large NP-Problem it must has an algorithm for verifying the solution, with complexity no more than polynomial time, otherwise there is not support to accept its solution. Therefore, it is a reasonable step towards the design of a polynomial algorithm able to solve any NP-Problem to try to design first the checking algorithm of the solution. Also, the design of an algorithm is based in structures and properties from the problem, and it is not always easy to relate the length's data input with the execution time. It is necessary to explore problem, data, and properties to understand and design an algorithm.

Definition 1.2. Notation

1. The real number are denoted by $\mathbb{R}$, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\inf, \inf\}$. Also, natural number are denoted by $\mathbb{N}$.

2. LJP_n stands for the Problem of Searching Optimal Geometrical Structures of n particles under the Lennard-Jones Potential.

3. GAP_n is the General Assign Problem of size n.

4. TSP_n is the traveller salesman problem of size n.

5. Euclidian mD TSP_n stands for the version of the TSP_n where the vertices are like cities in a real metric space of m-dimension, i.e., the vertices are points in $\mathbb{R}^m$.

6. The Research Space of a problem P_n is all valid and possible entries that comply with the properties of P_n.

7. Integer formulation are denoted by IP, where P denotes GAP or TSP.

8. The complexity is denoted by $\mathbf{O}(f(n))$. Polynomial time means n^p, $p < n$ where n is the length's data input, and p is a positive integer.

The NP-Problems are isomorphous and it is worth to study in deep some of them for understanding and take advantage of their properties, restrictions and structures. This is the reason to analyze LJP_n and GAP_n. The former is a particular case of the Searching of the Optimal Geometrical Structures of clusters of n particles under the potential function of Lennard-Jones. GAP_n is the general formulation of the TSP_n. The traveller salesman problem has two interesting cases Euclidian and arbitrary. Here an arbitrary TSP_n means that edges' cost are not from a norm or distance of the vertices as a points in $\mathbb{R}^m$ but from arbitrary real values ≥ 0. The two next section cover an study of these NP-problems. Section 2 describes the state of the art of LJP_n. GAP_n and its related problems are cover in section 3. The section 4 has the integer formulation of GAP_n. The section 5 presents the computational model of the Turing Machine and states a relation to an algorithm for checking the solution of GAP_n. The section 6 describes the main results of the complexity of GAP_n, and the last section contains conclusions and future work.

2 The Lennard-Jones Problem

The chemistry, physics, and applied mathematics academic community had dedicated thousand of articles because the importance of understanding of how the nature builds matter or beings from particles. The Searching of the Optimal Geometrical Structures of n particles consists to determine under a pairwise potential the geometrical shape of this finite group of n particles, (also, the result is called an optimal cluster of size n) that correspond to the minimum potential. There are many pairwise potential functions Buckingham (BU) Potential, Kihara Potential, Lennard-Jones (LJ) potential, Morse Potential, and so on. There are many methods to address this problem and some authors had been found that for LJ clusters of n (≤ 147) their algorithms have polynomial complexity but no generalization have claimed for large cluster's size.

The putative optimal LJ clusters (potential and particles' 3D coordinates) had been shared by the Wales, Doye et al. in their WWW site "The Cambridge Cluster Database (CCD)" (see [15]), Shao's articles(see [5, 4, 7, 11, 12, 13, 16]), and Barrón et. al's articles (see [3, 10].

In my experience, the success of some methods are based in the use of lattices to select an initial cluster of n particles and then, a minimization or relaxation procedure applied to it to estimate its minimum potential. I share this idea with Shao in a personal communication and he and his group achieve to repeat the previous results for $n \leq 309$ and states the new putative optimal clusters up to 1610 particles. Northby in his seminal article [9] stated the conjecture of the existence of a growing sequence in the lattice IC but the putative optimal clusters up to 1610 particles show that it is a local property without possibility to generalize to other lattices. There is also possible that even clusters with big magic number could be not longer the putative optimal cluster because of the central vacancy. Finally, the experience given by the putative optimal clusters up to 1610 do not provide general properties to reduce the alternatives to find out optimal cluster in an efficient way.

The Methods based in lattices drive my study towards this type of research space, using a version of a genetic algorithm adapted to the lattice IF in the article "Minimum search space and efficient methods for structural cluster optimization" was possible to repeat Shao group's results and found out new clusters, and also to state that the lack of central particle starts in the clusters 542,543,546,547,548 of the shell 309-561 and not as Shao states in the shell 561-923. The following proposition from [2]

Proposition 2.1. *Exist a discrete set, Ω, where $\forall j \in N$, $j \geq 2$, where the potential of the Searching Optimal Clusters of size j in a Discrete Space for an XX potential has the same optimal value of Searching Optimal Clusters of size j in a Continues Space for the same potential. The XX potential function complies with*

1. $\lim_{r_{i,j} \to 0} VXX(r_{i,j}) = \infty$.

2. $\nabla^2 VXX(x^*)$ semi-positive, $\|\nabla VXX(x^*)\| \ll 1$ and $\dfrac{\|\nabla VXX(x^*)\|}{|VXX(x^*)|} < \delta_0$, where $0 < \delta_0 \ll 1$

XX could be BU or LJ potential.

The previous proposition depicts a very populated research space where the process of the selection of particles from Ω in order to match of the results of a continuous minimization from an initial cluster from a lattice does not seem to be a very efficient option for designing an algorithm to solve the LJP_n. Therefore the lattice IF (which is combination of the lattice IC and the lattice FC) was proposed. Besides, that lattice IF unifies and avoid to use other lattices, it is not a simple research space. Some putative optimal clusters are not centered in the origin of the lattice IF. This cause that the selection of an initial cluster of size n from the lattice IF is not a simple process and depends where to put the center of the selection. In fact, the conjecture if all optimal clusters for any size exists inside of the lattice IF is still opens.

On the other hand, the existence of Ω point out to other possible lattices. One option is, the lattice CB which is a cubic lattice with edges of size $\frac{1}{2}d^*$ and it has a centered particle in (0,0,0), where $d^* = \sqrt[6]{2}$.

Proposition 2.2. *Any shape of n particles with edges $\approx d^*$ can be approximated from the lattice CB.*

Proof. Any shape is bounded by the well known Kissing Number in 3D. The unit centered in the origin cube of edge $= d^*$ of the lattice CB has 27 particles, the Kissing Number in 3D states that 12 spheres can be allocated around a centered particle. By example, the icosahedron's edges $= d^*$ has a centered particle with 12 neighbors and the other external twelve particles have 6 particles (five externals and the centered) and these particles can be selected from the 27 particles of the unit CB cube. Therefore there are sufficient neighbors to approximate any shape with edges $\approx d^*$ with n particles from the cubes centered in the origin of the lattice CB. $\square$

The figure 1 depicts the initial CB cluster, and optimal cluster of 13 particles.

With the lattice CB instead of the lattice IF a genetic algorithm was able to repeat the known results. The complexity of such algorithm is due to the selection of clusters from the lattice CB procedure and the genetic minimization procedure. The later procedure can guarantee by elitism that the cluster selected as optimal is the best cluster for the set of initial clusters given by the former procedure. However if this is the case there is not longer needed to use, other mechanism than the minimization procedure applied to all clusters given for the selection procedure. Therefore the cost of minimization over a subset of clusters selected from CB is proportional to the number of clusters by the computational cost of the minimization procedure. The minimization procedure complexity depends of selected optimization algorithm, in our genetic algorithm the Conjugated Gradient Method (see [6]) is used. It can approximate the minimum in at most 3n iterations (the complexity of minimization is due to the size of the cluster by 3, the dimension of the 3D particles' coordinates).

The lattice CB provides a search space where the selected clusters are centered in the origin, this means that a centered sphere with an appropriate ratio can give a sufficient research space to determine inside the optimal clusters until some size. Examples of optimal clusters for 38, 75, and 1,600 as wired black figures with the initial particles in green from lattice CB are depicted in Fig.2. The convexity and local continuity of the LJ potential guaranty that for the cluster with $n = 2, 3, 4$ with edges $= d^*$ are global minima cluster with first and second optimal condition satisfies. The gradient of the LJ potential is equal 0 and the Hessian of the LJ potential is definite positive but for $n > 4$ the gradient of the LJ potential is not null.

On the other hand, if the exploration is full in the appropriate centered sphere of the lattice CB, the optimal clusters are not more putative, they are global optimal clusters. The next proposition define a research subspace of the lattice CB for the determination of the global optimal clusters inside of it.
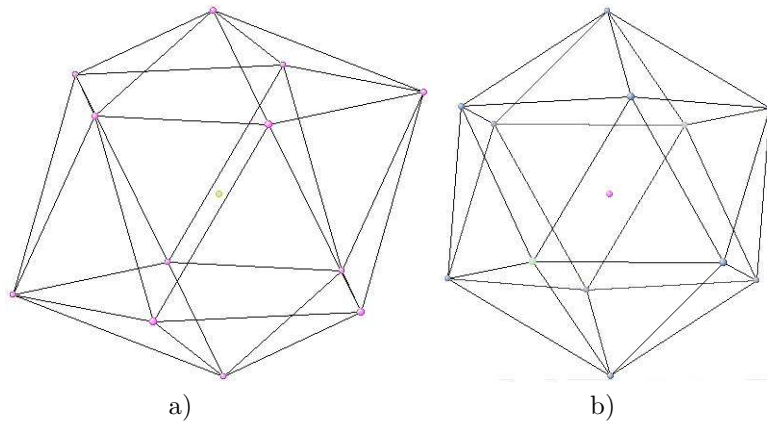
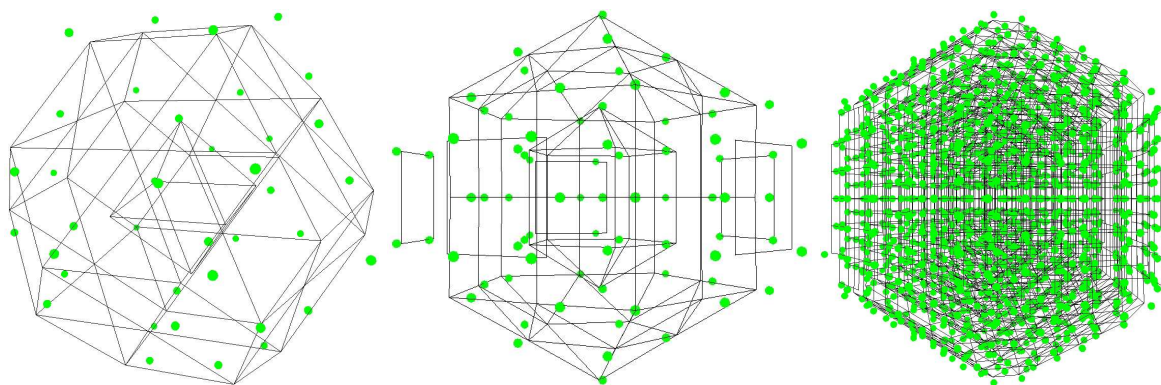Figure 1: a) Initial CB cluster, and b) Optimal cluster of 13 particles.

Figure 2: Optimal and initial clusters for 38, 75, and 1,600 particles.
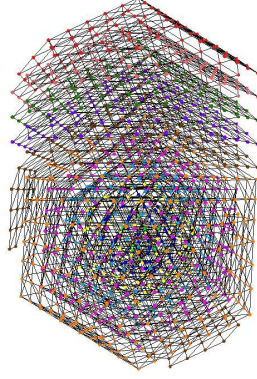
Figure 3: MIF1739 contain C_n^*, clusters with minimum LJ potential for $n = 2, \ldots, 1000$.

Proposition 2.3. *A cluster of n (> 4) particles with edges $\geq d^*$ has a descend property and a minimum over the centered cluster's diameter for the LJ potential in an centered in the origin CB sphere with double ratio determined by the ratio of the IC cluster over the so called magic number.*

Proof. Given n, the ratio of the cluster of the magic number can be determined by the formula $n_r = \frac{10}{6}r(r + 1)(2r + 1) + 2r + 1$ for the first $r \in \mathbb{N}$ such that $n_r > n$. Then the ratio of the sphere is $2rd^*$. Now, the diameter of a cluster inside of the sphere could be as big as $4d^*$. A cluster with large diameter has less edges of size $d*$ between particles than a cluster with smaller diameter, therefore $LJ(C_n(d_1)) > LJ(C_n(d_2))$ if $d_1 \geq d_2$. Finally, the minimum LJ potential of a cluster of n particles is reached over the finite possible selection of n particles in the CB sphere of ratio $2rd^*$. $\qquad \square$

This research subspace from the lattice CB are quite different from the minimum research space of the lattice IF, MIF1737 is depicted in figure 3. Also, the other difference is that all clusters can be selected with one parameter, the ratio of the centered CB sphere instead of two parameters, the ratio and the origin of an sphere in the IF lattice. So far, only the centered CB spheres with ratio > 0 are needed for searching optimal clusters of size n contained in a ratio of $2rd^*$, where r is such that $n \geq n_r$. This avoid the cone ice cream shape of the search space based on the lattice IF. With CB spheres to build an initial cluster is only necessary to pick up particles but even with trying to take advance of the symmetry and the previous and posterior optimal clusters of a given one, there is not possible to reduce the number of combinations to explore clusters of size n inside of a sphere with a guaranty that the exploration is complete. Different shapes of clusters with and without protuberances, the elasticity of the particles' position, and the existence of holes makes impossible to reduce combinations by symmetry. The shapes of putative optimal clusters in the range of the known putative optimal clusters from 2 to 1612, shows that they do not share general geometrical characteristics. The special extreme cases of selecting particles occur when the size of a cluster is around of the value of a magic number. The following proposition shows that for these cases, the complexity of generating initial cluster of n particles is exponential.

Proposition 2.4. *Given a set of M particles the selection of n particles with $M \gg n$ has exponential complexity.*

Proof. $\binom{M}{n} = \frac{M!}{(m-n)!n!} = \frac{M(M-1)\cdots 2\cdot 1}{n(n-1)\cdots 2\cdot 1} \geq \left(\frac{M}{n}\right)^n$, $M \gg n$. Taking $k = \text{fix}\left(\frac{M}{n}\right)$, k is an integer and the complexity is greater than k^n. $\qquad \square$

This proposition point out that the complexity grows exponentially because there are many alternatives for picking initial particles even if this selection is performed into an appropriate reduced research space. This reduced space is a centered sphere of the lattice CB.

In the next section a similar analyzes to reduce the research space is applied to NP problems with data structure given by a complete graph.

3   The General Assign Problem

The well known Travel Salesman Problem (TSP) is a case of an Assign General Problem(GAP). In this section the analysis of the GAP is presented. First, the results and properties are presented.

The GAP_n consists a complete graph, a function which assigns a value for each edge, and objective function, $GAP\_n = (G_n, c, f)$, where $G_n = (V_n, A)$, $V_n \subset \mathbb{N}$ $A = \{(i, j) \mid i, j \in V\}$, $c : V \times V \to \overline{\mathbb{R}}$, and f is a real function for the evaluation of any path of vertices. Solving a GAP means to look for a minimum or a maximum value of f over all cycles of G_n. Note that $c(i, i) = \inf$, and $c(\cdot, \cdot)$ can be see as a matrix of $\overline{\mathbb{R}}^{n \times n}$.

Remark 3.1. Some properties and characteristics are:

1. The complete property of G_n means that $\forall i, j \in V_n, \exists e = (i, j) \in A$.

2. G_n is a directed graph, which means that edges are a ordered pair.

3. A path of vertices is a sequence of vertices of V_n.

4. A cycle is a path of different vertices but the first and the last vertex are the same. Hereafter, a complete cycle is a cycle containing the n vertices of V_n. Also, a complete cycle has n edges.

5. A TSP_n is a case of a GAP_n where $c(i, j) = c(j, i)$ (the cost matrix is symmetric) and $c(i, j) \geq 0, \forall i, j$, and finally f is the sum of the edges' cost of a cycle.

6. A mD Euclidian TSP_n is a special case of a TSP_n where $c(i, j) = \text{distance}(i, j)$ $\forall i, j \in V_n$, and the vertex $i \in \mathbb{R}^m, m \geq 2, \forall i = 1, \ldots, n$.

Proposition 3.2. *Any GAP_n has a solution.*

Proof. Without lost of generality, it is assumed to look for a minimum. f is evaluated on all the cycles of G_n. The cycles corresponds to the finite permutations of the n vertices of V_n. Then, the image of f is discrete, therefore f reach its minimum in a cycle. $\qquad\square$

Proposition 3.3. *For f as minimum or maximum edges cost, then the cycle of the GAP_n which is the solution can be found in polynomial time.*

Proof. These are trivial cases where the complexity is at most $n(n-1)$ which corresponds to search into the entries of the cost matrix $c(i, j)$. $\qquad\square$

Notes. The last proposition defines in clear way what an algorithm must look for, but how to calculate and prove the optimality is quite different. Nevertheless, for any objective function f, if we have the discrete image of f, GAP_n is a computable problem and the minimum selection procedure gives the solution. The minimum selection procedure in mathematical notation is:

$$y^* = \arg \min_{y=(v_1, v_2, \ldots, v_n, v_1), v_1 \in V_n, v_i \in V_n, v_i \neq v_j, 1 \leq i < j \leq n} \{f(y) \mid GAP_n = (G_n, c, f), G_n = (V_n, A)\}$$

Properties of GAP_n derived from its graph's properties are presented in the following propositions.

Proposition 3.4. *Any GAP_n has*

1. $n(n-1)$ edges.

2. $(n-1)!$ complete cycles. The complete cycles can be enumerated in n different ways. Hereafter, a cycle means a complete cycle.

3. For $n \geq 4$, the maximum number of coincident edges of two different complete cycles is $(n-3)$.

Proof.   1. By induction, it is immediately. But the next data structure helps to prove. For computational purposes a function to identify the edges with integer values, $(1,2) \leftrightarrow 1$, $(2,1) \leftrightarrow -1$, $(1,3) \leftrightarrow 2$, $(3,1) \leftrightarrow -2$, ..., is also depicted in the following matrix:

$$
\begin{array}{c|ccccc}
Vertex & 1 & 2 & 3 & \ldots & n \\
\hline
1 & X & 1 & 2 & \ldots & (n-2)(n-1)/2+1 \\
2 & -1 & X & 3 & \ldots & (n-2)(n-1)/2+2 \\
3 & -2 & -3 & X & \ldots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
n & -(n-2)(n-1)/2+1 & -(n-2)(n-1)/2+2 & \ldots & \ldots & X
\end{array}
\tag{3.1}
$$

The matrix's elements correspond to the edges, but the diagonal's elements, which are marked by X, therefore the number of edges are $n^2 - n = n(n-1)$.

2. The first vertex and the last vertex are the same, therefore there are $(n-1)!$ different sequences of vertices (paths) of the remained (n-1) vertices. On the other hand, the first vertex can be selected in n different ways, therefore the complete cycles can be enumerated in n different ways.

3. Without lost of generality the complete cycles can be depicted in descent order. Then two consecutive paths are $(n, n-1, \ldots, 4, \mathbf{3,2,1,n})$, and $(n, n-1, \ldots, 4, \mathbf{3,1,2,n})$ which match on $(n-3)$ edges. $\square$

Some important differences between GAP_n and TSP_n are in the following proposition.

Proposition 3.5. *The objective function f is c, and let $c(p)$ be the cost function of p, i.e., it is given by the summation of edges cost of the consecutive pairs of vertices of p. Let l be the length path function, i.e., $l(p) = k$ where p is a path, and k is the number of vertices of p.*

1. *For TSP_n, c is monotonically increasing in the following way, $c(p_1) \leq c(p_2)$ where the sequence of vertices of p_1 is a subsequence of p_2 (hereafter, p_1 is a sub-path of p_2), and $l(p_1) < l(p_2)$.*

2. *For GAP_n, c is not monotonically increasing as in TSP_n.*

3. *Let $c' \in \mathbb{R}$, then for TSP_n, $P(c(p_2) \geq c' | c(p_1) = c') = 1$ where P is the probability function, p_1 is a sub-path of p_2.*

4. *Let $c' \in \mathbb{R}$, then for GAP_n. $P(c(p_2) \leq c' | c(p_1) = c') > 0$ where P is the probability function, p_1 is a sub-path of p_2.*

Proof.   1. It is monotonically increasing because $c(i,j) \geq 0$, and $c(p_2)$ has some edges' cost more than $c(p_1)$.

2. For an arbitrary GAP_n, if $c(\cdot)$ is monotonically increasing then it is a TSP_n.

3. For TSP_n, the monotonically increasing of $c(\cdot)$ implies that if a sub-path reach a value c' for sure any path containing it has a greater or equal value than c'.

4. Because the previous case, $c(\cdot)$ for GAP_n is not monotonically increasing. Then the probability of a descend from a reference value $c(p_1) = c'$ on a path p_2, where p_1 ia sub-path of p_2 is not zero. $\square$

Definition 1. The function to identify the edges is $e : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ (it was depicted in equation. 3.1)

$$
e(i,j) = \left\{ \begin{array}{ll}
i < j & (j-2)(j-1)/2 + i \\
\text{otherwise} & -(i-2)(i-1)/2 + j
\end{array} \right.
$$

The Research Space of GAP_n is finite and numerable, and it has $(n-1)!$ elements and can be enumerated in n different ways (see Prop. 3.4). An example of an enumeration in descendent order of the vertices is the following.

$$
\begin{array}{ccc}
\mathbb{N} & & \text{Cycle} \\
1 & \leftrightarrow & (n, n-1, \ldots, 2, 1, n) \\
2 & \leftrightarrow & (n, n-1, \ldots, 1, 2, n) \\
\vdots & \vdots & \vdots \\
(n-1)! - 1 & \leftrightarrow & (n, 1, \ldots, n-1, n-2, n) \\
(n-1)! & \leftrightarrow & (n, 1, \ldots, n-2, n-1, n)
\end{array}
$$

Hereafter, this numeration is assumed. For any GAP_n, given n and $j > 0$, a natural number between 1 and $(n-1)!$, the next algorithm computes the corresponding cycles assuming descending order.

Algorithm 2. **Input:** j, n. **Output:** y cycle of GAP_n (array of $\mathbb{N}^{n+1}$)

   1. $V_a := (n-1, n-2, \ldots, 2, 1)$; (array of $\mathbb{N}^n$)

   2. $y(1) = n$;

   3. $y(n+1) := n$;

   4. $j_a := j - 1$;

   5. $n_r := n - 1$;

   6.   **for** $k := 1$ **to** $n - 3$ {

   7. $d := (n - k - 1)!$;

   8. $r := \text{fix}(j_a/d) + 1$;

   9. $j_a := \text{mod}(j_a, d)$;

  10. $y(k+1) := V_a(r)$;

  11. $V_2 := (1, \ldots, n_r - 1)$; (array of $\mathbb{N}_r^n - 1)$)

  12. $k_2 := 1$;

  13.   **for** $k_3 := 1$ **to** n_r {

  14.   **if** $r \neq k_3$ **then**

  15. $V_2(k_2) := V_a(k_3)$;

  16. $k_2 := k_2 + 1$;

  17.   } (end for k_3)

  18. $n_r := n_r - 1$;

  19. $V_a := V_2$;

  20.   } (end for k)

  21.   **if** $j_a == 0$ **then**

  22. $y(n-1) := V_a(1)$;

  23. $y(n) := V_a[2]$;

  24.   **else**

25. $y(n-1) := V_a(2);$

26. $y(n) := V_a[1];$

The next propositions shows that the solution of any GAP_n can be mapped to first position of a given enumeration of the research space of an equivalent GAP_n.

Proposition 3.6.
Let be $[i_k]_{k=1}^{n+1}$ a given cycle of a GAP_n. Then $\exists! \; m : [1, \ldots, n] \to [1, \ldots, n]$ such that the cost of the cycles do not change for an equivalent GAP'_n with cost function given by $c \circ m^{-1} = c(m^{-1}(\cdot))$ and $[m(i_k)]_{k=1}^{n+1} = [n, n-1, \ldots, 1, n]$, which corresponds to the first cycle of the descent enumeration of the vertices beginning with n.

Proof. Note that the first n vertices of the given cycle $[i_k]_{k=1}^{n+1}$ correspond to a permutation of $[1, 2, 3, \ldots, n]$. Therefore the unique m is the mapping $i_1 \leftrightarrow n$, $i_2 \leftrightarrow n-1$, ..., $i_n \leftrightarrow 1$. By construction the cost in the equivalent GAP'_n is the original cost of the GAP_n. Let be $[l_k]_{k=1}^{n+1}$ an arbitrary cycle of the equivalent GAP'_n, its cost is given by $f(c(m^{-1}([l_k]_{k=1}^{n+1}))) = f(c([j_k]_{k=1}^{n+1}))$, where $m^{-1}([l_k]_{k=1}^{n+1}) = [j_k]_{k=1}^{n+1}$. □

Proposition 3.7.
Let be $[i_k]_{k=1}^{n+1}$ the solution cycle of a GAP_n. Then $\exists! \; m : [1, \ldots, n] \to [1, \ldots, n]$ such that $[n, n-1, \ldots, 1, n]$ is the solution of the equivalent GAP_n, which corresponds to the first cycle of the descent enumeration of the vertices beginning with n.

Proof. Let be $m(\cdot)$ constructed as in the previous proposition. The cost function of the equivalent GAP_n is $c(m^{-1}(\cdot))$. Let be $[l_k]_{k=1}^{n+1}$ an arbitrary cycle in the equivalent GAP_n, with $m^{-1}([l_k]_{k=1}^{n+1}) = [j_k]_{k=1}^{n+1}$. Therefore the optimality follows from $f(c(m^{-1}([n, n-1, \ldots, 1, n]))) = f(c([i_k]_{k=1}^{n+1}))$, and $f(c([i_k]_{k=1}^{n+1})) \leq f(c([j_k]_{k=1}^{n+1})) \; \forall [j_k]_{k=1}^{n+1}$ cycle of GAP_n. □

Proposition 3.8. *Let be GAP_n such that $c(i, j)$ is given by the following matrix*

$$\left\{ \begin{array}{ccccc} & 1 & 2 & \cdots & n-1 \\ n & & n \cdot 2 & \cdots & n \cdot n - 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n^{n-2} & n^{n-2} \cdot 2 & \cdots & & n^{n-2} \cdot (n-1) \\ n^{n-1} & n^{n-1} \cdot 2 & \cdots & n^{n-1} \cdot (n-1) & \end{array} \right\}$$

then it has a unique solution and all cycles have different cost.

Proof. The cycles of GAP_n correspond to a unique combinations of its edges. The elements of the matrix $c(i, j)$ are all different monomials on the numerical base n. Therefore, any cycle has a unique cost given by its unique polynomial combination of n terms $c(i, j)$ on the numerical base n. □

4   Integer Problem Formulation of GAP

A given a GAP_n can be formulated as an integer problem (IPGAP_n) as follow:

$$\min \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

s.t.

$$\sum_{j=1}^n c_{ij} x_{ij} = 1, \forall i = 1, \ldots, n,$$
$$\sum_{i=1}^n c_{ij} x_{ij} = 1, \forall j = 1, \ldots, n,$$
$$\sum_{i=1}^n c_{ii} x_{ii} = 0,$$
$$\{0, 1\} \in x_{ij}, \forall i, j,$$

where c_{ii} are the entries of matrix of cost, and x_{ij} are the variables associated to an edge with vertices i, j.

Proposition 4.1. *A cycle of $\text{GAP}_n \Leftrightarrow$ a feasible point of IPGAP_n.*

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Proof. $\Rightarrow$. Let $v_1, v_2, \ldots, v_n, v_1$ the vertices a cycle of GAP_n. The corresponding point IPGAP_n is given by

$$x_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } i = v_k \text{ and } j = v_{k+1}, k = 1, \ldots, n \\ 0 & \text{otherwise.} \end{array} \right. .$$

For any $i = 1, \ldots, n$, it exists a unique $v_k = i$, then $\sum_{j=1}^{n} x_{ij} = x_{v_k v_{k+1}} = 1$. For any $j = 1, \ldots, n$, it exists a unique $v_{k+1} = j$, then $\sum_{i=1}^{n} x_{ij} = x_{v_k v_{k+1}} = 1$. Finally, a cycle has different consecutive vertices, $\sum_{i=1}^{n} c_{ii} x_{ii} = 0$. Therefore $\{x_{ij}\}$ is feasible.

$\Leftarrow$. Let be $\{x_{ij}\}$ a feasible point of IPGAP_n. This means that there are zeros in the diagonal and there is a unique 1 by column and by row in $\{x_{ij}\}$. Then the corresponding cycle is constructed by taking

$$v_{i_k} = \left\{ \begin{array}{l} v_1 = 1 \\ v_{k+1} = j \end{array} \right. \quad \text{where } x_{v_k j} = 1, k = 1, \ldots, n$$

By induction over n. For $n = 2$ the unique cycle of GAP_2 corresponds $1, 2, 1$ given by the previous formula on the matrix of the feasible point of IPGAP_2 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For $n = 3$ the two cycles of GAP_3 corresponds $1, 2, 3, 1$, and $1, 3, 2, 1$ are given by the previous formula on the matrices of the feasible points of IPGAP_3 $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Also the number of feasible points for IPGAP_3 is $2 * 1 = 2!$. The hypothesis for n is that all $(n-1)!$ feasible points give all the cycles. For $n+1$, first a feasible point of IPGAP_n is taken, then a cycle is constructed $v_1, v_2, \ldots, v_n, v_1$, also a column, and a row are added. On the resulting matrix $\{x_{ij}\}_{i,j=1,\ldots,n,(n+1)}$, the unique entry for which $x_{in} = 1$ is set to zero, and for that i, the entry $x_{i(n+1)}$ is set to 1, and finally the entry $x_{(n+1)n}$ is set to 1. With these changes the point $\{x_{ij}\}_{i,j=1,\ldots,n,(n+1)}$ is a feasible point of $\text{IPGAP}_{(n+1)}$, and it has a corresponding cycle of $\text{GAP}_{(n+1)}$ $v_1, v_2, \ldots, i, n+1, n, \ldots, v_n, v_1$. The last step can be done n times because $x_{(n+1)(n+1)} = 0$. Therefore $n!$ is the total number of feasible points of $\text{IPGAP}_{(n+1)}$. $\square$

Remark 4.2. By the previous proposition, the minimum number of vertices in order to change a cycle are two, this means 3 edges, or in term of IPGAP_n, 3 variables of x_{ij} need to be replaced to pass from one feasible point to another. This means that a pivot method for solving IPGAP_n needs to change at the same time at least three active variables. For TSP_n the cost function is monotone, this can help to solve the problem with pivot's strategies with limited variables (see proposition 3.5). For GAP_n the possibility of missing the solution using pivot's strategies limited by the number of variables makes hard to solve GAP_n by integer programming methods.

Proposition 4.3. *The GAP_n and IPGAP_n are equivalent.*

Proof. It is immediately from the previous proposition. $\square$

To visualize the cost matrix, two linear transformation are used to map any cost matrix of GAP_n into $[-1, 1]^n$, or $[0, 1]^n$. Scale transformation:

$$c'(i,j) = \left\{ \begin{array}{ll} 1 & \text{if } c(i,j) = \inf \\ c(i,j)/s^+ & \text{if } c(i,j) \geq 0 \\ c(i,j)/s^- & \text{otherwise} \end{array} \right.$$

where $s^+ = \max_{i,j=1,\ldots,n,c(i,j)\neq\inf} c(i,j)$, and $s^- = \min_{i,j=1,\ldots,n} c(i,j)$.
Scale and translation transformation:

$$c'(i,j) = \left\{ \begin{array}{ll} 1 & \text{if } c(i,j) = \inf \\ (c(i,j) - m)/s & \text{otherwise} \end{array} \right.$$

where $m = \min_{i,j=1,\ldots,n} c(i,j)$, $M = \max_{i,j=1,\ldots,n,c(i,j)\neq\inf} c(i,j)$, and $s = M - m$.

Section 6 contains images of matrices using the last transformation. The idea is to study the cost matrix and an associate vertex's index matrix to explore pivot's strategies. Because, the diagonal of the cost matrix is inf the diagonal was omitted for an easy interpretation on a gray scale of color $[0, 1]$, where minimum values correspond to black (0), and maximum values correspond to white(1). A sorted matrix $\mathcal{M}$ depicts a relation between edge's cost and vertex's number. By the moment, pivot's strategies are not proposed but algorithm 9 can be modified for integer programming approach. The next proposition presents a version of a necessary optimality condition for IGAP_n.

Proposition 4.4. *Given $IGAP_n$. x_{ij}^* is its solution, if for any pivot's strategy that suggest any set of x_{ij} feasible (i.e., a sub-path or a cycle of the equivalent GAP_n) with cost less than $t^* = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}^*$, they can not be inserted to replace appropriately the variables of x_{ij}^* with diminish t^*.*

Proof. It is immediately by the global optimality of y^*. □

The next section presents a well known computational model to apply for defining an algorithm to verify the solution of GAP_n.

5   Turing Machine and Finite Automata for GAP_n

This section relates two simple computational models to resolution of GAP_n. The section 6 will present the algorithm "The Cycle generator" that is a Turing Machine (TM). To justify and analyze the complexity of an algorithm for checking the solution of GAP_n a simple variations of the problem to locate a mark on an infinity or finite tape are presented. This is done in detailed way to distinguish the importance of the existence of properties to analyze problems, constrains, data, and structures.

P1. It consists to locate a mark in an infinity tape (or memory).

The importance of properties and well defined hypothesis is quite important. What if for P1 there is no mark! No matters Finite Automata (FA) or TM the problem is not computable. Hereafter, the existence of a mark is assumed. Other practical matter is how the mark is putted, i.e., how the problem can be constructed, what process or person can manages an infinity tape. But this is done by a assuming and accepting, that this problem is a mental experiment.

P1 is perhaps the ideal, easy and simple problem that a TM can compute but a FA can not. There are two reason for the success of the TM over the FA model for this problem, the first one is the ability to move forward and backward, the second is the interaction with the tape or memory that allow it to remember what memory has been explored. It can be argued that a FA can not be used for this problem but for the problem that the mark is on the direction that the FA can analyze. If FA depends from an special configuration that need to know in what direction is the mark is not the same problem, knowledge has been added in order that the FA can solve a similar but it is a different problem to locate a mark in a given direction.

P2. It consists to locate a mark from the origin of an infinity tape.

Now, for solving P2, a FA is sufficient. A TM can solve the problem but no memory interaction and backward are needed.

P3. It consists to locate a mark from the origin of an finite tape. Again, for solving P3, a FA is sufficient. A TM can solve the problem but no memory interaction and backward are needed.

A TM has a halting property no mater where the mark is located for the three problems. However, there is a crucial difference, because computability, i.e., that it is for sure that a TM can always give a solution for these problems. A FA can only solve P2 and P3 with assuming that there is a mark. It can be also showed that the TM has not advantages using its ability to interact with the tape than a FA. The simple algorithm to locate a mark is:

1. if the mark is found then "here is the mark" stop,

2. otherwise if there is an available memory's space then skip,

3. otherwise stop "there is not mark".

P4. This problem consist in determine the minimum of a finite set of marks (each one has a associate an integer value) on a finite memory.

For P4, there is a TM, that looks the memory, then if there is a mark then stores and keep this mark when it corresponds to the minima, until all memory is reviewed. The main hypothesis are a finite number (but unknown number) of marks, and each mark correspond to an integer value. The existence of the minima is guaranty by elemental Mathematical Analysis, therefore P4 is computable. Without more assumptions this last problem can not finish before to look for all the memory. Its complexity depend of the length of the tape. Moreover, the previous TM that can compute P4, and there is no properties to gain efficiency. If an TM claims to solve P4 in efficient time for any instance of P4. A practical numerical test can be formulated for verification where the claimed TM must has a probability of finding the solution equal one. If this is the case, any arbitrary set of
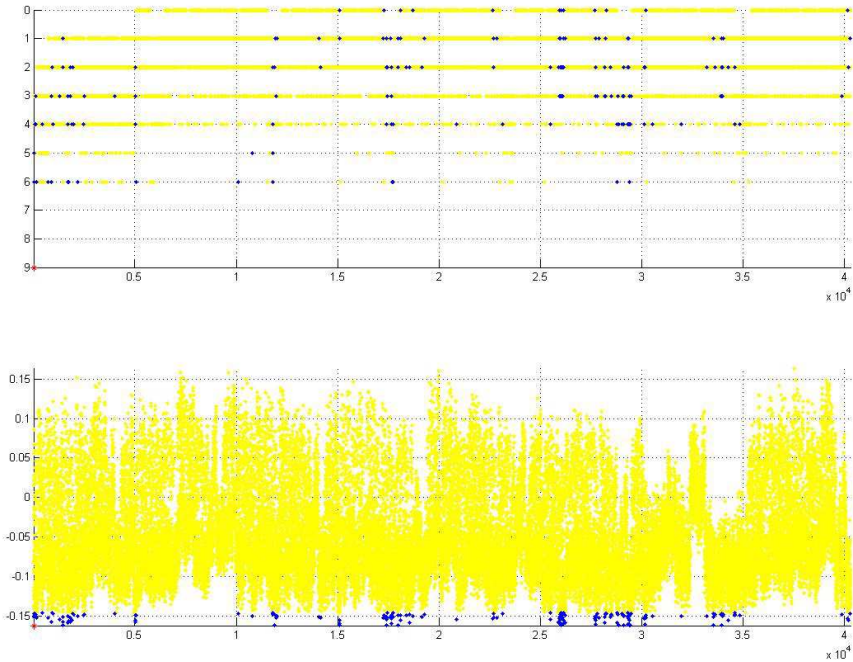
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Figure 4: Edges' cost that coincide with the edges' cost of the optimal cycle, and cost each cycle numerated from 1 to 8! for an example of TSP$_9$. The first cycle is the solution, and it is depicted in red. Blue dots depicts cycles with cost closed to the optimal cost.

arbitrary instances of P4 selected for this test are related but arbitrary! What if with appropriate assumptions or properties it is possible to formulate algorithms with high probability to determine a putative solution founded in efficient time. This opens an interesting line of research for probabilistic algorithms, they trade off between putative or approximate the solution and early stop determined by efficiency (no more than polynomial iterations), this type of algorithm must be addressed and studied but this is out of the scopes of this paper.

1. if the mark is found then "here is the mark" stop,

2. otherwise if there is an available memory's space then skip,

3. otherwise stop "there is not mark".

These computational models are without complicated instructions but simple transitions driven by the input data. Its complexity is not related to the number of transitions but by the length of the tape or memory until finish to reviewing it. Without properties or conditions, they can be used instead of a more elaborated algorithm.

Here, in order to solve GAP$_n$ an algorithm could keeps a record of the generated cycles but the number of marks requires exponential memory's size. This is elaborated in following paragraphs. First, the lack of a local vicinity optimal property on the cycles' cost makes not necessary to keep a record of the putative optimal cycles but to keep the best one. The information of a local optimal cycle could not be useful to predict possible paths to pursue the global optimal. Therefore a TM capable to generate cycles and keep the putative optimal cycle is sufficient.

On the other hand, for an special cost matrix, i.e., matrix with properties, it is highly possible to build an ad-hoc algorithm, that could solve and demonstrate the solution, and if matrix's properties are related to the putative solution then a record of the algorithm steps could improve the research, and provides the justification of the solution. For GAP$_n$ any property or relation in the cost matrix can be nullified by an arbitrary combination of its coefficients for large and arbitrary problems of it.
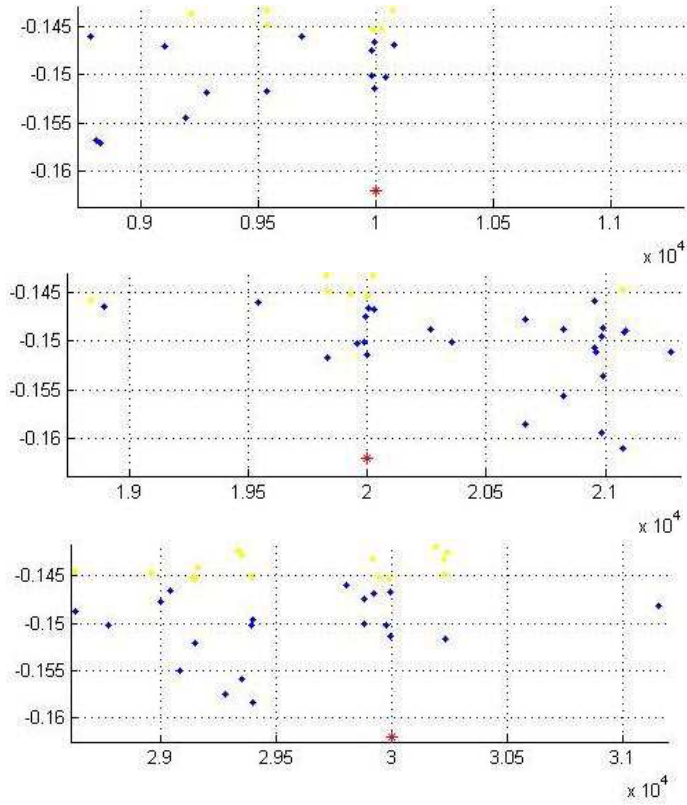
Figure 5: Optimum cycle in positions = 10,000, 20,000, 30,000 of the numeration from 1 to 8! corresponding to cycles on descending order of the algorithm 2 for an example of TSP_9. The solution is depicted in red. Blue dots depicts cycles with cost closed to the optimal cost.
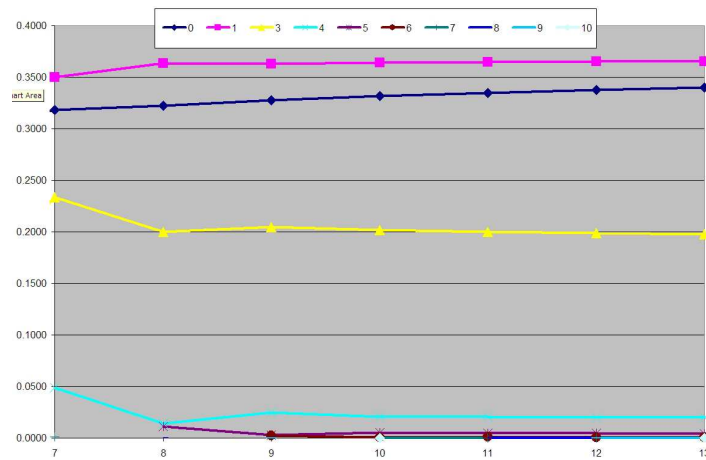

Figure 6: Edges' cost that coincide with the edges' cost of the optimal cycle for GAP_n, $n = 7, 8, 9, 10, 11, 12, 13$.

Figure 4 depicts the landscape of TSP_n, i.e, the cycle's cost versus cycle's number in descending order (see proposition 3.7). Blue dots depict cycles with closed values to the optimal. Notes that the blue dots are spread, and there are some blue dots with zero coincidences with the edges' cost of the optimal solution (examples of this are depicted in the upper figure over the same vertical position of the cycles (marked on blue) with cost closed to the solution). The optimal cycle was mapped to first position (see proposition 3.7) and its is depicted by a red dot in the origin. It is noted that the colored dots (including the optimal) are not a local minima and they could be closed or far away between them. Also, fig. 5 depicts how the optimums vicinity change when the same optimum cycle is mapped on different positions of the descending order of the algorithm 2, the cycles' cost are the same but in different positions. The optimums vicinity is not the same besides the cost matrix $C(i,j)$ is permuted according to these cycles' numeration. The change affects the neighbors' cost of the optimal cycle because, they share sub-paths but in different orders of the vertices by the descending numeration.

The number of cycles grows exponentially and each cycle inherent by the enumeration of the branches edges' cost but not necessarily to form a global minima as it is depicted in the landscape of the fig. 4, and fig. 5.

It is possible to solve GAP_n until $n = 14$ in a home made computer (These numerical experiments run in a AMD Athlon 64X2 Dual core processor 5200+, 2.6GHz, 3.25 GB of RAM) by a direct evaluation of the 13! different cycles computed by algorithm 2 (some adjust are need to use large integer numbers) but the execution's time grows as it expects exponentially from few seconds ($n = 4, 5, 6, 7, 8, 9$), 1.1 minutes ($n = 10$), 13.7 minutes ($n = 11$), 2.70 hours (n=12), 1.5 days ($n = 13$), 0.66 months ($n = 14$), and 0.82 years ($n = 15$). The solutions are verifying at the same time by this exhaustive approach but it is completely impractical.

However, studying these small cases fig. 6 is constructed. It depicts an remarkable relation between the edge's cost of the different cycles respect to the first cycle. It is interesting to note that the number of edge's cost that no coincide at all or coincide with only one of the edges cost of the first cycle has a tendency approximately of 33% and 36% respectively with a slightly growing(zero coincidence is the dark blue line, one coincidence is the fuchsia line in the fig. 6). This means 69% of the cycles of GAP_n could not share or they are not related to the current putative optimal cycle. To keep track in the memory of the visited cycles requires to have a memory's size of $69\%(n-1)!$, which is a huge requirement. This requirement makes not possible to keep track of the visited cycles.

If it is possible to cut branches of the complete graph to reduce the research space of a GAP_n, then it is not necessary to use "if" to cut branches that can not provide the solution. Even more, there is not property or justification to define an early stop after reach a putative solution for an arbitrary and large GAP_n without special properties. In fact, because GAP_n's objective cost function is not monotonically increasing but oscillating, skip cycles could drive to miss the solution, it is probable that a reference cost diminish on a large path of a given sub-path (see proposition 3.5). For TSP_n an "if can be useful to cut cycles with sub-path's cost above the current putative optimal cycle's cost. Also, this descending property is the track to prove and justify the optimality of the last stored optimal cycle.

It is a TM the appropriate computational model for a simple algorithm to explore at full the GAP_n's research space or a reduced research space of it. In the next section the generator of putative cycles for GAP_n is a TM. It generates by enumeration of the vertices all cycles remaining in the reduced research space, and the complexity only relates to the number of them.

6 Reducibility

The main concerns in designed an efficient algorithm is to take advantage of the properties of the problem with an appropriate data structures. The proposition 3.7 shows that a GAP'_n can be isomorphically transformed in an equivalent GAP_n with its solution on the first position under a given enumeration of the cycles, i.e., the research space of $GAPn$ is finite, numerable, and the optimal cycle could be on any position of the natural interval $[1, (n-1)!]$, by example, the first cycle (see section 3.4).

For the following it is assumed that a computer machine has sufficient memory and processors. Also, it is able to perform sum of arbitrary size as one operation, and it has simple flow control structures such as "while", and "if".

Definition 3. Given a GAP_n and $\{a_i \in \mathbb{N}\}_{i=1}^n$ where a_i is the number of edges consider relevant that connect the vertex i with the other vertices. This collection is called alternatives.
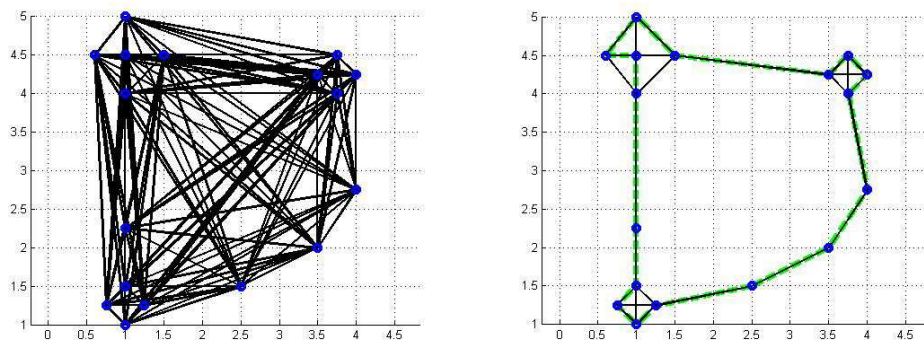
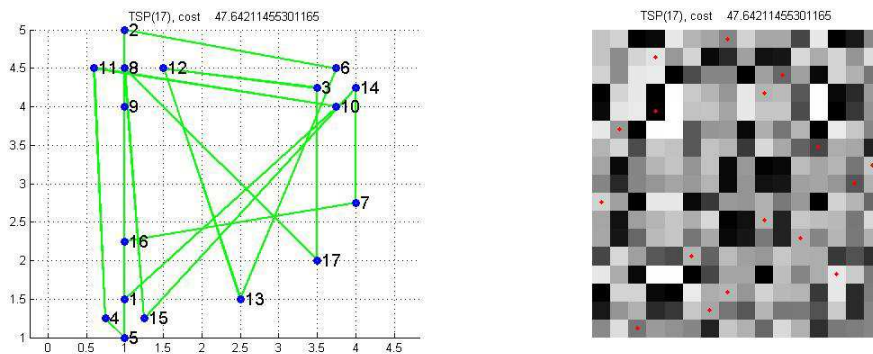Figure 7: Complete and reduced equivalent graph of an 2D Euclidian TSP$_{17}$.

Figure 8: Initial numeration, an initial cycle (green lines), and cost matrix with initial cycle (red dots) for the example 2D Euclidian TSP$_{17}$.
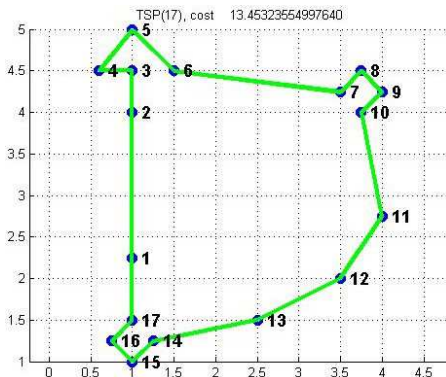

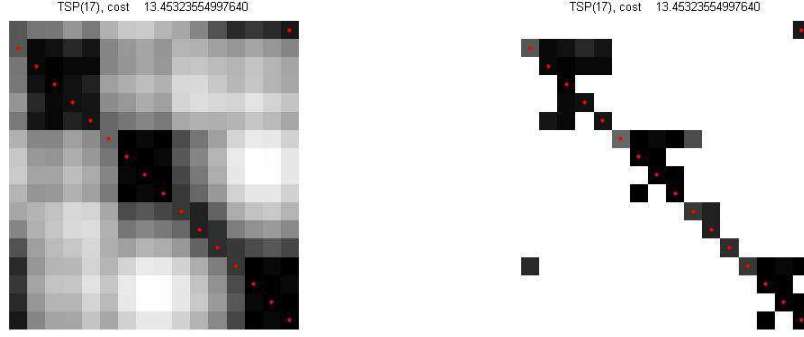Figure 9: Solution (green lines) of the example 2D Euclidian TSP$_{17}$.

Figure 10: Cost matrix enumerated by the solution (red dots) and reduced equivalent cost matrix for the example 2D Euclidian TSP_{17}.

A tube is a set of vertices of a given a GAP_n where the number of edges consider relevant that connected them and the rest of the vertices are less than 3. T denotes the number of vertices of the tubes of a given GAP_n.

The next definition comes from the Multiplication Rule.

If exists $p \ll n$ such that

$$\frac{\sum_{i=1}^{n} \log_2(a_i)}{\log_2(n-T)} \quad \leq \quad p \tag{6.1}$$

then GAP_n is reducible and its research space has polynomial size.

The figure 7 corresponds to an instance of the 2D Euclidian TSP_{17}, where a vertex represents a city in a given 2D position and the edges' cost are the Euclidian distance between them. The left figure depicts the complete graph of the 2D Euclidian TSP_{17}. The right figure depicts the reduced graph where the alternatives (lines in black) for verifying the solution (lines in green) are defining by appropriately using the triangle inequality to avoid considering large edges. Therefore, this picture has three tubes that connect three clouds of vertices. The introduction of T is necessary to support in clear way that the complexity depend strongly of the alternatives, the tubes do not increase the complexity for verifying the solution. They are vertices connected by the best possible edge's cost, i.e., these sets of vertices can be considered connected without an exploration of alternatives because they do not have other choice. In this example using the numeration of the fig. 9, the edges fixed, i.e., the tubes are $(17, 1, 2, 3)$, $(6, 7)$, and $(11, 12, 13, 14)$. In this case for verifying the solutions the parts to be considered are the three clouds $\{3, 4, 5, 6\}$, $\{7, 8, 9, 10\}$, and $\{14, 15, 16, 17\}$. This motivate the next definition for parallel computing.

Definition 4 (The parallel complexity version of GAP_n). GAP_n is reducible if exists $p \ll n$ such that

$$\frac{\max_{k=1,\ldots,K} \sum_{i=i_k}^{j_k} \log_2(a_i)}{\log_2(n-T)} \leq p.$$

where K is the number of clouds, $\{i_k, \ldots, j_k\}$ are the vertices of cloud $k = 1, \ldots, K$.

For the 2D Euclidian TSP_{17} the grade for verifying the solution in sequential mode is ≈ 2.934 and in parallel is ≈ 1.386, where $n = 17$, $T = 11$, and the clouds' total alternatives are $12, 4, 4$ respectively.

The exploration of the irrelevance of the alternatives by properties helps to define an algorithm for solving NP problems. An example of this is the triangle inequality, and the monotony of the cost function (as the distance between cities) for the mD Euclidian TSP_n. This is not aim in this article because we are exploring to build an algorithm for verifying the solution of NP problems.

Let be $\mathcal{M}$ an array structure as follows

c_{i_1,j_1^1}, v_1^1	$\cdots$	$c_{i_1,j_{n-1}^1}, v_{n-1}^1$
$\vdots$	$\vdots$	$\vdots$
c_{i_n,j_1^n}, v_1^n	$\cdots$	$c_{i_n,j_{n-1}}, v_{n-1}^n$

where the columns have the edge's cost and edge's vertex, they are in ascending order by the values of $c(i_k, j)$ (edge's cost) for each row of the vertex i_k. Hereafter, $\mathcal{M}(i, j).c$ denotes the corresponding edge's cost, and $\mathcal{M}(i, j).v$ denotes the corresponding vertex.

An algorithm to build $\mathcal{M}$ is

Algorithm 5. **Input:** Unsorted $\mathcal{M} = [c(i, j), j | c(i, j) \neq \inf]$, $i = 1, \ldots, n; j = 1, \ldots, n$.
    Output: Sorted $\mathcal{M}$, y the optimal cycle when the first column's indexes of the sorted $\mathcal{M}$ conform a cycle.

1.   **for** i:= 1 **to** n

2.   sort in ascending order the row i of $n - 1$ elements of $\mathcal{M}$ by $\mathcal{M}(i, \cdot).c$.

3. $y := 0$; (array of $\mathbb{R}^{n+1}$)

4. $v := 0$; (array of $\mathbb{N}^n$)

5. $i_d := 0$; (cycle's next vertex)

6.   **for** $j := 1$ **to** n {

7. $i_d := \mathcal{M}(1, j).v$;

8.   **if** $v(i_d) == 0$ **then**

9. $v(i_d) := 1$;

10. $y(j) := i_d$;

11.   **else**

12.   "No optimal cycle";

13.   stop;

14.   } (end for j)

15. $y(n + 1) := y(1)$;

16.   "Optimal cycle is", y;

Using the Quick Sort the complexity of the previous algorithm is $\mathbf{O}(n^2 \log_2(n))$.

The next algorithm is a greedy algorithm to compute an initial cycle using sorted $\mathcal{M}$.

Algorithm 6. **Input:** Sorted $\mathcal{M} = [c(i, j), j]$, $i = 1, \ldots, n; j = 1, \ldots, n$. **Output:** y a cycle.

1. $y := 0$; (array of $\mathbb{R}^{n+1}$)

2. $v := 0$; (array of $\mathbb{N}^n$)

3. $i_d := 0$; (cycle's next vertex)

4.   **for** j:= 1 **to** n {

5. $i_d := \mathcal{M}(1, j).v$;

6.   **if** $v(i_d) == 0$ **then**

7. $v(i_d) := 1$;

8. $y(j) := i_d$;

9.   **else**

10.   break;

11.   } (end for j)

12. $r := n - \mathrm{sum}(v)$;

13.   **if** $r == 0$ **then**

14. $y(n+1) := y(1)$;

15.   "The optimal cycle is", y;

16.   stop;

17.   **for** $k:= 1$ **to** r {

18.   **for** $l:= 1$ **to** $n - 1$ {

19. $i_t := \mathcal{M}(i_d, l).v$;

20.   **if** $v(i_t) == 0$ **then**

21. $i_d := i_t$;

22. $v(i_d) := 1$;

23. $y(j) := i_d$;

24. $j := j + 1$;

25.   } (end for l)

26.   } (end for k)

27. $y(n+1) := y(1)$;

28.   "An initial cycle is", y;

The complexity of the previous algorithm is $\mathbf{O}(n^2)$. It takes advantage of the sorted $\mathcal{M}$ picking first the vertices with small edge's cost than the vertices with large edge's cost. The algorithm's behavior depends of the numeration of the vertices that affect the cost order of the rows. It is no invariant, different equivalent GAP$_n$ (see proposition 3.6) could give different initial cycles.

Proposition 6.1. *Given GAP$_n$, if the sorted $\mathcal{M}$ has in its first column all vertices then these vertices conform a cycle, which is the solution. Furthermore, this is done in polynomial time using the previous algorithm or the algorithm 5.*

Proof. First, the optimality of the cycle comes from the vertices of first column of $\mathcal{M}$. Given any cycle of vertices $(v_1, v_2, \ldots, v_n, v_1)$, $v_i \neq v_j$, $1 \leq i < j \leq n$, its cost is given by

$$\sum_{k=1}^{n} c(v_k, v_{k+1}).$$

But each term $c(v_k, v_{k+1}) \geq c(v_k, j_k^*)$, where j_k^* is a vertex on the first column of $\mathcal{M}$. Then

$$\sum_{k=1}^{n} c(v_k, v_{k+1}) \geq \sum_{k=1}^{n} c(v_k, j_k^*).$$

The right side is the cost of the cycle that corresponds to the vertices of the first column of $\mathcal{M}$.
The complexity of the algorithm 5, or the algorithm 6 is bounded by $\mathbf{O}(n^3)$. $\qquad\square$

Proposition 6.2. *Given GAP$_n$, a sufficient condition for computing in polynomial complexity its solution is that the matrix cost $c(i, j)$ has a minimum by row and the corresponding columns index of each row cover all vertices.*
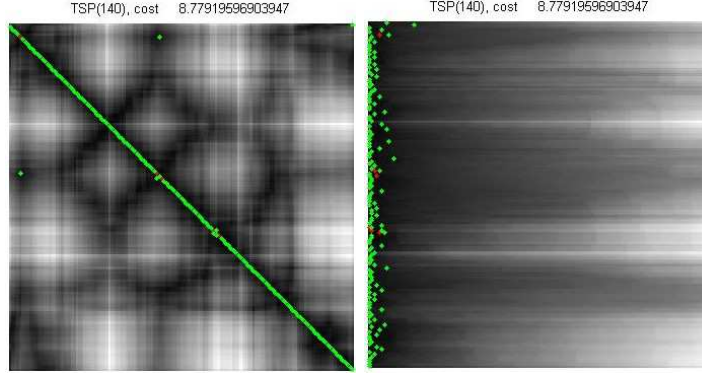
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Figure 11: On the left cost matrix, and on the right sorted cost of $\mathcal{M}$ for the example of the 2D Euclidian TSP$_{140}$. The red dots are the cycle's frontier (with cost 8.7791), and the green dots are an above cycle with cost 8.8880.

Proof. This follows from the previous proposition. □

Remark 6.3. Because the first column of the vertices' number of sorted $\mathcal{M}$ determines that there is only one alternative by row to define the optimal cycle, the right side of the inequality 6.1 (tubes are not considered, i.e., $T = 0$) imply $p \geq 0 = \frac{\sum_{i=1}^{n} \log_2(1}{\log_2(n)} = \frac{\sum_{i=1}^{n} \log_2(a_i)}{\log_2(n)}$. Therefore, the verification of the optimal cycle is not necessary, it is given by the dominance property of the first column edge's cost and vertex's number of the matrix $\mathcal{M}$. Zones of this type in a GAP$_n$ are considered tubes.

Definition 7. Given a GAP$_n$, and after computing the sorted $\mathcal{M}$, the vertices of any cycle define a frontier on $\mathcal{M}$.

1. A cycle is below if all its costs are equal or less than their corresponding cost of the frontier, i.e., they are on the left side of the frontier.

2. A cycle is above if all its costs its costs are equal or greater than their corresponding cost of the frontier, i.e., they are on the right side of the frontier.

3. A cycle is oscillating if it has costs on both sides of the frontier.

Figure 11 depicts an example of the 2D Euclidian TSP$_{140}$. Taking in consideration, that when the vertices coincide only the green dots are visible. The left picture only shows that a cycle (green dots) does not coincide with the diagonal (the optimal cycle) but the right picture implies that the green dots are an above cycle, therefore its cost is greater than the cycle's cost of red dots. The right picture has the sorted cost of $\mathcal{M}$ where the frontier (red dots) is below of the cycle with green dots. Then the green dots are an above cycle because there are not any green dots below of the red dots. Finally, the red dots are the optimal cycle of the 2D Euclidian TSP$_{140}$ because there is not below cycles of it as frontier.

Proposition 6.4. *Given GAP$_n$, and after computing the sorted $\mathcal{M}$. A necessary condition for a cycle to be a possible solution of GAP$_n$ is that there is not cycles strictly below of it as a frontier.*

Proof. By the definition, if there is a cycle strictly below, then the frontier cycle has a cost greater than it. Therefore the frontier cycle is not a candidate to be a minimum, i.e., it is not longer a solution. □

Remark 6.5. The importance of the previous proposition is that it is possible to use this necessary condition on the oscillating cycles of a given frontier. Figure 13 depicts the frontier (red dots), and oscillating cycle (green dots) with cost 204.2770. Where the cost coincide, only the green dots are visible. This example shows how sensitive is GAP$_n$, many green dots are above, on, and below of the red dots, and cost of these cycles is similar.The main cause of this is that the frontier is shifted to the right position in the image of the sorted $\mathcal{M}$, and the frontier cycle contains edges with cost negative and positive.

In short, above cycles must be discarded, below cycles must not exist, therefore, an algorithm needs only to focuss in the oscillating cycles of the sorted $\mathcal{M}$, and if the putative optimal cycle is not, then it discards the
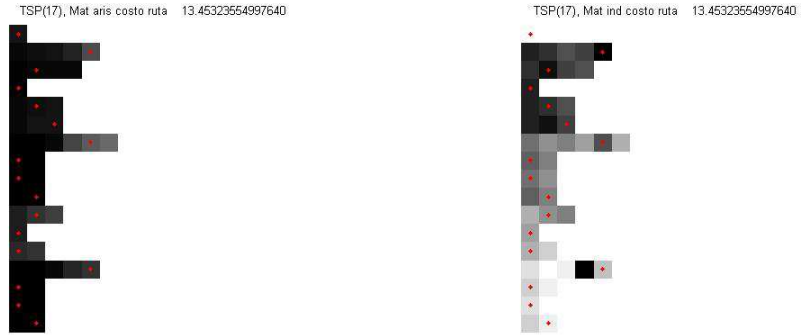
Figure 12: On the left the sorted cost matrix, and on the right their vertices of the sorted $\mathcal{M}$ for the example of the 2D Euclidian TSP$_{17}$.
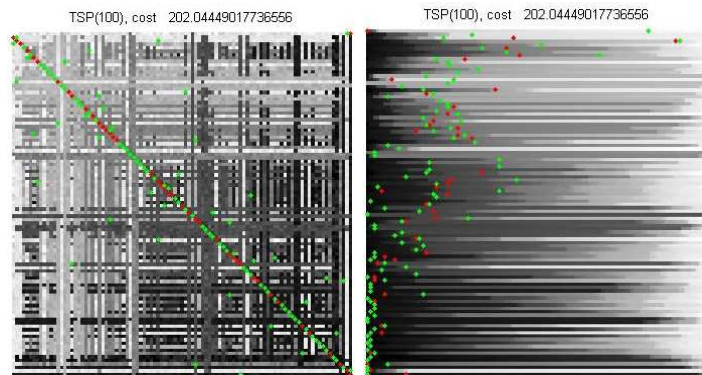

Figure 13: On the left cost matrix, and on the right sorted cost of $\mathcal{M}$ for the example GAP$_{100}$. The red dots are the cycle's frontier (with cost 202.0444), and the green dots are a oscillating cycle with cost 204.2770.

previous and keeps the best cycle. This means that the complexity of solving or checking the solution of a GAP_n is related to the maximum between the number of oscillating cycles (as the reduced research space) and $\mathbf{O}(n^3)$ (the cost of computing sorted $\mathcal{M}$).

It is also possible to combine the necessary condition with properties of special matrix $c(i, j)$ but this is not the general case. Hereafter, the matrix $c(i, j)$ is arbitrary for any large and arbitrary GAP_n.

For the matrix $c(i, j)$ we can compute $\mathcal{M}$ and in order to estimate a reduce research space for the cycles around the candidate a solution that define the current frontier this can be done by focus to the cost around and below of the cost of frontier cost in $\mathcal{M}$. By taking the vertices in $\mathcal{M}$ around the frontier we get a reduced research space that it is possible to contain the solution of the complete problem GAP_n. For special matrices $c(i, j)$ could be possible to determine the reduced research space with the guaranty that the solution is in it. This means only to take in consideration the edges (and their vertices) such that

$$c(v, j) \quad \leq \quad c_{vk}$$

where $c_{vk} = \max\{c(v, k)(1 - \varepsilon_{r_v}), c(v, k)(1 + \varepsilon_{r_v})\}$, v, k are consecutive vertices in the optimal cycle defining the frontier, and $-1 \leq \varepsilon_{r_v}$, it depends of the vertex's vicinity. In the examples, $-0.9 \leq \varepsilon_{r_v} \leq 0.6$. The algorithm 10 estimates the vertices' alternatives and ε_{r_v}. Figure 19 depicts alternatives of each vertex as magenta dotted lines.

Remark 6.6. The previous inequality keeps vertices that could belongs to oscillating cycles. v_{vk} is shifted towards the positive direction to accept all vertices with cost below of the corresponding frontiers vertex that could form a cycle with cost less or equal than frontier's cost. An example of an special matrix is a cost matrix of an mD Euclidian TSP_n where the cost correspond to a distance or a norm. The discrete distance gives trivial TSP_n problems with matrix's cost full of zeros and ones. They are trivial because the solution is rapidly and easily founded by the algorithm 6, only two columns needed to consider to found a the optimal cycle, i.e, the cycle with the maximum number of edges' cost zero. The verification of the solution by the algorithm 9 is on linear time to prove in this case that there are not cycles below of the frontier. The dimension of the space is significative for the number of alternatives to consider for updating a given path. Even, that there are a good algorithms for building a triangulation for sets of points in $\mathbb{R}^m$, $m > 2$, It becomes more difficult to build a good initial vertices' numeration as the m grows, and it is not easy to see what vertex is the next one as in the plane.

The next algorithm estimates a lower bound of the size of the reduced research space for a given frontier.

Algorithm 8. **Input:** y^* the putative optimal cycle that defines the current frontier. $c(i, j)$ the edges' cost matrix. **Output:** minimum p, a possible polynomial grade (see inequality 6.1), and A lower bound of the number of alternatives, i.e., lower bound of the reduced research space for a given frontier and a given set of $\{\varepsilon_{r_v}, v = 1, \ldots, n\}$.

1. $t := 0$; (array of $\mathbb{R}^{n \times n}$)

2. $t(1, n) := 1$;

3. **for** $i := 2$ **to** n {

4.   **for** $j := $ i **to** n-1 {

5. $k := y^*(i)$;

6. $v_{ik} := \max\{c(i, k)(1 - \varepsilon_{r_i}), c(i, k)(1 + \varepsilon_{r_i})\}$;

7.   **if** $c(i, j) \leq v_{ik}$ **then** (see inequality 6.2)

8. $t(i, j) := 1$

9.   } (end for j)

10. } (end for i)

11. $a := 0.0$; (array of $\mathbb{R}^n$)

12. **For** i := 1 **to** n

13. $a(i) = \max\{1.0, \sum_{j=1}^{n} t(i,j)\};$

14. $A := 1.0;$

15. $p := 0.0;$

16.   **For** $i := 1$ **to** n {

17. $A := A * a(i);$

18. $p := p + \log_2(a(i));$

19.   } (end for i)

20. $p := \frac{p}{\log_2(n)}.$

Note that $y^* = [n, n-1, \ldots, 1, n]$ by the numeration of the cycles on descending order algorithm 2, $t(1, n) := 1$, i..e., the cycles starts on n and finish on n for counting the cycles in descending order. Because $a(i) \neq 0, \forall i = 1, \ldots, n$ by construction, and because the graph G_n of GAP_n is complete, this algorithm gives always an estimation of $A \neq 0$ and $p \neq 0$. Also, it is possible to consider the lower edges' cost matrix. In the case where is no reduction of alternatives, the previous algorithm gives $A = (n-1)!$ and $p = \frac{\log_2(A)}{\log_2(n)}$. For future work, it is possible to create a more precise algorithm. For example, taking $a(i)$ as the number of vertices that comply inequality 6.2 without repeating them provides other estimation of A and p.

Proposition 6.7. *Given GAP_n, and after computing the sorted $\mathcal{M}$. A necessary condition for a cycle to be a possible solution of GAP_n is that there is not a cycle below of it as a frontier.*

Proof. By the definition, if there is a cycle with some vertices below and other on the frontier, then the frontier cycle has a cost greater than it. Therefore the frontier cycle is not a candidate to be a minimum, i.e., it is not longer a solution. $\square$

Remark 6.8. For GAP_n and the solution's frontier, strictly below and above cycles could not exists, all cycles could be oscillating. For TSP_n is possible that exists oscillating cycles.

Figure 12 depicts the case of the example of the 2D Euclidian TSP_{17} (Fig. 9 depicts the solution with green lines) where the frontier for the solution is marked by red dots. Because many dots of the frontier are on the left border the alternatives of many vertices are less than 2 implying that the solution is easy to verify and estimate, i.e., this is a case of very few alternatives to explore. Note that also the tube given by the vertices $3, 2, 1, 17$ is easily to identify on the picture. The cost of the edge $1, 17$ is marked by the red dot on upper left. It is on the left of the first row because this cost is less than all the edges' cost of the vertex 1. The cost of the edge $2, 1$ is on the second row, four column because it is greater than the cost of the edges $3, 2$, $4, 2$, $6, 2$, and $5, 2$. The red dot of the cost of the edge $3, 2$ on the third row is on second column because it is greater than edge's cost $4, 3$. On the other hand, the picture on the left denotes also the frontier of the solution on the gray image of the vertices' number. The first row has the red dot of edge's cost $1, 17$ over a white square that correspond to the color of the vertex 17. On the second row there is four gray squares corresponding to the color of the vertices $3, 4, 6, 5$ and the red dot of the edge cost is on a black square that corresponds to the vertex 1. Note that over the columns the red dots goes from white-black gray fading to white because the color of the vertex of the solution goes from $(17, 16, 15, \ldots, 2, 1, 17)$. These two pictures depicted the relation between the edges' cost and the vertices' number of the solution. There is a possible efficient verification of the solution when the cost of the solution are on closed to the left column and the vertices' number creates a pattern white-black gray fading to white, this combination point out that the alternatives for most of the vertices are very few. For the example of the 2D Euclidian TSP_{17} the other solution is the inverse cycle $(1, 2, 3, \ldots, 16, 17, 1)$. These solutions are equals because the cost matrix is symmetric.

The Figure 14 depicts an example of GAP_{100} where the frontier given by the solution is scattered. The image of the cost matrix on the left depicts on many rows an oscillation of gray squares from black to gray, this means that many vertices have edge's cost worthwhile to explore. There are very few row with uniform gray color. On the images of the sorted $\mathcal{M}$, there are many red dots of the frontier to the right with a fading sequence of gray squares, it means edges' cost less than the cost of the edge's cost of the solution. Finally the vertices' number
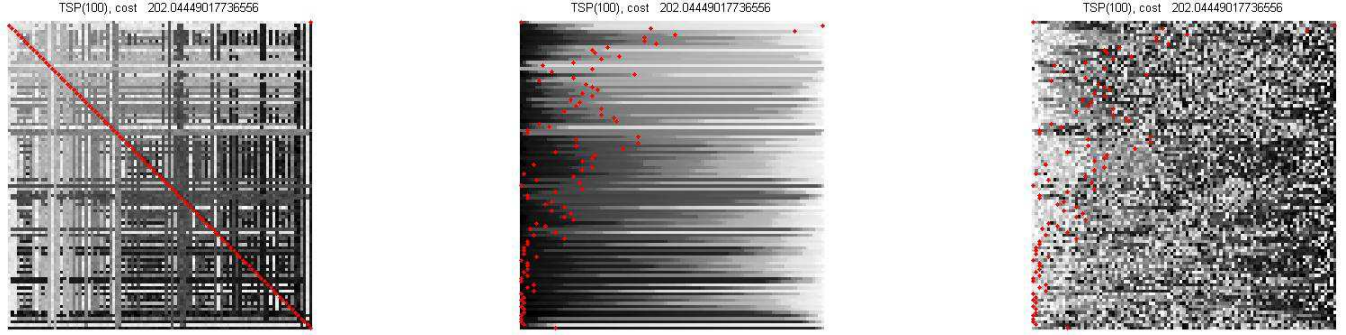
Figure 14: Example of an GAP$_{100}$ with arbitrary cost matrix, by construction there is not pattern to point out a reduction of vertices' alternatives From left to right, Cost matrix, sorted cost matrix, and their vertices of the Sorted $\mathcal{M}$.
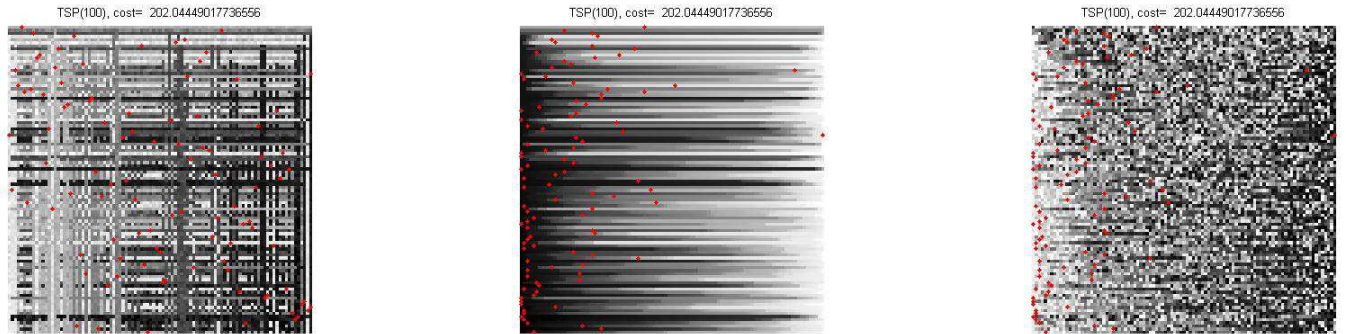

Figure 15: Example of an GAP$_{100}$ with arbitrary cost matrix. Here the matrices are sorted by columns and by row taking the first column's cost, from left to right, Cost matrix, sorted cost matrix, and their vertices of $\mathcal{M}$.

picture depicted a random mosaic, it means that many vertices are related to many vertices that could give a reduction of the cycle cost but the research space is not reduced. The Figure 15 depicts the previous matrices sorted by the first column's cost. These figures makes more clear the existence of oscillating cycles around the solution and also depicts that the frontier in sorted $\mathcal{M}$ is not on the left side but on the middle.

An example not small is 2D Euclidian the TSP$_{140}$ depicted in the Fig. 16. The pattern of the frontiers solution (red dots) is to left on both images of the sorted cost matrix, and their vertices of the Sorted $\mathcal{M}$. This means a reduction of the research space to verify (or even to possibly estimate the true solution) the solution. To compute the solution of this example the algorithm is the following. The main flow chart structure is depicted in the Fig.17.

Algorithm 9. **Input:** GAP$_n$, $c(\cdot)$ cost function for cycles. **Output:** y^* cycle solution, and sorted matrix $\mathcal{M}$.

1. Estimate for a greedy algorithm a possible solution. Let be y^* a putative cycle solution.

2. Reorder and renumbering the vertices and cost matrix for the solution. The first cycle $n, n-1, \ldots, 2, 1, n$.

3. Compute sorted $\mathcal{M}$.

4. Set the frontier, i.e., Compute the reduced research space by estimate the vertices' alternatives for the current minimum cycle by equation 6.2.

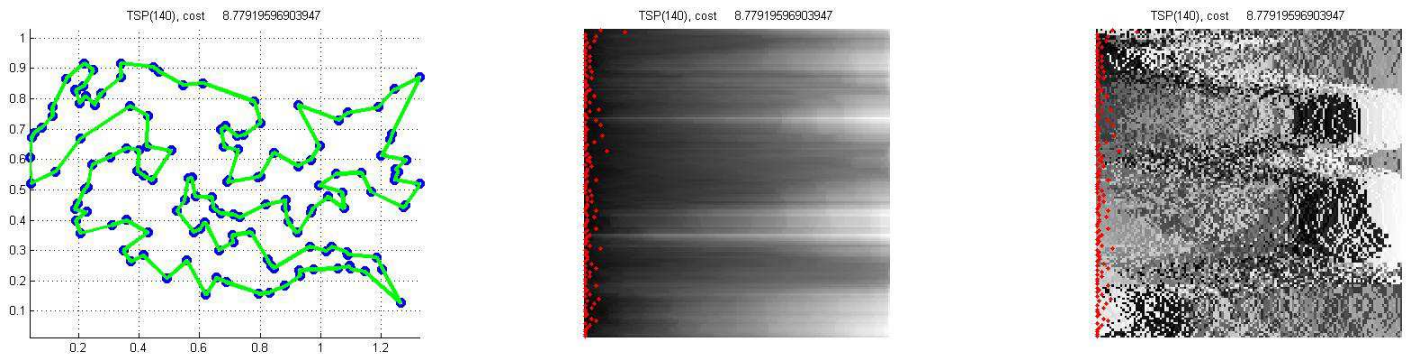5. **repeat** for the current frontier generate y_k (a cycle)

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Figure 16: Example of an 2D Euclidian TSP$_{140}$. There is a pattern to point out a reduction of vertices' alternatives From left to right, cycle with the minimum cost, sorted cost matrix, and their vertices of the sorted $\mathcal{M}$.
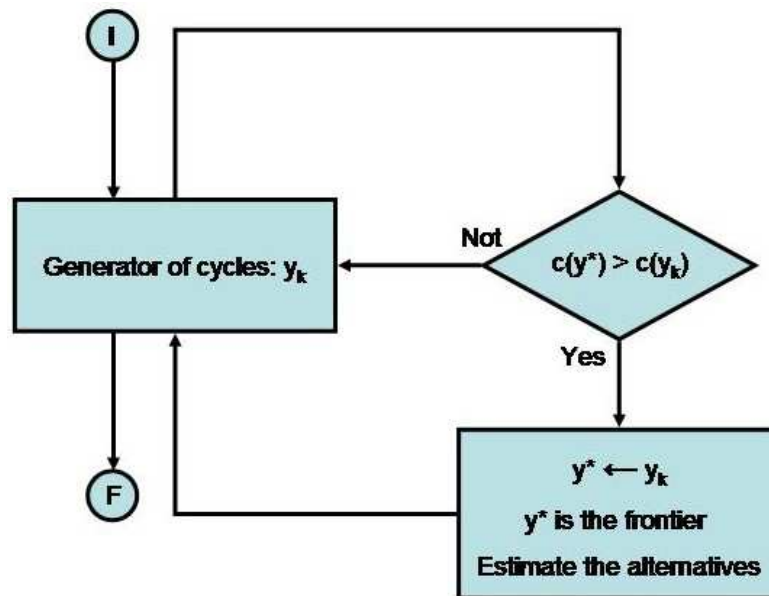

Figure 17: Flow chart for the verification and estimation of the solution on a reduced research space.

6.   **if** $c(y^*) > c(y_k)$ **then**

7. $y^* := y_k$

8.   Update the frontier and estimate the vertices' alternatives.

9.   **until** there is not changes and all possible cycles for the last reduced research space are generated. (end repeat)

An example of a greedy algorithm for step 1 is given in algorithm 6. The complexity of the previous algorithm for verifying the solution (assuming that it is the solution) is **O** (max greedy algorithm, sort of $\mathcal{M}$, one estimation of the reduced research space, while for the generator of cycles). All but the while for the generator of cycles are polynomial. The generator of cycles is a TM for keeping the minima memory's interaction needed only to preserve the optimal solution, therefore this loop corresponds to a TM when the candidate cycle does not change, there is not necessary to repeat reorder and sort of $\mathcal{M}$ for the new solution, and the estimation of the alternatives of the new solution, ie.e, the estimation of the reduced research space. Nevertheless its complexity depend of the number of cycles generated. The computation of the cycle's cost is assumed in one machine's operation. The only way to guaranty the algorithms complexity as polynomial is if the reduced search space does not change and it has no more n^p possible cycles. It is easy to compute cycles by arrays containing the number of the vertices of the reduced research space, also it is possible to estimate cycles by a backtracking algorithm. Nevertheless, the complexity of this loop of the generator of cycles has a lower bound given by the product of the alternatives that conform different cycles (i.e., paths with $n+1$ vertices without repeating vertices but the first and the last one). It is possible to define a practical early stop over an accepted reduction given by an empirical selection of ε_{r_v} of this algorithm, i.e., for a given ε_{r_v} if there is not changes of the frontier then, the frontier is the local solution accepted as the global putative solution. It corresponds to the local minimum of the reduced research space for all cycles generated by the alternatives given by such ε_{r_v} and such frontier. The next proposition exploits such idea together with the 2D Euclidian properties, and it is proven that the reduced research space keeps the global solution.

Proposition 6.9. *The 2D Euclidian TSP$_n$ of the NP-Class have an polynomial algorithm for checking their solution.*

Proof. For few vertices less than 10 is trivial to find the solution in polynomial time, and no matter if the vertices are closed or disperse the sorted $\mathcal{M}$ has the solution's frontier closed to the left, and ε_{r_i} have appropriate values. Let assume an instance of a problem 2D Euclidian TSP$_n$ and its solution. Using the algorithm 5 the sorted $\mathcal{M}_n$ is computed. Now, a vertex is given and the cost matrix is updated for an 2D Euclidian TSP$_{n+1}$. Three cases are need to be considering in 2D for any flat figure limited by vertices. The new vertex is in or out of the vertices' hull. if it is out, by the minimum distance from this point to vertices' hull in 2D this vertex can only add at most 3 cost relevant at most 3 vertices which are closed to the minimum distance to it. If the vertex is inside, then considerer a triangulation of the vertices' hull. The new vertex must be in a triangle or on triangle's edge. Any way,in the three cases, it can be added to keep a triangulation between the vertices of the vertices' hull of the given 2D Euclidian TSP$_n$. Now the solution of the 2D Euclidian TSP$_n$ is marked on the triangulation of the $n+1$ vertices and this solution is altering to include the new vertex changing the closed pair of vertices to include the new one without crossing edges and the numeration is changed to include the new vertex to reflect its vicinity and also if it is necessary adjust ε_{r_v} to keep this vicinity. Now the algorithm 9 can be used to estimate the solution for this 2D Euclidian TSP$_{n+1}$ with the updated $\mathcal{M}_{n+1}$ and the suitable initial solution is the vertices' numeration including the new vertex. The alternatives to verify the 2D Euclidian TSP$_{n+1}$ are limited to the new vertex's alternatives of its vicinity vertices of the previous solution of the 2D Euclidian TSP$_n$. $\square$

Remark 6.10. The previous proposition states that a growing strategy vertex by vertex can be used not to solve but to verify efficiently the 2D Euclidian TSP$_n$'s solution. Proposition 3.5 shows that for the TSP$_n$, it is possible to stop computing paths in the same branch when a reference value is reached, but the idea of the reduction of the research space, also does not consider such paths, therefore the algorithm 9 does not need to include or use this property of the TSP$_n$, i.e. it is not necessary an "if inside of cycles generator when $c(p_2) \geq c^*$, $c(y^*) = c^*$ where p_1 is a path of TSP$_n$. The algorithm 9 is not such algorithm as it is depicted here. For the 2D Euclidian TSP$_n$ examples of this paper, I did manually ad-hoc initial solution with its enumeration of the vertices (a greedy manual
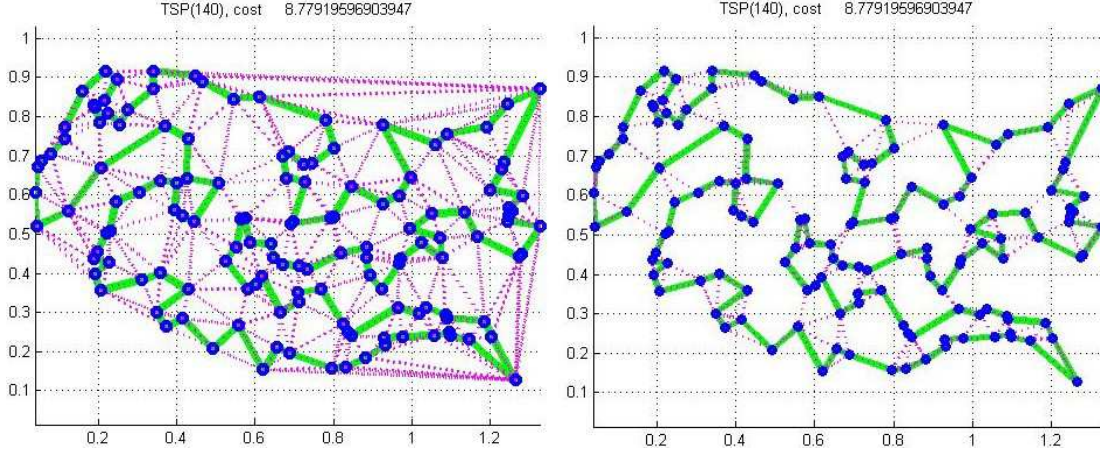
Figure 18: Triangulation (magenta dotted lines) complete and partial for the example 2D Euclidian TSP_{140}, the optimal cycle is depicted by the green lines, and the vertices are depicted the blue dots.

process), after this, a reorder of the cost matrix to build an equivalent GAP_n (see proposition 3.7), and with the algorithm 6 an initial cycle was computed, finally, after that to verify and compute the solution of the 2D Euclidian TSP_n examples the algorithm 9 was applied considering neighbors with $\varepsilon_{r_i} \in [-0.9, 0.6]$. It is not be confused that the manually reduction described above missed or assumed the global optimal cycle of examples of 2D Euclidian TSP_n, it is obvious that the global optimal solution is a local optimal solution as necessary condition. Therefore, the previous proposition shows that the reduced search space allows to verify the locally of the putative solution, and with a carefully reduction of the alternatives based in the 2D Geometry, the global solution remains inside of the alternatives of each putative cycle founded. Figures 18, and 19 depict examples of appropriate complete and partial triangulation that they keep inside the solutions.

The key to build a efficient algorithm for varying the solution is the triangle inequality property and the monotony of a finite norm in 2D that allows to have locally vertices' vicinity and to keep a triangulation, and in 2D a few paths closed from where it is possible to obtain an appropriate numeration of vertices. The triangle inequality helps to walk by short edges avoiding large diagonal (see proposition 6.11), and helps to define ε_{r_i} to find an appropriate local vertex's vicinities (see fig.19). The parameter ε_{r_i} correspond to the vertex i. The next algorithm estimates the alternatives, and the parameters ε_{r_i} by selecting at least two alternatives.

Algorithm 10. **Input:** GAP_n, $y = (n, 1, 2, \ldots, n-1)$ the vertices of the putative cycle solution, and $c(\cdot, \cdot)$ cost matrix. **Output:** $a \in \mathbb{N}^{n \times (n-1)}$ (v's alternatives, i.e., vertex related with v by the $c(v,\text{vertex})$ closed to $c(v, y(v))$), $\varepsilon_r \in \mathbb{R}^n$ (array of the estimations of ε_{r_v}).

1. **for** v:=1 **to** n {

2. $\varepsilon_r(v) := -1.0$;

3.   **while** (True) {

4. $c_{vk} := c(v, y(v))(1.0 + \varepsilon_r(v))$;

5. $t := 0; t \in \mathbb{N}^{(n-1)}$ (array of vertices)

6. $t = \text{find}(c(k, :) \leq c_{vk}) \cup y(v)$; (vertices of the v's vicinity)

7. $n_t := \text{length}(t)$;

8.   **if** $n_t > 1$ **then**

9.   break;
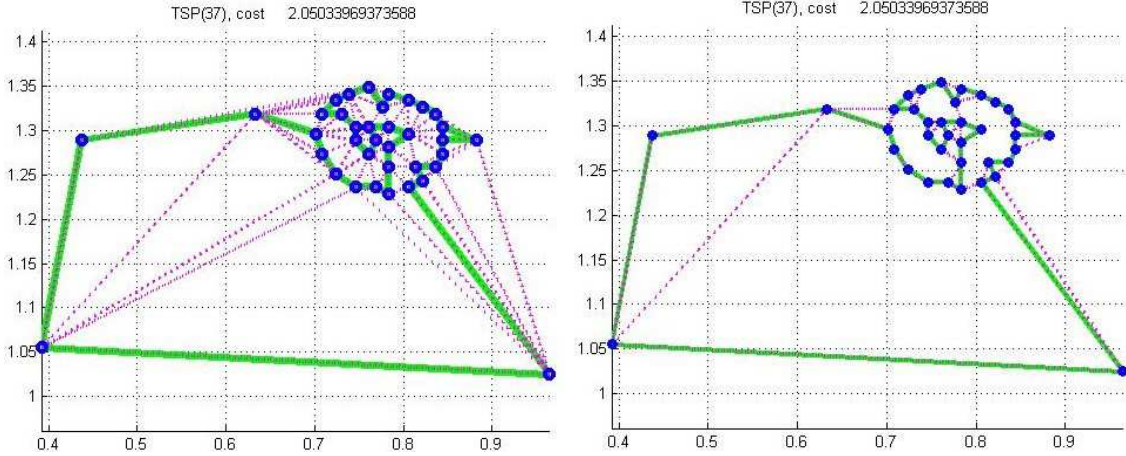
10.   **if** $n_t > (n-1)$ **then**

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Figure 19: Triangulation (magenta dotted lines) complete and partial for the example 2D Euclidian TSP$_{37}$, the solution is depicted by the green lines, and the vertices are depicted by the blue dots.
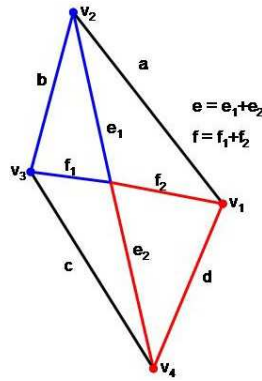

Figure 20: Arbitrary quadrilateral.

11. $\varepsilon_r(v) := \varepsilon_r(v)$ - 0.01;

12.   **else**

13. $\varepsilon_r(v) := \varepsilon_r(v) + 0.01$;

14.   } (end while)

15. $a(v, :) := t$; (store v's alternatives)

16.   } (end for)

The following proposition shows that for any quadrilateral, its perimeter is the lowest cycle'length.

Proposition 6.11. *Given four points in 2D as vertices, its quadrilateral's perimeter is lower than the cycle's Euclidian length using the diagonals and the opposites sides.*

Proof. Figure 20 depicts an arbitrary quadrilateral with vertices v_1, v_2, v_3, and v_4, and edges' Euclidian length a, b, c, and d. Its diagonals' length are e, and f. Without loss of generality the opposites sides are a, and c. The next relations are valid by triangle inequality for the blue triangle, and the red triangle:

$$
\begin{aligned}
b &\leq e_1 + f_1 \\
d &\leq e_2 + f_2.
\end{aligned}
$$

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Then, these relations and the quadrilateral's perimeter comply:

$$a + b + c + d \leq a + e_1 + f_1 + c + e_2 + f_2 = a + e + c + f$$

<div align="right">□</div>

In this case the verification of the solution is locally around of the oscillating cycles closed to the frontier, as the a vertex is far away of a local vicinity the monotony of the distance allows that such vertices can be ignoring because they include pair of vertices with higher cost. Finally the given cycles by the initial enumeration and the cycles computed do not have crossing edges, this means that all edges of the solution does not cross the edges of an appropriate triangulation (see figures 19, and 18). This is the case for Euclidian length, $d(x,y) = \sqrt{x^2 + y^2}$ but $d_{\inf}(x,y) = \max(|x|, |y|)$ allows crossing edges because diagonal and sides have the same length in a quadrilateral. The symmetry of the matrix cost is not used in any way in the process of verification. Other way to see that the 2D Euclidian TSP_n can be solved in polynomial time is the property that its sorted $\mathcal{M}_{n+1}$ has a left frontier (this mean very few alternatives around of the solution) and there is always a vertices' numeration which it takes in consideration a closed vertex with a consecutive number, so the cycle's numeration is consecutive and correspond to the first cycle in the enumeration given by the proposition 3.4. Also, its integer formulation can be also solve in polynomial time, and pivot strategies get advantage of the geometric properties (the variables of a pivot for TSP_n are limited, see proposition 3.5), which are reflected on the structure of the sorted $\mathcal{M}_{n+1}$ and its frontier. But in 3D or higher dimensions the initial enumeration is not easy to do, and could be many paths around a new vertex, even so, in numerical experiments of mD Euclidian TSP_n ($m \geq 3$), its the sorted $\mathcal{M}_{n+1}$ has a left frontier lightly shifted to the right. The reduction of the alternatives in the 2D Euclidian is limited by the number of neighbors that a vertex can have in a plane to keep a triangulation that does not change the global solution but reduce the alternatives to preserve the solution in a local or reduced research space. Other property in the solutions of 2D Euclidian TSP_n is that the cycles have not cross over so the new vertex has limited way to be connected. In 3D the Euclidian TSP_n has a growing alternatives quite different. A vertex could has from one to 12 neighbors and many path around it but a cheap triangulation opens the possibility to formulate an efficient algorithm for solving it.

On the other hand, the integer programming version IPGAP_n depends on pivot strategies to explore the feasible points (see [8]) x_{ij}. For an arbitrary cost matrix without a pattern ,the exploration even by oracles' strategies are not suitable because the set of values of the edges' cost has not a relation or property. On the section 4 was noted that at least two vertices are need to change for three x_{ij} by three x_{ij} for two feasible points. A pivot strategy of using two vertices is not sufficient to explore all feasible points, a cycle could by different of other cycle for more than two vertices. The cycles obtained over a given cycle of GAP_n by using an strategy of two vertices is $2!(n-2)$, $n \geq 3$; for 3 vertices the cycles are $3!(n-3)$ $n \geq 4$ but the previous ones are cover. For k vertices the cycles are $k!(n-k)$ $n \geq k+1$ over a given cycle. By the proposition 3.5 to limit the number of variables on a pivot's strategy could cause to miss the solution (the cost function of GAP_n is oscillating). The limits is an strategy of using $n-1$ active variables but this means to explore all the $(n-1)!$ cycles of the equivalent GAP_n.

This point could be debatable but the complexity of matrix cost of a given GAP_n can grows. This can be done by assigning the edges' cost as a random field or white noise with values in $[0, 1]$ or by mixing vertices at different but similar distances (distance as edge's cost). Then without properties it is not possible to reduce the research space, and properties can not be inherited. For example, take an 2D Euclidian TSP_n and add m arbitrary vertices and their costs, it loses its triangulation, i.e., the new m vertices and the n original vertices are not longer related by the Euclidian distance but arbitrary values let say in the interval $[-v, v]$, $v > 0$, the resulting problem is a GAP_{m+n} with $m \gg n$ and without the properties of an Euclidian Space. Also, a GAP_k with cost matrix with values on $[-\varepsilon/k, \varepsilon/k]$ has cycles' cost bounded by $[-\varepsilon, \varepsilon]$ and ad-hoc probabilistic algorithm could give a putative solution in polynomial time but if new m vertices and their arbitrary costs are added, the solution of the GAP_k and properties what helps to solve it, they do not contribute and be out of the solution in many cases for arbitrary and large GAP_{k+m}.

Proposition 6.12. *An arbitrary and large GAP_n of the NP-Class has not an polynomial algorithm for checking their solution.*

Proof. If such algorithm exist for any problem size m. Then it can verify the solution of any problem GAP_n with $n \gg m$. Let the algorithm to estimate the solution of an arbitrary GAP_n, constructed by an arbitrary combination

of vertices with arbitrary edges' cost. This means that the cost matrix has not properties or relations between the edges' cost of the vertices of GAP_n. Now for the putative solution, reorder and renumbering the vertices and cost matrix such that the first cycle $n, n-1, \ldots, 2, 1, n$ is the solution. Compute sorted $\mathcal{M}$ and looks the images of the sorted cost matrix, and their vertices of $\mathcal{M}$. If the there is a pattern then the matrix is not arbitrary, which is not the case by the construction of the cost matrix of GAP_n. If the frontier is on almost over the left size then the values of the cost matrix by row are related by an order, which is not the case by the construction of GAP_n. Therefore, most of vertices of the frontier are not close to the left side. The lack of properties of the given cost matrix does not allow to use other algorithm but the depicted by the Fig. 17. But this means that even in an optimism case for at least tree alternatives for each vertex, the complexity of the loop to verify the solution is $\mathbf{O}(3^n)$. There is not exist such efficient algorithm. $\qquad\square$

The proposition shows that an algorithm could claim and maybe solve a NP problem but the verification without properties over an a worst case with arbitrary data can not be demonstrated in polynomial time, and if the research space could be reduced then input problem is not a arbitrary worst case with many oscillating cycles.

Conclusions and future work

Here the concept of a tube and the parallel solution were drafted as an example of a characteristic that in order to estimate the complexity does not relates to the complexity of the input data's size. This means that there are easy to solve instances of TSP_n and GAP_n. The images of the frontier depicts and corroborates the existence of this tubes for the examples of 2D Euclidian TSP_n but there are more work to do. The reducibility of 2D Euclidian TSP_n allows to verify its solution in polynomial time. It is consequence of the 2D geometric restriction, the monotony of a distance and their geometric properties. For LJP_n the reducibility allows to define the lattice CB but arbitrary and large optimal clusters can not be efficiently estimated. I suspect and has not proved that an algorithm to solve an arbitrary mD Euclidian TSP_n $(m > m_0)$ has exponential complexity starting with some $m_0 > 4$, but if the vertices can be numerated, and the initial cycle has a small cost then the complexity for verifying or updated the initial putative cycle could be limited to polynomial time by geometric properties. The key is to have an appropriate way to numerate, and also, to reduce the alternatives by the geometric properties. It could be possible to find a reduced research space using an appropriate triangulation so that the global solution does not escape. But Triangulation algorithms has $\mathbf{O}\left(n^{\frac{m-1}{p}}\right)$, $\forall p \in [2, m-1]$ (see [1]). Therefore, it is highly possible that it exists an algorithm to solve arbitrary and large mD Euclidian TSP_n with large polynomial time until some finite $m > 0$. There is not an efficient algorithm for solving 2D Euclidian TSP_n but it is highly possible to build it. However, for arbitrary and large GAP_n, it does not exist an efficient algorithm for solving it.

Moreover, even with the reduction of the research space to the lattice CB, it was expected that LJ problem or the Searching of the Optimal Geometrical Structures of clusters of n particles remains out of the existence of an efficient algorithm but now it is proved that it has exponential complexity. On the other hand, the putative optimal cluster that I tested in the appropriate reduced lattice CB from 2 to 55 particles are now considered global optimal clusters for the LJ potential. There are more work to verify the long list of putative optimal clusters from 56 to 1501 particles given by Shao or The Cambridge Cluster Database (CCD), i.e., to shift them from putative to global optimal clusters.

The line to divide when is possible to verify in polynomial time the solution is given by the frontiers position on the sorted matrix $\mathcal{M}$. This brings the solution of Conjecture of the P-Class and the NP-Class, the later has not polynomial complexity in the general or worst case scenario (large and arbitrary NP problems) but exponential, therefore, the Classes P and NP are different except for trivial cases when it is possible to verify the solution by the reducibility. It took a while to understand and accept that they lack of properties do not allow to design an algorithm able to tackle the growing of the complexity in term of the input data's size. Algorithms are based on properties or data relations given or constructed by organizing the data on appropriated data structures as the algorithm progress. Here the problems of class NP are not only a worst cases but without reducibility, arbitrary and large ones.

The result shows that does not exist a general property that allows to solve in polynomial time all NP problems, therefore is important to continuing developing specific and ad-hoc algorithms for each type of the NP problems. Also, it showed that is important in the design of these algorithms to consider the problem's properties and

characteristics as insights to save operations. The polynomial reduction procedure states that is possible to solve 2D Euclidian TSP_n by changing the cost matrix to an structure based in few alternatives, i.e., as a rectangular matrix with few columns to consider or an appropriate triangular sub-graph of its graph G_n.

The local or global dilemma, could be addressing as in here by defining an appropriate search space where a local not heavy time consuming algorithm could be sufficient for many scientific, technological, and engineering issues. In fact, the landscape of the research space of the NP Problems allows stationary properties from where is quite difficult to escape for improving the putative solution. The evolution and the living beings are the kind of solution of problems of this class. There are many examples where the solution is know but we can not justify or demonstrate and reproduce such results. Too many combinations, relations, variables, restrictions, and process are perfectly working every day in a biological machine but we are not yet able to understand them. For the NP problems the landscape of the research space could have one or many (exponential) similar results but the solution could be slightly different. We are yet able to find out these slightly but important solutions because our computers technology has time limitations. Practical limitation are the main argument for an early stop. However, besides that is not worth to expend exponential time for look for the solution, it is possible that this slightly difference could be the answer for the life mechanism or other complex biological process. The Lennard-Jones problem is an example where the meaning of practical purposes for limiting the time for determining the optimal geometry for cluster of n particles is debatable. The prediction of real cluster structures and the possibility to help to understand the cooling process of real cluster for this simple NP problem keeps many researchers busy and greedier for new results.

For the future, new computing technology and models are needed, it will be very interesting to define quantum computing models to look for establishing novel possible efficient perspectives to solve the problems of the NP-Class and exponential ones.

Finally, this novel formulation solves the Noted Conjecture of the NP-Class and brings a promising theoretical perspective to address the construction of efficient algorithm for arbitrary problems.

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