

An Extension of the Permutation Group Enumeration Technique (Collapse of the Polynomial Hierarchy: $\mathbf{NP} = \mathbf{P}$)

Javaid Aslam* jaslamx@yahoo.com

October 28, 2018

Abstract

The distinguishing result of this paper is a \mathbf{P} -time enumerable partition of all the $n!$ possible perfect matchings in a bipartite graph. This partition is a set of equivalence classes induced by the missing edges in the potential perfect matchings.

We capture the behavior of these missing edges in a polynomially bounded representation of the exponentially many perfect matchings by a graph theoretic structure, called MinSet Sequence, where MinSet is a \mathbf{P} -time enumerable structure derived from a graph theoretic counterpart of a generating set of the symmetric group. This leads to a polynomially bounded generating set of all the classes, enabling the enumeration of perfect matchings in polynomial time. The sequential time complexity of this $\#\mathbf{P}$ -complete problem is shown to be $O(n^{45} \log n)$.

And thus we prove a result even more surprising than $\mathbf{NP} = \mathbf{P}$, that is, $\#\mathbf{P} = \mathbf{FP}$, where \mathbf{FP} is the class of functions, $f : \{0, 1\}^* \rightarrow \mathbb{N}$, computable in polynomial time on a deterministic model of computation.

Keywords: *Perfect Matching, Permutation Group Enumeration, Counting Complexity, NP-Completeness, Polynomial Hierarchy.*

1 Introduction

Enumeration problems [GJ79] deal with counting the number of solutions in a given instance of a search problem, for example, counting the total number of perfect matchings in a bipartite graph. Their complexity poses unique challenges and surprises. Most of them are \mathbf{NP} -hard, and therefore, even if $\mathbf{NP} = \mathbf{P}$, it does not imply a polynomial time solution of an \mathbf{NP} -hard enumeration problem.

\mathbf{NP} -hard enumeration problems fall into a distinct class of polynomial time equivalent problems called the $\#\mathbf{P}$ -complete problems [Val79b]. As noted by Jerrum [Jer94], problems in $\#\mathbf{P}$ are ubiquitous- those in \mathbf{FP} are more of an exception. What has been found quite surprising is that the enumeration problem for perfect matching in a bipartite graph is $\#\mathbf{P}$ -complete [Val79a] even though the associated search problem has long been known to be in \mathbf{P} [Kuh55, Ege31, Edm65].

Enumeration of a permutation group has long been known to be in \mathbf{FP} ([But91, Hof82]). The basic technique for enumerating a permutation group G (any subgroup of the symmetric group S_n) is based on creating a hierarchy of the *Coset Decompositions* over a sequence of the subgroups of G , where the smallest subgroup is the trivial group I .

A Coset Decomposition of G is essentially a set of equivalence classes defining a *Partition* of G for a subgroup H of G , induced by a set of group elements called *Coset Representatives*(CR). Here each element ψ in CR represents a unique subset of G , called Coset of H in G , obtained by multiplying each element in H by ψ , in certain (right or left) order. For the symmetric group, S_n , the partition hierarchy for a fixed subgroup sequence is shown as an n -partite directed acyclic

*©Copyright Javaid Aslam 2009-2017, Santa Clara, CA 95054 All rights reserved.

graph in Figure 1, where the nodes in each partition are the elements in CR representing the subsets of a group $G^{(i)}$ in the subgroup sequence $G^{(0)} > G^{(1)} \dots > G^{(n)}$. The edges represent a disjoint subset relationship.

The enumeration technique for perfect matchings extends the above coset decomposition scheme by further partitioning

each coset into a family of polynomially many equivalence classes. This extended partition hierarchy (Figure 2) then captures the perfect matchings as an equivalence class in this partition, where each such class allows the P-time enumeration uniformly for all $n \geq 3$.

The associated equivalence relation over a coset is induced by a graph theoretic attribute called *edge requirements* which confirms a potential perfect matching subset in each equivalence class.

The hierarchy of the various classes for a bipartite graph holds the following containment relationship:

$$S_n \supset \text{Coset} \supset [\text{other equiv classes}] \supseteq \text{Perfect Matchings}$$

The extended partition hierarchy contains “other equivalence classes” *CVMP Sets* and *MinSet Sequences*, described below.

We map a specific generating set of the symmetric group S_n to a graph theoretic “generating set”, such that each coset representative of a (group, subgroup) pair is mapped to a set of graph theoretic coset representatives. This mapping is then used to construct a *generating graph* for generating all the perfect matchings as directed paths in the generating graph which is a directed acyclic n -partite graph of size $O(n^3)$.

Each perfect matching in a bipartite graph with $2n$ nodes is expressed as a unique directed path of length $n-1$, called Complete *Valid Multiplication Path* (CVMP) in the generating graph. The condition for a CVMP of length $n-1$ to represent a unique perfect matching in the given bipartite graph is captured by an attribute of the CVMP, called *Edge Requirement* (ER).

The graph theoretic coset representatives induce disjoint subsets of the Cosets, called CVMP Sets, an equivalence class containing the CVMPs.

Each CVMP Set is further partitioned into polynomially bounded classes called *MinSet Sequences* induced by the ER of each CVMP, where a MinSet is the set of all Valid Multiplication Paths (VMPs) of common ER.

A judicious choice of the common ER of these VMPs allows a MinSet and any sequence of the MinSets to be P-time enumerable, and which makes the perfect matchings also P-time

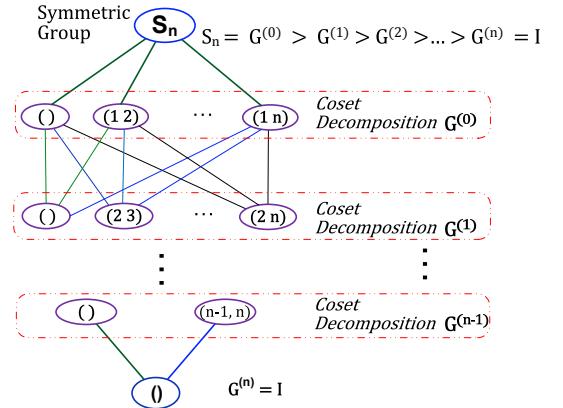


Figure 1: A Hierarchy of the Coset Decompositions of S_n

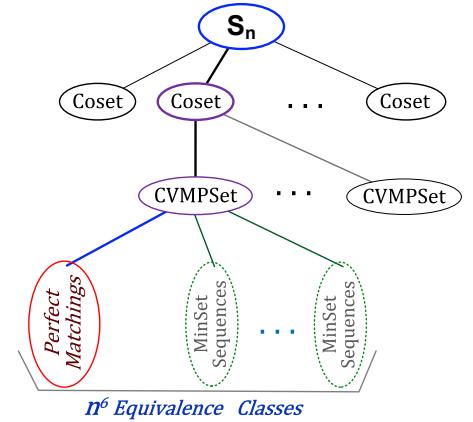


Figure 2: The Extended Partition Hierarchy

enumerable as follows.

These MinSet sequences can be viewed as an instance of a perfect matching subproblem, where a sequence containing only one MinSet would represent a set of perfect matchings when each CVMP is of length $n-1$ with common $ER = \emptyset$.

There are exponentially many MinSet sequences in a CVMPSet, and all of them can be covered by only $O(n^6)$ unique prefix MinSets. And thus these unique prefixes induce polynomially many equivalence classes each containing exponentially many MinSet sequences (Figure 3).

Further, each class size decreases exponentially with n , and thus this hierarchy enables enumeration of all the equivalence classes in polynomial time.

Main

results of this paper are summarized as follows:

1. The Mapping : Section 3 develops the concepts leading to the graph theoretic counterparts of the permutation group generating set. It defines the mapping (Lemma 3.5) of a specific generating set of S_n to a set of graph theoretic structures.
2. The Extended Partition Hierarchy: Section 4 describes how to partition all the potential perfect matchings into a set of equivalence classes induced by the missing edges in all the potential perfect matching in the given bipartite graph.
Lemma 4.18 states how exponentially many MinSet sequences in any CVMPSet are partitioned into polynomially many equivalence classes of polynomially many prefixes to the MinSet sequences.
3. Polynomial Time Enumeration: Algorithm 5.1 describes a high level logic to count all the perfect matchings in time $O(n^{45} \log n)$.

Section 6 provides the conclusion, collapse of the Polynomial Time Hierarchy.

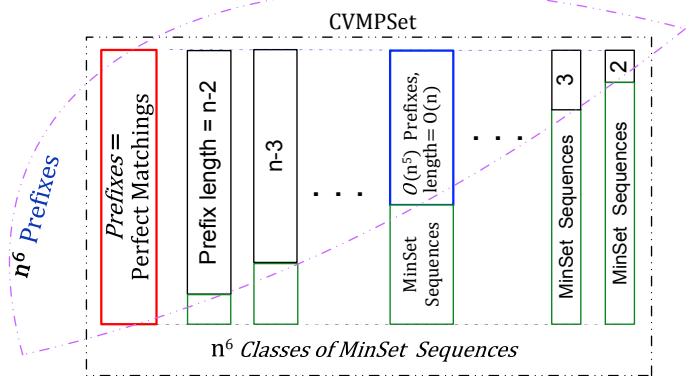


Figure 3: Partition of a CVMPSet into MinSet Sequences

2 Preliminaries: Group Enumeration

The following concepts can be found in many standard books (including [But91]) on permutation group theory. The notations and definitions used here are taken mostly from [Hof82].

2.1 The Permutation Group

Let G be a finite set of elements, and let “.” be an associative binary operation, called *multiplication*. Then G is a group if satisfies the following axioms:

1. $\forall x, y \in G, x \cdot y \in G.$
2. there exists an element, $e \in G$, called the identity, such that $\forall x \in G, x \cdot e = e \cdot x = x.$
3. $\forall x \in G$, there is an element $x^{-1} \in G$, called the inverse of x , such that $x \cdot x^{-1} = x^{-1} \cdot x = e.$

Let H be a subgroup of G , denoted as $H < G$. Then $\forall g \in G$ the set $H \cdot g = \{h \cdot g | h \in H\}$ is called a right *coset* of H in G . Since any two cosets of a subgroup are either disjoint or equal, any group G can be partitioned into its right (left) cosets. That is, using the right cosets of H we can partition G as:

$$G = \biguplus_{i=1}^r H \cdot g_i \quad (2.1)$$

The elements in the set $\{g_1, g_2, \dots, g_r\}$ are called the *right coset representatives* (AKA a *complete right traversal*) for H in G .

In this paper we will deal with only one specific type of finite groups called *permutation groups*. A *permutation* π of a finite set, $\Omega = \{1, 2, \dots, n\}$, is a 1-1 mapping from Ω onto itself, where for any $i \in \Omega$, the image of i under π is denoted as i^π . The product of two permutations, say π and ψ , of Ω will be defined by $i^{\pi\psi} = (i^\pi)^\psi$.

A permutation group contains permutations of a finite set Ω where the binary operation, the multiplication, is the product of two permutations. The group formed on all the permutations of Ω is a distinguished permutation group called the *Symmetric Group* of Ω , denoted as S_n .

We will use the cycle notation for permutations. That is, if a permutation $\pi = (i_1, i_2, \dots, i_r)$, where $i_x \in \Omega$, and $r \leq n$, then $i_x^\pi = i_{x+1}$, for $1 \leq x < r$ and $i_r^\pi = i_1$. Of course, a permutation could be a product of two or more disjoint cycles.

2.2 The Enumeration Technique

A permutation group enumeration problem is essentially finding generators for all the permutations in the group, and thus leading to finding *order* of the group. It can also be viewed as an enumeration problem corresponding to any search problem [GJ79] over a finite universe.

A *generating set* of a permutation group G is defined to be the set of permutations, $K \subset G$, such that all the elements in G can be written as a (polynomially bounded) product of the elements in K .

Let $G^{(i)}$ be a subgroup of G obtained from $G < S_n$ by fixing all the points in $\{1, 2, \dots, i\}$. That is, $\forall \pi \in G^{(i)}$, and $\forall j \in \{1, 2, \dots, i\}$, $j^\pi = j$. Then it is easy to see that $G^{(i)} < G^{(i-1)}$, where $1 \leq i \leq n$ and $G^{(0)} = G$. The sequence of subgroups

$$I = G^{(n)} < G^{(n-1)} < \dots < G^{(1)} < G^{(0)} = G \quad (2.2)$$

is referred to as a *stabilizer chain* of G .

The stabilizer chain in (2.2) gives rise to a generating set given by the following Theorem [Hof82].

Theorem 2.1. [Hof82] Let U_i be a set of right coset representatives for $G^{(i-1)}$ in $G^{(i)}$, $1 \leq i \leq n$. Then a generating set K of the group $G < S_n$ is given by

$$K = \bigcup_{i=1}^n U_i, \quad (2.3)$$

Group enumeration by a generating set creates a canonic representation of the group elements, i.e., a mapping f defined as

$$f : \bigtimes_{i=n}^1 U_i \rightarrow G = \{\psi_n \psi_{n-1} \psi_{n-2} \dots \psi_i \psi_{i-1} \dots \psi_2 \psi_1 \mid \psi_i \in U_i\}. \quad (2.4)$$

The order $|G|$ can then be easily computed in time $O(n^2)$ by

$$|G| = \prod_{i=1}^n |U_i| \quad (2.5)$$

These generating sets are not unique, and the one we are interested in is derived from those coset representatives that are transpositions (except for the identity). That is, for $G = S_n$ and for the subgroup tower in Eqn. (2.2), the set of coset representatives U_i are [Hof82]

$$U_i = \{I, (i, i+1), (i, i+2), \dots, (i, n)\}, \quad 1 \leq i < n. \quad (2.6)$$

Then the generating set K of S_n is given by

$$K = \bigcup U_i = \{I, (1, 2), (1, 3), \dots, (1, n), (2, 3), (2, 4), \dots, (2, n), \dots, (n-1, n)\} \quad (2.7)$$

The partition hierarchy of the coset decompositions for S_n is shown above in Figure 1.

Example

All the coset representatives U_i for the stabilizer chain (2.2) of the symmetric group S_4 are shown in Table 1. Each permutation in S_4 can be expressed as a unique ordered product, $\psi_4\psi_3\psi_2\psi_1$, of the four permutations $\psi_1 \in U_1$, $\psi_2 \in U_2$, $\psi_3 \in U_3$ and $\psi_4 \in U_4$.

U_1	U_2	U_3	U_4
$\{I, (1, 2), (1, 3), (1, 4)\}$	$\{I, (2, 3), (2, 4)\}$	$\{I, (3, 4)\}$	$\{I\}$

Table 1: The Generators of S_4

For example, the permutation $(1,3,2,4)$ in S_4 has a unique canonic representation, $\psi_4\psi_3\psi_2\psi_1 = I * (3, 4) * (2, 4) * (1, 3)$; the element $(1, 2)$ is represented as $I * I * I * (1, 2)$. Also, note that under this enumeration scheme the order of S_4 is the product, $|U_1| * |U_2| * |U_3| * |U_4|$.

Perfect Matchings in a Bipartite Graph

Let $BG = (V \cup W, E)$ be a bipartite graph on $2n$ nodes, where, $|V| = |W|$, $E = V \times W$ is the edge set, and both the node sets V and W are labeled from $\Omega = \{1, 2, \dots, n\}$ in the same order. Under such an ordering of the nodes, the node pair $(v_i, w_i) \in V \times W$ will often be referred to as simply the *node pair at position i* in BG .

A *perfect matching* in BG is a subset $M \subseteq E$ of the edges in BG such that each node in BG is incident with exactly one edge in M . A perfect matching M in BG represents a permutation π in S_n , and hence a 1-1 onto correspondence in a natural way. That is, for each edge $(v, w) \in M \iff v^\pi = w$.

We will use the above group generating set concepts in developing a combinatorial structure, i.e., a graph theoretic analog of the generating set K , for generating all the perfect matchings in a bipartite graph.

3 The Mapping: Graph Theoretic Generators

In this Section we develop the basic framework for defining the partition hierarchy of the Symmetric group applicable to the perfect matchings in a bipartite graph. We will develop the concepts leading to the graph theoretic counterparts of the permutation group generating set. Specifically, we choose the generating set of the Symmetric group, S_n , to be the set of transpositions given by Eqn. (2.7), with the *Right Coset Decomposition* (Eqn. (2.1)) for group enumeration.

First we develop the concepts for defining the permutation multiplication in a bipartite graph.

3.1 Permutation Multiplication in a Bipartite Graph

Let BG' be a bipartite subgraph of $K_{n,n}$ on $2n$ nodes, and let $E(\pi)$ denote a perfect matching in BG' realizing a permutation $\pi \in S_n$.

Theorem 3.1. *Let $\pi \in S_n$ be realized as a perfect matching $E(\pi)$ in BG' . Then for any transposition, $\psi \in S_n$, the product $\pi\psi$ is also realized by BG' iff BG' contains a cycle of length 4 such that the two alternate edges in the cycle are covered by $E(\pi)$ and the other two by $E(\pi\psi)$.*

(The proof is in Appendix A.1)

Examples.

Figure 4(a) shows an instance of BG' having two perfect matchings, and realizing the product $\pi\psi$ of the permutations $\pi = I$ (identity permutation) and $\psi = (2, 3)$. Figure 4(b) shows the two perfect matchings, realizing π and $\pi\psi$, and the associated graph cycle $(1, 2, 4, 3)$, representing the multiplier $(2, 3)$, responsible for the multiplication.

Figure 4(c) shows the same multiplication $\pi\psi = (1, 2, 4, 3, 5) * (2, 3)$ as a cascade of the two associated perfect matchings in the two bipartite graphs.

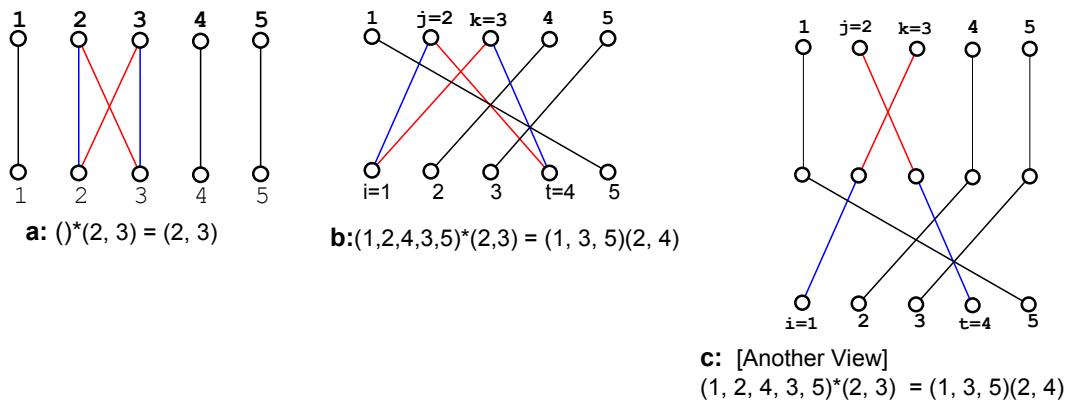


Figure 4: Permutation Multiplication in a Bipartite Graph

3.2 Graph Theoretic Coset Partition

We can now define a set of graph theoretic counterparts of the Coset Representatives, called *partition representatives*. These are obtained by mapping the symmetric group S_n generators in (2.7) to a set of edge pairs in $K_{n,n}$ with the help of Theorem 3.1.

Let $\mathbb{M}(BG')$ denote the set of permutations realized as perfect matchings in a bipartite graph BG' . Let BG_i denote the sub (bipartite) graph of $BG = K_{n,n}$, induced by the subgroup $G^{(i)}$, such that $\mathbb{M}(BG_i) = G^{(i)}$. That is, $\forall t \in \{t \mid 1 \leq t \leq i\}$, there is exactly one edge $v_t w_t$ incident on the nodes v_t and w_t . Moreover, BG_i contains a complete bipartite graph, $K_{n-i, n-i}$, on the nodes at positions $i+1, i+2, \dots, n$.

The following is a direct Corollary of Theorem 3.1, and is the basis for mapping the Coset representatives in U_i to a set of edge pairs in bipartite graph $K_{n,n}$.

Let $K_{n,n} = (V \cup W, V \times W)$, where $V = W = \{1, 2, \dots, n\}$.

Let $(v_i w_k, v_t w_i) \in (V \times W, V \times W)$ be an ordered pair of edges in $K_{n,n}$, where $n \geq t > i$.

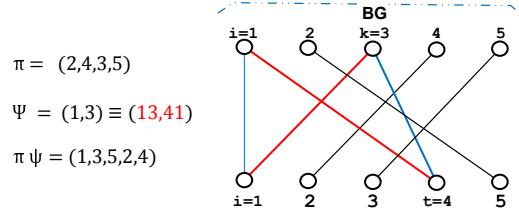


Figure 5: Multiplication by a Coset Representative $\psi = (1, 3)$

Corollary 3.2. Let $BG = K_{n,n}$ be a complete bipartite graph, and U_i be a set of right coset representatives of $G^{(i)}$ in $G^{(i-1)}$ given by Eqn. (2.6). Then, for each $\psi = (i, k) \in U_i$, and for each $\pi \in G^{(i)}$, $\pi\psi$ is realized by BG_{i-1} using a unique edge pair, $a_i(\pi, \psi) = (v_i w_k, v_t w_i)$ in BG_{i-1} , which forms a cycle of length 4 with $E(\pi)$ and represents ψ for a given π , such that $i^\psi = k = t^\pi$.

When the coset representative ψ is an identity, we have a special case of the above behavior where the edge pair $a_i(\pi, \psi)$ reduces to one edge $v_i w_i$ for each $\pi \in G^{(i)}$.

Definition 3.3. A partition representative, $g(i)$, $1 \leq i \leq n$, for the subgraph BG_i in $K_{n,n}$ is defined as:

$$g(i) \stackrel{\text{def}}{=} \{(ik, ti) \mid k, t \in \{i+1, \dots, n\}\} \cup \{(ii, ii)\}, \quad (3.1)$$

where (v_i, w_k, v_t, w_i) is a cycle of length 4 in $K_{n,n}$.

The following Property states how each $g(i)$ is partitioned into $n - i + 1$ equivalence classes induced by each $\psi \in U_i$.

Property 3.4. For each $(\pi, \psi) \in G^{(i)} \times U_i$, if $\psi = (i, k)$ is fixed, then the subset of $g(i)$ that can realize $\pi\psi$ is:

$$\{(ik, ti) \mid i^\psi = k = t^\pi\}, \text{ where } k, t \in \{i+1, \dots, n\} \quad (3.2)$$

The following Lemma states the exact mapping, showing how $g(i)$ is effectively the graph theoretic counterpart of U_i , where the element $(ii, ii) \in g(i)$ is the ID element corresponding to $I \in U_i$. We will denote each $(ii, ii) \in g(i)$ by I .

Lemma 3.5. *There exists a 1-1 mapping*

$$h : G^{(i)} \times U_i \longrightarrow g(i) \times M(BG_i),$$

s.t., $\forall (\pi, \psi) \in G^{(i)} \times U_i$, $\pi\psi$ is realized by a unique pair $(x_i, pm_i) \in g(i) \times M(BG_i)$ using a unique cycle (v_i, w_k, v_t, w_i) of length 4 in BG_{i-1} , defined by $x_i = (ik, ti)$, such that the edge pair x_i is covered by $\pi\psi$, and the other two alternate edges in the cycle are covered by π .

When $\psi = I$, the identity in S_n , the cycle collapses to one edge $x_i = (ii, ii)$ covered by π and $\pi\psi$ both.

Proof. Follows from Corollary 3.2 and noting that the ID element in U_i is not mapped by a cycle in $K_{n-i+1, n-i+1}$ but corresponds to the edge $(v_i, w_i) \in V \times W$. □

Now we can define a generating set for enumerating all the perfect matchings in $K_{n,n}$, analogous to a generating set for the Symmetric group S_n .

Let $\psi(a_i)$, $a_i \in g(i)$, denote a transposition $\psi \in U_i$ realized by the edge pair a_i .

Definition 3.6. A generating set, denoted as $E_M(n)$, for generating all the $n!$ perfect matchings in a complete bipartite graph $K_{n,n}$ is defined as

$$E_M(n) \stackrel{\text{def}}{=} \bigcup_{i=1}^n g(i) \quad (3.3)$$

3.2.1 The Multiplicative Behavior of the Generators

Group enumeration by a generating set creates a canonic representation of the group elements, i.e., a mapping f defined as

$$f : \bigtimes_{i=1}^n U_i \rightarrow \{\psi_n \psi_{n-1} \psi_{n-2} \cdots \psi_i \psi_{i-1} \cdots \psi_2 \psi_1 \mid \psi_i \in U_i\} = G. \quad (3.4)$$

This suggests the need for a set of similar rules that would govern the multiplication of $g(i)$ elements. The binary relations in the next sub sections are aimed to do precisely that.

Binary Relations over the Generating Set E_M

We now formulate two binary relations, R and S , over the generating set E_M . These two relations will be used to capture a far more complex relationship than what exists in a permutation group generating set in order to enumerate the group elements.

First we define some more terms.

Let $\pi(a_i)$, $a_i \in g(i)$, represent a class of permutations defined as follows.

$$\pi(a_i) \stackrel{\text{def}}{=} \pi \mid \pi \in G^{(i-1)} \text{ and } E(\pi) \text{ covers } a_i. \quad (3.5)$$

For brevity we will often qualify a permutation $\pi \in G$ as “ π covers a set of edges e ” whenever the corresponding perfect matching, $E(\pi)$ in $K_{n,n}$ covers e .

Let $\psi(a_i)$ denote the coset representative of $G^{(i)}$ in $G^{(i-1)}$ realized by the edge pair a_i for some $\pi \in G^{(i)}$ such that $\pi(a_i) = \pi\psi(a_i) \in M(BG_{(i-1)})$.

Corresponding to the identity coset representative $I \in U_i$ we will call the edge pair $(v_i w_i, v_i w_i)$ at node pair i as *identity* edge pair, denoted by id_i .

The Transitive Relation R over $E_M(n)$

The following definition of the relation R specifies the condition under which two coset representatives, $\psi(a_i)$ and $\psi(b_j)$, corresponding to the two edge pairs $a_i \in g(i)$ and $b_j \in g(j)$, $i < j$, may realize the product, $\pi(b_j)\psi(a_i)$ by the bipartite graph $BG = K_{n,n}$.

Definition 3.7. Two edge pairs $a_i \in g(i)$ and $b_j \in g(j)$, $1 \leq i < j \leq n$, are said to be related by the relation R , denoted as $a_i R b_j$, if $\forall \pi(b_j) \in G^{(j-1)}, \exists \psi(a_i) \in G^{(i-1)}$, such that $\pi(b_j)\psi(a_i) \in G^{(i-1)}$ is realized by $K_{n,n}$, conforming to Corollary 3.2.

Example.

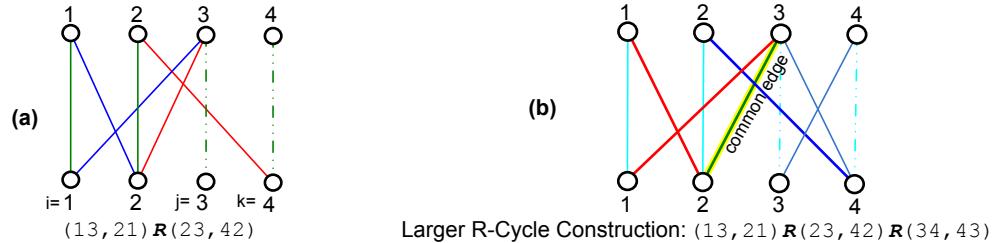


Figure 6: The Edge Pairs Forming the Cycles and the Relation R

Lemma 3.8. *The relation R is transitive.*

Proof. Let $a_i R b_j$ and $b_j R c_k$.

First we consider the case when each of these edge pairs, (a_i, b_j) and (b_j, c_k) , are related by the virtue of two graph cycles, C_{ab} and C_{bc} respectively, satisfying the condition of Corollary 3.2.

Then, for $j - i = 1$, it is easy to see that we can merge C_{ab} and C_{bc} by deleting the edge, say $e \in b_j$, common to C_{ab} and C_{bc} , and replacing it with the 2 edges in a_i [Figure 6(b)].

Therefore, $a_i R c_k$.

Clearly this process of merging the two cycles can be iterated for each (j, k) , $i < j < k \leq n$. The result then follows from the induction on $j - i$.

In the event that $a_i = I$, the transitivity is trivial as I would multiply each permutation.

In the event that ψ in the above definition is composed of 2 or more disjoint permutation cycles, the corresponding graph cycles are also disjoint and hence their composition is always valid.

Therefore, $a_i R c_k$.

□

The Disjoint Relationship

Definition 3.9. Any two edge pairs a and b in E_M are said to be *disjoint* if (i) the corresponding edges in the bipartite graph BG are vertex-disjoint, and (ii) aRb is false.

When the disjoint edge pairs a and b belong to two adjacent node partitions, i.e., $a \in g(i)$ and $b \in g(i+1)$, $1 \leq i < n$, a and b are said to be related as aSb .

3.3 The Generating Graph

We now develop graph theoretic concepts to represent each permutation in S_n by a directed path in a directed acyclic graph, called *generating graph*, denoted as $\Gamma(n)$. This *generating graph* models the multiplicative behavior of the elements in the set E_M (Eqn. (3.3)).

Definition 3.10. The generating graph $\Gamma(n)$ for a complete bipartite graph $K_{n,n}$ on $2n$ nodes is defined as

$$\Gamma(n) \stackrel{\text{def}}{=} (V, E_R \cup E_S),$$

where $V = E_M = \cup g(i)$ (Eqn. (3.3)),

$$E_R = \{a_i a_j \mid a_i R a_j, a_i \in g(i), a_j \in g(j), |a_i R a_j| = 1, 1 \leq i < j \leq n\}, \text{ and}$$

$$E_S = \{b_i b_{i+1} \mid b_i S b_{i+1}, b_i \in g(i) \text{ and } b_{i+1} \in g(i+1), 1 \leq i < n\}.$$

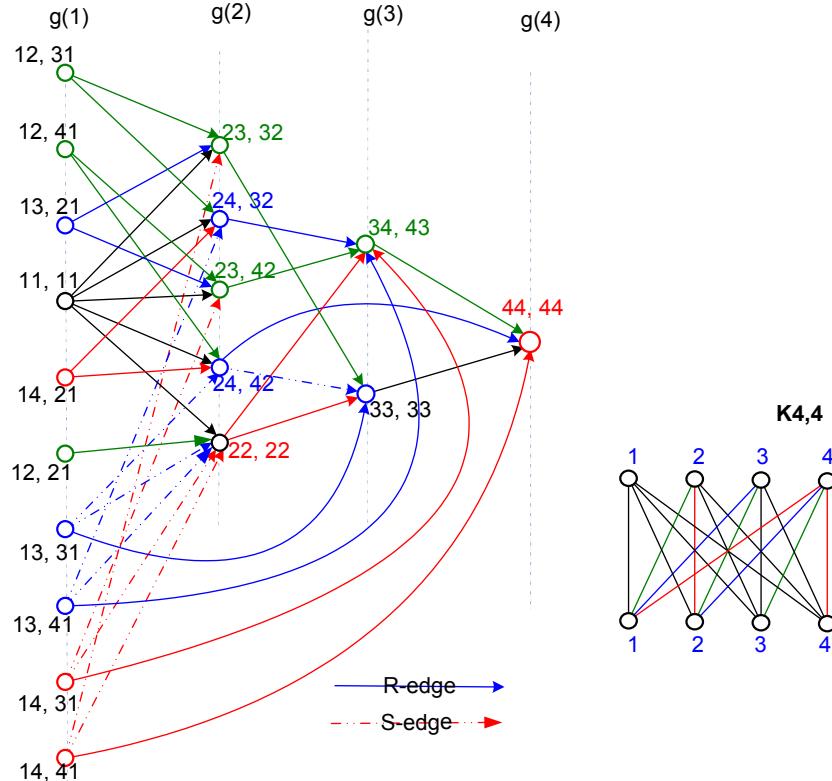


Figure 7: The Generating Graph $\Gamma(4)$ for $K_{4,4}$

Thus the generating graph is an n -partite directed acyclic graph where the nodes in the partition i are from $g(i)$, $1 \leq i \leq n$ (Eqn. (3.1)), representing the right coset representative U_i of $G^{(i)}$ in $G^{(i-1)}$, and therefore, are labeled naturally by the corresponding edge pairs, $g(i)$.

The edges in $\Gamma(n)$ represent either the transitive relation R (by a solid directed line) between the two nodes, or the *disjoint* relationship between the two nodes (by a dotted directed line) in the adjacent partitions. Each edge is a directed edge from a lower partition node to the higher partition node. Figure 7 shows a generating graph $\Gamma(4)$ for the complete bipartite graph $K_{4,4}$.

The edges in E_R will be referred to as *R*-edges. Similarly, the edges in E_S will be referred to as *S*-edges. An *R*-edge between two nodes that are not in the adjacent partitions will be called a *jump* edge, whereas those between the adjacent nodes will sometimes be referred to as *direct edges*. Moreover, for clarity we will always represent a jump edge by a solid curve.

Definition 3.11. An *R*-path, $a_i, a_{i+1}, \dots, b_j \in \Gamma(n)$ is a sequence of adjacent *R*-edges between the two nodes $a_i, b_j \in \Gamma(n), j > i$ such that $a_i R b_j$.

Note: The treatment of a “path” formed by a sequence of adjacent *R* and *S*-edges (generally called as *RS*-path) is more complex and will be discussed in the next sub Section.

Property 3.12. The following attributes of the generating graph $\Gamma(n)$ follow from the Properties stated in Appendix A.4.

$$\text{Total number of nodes in the partition } i \text{ is } |g(i)| = (n - i)^2 + 1, \quad 1 \leq i \leq n \quad (3.6a)$$

$$\text{Total number of nodes in } \Gamma(n) = O(n^3) \quad (3.6b)$$

$$\text{Maximum } R\text{-outdegree of any node (except the ID) at partition } i = n - i \quad (3.6c)$$

$$\text{Maximum } S\text{-outdegree of any node at partition } i = (n - i - 2)^2 + 1, \quad 1 \leq i < n - 1 \quad (3.6d)$$

$$\text{Total number of } R\text{-edges in } \Gamma(n) = O(n^4) \quad (3.6e)$$

$$\text{Total number of } S\text{-edges in } \Gamma(n) = O(n^5) \quad (3.6f)$$

Definition 3.13. An *R*-path, $a_i, a_{i+1}, \dots, b_j \in \Gamma(n)$ is a sequence of adjacent *R*-edges between the two nodes $a_i, b_j \in \Gamma(n), j > i$ such that $a_i R b_j$.

Length of an *R*-path aRb is the number of adjacent *R*-edges between the node pair $(a, b) \in \Gamma(n)$.

Two *R*-edges, $x_i x_t$ and $x_j x_t$, $i \neq j$, meeting at a common node $x_t \in \Gamma(n)$ are said to be *disjoint* if the associated cycles in $K_{n,n}$ are vertex disjoint.

Now we can define a distinguished path called *valid multiplication path* (VMP) in a generating graph $\Gamma(n)$, in order to represent a perfect matching in a bipartite graph.

3.3.1 Valid Multiplication Path (VMP)

Definition 3.14. Let $p = x_i x_{i+1} \cdots x_{j-1} x_j$ be any path formed by adjacent *R*- and *S*-edges in $\Gamma(n)$ such that exactly one node x_r is covered in each node partition r , where $x_r \in g(r)$, $1 \leq i \leq r \leq j$. Then p is a *valid multiplication path* if $\forall (x_r, x_s)$ on p , $s > r$, we have either $x_r R x_s$ or the edge pairs x_r and x_s , are vertex-disjoint in $K_{n,n}$ and $x_r R x_s$ is false.

Further, p is a *complete valid multiplication path* (CVMP) if every *R*-edge, $x_t R x_r$ (direct or jump edge) in p , $i \leq t < r \leq j$, is covered by p .

Note that a *VMP*, $p' = x_i x_{i+1} \cdots x_{j-1} x_j$, is any sub-path of a *CVMP*, $p = x_r x_{r+1} \cdots x_{n-1} x_n$ which contains p' .

The Multiplying DAG: General Specification for Multiplying two Nodes

We can now define an inductive structure called *Multiplying Directed Acyclic Graph* (abbr. *mdag*) that can be used to specify a VMP of length $r + 1$ using a VMP of length r .

Definition 3.15. A *Multiplying Directed Acyclic Graph* (*MDAG*), denoted as $\text{mdag}(x_i, x_{i+1}, x_t)$, is a general specification for “multiplying” two nodes x_i and x_{i+1} in adjacent node partitions where $x_i S x_{i+1}$, and $x_i R x_t$ defines an *R-edge* such that all three nodes, x_i , x_{i+1} and x_t are covered by a common VMP. Clearly, in the extreme case $\text{mdag}(x_i, x_{i+1}, x_t)$ reduces to an *R-edge* defined by $x_i R x_{i+1}$, with $x_{i+1} = x_t$.

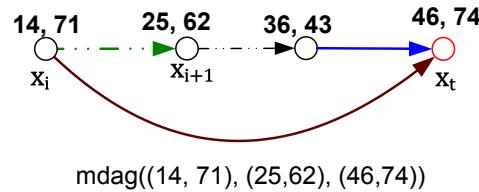


Figure 8: A Simple mdag

3.3.2 Perfect Matching Represented by a CVMP

The following Lemma follows directly from Lemma A.12

Lemma 3.16. Every *CVMP*, $p = x_1 x_2 \cdots x_{n-1} x_n$ in $\Gamma(n)$, of length $n - 1$, where $x_r \in g(r)$, $r \in [1..n]$ represents a unique permutation $\pi \in S_n$ realized as a perfect matching $E(\pi)$ in $K_{n,n}$ given by

$$\pi = \psi(x_n)\psi(x_{n-1}) \cdots \psi(x_2)\psi(x_1), \quad (3.7)$$

where $\psi(x_r) \in U_r$ is a transposition defined by the edge pair x_r , and U_r is a set of right coset representatives of the subgroup $G^{(r)}$ in $G^{(r-1)}$ such that $U_n \times U_{n-1} \cdots U_2 \times U_1$ generates S_n .

4 The Extended Partition Hierarchy

We extend the permutation group enumeration technique to the set of perfect matchings by essentially identifying two additional levels of the partition hierarchy of S_n , consisting of *CVMP Sets* and *MinSet sequences*:

$$S_n \supset \text{Coset} \supset \text{CVMPSet} \supseteq \text{MinSet Sequences}.$$

In this Section we develop these two key structures and the associated enumeration algorithms for the perfect matchings.

4.1 The CVMPSet

A CVMPSet is essentially a collection of CVMPs to represent the partition of a Coset of a subgroup in S_n . First we define a more general structure, the VMP Set, a set of all the VMPs.

Let $mdag(a_i) = mdag(a_i, x_{i+1}, x_r), r > i + 1$, denote a family of mdags. We also note that each mdag, $mdag(a_i, x_{i+1}, x_r)$, can reduce to an R -edge when $a_i R x_{i+1}$.

For brevity we will use the notation m_i to represent an mdag, $mdag(a_i)$ at some node a_i in the node partition i of $\Gamma(n)$ for a bipartite graph on $2n$ nodes.

Definition 4.1. A $VMPSet(m_i, m_j)$ is a set of all the VMPs between an mdag pair (m_i, m_j) , in the node partition pair $(i, j), 1 \leq i < j \leq n$, in the generating graph $\Gamma(n)$.

Representation of a VMPSet

A polynomial size representation of a VMPSet, $VMPSet(m_i, m_j)$, is a subgraph of the generating graph $\Gamma(n)$. This subgraph contains all the VMPs between the mdags $(mdag(a_i), mdag(b_j))$ at the node pair (a_i, b_j) .

A data structure, for representing a $VMPSet$ is a collection, called "Struct", of various primitive components defined using Algol kind of notation as follows.

```
EdgesAtNode (Node, {incident edges});
NodePartition Array[ ] of EdgesAtNode;
```

```
VMPSet(mdag<aij>) = Struct {
    MdagPair (mdag<ai>, mdag<bj>);
    PartitionList Array[i .. (j+1)] of NodePartition;
    Count integer; // the count of all the contained VMPs
}
```

(4.1)

A CVMPSet can analogously be represented by the same structure.

The following Figure [9] shows a subset of a CVMPSet between two fixed mdags at nodes (16,31) and (57,85).

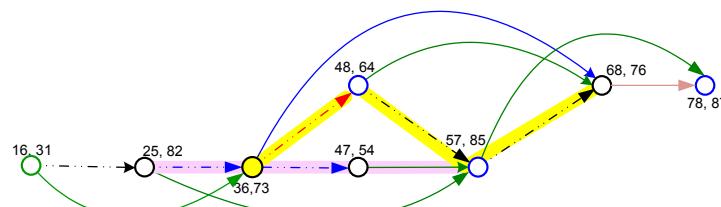


Figure 9: A Subset of CVMPSet($mdag((16, 31))$, $mdag((57, 85))$)

Product of two VMPs

Let $q \in CVMPSet(x_{i+1}, x_{n-1})$ and $\pi(q)$ be the corresponding permutation in $G^{(i)}$, where $x_i \in g(i)$.

Then by $x_i \cdot q$ we imply the product $\pi(q)\psi(x_i) \in G^{(i-1)}$, where $x_i \cdot q \in CVMPSet(x_i, x_{n-1})$.

Further, by induction, we can use the term product $p \cdot q$ of two adjacent VMPs, (p, q) , to denote the multiplication of the associated permutations, noting that when $p = mdag(x_i, x_{i+1}, x_{i+2})$ it is same as $x_i \cdot q$.

The product of two VMPSet then can be defined as,

$$VMPSet(m_i, m_t) \cdot VMPSet(m_t, m_j) \stackrel{\text{def}}{=} \{p \cdot q \mid (p, q) \in VMPSet(m_i, m_t) \times VMPSet(m_t, m_j)\} \quad (4.2)$$

Note that again, all $q \in VMPSet(m_t, m_j)$ cannot be multiplied by all $p \in VMPSet(m_i, m_t)$.

CVMPSet Properties

Let x_i be any node in the i th node partition $g(i)$ of the generating graph $\Gamma(n)$, where $1 \leq i \leq n - 1$. Let $\psi(x_i)$ and $\psi(y_i)$ be the two coset representatives in U_i realized by the nodes x_i and y_i respectively, where U_i is the set of coset representatives for the subgroup $G^{(i)} < G^{(i-1)} < S_n$.

Property 4.2. All CVMP sets, $CVMPSet(mdag < x_i >, mdag < x_{n-1} >)$, are mutually disjoint for each mdag pair $(mdag < x_i >, mdag < x_{n-1} >)$ and each i , $1 \leq i \leq n - 2$.

Proof. Note that each CVMP set, $CVMPSet(mdag < x_i >, mdag < x_{n-1} >)$, is uniquely defined by its two mdags. Moreover, every $x_i \in g(i)$ as well as every R-edge and every mdag at x_i are unique. And therefore, if every CVMP, $p \in CVMPSet(mdag < x_i >, mdag < x_{n-1} >)$, is unique, so are all $x_i \cdot p \in CVMPSet(x_i, x_{n-1})$. That is, for each $(x_i, y_i) \in g(i)$ and for each $p_r \in CVMPSet(mdag < x_{i+1} >, mdag < x_{n-1} >)$,

$$x_i \neq y_i \implies x_i \cdot p_r \neq y_i \cdot p_r, \text{ and}$$

$$p_1 \neq p_2 \implies x_i \cdot p_1 \neq x_i \cdot p_2.$$

And hence, by induction, the disjointness of $CVMPSet(mdag < x_i >, mdag < x_{n-1} >)$ follows from the disjointness of the contained CVMPs. □

Property 4.3. *There are at the most $O(n^5)$ CVMP sets, $CVMPSet(mdag < x_i >, mdag < x_{n-1} >)$, between any node pair (x_i, x_{n-1}) in a generating graph $\Gamma(n)$.*

Proof. We simply note that there are $O(n^2)$ nodes $x_i \in g(i)$ at any node partition i in $\Gamma(n)$, and there are $O(n^3)$ mdags at each node. Moreover, there are only two nodes in the node partition $n - 1$. □

4.1.1 Partition of a Coset into CVMPSets

Lemma 4.4. *Each Coset $G^{(i)}\psi_k$ of a subgroup $G^{(i)} < G^{(i-1)}$ can be partitioned into a family of equivalence classes, viz., the CVMPSets, as follows:*

$$G^{(i)}\psi_k = \biguplus_{\substack{x_{n-1}, \\ \psi(a_i) = \psi_k}} CVMPSet(mdag\langle a_i \rangle, mdag\langle x_{n-1} \rangle), \quad (4.3)$$

where $\psi_k = (i, k) \in U_i$, $a_i \in g(i)$, $x_{n-1} \in g(n - 1)$, $1 \leq i < n - 1$.

Proof. The proof follows essentially from Property 4.2 and an observation that the multiplication $mdag\langle a_i \rangle \cdot cvmp$ where $cvmp \in CVMPSet(mdag\langle a_{i+1} \rangle, mdag\langle x_{n-1} \rangle)$, defines an equivalence relation over the coset $G^{(i)}\psi_k$.

Clearly, each distinct $cvmp$ in a $CVMPSet(mdag < a_{i+1} >, mdag < x_{n-1} >)$ represents a unique permutation $\pi \in G^{(i)}$. By the definition of a CVMPSet we also know that it contains precisely those CVMPs from $CVMPSet(mdag\langle a_{i+1} \rangle, mdag\langle x_{n-1} \rangle)$ which can be multiplied by a common mdag at the node a_i .

Since the set $\{a_i | \psi(a_i) = \psi_k\}$ represents the coset representative $\psi_k \in U_i$, each CVMP in $CVMPSet(mdag\langle a_i \rangle, mdag\langle x_{n-1} \rangle)$ represents the product $\pi\psi(a_i)$ in the coset $G^{(i)}\psi(a_i)$. Therefore, a union of all these (disjoint) subsets satisfying $\psi(a_i) = \psi_k$ gives $G^{(i)}\psi(a_i)$. □

Note. For each node $a_i \in g(i)$ there are $O(n^3)$ distinct $mdag < a_i >$ over which the above union operation in (4.3) is performed.

4.1.2 Coverage of the Symmetric Group S_n

Property 4.5. *All the CVMPSets of length $n - 1$ in $\Gamma(n)$ jointly contain $n!$ unique CVMPs representing precisely the $n!$ permutations in S_n . That is,*

$$\biguplus_{\substack{a_1 \in g(1), \\ x_{n-1} \in g(n-1)}} CVMPSet(mdag\langle a_1 \rangle, mdag\langle x_{n-1} \rangle) = S_n \quad (4.4)$$

Proof. The proof follows from (4.3) of Lemma 4.4.

Note that $g(1)$ is the set of all the edge pairs that collectively represent the *generators* (elements in the set of right coset representatives), U_1 , and further all the associated CVMP sets are mutually disjoint (Property 4.2).

Therefore,

$$S_n = \biguplus_{\psi_k \in U_1} G^{(1)}\psi_k = \biguplus_{\substack{a_1 \in g(1), \\ x_{n-1} \in g(n-1)}} CVMPSet(mdag\langle a_1 \rangle, mdag\langle x_{n-1} \rangle).$$

□

4.1.3 Counting Perfect Matchings in $K_{n,n}$

To demonstrate the basic counting technique, we first present a counting algorithm that enumerates seemingly a trivial set, viz., the set of all $n!$ perfect matchings in $K_{n,n}$ in polynomial time. We will then augment the same algorithm to allow the counting in any bipartite graph, while still maintaining the polynomial time bound. The following algorithm describes the high level steps which realize the group enumeration technique as captured by Property 4.4 and 4.5.

Algorithm 4.1 CountAllPerfectMatchings ($K_{n,n}$)

Input: generating graph $\Gamma(n)$;
Output: count of Perfect Matchings in $K_{n,n}$;

Step (a): Inductively compute all CVMPSets

```

1: for node partitions  $i = n - 3$  to 1 do
2:   for all nodes  $a_i \in g(i)$  do
3:     for all  $(a_{i+1}, x_{n-1}) \in g(i+1) \times g(n-1)$  do
4:       compute all  $CVMPSet(mdag\langle a_i \rangle, mdag\langle x_{n-1} \rangle)$ 
         $= \{mdag\langle a_i \rangle \cdot CVMPSet(mdag\langle a_{i+1} \rangle, mdag\langle x_{n-1} \rangle)\};$ 
5:     end for
6:   end for
7: end for

```

Step (b): Sum the counts in each CVMPSet

the Count = $\sum_{(a_1, x_{n-1})} CVMPSet(mdag\langle a_1 \rangle, mdag\langle x_{n-1} \rangle) \cdot Count$

As we shall see that the most complex step in the above algorithm 4.1 is the step (4), incrementing a $CVMPSet(mdag\langle a_{i+1} \rangle, mdag\langle x_{n-1} \rangle)$ by an $mdag\langle a_i \rangle$.

The Disjoint Subsets of CVMP Sets

Unlike the product $\pi\psi$ in a coset $G^{(1)}\psi$ of the group $G^{(1)}$, most $x_i \in g(i)$ *cannot* multiply all the CVMPs in any $CVMPSet(x_{i+1}, x_{n-1})$, but only a subset thereof induced by x_i . Therefore, we need to find the disjoint subsets of $CVMPSet(m_{i+1}, m_{n-1})$ in which all CVMPs can be multiplied

by a common $x_i \in g(i)$. The structure, $\text{prodVMPSet}(m_i, m_t, m_{n-1})$, defined below is one accomplishes this capability.

Let $x_r \in g(r)$ be node in the node partition i of $\Gamma(n)$ for $1 \leq r \leq n$; let $m_r = \text{mdag}(< x_r >)$ denote an mdag at the node x_r , and let $\text{CVMPSet}(m_i, m_{n-1})$ be any subset of the coset $G^{(i)}\psi_k$ in (4.3). We define $\text{prodVMPSet}(m_i, m_t, m_{n-1})$ as a family of disjoint subsets of $\text{CVMPSet}(m_i, m_{n-1})$ as follows.

Definition 4.6. Let $m_t = \text{mdag}(x_t, x_{t+1}, y_t)$, $1 < t \leq n-1$. Then

$$\begin{aligned} \text{prodVMPSet}(m_i, m_t, m_{n-1}) &\stackrel{\text{def}}{=} \{p \in \text{CVMPSet}(m_i, m_{n-1}) \mid p \text{ covers } m_t \text{ such that} \\ &\quad \text{the node } x_{t+1} \text{ in } m_t \text{ is incident by } R\text{-edges, } R_{x_{t+1}}, \text{ with common } SE(R_{x_{t+1}})\} \end{aligned} \quad (4.5)$$

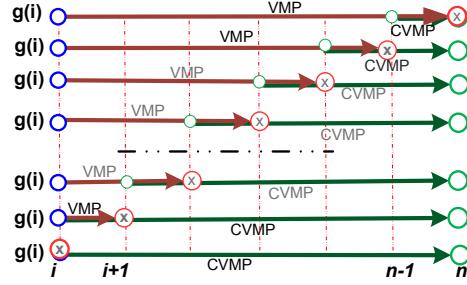


Figure 10: $\text{prodVMPSet}(m_i, m_t, m_{n-1})$: Disjoint Subsets of $\text{CVMPSet}(m_i, m_{n-1})$

Remark. In general a node x_{t+1} in m_t covered by p can have 0 to 2 R -edges incident upon it. Thus for each m_t , there are 4 possible disjoint subsets induced by the 2 incident R -edges. The following Property follows directly from the above definition.

Property 4.7. For each fixed t , $1 \leq t \leq n-1$, the associated disjoint subsets, $\text{prodVMPSet}(m_i, m_t, m_{n-1})$, are related to $\text{CVMPSet}(m_i, m_{n-1})$ as follows:

$$\text{CVMPSet}(m_i, m_{n-1}) = \biguplus_{m_t} \text{prodVMPSet}(m_i, m_t, m_{n-1}) \quad (4.6)$$

Property 4.8.

$$\text{prodVMPSet}(m_i, m_t, m_j) = \text{VMPSet}(m_i, m_t) \bullet \text{VMPSet}(m_t, m_j) \quad (4.7)$$

Proof. Follows from the definition of the product of two VMPS in (4.2). □

4.2 Perfect Matching Generators

In this section we develop generators for a subproblem of perfect matchings, the MinSet sequences. This subproblem is an equivalence class of the $n!$ perfect matchings, represented as a sequence of a P-time enumerable structure called MinSets which is induced by the missing edges in each potential perfect matching.

We first define MinSets which are collections of VMPs induced by the “missing” edges, and which in turn are modeled as an attribute called *Edge Requirements*(ER) of a CVMP that would represent an existing perfect matching under certain conditions on the ER. These MinSets become generators for the MinSet Sequences, and hence for the perfect matchings.

4.2.1 Edge Requirements: the CVMP Qualifier

Edge Requirement is an algebraic formulation of the perfect matching behavior that every node in the graph is incident with exactly one edge, i.e., the matched edge. To define the edge requirement of a CVMP (VMP), p , we will define two terms, viz. the *edge requirement*, $ER(x_i)$, of a node, x_i in p , and *surplus edge*, $SE(x_i x_j)$, of an R -edge $x_i x_j$.

Let $p = x_1 x_2 \cdots x_{n-1} x_n$ be CVMP in $\Gamma(n)$ for a bipartite graph BG . The (initial) Edge Requirement, $ER(x_i)$ of a node $x_i \in g(i)$ in p is

$$ER(x_i) \stackrel{\text{def}}{=} \{e \mid e \in x_i \text{ and } e \notin BG\} \quad (4.8)$$

x_i represents an initial assignment of the matched edges incident on the node pair (v_i, w_i) .

We first consider the perfect matchings in $K_{n,n}$. There are exactly two ways each node pair $(v_j, w_j) \in K_{n,n}$ can be incident with 2 matched edges in a perfect matching in $K_{n,n}$:

1. The node pair (v_j, w_j) is incident with an edge pair represented by $x_j \in g(j)$, and
2. one or both the edges in x_j incident on (v_j, w_j) are replaced by 2 alternate edges of a cycle formed between $x_i \in g(i)$ and x_j such that $x_i R x_j$, where x_i is in p .

For each instance of an R -edge $x_i R x_j$ in p , the edge in the edge-pair x_j being replaced by the R -cycle $x_i R x_j$ will be called a *Surplus Edge*, $SE(x_i x_j)$, and is defined as follows:

$$SE(x_i x_j) \stackrel{\text{def}}{=} \text{the edge } e \in x_j \text{ covered by the associated } R\text{-cycle defined by } x_i R x_j. \quad (4.9)$$

When the given graph is not a complete bipartite graph, the edge requirement of a node x in $\Gamma(n)$ can be met by the surplus edge(s) as determined by the R -edges incident on x . For example, in Figure 11, for the CVMP $p = (12, 31) \cdot (24, 32) \cdot (34, 43) \cdot (44, 44)$, the initial ER of various nodes on p , i.e., $\{44, 34, 32\}$, is satisfied by the SE of the incident R -edges.

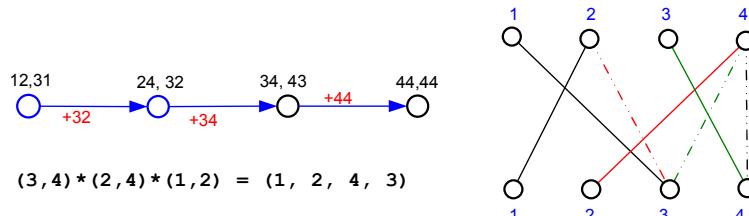


Figure 11: A Perfect Matching: satisfying the Edge Requirements

The Edge Requirement $ER(p)$ of a VMP, p in $\Gamma(n)$, is the collection of each of the nodes' Edge Requirement that is not satisfied by the SE of the R -edge incident on that node. That is,

$$ER(p) \stackrel{\text{def}}{=} \bigcup_{x_i \in p} ER(x_i) - (\{SE(x_j x_k) \mid x_j, x_k \in p\} \cap (\bigcup_{x_i \in p} ER(x_i))) \quad (4.10)$$

(The intersection in the second term is needed simply to avoid any “negative edge requirements” resulting from any redundant edges incident on any of the nodes.)

As a VMP p in $\Gamma(n)$ grows, its edge requirements $ER(p)$ can change. We will show later (Lemma 4.10) that the edge requirements of a VMP, p in $\Gamma(n)$ is null iff p leads to a perfect matching in BG' . Note that $ER(p)$ is specific to a given bipartite graph BG' since the ER of a node in p depends on BG' , whereas SE of an R -edge is fixed for all BG' .

The following corollary follows directly from Lemma A.13

Corollary 4.9. *Every CVMP, $p = x_1 x_2 \cdots x_{n-1} x_n$ in $\Gamma(n)$, represents a unique perfect matching $E(\pi)$ in $K_{n,n}$ given by*

$$E(\pi) = \bigcup_{x_i \in p} x_i - \{SE(x_j x_k) \mid x_j Rx_k, (x_j, x_k) \in p\}. \quad (4.11)$$

The Condition for a CVMP to be a Perfect Matching

Lemma 4.10. *Let $p = x_1 x_2 \cdots x_{n-1} x_n$ be a CVMP of length $n - 1$ for a bipartite graph BG' . Then $ER(p) = \emptyset \iff E(\pi)$ is a perfect matching in BG' given by (4.11), where π is given by (3.7).*

Proof. The expression for $ER(p)$ in (4.10) can be re-written as:

$$\begin{aligned} ER(p) &= (\bigcup_{x_i \in p} ER(x_i)) \cap (\{e \mid e \in x_i \in p\} - \{SE(x_j x_k) \mid x_j, x_k \in p\}) \\ &= (\bigcup_{x_i \in p} ER(x_i)) \cap E(\pi) \end{aligned}$$

Therefore, $ER(p) = \emptyset$ iff either

- (1) $\forall x_i \in p, ER(x_i) = \emptyset$, or
- (2) $\forall e \in E(\pi), e \notin \cup ER(x_i)$, and hence $e \in BG'$.

Thus both cases lead to $E(\pi)$ being realized by BG' .

□

A Pattern of ER Satisfiability

Following the enumeration scheme in A.15, each step of constructing incrementally larger length CVMPSet can reduce the *ER* of its member CVMPs by at the most one edge. For example, each perfect matching in $CVMPSet(m_1, m_{n-1})$ can allow at the most one missing edge in the perfect matchings in BG_1 as illustrated in Figure 12 below.

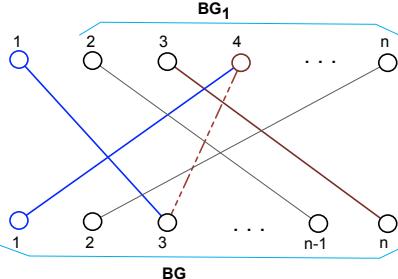


Figure 12: *Incremental Reduction of ER*

The following Property is a consequence of Lemma A.15.

Property 4.11. *For every $(x_1, p_2) \in g(1) \times CVMPSet(m_2, m_{n-1})$, if $x_1 \cdot p_2$ is a perfect matching in BG , then $|ER(p_2)| \leq 1$.*

Proof. The permutation group enumeration (Lemma A.15) requires that each $p_2 \in CVMPSet(m_2, m_{n-1})$ be multiplied by exactly one generating element $x_1 \in g(1)$ in order to generate another member $x_1 \cdot p_2$ in $CVMPSet(m_1, m_{n-1})$. Therefore, $ER(x_1 \cdot p_2) = \emptyset \implies |ER(p_2)| \leq 1$. \square

4.2.2 MinSets: The VMPs of Common ER

Counting all the VMPs of a common ER for any given bipartite graph cannot be done in polynomial time because of the possibility of exponentially many ER sequences over a VMPSet. However, for certain small fixed patterns of the ERs, the VMPSet can be counted in polynomial time.

The above Property 4.11 drives the definition of an ER-constrained set, called MinSet, which has a common ER for all the contained VMPs. A $CVMPSet(m_1, m_{n-1})$ can then be expressed as a polynomially bounded set of a sequence of P-time enumerable MinSets.

Let $ER^p(x_j)$ denote the ER of a node x_j covered by a VMP, p .

Definition 4.12. A $MinSet(m_i, m_j)$, $1 \leq i < j \leq n - 1$, is the largest subset of $VMPSet(m_i, m_j)$, where each $p \in MinSet(m_i, m_j)$ has a common ER, $ER(p)$, such that

$\forall (p, x_k) \in MinSet(m_i, m_j)$, the common ER, $ER^p(x_k) = \emptyset$ except for the 3 common nodes, x_i , x_{i+1} , and x_{j+1} , in 3 distinguished node partitions (i , $i + 1$, and $j + 1$) (Fig 13).

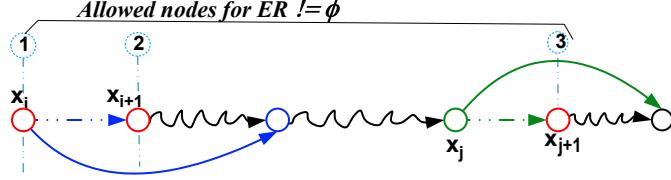


Figure 13: An Abstract MinSet: $\text{MinSet}(\text{mdag} < x_i >, \text{mdag} < x_j >)$

Representation of a MinSet

A MinSet has a representation similar to that of a VMPSet except for the additional attributes for the common ER and the incident R -edges. For notational convenience, the common ER at the 3rd node x_{j+1} is not being captured, although it can differ between any two $\text{MinSet}(\text{mdag} < x_i >, \text{mdag} < x_j >)$.

*EdgesAtNode (Node, {incident edges});
NodePartition Array[] of EdgesAtNode;*

```

MinSet(mdag<ai>, mdag<bj>) = Struct {
    MdagPair (mdag<ai>, mdag<bj>);
    PartitionList Array[i .. (j+1)] of NodePartition;
    //ER at 3 distinguished node positions
    CommonER ER(xi);
    CommonER ER(xi+1);
    CommonER ER(xj+1);
    Count integer; // the count of all the contained VMPs
}
(4.12)

```

Definition 4.13. A $\text{MinSet}(m_i, m_t, m_j)$, $1 \leq i < j \leq n - 1$, is a distinguished $\text{MinSet}(m_i, m_j)$ such that

- (1) $\text{MinSet}(m_i, m_t, m_j) \subseteq \text{prodVMPSet}(m_i, m_t, m_j)$, and
- (2) $ER(m_t) = ER(x_{t+1})$, where $m_t = \text{mdag}(x_t, x_{t+1}, x_r)$.

Fact 4.14. For $K_{n,n}$,

$$\text{CVMPSet}(m_i, m_{n-1}) = \text{MinSet}(m_i, m_{n-1}).$$

Notation: The labeling of nodes and edges in $\Gamma(n)$

Assuming the nodes in $K_{n,n}$ are labeled from N using decimal numbers, a node $(iv, wi) \in \Gamma(n)$ is then labeled as $i.v, w.i$, while the R -edges $((iv, wi), (wv, tw))$ are labeled by $+w.v$, where “.” is used as a delimiter to separate the node labels. When the node numbers are 0, 1, 2, … 9, we will ignore this delimiter “.”.

4.2.3 Finding the MinSets of a CVMPSet

We begin with an example CVMPSet, showing how do the various MinSets compose a CVMPSet. We show how a MinSet is determined for the various values of ERs of x_4 and x_6 in the subset of a $CVMPSet(m_1, m_5)$ in a bipartite graph. We consider two cases: a) when $ER(x_6) = \{68\}$, and b) when $ER(x_4) = \{48\}, ER(x_6) = \{68\}$.

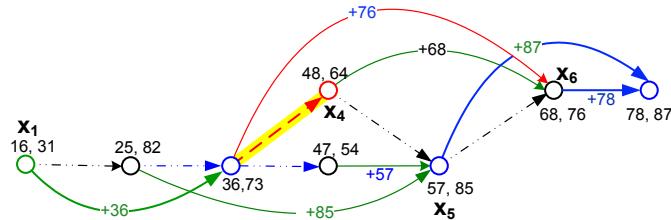
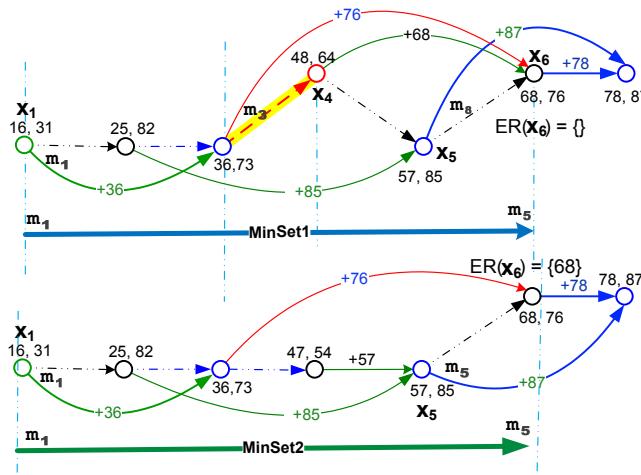


Figure 14: A CVMPSet Subset with 2 CVMPs

VMPSet Partition 1: $ER(x_6) = \{68\}$

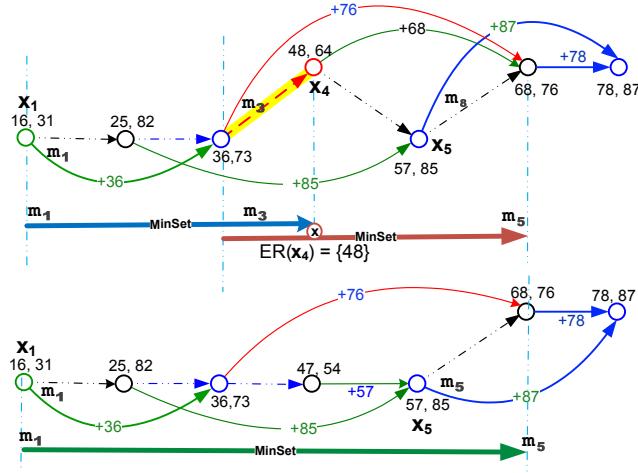
Here the two MinSets differ in the common R -edge incident at x_6 .

$$CVMPSet(m_1, m_5) = MinSet(m_1, m_5) + MinSet(m_1, m'_5)$$



VMPSet Partition 2: $ER(x_4) = \{48\}, ER(x_6) = \{68\}$

$$VMPSet(m_1, m_5) = MinSet(m_1, m_3) \cdot MinSet(m_3, m_5) + MinSet(m_1, m_5)$$



Lemma 4.15. For each $VMPSet(m_r, m_s)$ there are at the most 4 MinSets, $MinSet(m_r, m_s)$.

Proof. By Definition 4.13, for a fixed (m_r, m_s) , two MinSets in $\{MinSet(m_r, m_s)\}$ can differ only due to 3rd distinguished node because of the incident R -edges can make its ER to be different. For any other node at positions $\{r+2, r+3, \dots, s\}$, a non-null ER would imply MinSets of shorter length.

Moreover, the first two nodes do not have any R -edges incident on them. Thus, the 3rd node is the only one left that can induce subsets of $VMPSet(m_r, m_s)$ due to its two possible ERs resulting from the incident R -edges.

Now the ER of this 3rd distinguished node, $ER(x_{s+1})$, can take at the most 4 values corresponding to the 4 possible subsets of the edge pair $x_{s+1} \in g(s+1)$ in m_s . Therefore, for any fixed (m_r, m_s) ,

$$|\{MinSet(m_r, m_s)\}| \leq 4$$

□

Remark. Clearly, any $VMPSet(m_r, m_s)$ of length more than 3 can have an empty MinSet, $MinSet(m_r, m_s)$, because of the non null edge requirements.

Lemma 4.16. The maximum number of $MinSet(m_i, m_t, m_{n-1})$, $1 \leq i \leq t \leq n-1$, for a given $CVMPSet(m_i, m_{n-1})$ is bounded by $O(n^6)$.

Proof. The bound follows from the bound on the maximum number of mdags covered by any $CVMPSet(m_i, m_{n-1})$ in a given node partition, where there can be $O(n^3)$ mdags, $mdag < x_i >$ at any node x_i .

□

4.2.4 The Structure of a CVMPSet Partition

We will now show the exact composition of a CVMPSet in terms of its components, the MinSets.

The Covering MinSet- a subset of CVMPSet

Now we show how each $CVMPSet(m_i, m_{n-1})$ can be partitioned into disjoint subsets which are equivalence classes represented by a sequence of MinSets. We first define this MinSet sequence and then show how does it cover a $CVMPSet(m_i, m_{n-1})$.

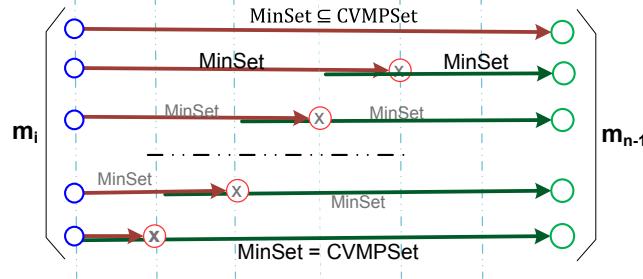


Figure 15: Sequences of 1-2 MinSets:
 $CVMPSet(m_i, m_{n-1}) = \biguplus MinSet(m_i, m_t) \cdot MinSet(m_t, m_{n-1})$

Let \prod denote the product of two or more adjacent MinSets, similar to the product of VMPSets.

Definition 4.17. A *covering minset*, $CMS_{it}(r)$, represents a subset of $VMPSet(m_i, m_t)$ by a sequence of r MinSets for the given $VMPSet(m_i, m_t)$. That is,

$CMS_{it}(r) \stackrel{\text{def}}{=} \{MinSet(m_i, m_{j_1}), MinSet(m_{j_1}, m_{j_2}), \dots, MinSet(m_{j_{r-1}}, m_t)\}$,
such that

$$\prod_{i_j \in I} MinSet(m_{i_j}, m_{i_{j+1}}) \subseteq VMPSet(m_i, m_t),$$

where $I = \{i, j_1, j_2, \dots, j_{r-1}\}$ is an index set representing the various node partitions induced by the $ER \neq \emptyset$ nodes in $VMPSet(m_i, m_t)$ such that $|I| = r$, $1 \leq r \leq t - i$.

The following Lemma 4.18 precisely states the composition of a $CVMPSet(m_i, m_{n-1})$ in terms of its MinSet sequences.

Lemma 4.18. Let $CMS_{in}(r)$ be a MinSet sequence of length r representing a subset of $CVMPSet(m_i, m_{n-1})$, where $1 \leq r \leq n - 2$, $1 \leq i \leq n - 2$.

Further, let $I = \{i, r_1, r_2, \dots, n - 1\}$ be an index set representing the indices to various node partitions induced by the $ER \neq \emptyset$ nodes in $CMS_{in}(r)$ such that $|I| = 1 + r$. Then, for all $i, 1 \leq i \leq n - 2$,

$$CVMPSet(m_i, m_{n-1}) = \biguplus_{r=1}^{n-2} \prod_{i_j \in I} MinSet(m_{i_j}, m_{i_{j+1}}) \quad (4.13)$$

(The proof is deferred until after the counting algorithm)

The Equivalence Class- MinSet Sequences

We define a new equivalence class induced by the following equivalence relation \mathfrak{R} over the set CVMPSet.

The Equivalence Relation \mathfrak{R}

Definition 4.19.

$$\forall (p_i, p_j) \in CVMPSet(m_i, m_{n-1}), p_i \mathfrak{R} p_j \iff \exists \text{ a prefix } MinSet(m_i, m_t) \text{ common to the sequences in (4.13) containing } p_i \text{ and } p_j.$$

Claim 4.20. *The relation \mathfrak{R} is an equivalence relation over the set $CVMPSet((m_i, m_{n-1}))$.*

Proof. Follows from Property 4.21. □

Property 4.21. *Every MinSet sequence $CMS_{in}(r)$ in $CVMPSet(m_i, m_{n-1})$ is unique.*

Proof. Note that each mdag in the mdag sequence $\{m_{i_j}, m_{i_{j+1}}, \dots, m_r\}$ induced by the nodes with $ER \neq \emptyset$ is unique because of the uniqueness of the missing edges in the bipartite graph. Now one can view each MinSet sequence, $CMS_{in}(r)$, as words composed out of unique mdags with $ER \neq \emptyset$, and without repetitions.

Thus, two MinSet sequences are identical iff the two sequences of the “delimiting” mdags are identical. Hence each sequence of unique mdags gives rise to a unique sequence in $CMS_{in}(r)$. □

Property 4.22. *There are $O(n^6)$ prefix MinSets, $MinSet(m_i, m_t)$, for any $CVMPSet(m_i, m_{n-1})$, covering all the exponentially many MinSet sequences in (4.13), where $1 \leq i < t \leq n - 1$.*

Proof. Note that for each fixed mdag m_i , $i < t \leq n - 1$, $|\{(m_i, m_t) \mid i < t \leq n - 1\}| = O(n^6)$, and $|\{MinSet(m_i, m_t)\}| = c|\{VMPSet(m_i, m_t)\}|$, $c \leq 4$ (Lemma 4.15). □

Partitions Induced by the MinSet Prefixes

The following Lemma is another version of (4.13) representing a CVMPSet partition induced by the prefix MinSets.

Let $PREFIX(l)$ denote a class of MinSets, $MinSet(m_1, m_l)$, containing VMPs of length l . Also, we define all MinSet sequences of zero length, $CMS(0)$, to be an identity, I , such that

$$PREFIX(l) \cdot \prod_{CMS(0)} MinSet(m_{i_j}, m_{i_{j+1}}) = PREFIX(l).$$

Lemma 4.23. Each $CVMPSet(m_1, m_{n-1})$ for a bipartite graph can be partitioned into $O(n^6)$ equivalence classes induced by the $O(n^6)$ prefix MinSets, $PREFIX(l)$, $2 \leq l \leq n-1$ in (4.13), such that:

$$CVMPSet(m_1, m_{n-1}) = \biguplus_{l=2}^{n-1} \left(\biguplus_{r=1}^{n-l-1} PREFIX(l) \cdot \prod_{\substack{CMS_{in}(r), \\ l=i_j \in I}}^{s_{r-1}} MinSet(m_{i_j}, m_{i_{j+1}}) \right), \quad (4.14)$$

where l , $2 \leq l \leq n-1$, is the first index in any index set, $I = \{1, s_1 = l, s_2, \dots, s_{r-1}, n-1\}$, representing all the node partitions induced by the $ER \neq \emptyset$ nodes in $CMS_{in}(r)$.

Proof. The proof follows from Lemma 4.18, Property 4.21 and Property 4.22. \square

The following figures ([Fig 16 and Fig 17]) depict the partition implied by (4.14), i.e., how $O(n^6)$ prefix MinSets induce polynomially many ($O(n^6)$) equivalence classes containing exponentially many $CMS_{in}(r)$ sequences in $CVMPSet(m_i, m_{n-1})$.

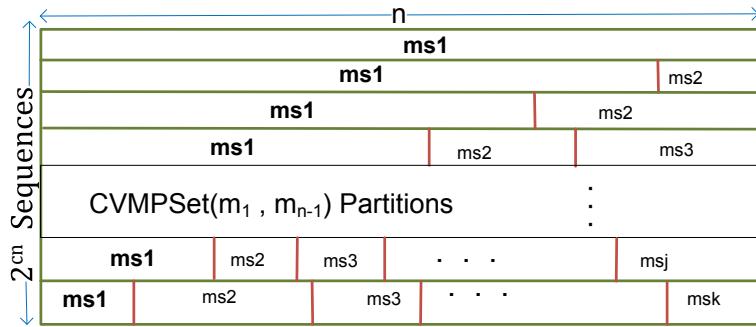


Figure 16: CVMP Set Partitions by $CMS(r)$ Sequences

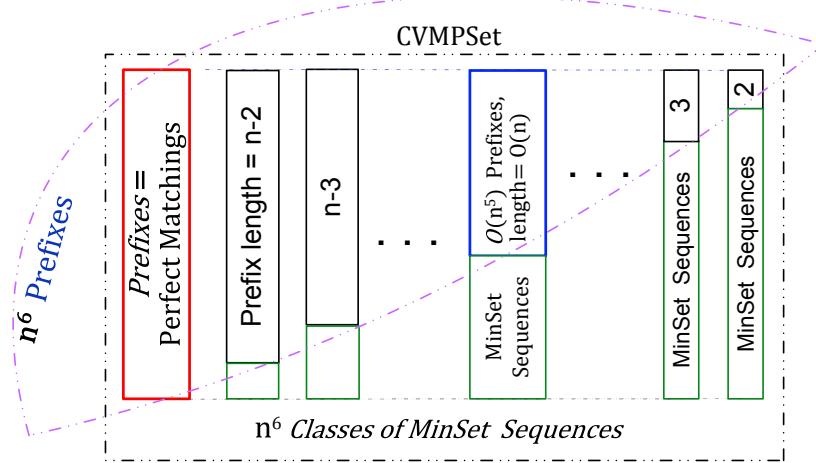


Figure 17: Partition of a CVMPSet Induced by the MinSet Prefixes

A Generating Set for the MinSet Sequences

Before we can describe the counting algorithm, we need one more generating set to consolidate the generation of all the MinSets shared by the various CVMPSets through their CMS partitions. To accomplish this we define a *generating set*, called *GMS*, for generating the Covering MinSets which constitute a partition of CVMPSet.

Definition 4.24. A generating set for MinSet sequences, $GMS(i, n)$, $1 \leq i \leq n - 2$, for a bipartite graph on $2n$ nodes is a set of MinSets defined as

$$GMS(i, n) \stackrel{\text{def}}{=} \{MinSet(m_r, m_s) \mid (r, s) \in [i \cdots n-2] \times [i+1 \cdots n-1], r < s\},$$

where $\{(m_r, m_s)\}$ covers $g(r) \times g(s)$.

Note that $GMS(i, n)$ contains all the $O(n^{11})$ MinSet prefixes for exponentially many sequences, $CMS_{in}(r)$, for all $CVMPSet(m_i, m_{n-1})$ in $\Gamma(n)$.

The following two properties follow from the definitions of GMS and CMS.

Property 4.25.

$$GMS(1, n) \supseteq \bigcup_{r, (m_1, m_{n-1})} CMS_{1n}(r) \quad (4.15)$$

Property 4.26. An upper bound on the size of $GMS(1, n)$ is $O(n^{12})$.

Proof. Note that $GMS(1, n)$ is precisely the set of all the $MinSet(m_r, m_s)$ which connect the $O(n^{12})$ pair of mdags, $\{(m_r, m_s)\}$, by all the VMPs in $\{VMPSet(m_r, m_s)\}$ in $\Gamma(n)$, where $(r, s) \in [1 \cdots n-2] \times [r+1 \cdots n-1]$.

Thus, this bound is same as the bound on the number of edges in a graph with $O(n^6)$ nodes. Therefore, by Lemma 4.15,

$$|GMS(1, n)| \leq 4|\{VMPSet(m_r, m_s)\}| \leq O(n^{12}).$$

□

5 A Polynomial Time Enumeration Algorithm

Based on the concepts of MinSet sequences developed in the previous Section, we have the following enumeration Algorithm 5.1 which counts all the perfect matchings in a given bipartite graph BG on $2n$ nodes, $n \geq 3$.

Algorithm 5.1 countPerfectMatchings(BG)

Input: a bipartite graph BG on $2n$ nodes, $n \geq 3$;

Output: count of the *perfect matchings* in BG ;

Step 0: Initialize- Compute the Initial Generating Set of all the MinSet Sequences

- 1: $i = n - 3$; // i is the current node partition;
 - 2: Compute the generating set $E_M = \{g(r) \mid 1 \leq r \leq n\}$;
 - 3: Compute the generating set $GMS(i+1, n) = \{\text{MinSet}(m_{n-2}, m_{n-1})\}$; // the set of all the MinSet Sequences; each $\text{CVMPS}(m_{n-2}, m_{n-1})$ is a $\text{MinSet} \in GMS(n-2, n)$, with a total count of 6 CVMPs.
-

Step 1: Count

if ($i = 0$) **then** // $GMS(1, n)$ may contain the set $\{\text{MinSet}(m_1, m_{n-1})\}$

$$\text{perfect matching count} = \sum_{\substack{ER=\emptyset, \\ (m_1, m_{n-1})}} \text{MinSet}(m_1, m_{n-1}) \cdot \text{Count};$$

return;

Step 2: Increment & Join the MinSet Sequences

$\text{incrementMSS}(GMS(i+1, n))$; // assuming $n \geq 3$

(Follows the structures in Figure 17)

decrement i;

repeat Steps 1-2;

End.

5.1 The Polynomial Time Bound

Claim 5.1. *The time complexity of Algorithm 5.1 is $O(n^{45} \log n)$.*

Proof. Although a tight upper bound would require more details of the algorithm, a fairly loose upper bound should be easy to establish as follows. Let $T(\text{ops})$ denote the time complexity of the operation ops .

We note that Step 2 calls Algorithm 5.2, $\text{incrementMSS}()$, having a time complexity $O(n^{44} \log n)$, and dominates:

$$T(\text{Step 2 : Increment}) = O(n^{44} \log n)$$

$$T(\text{Step 1 : Count}) = O(n^8).$$

Steps 1-2 are iterated $O(n)$ times, and thus the time complexity of the counting algorithm is $O(n^{45} \log n)$. □

5.2 Correctness of the Count

Lemma 5.2. All the perfect matchings in a bipartite graph BG on $2n$ nodes can be correctly enumerated by 5.1 in polynomial sequential time $O(n^{45} \log n)$.

Proof. The correctness of the count follows from the Lemmas 4.18 and 5.4 which prove the following two assertions:

1. All $\text{MinSet}(m_1, m_{n-1})$ with $ER = \emptyset$ are contained in $GMS(1, n)$.
2. The perfect matching count is:

$$\sum_{\substack{ER=\emptyset, \\ (m_1, m_{n-1})}} \text{MinSet}(m_1, m_{n-1}) \cdot \text{Count}$$

□

Proof. (of Lemma 4.18)

The proof is by induction on the length r of the MinSet sequence, $CMS_{in}(r)$. We will consider VMPSets as a general representation of CVMPSets.

Let $m_j = mdag(x_j, x_{j+1}, x_d)$, where $x_j \in g(j)$, $x_{j+1} \in g(j+1)$, and $x_d \in g(d)$.

Case: $r = 1$

The length of a MinSet sequence for a VMPSet(m_r, m_s) is one when all the VMPs in the MinSet are of the same length as those in VMPSet(m_r, m_s). This can happen when the $ER \neq \emptyset$ only at the 3 allowed nodes, viz., at x_i or at x_{i+1} in m_i and x_{j+1} in m_j . And then, we have 2 sub cases, that is, either

$$\text{VMPSet}(m_i, m_j) = \text{MinSet}(m_i, m_j), \text{ when } ER \neq \emptyset \text{ at } x_i \text{ or at } x_{i+1}, \quad (5.1a)$$

or

$$\text{VMPSet}(m_i, m_j) = \biguplus_{m_j} \text{MinSet}(m_i, m_j) = \biguplus_{m_j} \text{prodVMPSet}(m_i, m_j, m_j), \quad (5.1b)$$

when $ER(x_{j+1}) \neq \emptyset$ for some $p \in \text{VMPSet}(m_i, m_j)$ at the last node x_{j+1} in m_j .

Case: $r \leq 2$

Let $CMS_{ij}(r) = \{\text{MinSet}(m_i, m_t), \text{MinSet}(m_t, m_j)\}$, where the two MinSets have a common m_t with $ER(m_t) \neq \emptyset$, $i < t < j$. Then, the first MinSet could be a subset of $\text{VMPSet}(m_i, m_t)$, governed by (5.1b), whereas the second MinSet is the corresponding $\text{VMPSet}(m_t, m_j)$ by (5.1a). Therefore, for each such sequence we can apply the result of $r = 1$ to each MinSet and Property 4.7, (4.7) to obtain:

$$\begin{aligned} \text{MinSet}(m_i, m_t) \cdot \text{MinSet}(m_t, m_j) &\subseteq \text{VMPSet}(m_i, m_t) \cdot \text{VMPSet}(m_t, m_j) \\ &= \text{prodVMPSet}(m_i, m_t, m_j) \end{aligned}$$

Further we note that each m_t in each sequence of 2 MinSets must be disjoint with each other, irrespective of the node partition t to which m_t belongs. Or else, we will have $r > 2$ because of the additional $ER \neq \emptyset$ nodes common to the same MinSet.

Therefore,

$$\biguplus_{m_t} \text{MinSet}(m_i, m_t) \cdot \text{MinSet}(m_t, m_j) = \biguplus_t \biguplus_{m_t} \text{prodVMPSet}(m_i, m_t, m_j) \\ = \text{VMPSet}(m_i, m_j).$$

Induction: Sequence size $l = r + 1$

If the induction hypothesis is true for all sequence sizes, $l \leq r$, then each sequence of length $r + 1$ can be partitioned into 2 subsequences, $\text{MinSet}(m_i, m_t)$ and $\text{MinSet}(m_t, m_{n-1})$, each of length less than or equal to r .

Therefore, again applying the above cases for $r \leq 2$ we have

$$\text{CVMPSet}(m_i, m_{n-1}) = \biguplus_{m_t} \text{MinSet}(m_i, m_t) \cdot \text{MinSet}(m_t, m_{n-1}).$$

□

5.3 Examples- Incrementing and Joining the Adjacent MinSets

The following series of figures illustrate the Step 2: "Increment all the MinSet Sequences" of Algorithm 5.1. Figure 18 shows a bipartite graph with 3 perfect matchings and the associated CVMPs that will generate the perfect matchings. Figure 19 shows a very simple Increment & Join operation on the MinSets.

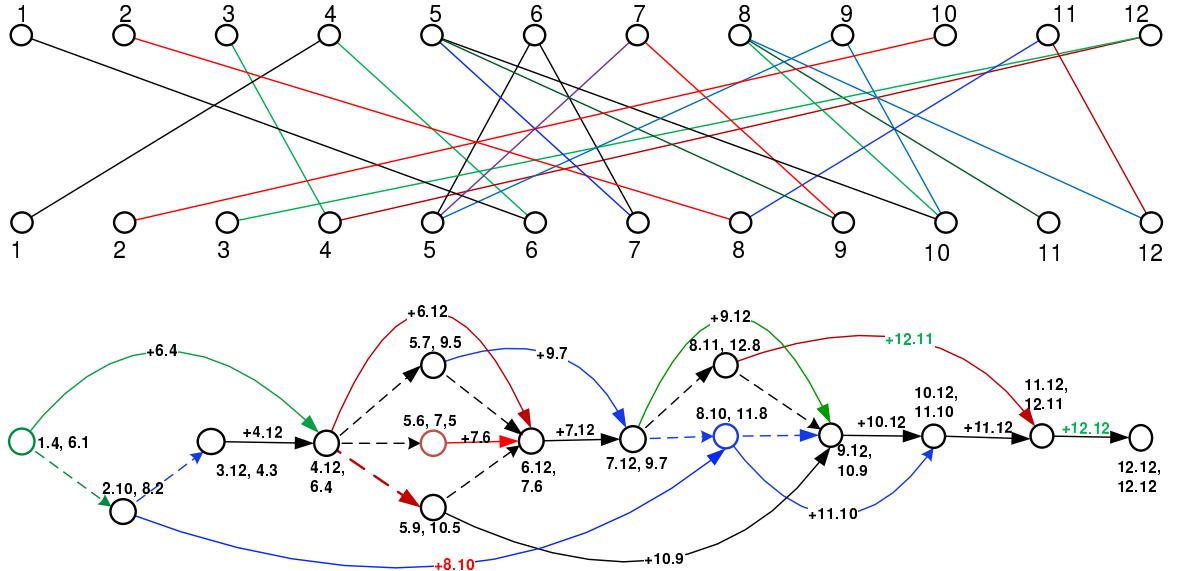


Figure 18: A Bipartite Graph and its CVMPs

Figure 19(a) shows two MinSets each having a common $ER = \{8.10\}$ because of the missing edge $(8, 10)$ in the bipartite graph.

Figure 19(b) shows an increment of the MinSet1 by $x_2 = (2.10, 8.2)$. This increment creates a jump R -edge between x_2 and x_8 , and thus satisfying the $ER = \{8.10\}$ of x_8 , and joining the two MinSets to produce $p = \text{MinSet1} \cdot \text{MinSet2}$, which contains 3 CVMPs. Note that $ER(p) = \emptyset$.

Figure 19(c) shows a further increment of the CVMPs in p by $x_1 = (1.4, 6.1)$. This gives a set of 3 CVMPs with null ER to produce 3 perfect matchings.

Note: The edge $(4, 6)$ in the bipartite graph BG is not needed for the resulting 3 perfect matchings but its presence simplifies the Step 2 of the algorithm.

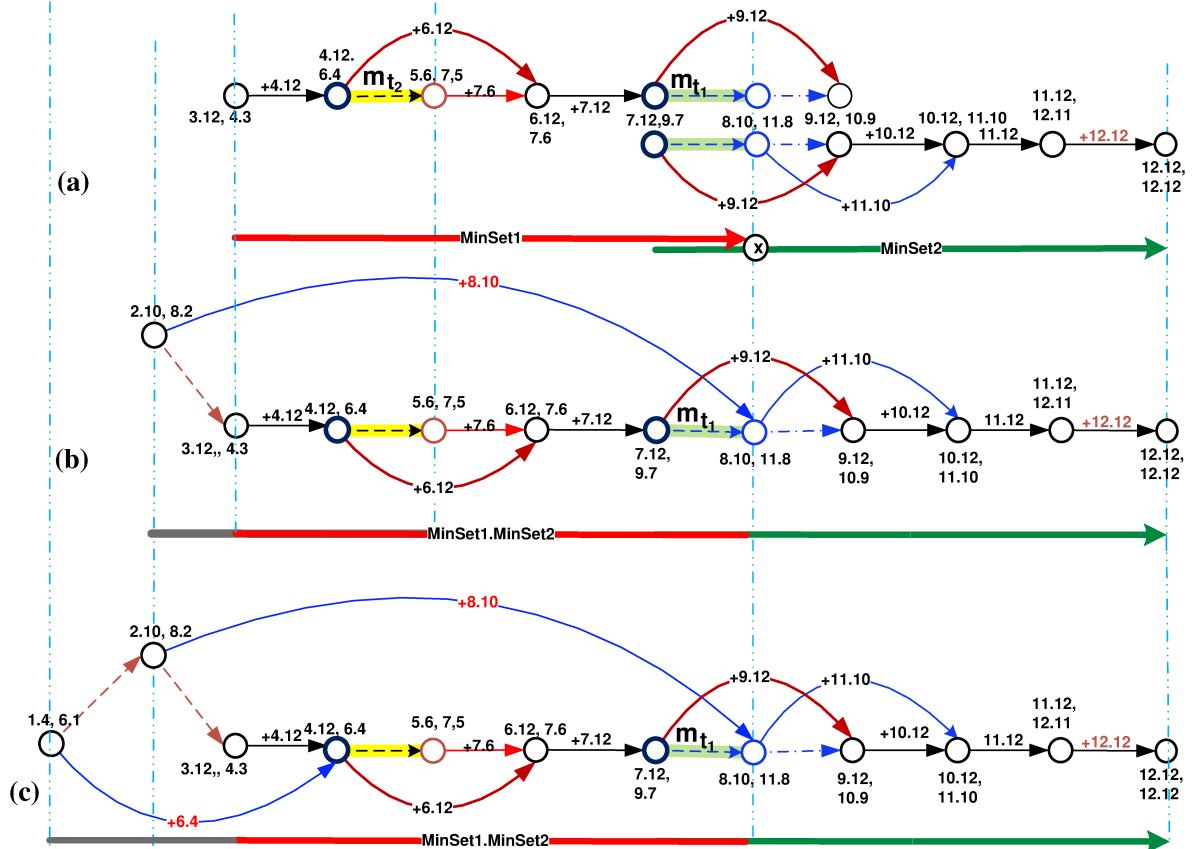


Figure 19: A Simple Increment & Join of 2 MinSets

5.4 The Basic Behavior of Incrementing a MinSet Sequence

The incrementing process can become more intricate if there are other MinSets, $\text{MinSet}(m_t, m_s)$, adjacent to $\text{MinSet}(m_{i+1}, m_t)$, which may also have to be joined with the prefix MinSet. A simplified view of this process is illustrated in the following Figures 20(a-c). These Figures show how a sequence of three adjacent MinSets is incremented and joined by the multiplying mdags at node partitions, $i = 2$ and $i = 1$.

Figure 20(a) shows a sequence of three MinSets, MS_1 , MS_2 and MS_3 .

Figure 20(b) shows that $x_2 = (2.9, 8.2)$ increments the prefix MinSet, MS_1 but with no join operation.

Figure 20(c) shows that $x_1 = (1.6, 5.1)$ increments the prefix MinSet, MS_1 and joins all three MinSets in the sequence.

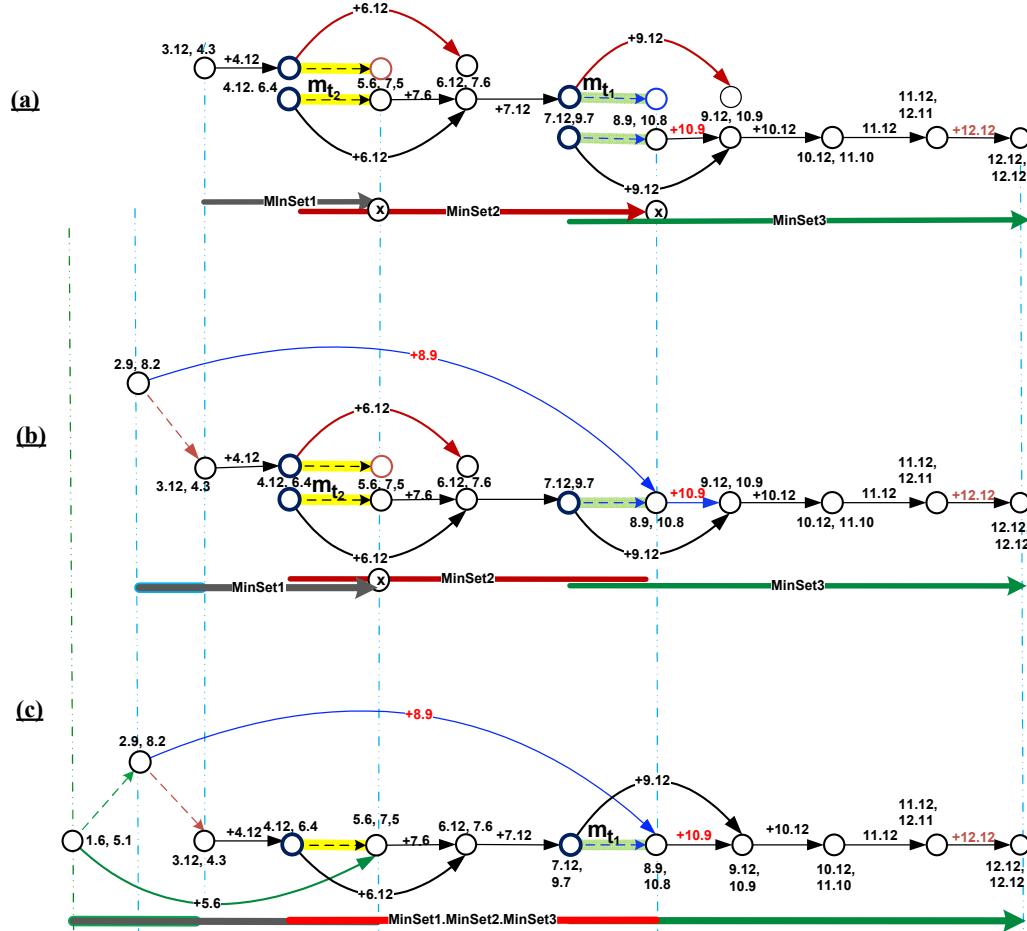


Figure 20: Incrementing & Joining the Adjacent MinSets

5.5 Incrementing the MinSet Sequences in GMS

The above counting algorithm 5.1 calls the following algorithm 5.2, *incrementMSS()*, to increment all the MinSet Sequences in $GMS(i, n)$. This algorithm uses two basic operations on MinSet, viz., increment a prefix MinSet, and join two or more adjacent MinSets into one, described by the associated algorithms in the next subsection.

Algorithm 5.2 *incrementMSS($GMS(i + 1, n)$)*

Input: $GMS(i + 1, n)$; // contains all sequences $CMS_{i+1,n}(r)$, $1 \leq r \leq n - 2$
Output: $GMS(i, n)$;

Step (a): Increment all Prefix MinSets, $\text{MinSet}(m_{i+1}, m_s) \in \bigcup CMS_{i+1,n}(r)$

```

1: for all  $x_i \in g(i)$  do
2:   for all  $s \in [i+2 \dots n-1]$ ,  $x_i R x_{t+1}$ ,  $s > t + 1$  do
3:     for all  $\text{MinSet}(m_{i+1}, m_s) \in \bigcup CMS_{i+1,n}(r)$  do //  $O(n^6)$  MinSets covers all  $CMS_{in}(r)$ 
4:       add  $\text{incrMinSet}(\text{mdag} < x_i >, \text{MinSet}(m_{i+1}, m_s))$  to  $GMS(i, n)$ ;
5:     end for
6:   end for
7: end for

```

Step (b): Join all the Sub Sequences Selected by $\text{MinSet}(m_i, m_t)$;

$$m_i = \text{mdag} < x_i >, m_t = \text{mdag} < x_t >, x_i R x_{t+1}.$$

```

1: for all  $x_i \in g(i)$  do
2:   for all  $\text{MinSet}(m_i, m_t) \in GMS(i, n)$  do
3:      $\text{updateSequence}(\text{MinSet}(m_i, m_t), GMS(i, n))$ ;
4:   end for
5: end for
6: return  $GMS(i, n)$ ;

```

5.5.1 The Time Complexity

Claim 5.3. *The time complexity of Algorithm 5.2 is $O(n^{44} \log n)$.*

Proof. It should be easy to see that the time in Step(b) dominates.

The For loop at line b(1) is iterated $O(n^2)$ times, and the For loop at line b(2) is iterated $O(n^6)$ times determined by the cardinality of $\{(m_i, m_t)\}$ for a given x_i .

The time complexity at line b(3) of *updateSequence()* is $O(n^{36} \log n)$ (Claim 5.7, Algorithm 5.5).

Therefore, the time complexity of the algorithm as determined by the Step(b) is

$$T(\text{Step}(b)) = O(n^8 * n^{36} \log n) = O(n^{44} \log n)$$

□

5.5.2 Correctness of Algorithm 5.2: incrementMSS()

Lemma 5.4. *For each $x_i \in g(i)$, $i \geq 1$ the Algorithm 5.2 increments $GMS(i, n)$ to $GMS(i - 1, n)$ to satisfy (4.13) of Lemma 4.18.*

Proof.

The correctness follows from the fact that each increment operation by $x_i \in g(i)$ applies either to a prefix or a subset of that prefix to each sequence of MinSets in $GMS(i, n)$. Moreover, all the $O(n^6)$ prefixes are incremented by the elements $x_i \in g(i)$.

Let $ps = MinSet(m_{i+1}, m_t)$ be a prefix to a sequence in $GMS(i, n)$, $i + 1 < t \leq n - 2$, and it is incremented by a unique $x_i \in g(i)$ in Step(a:4). Then, either $x_i \cdot ps$ is in a MinSet in $GMS(i - 1, n)$, or it exists as a new sequence of MinSets in $GMS(i - 1, n)$, with a new prefix MinSet for the original MinSet sequence.

Further, when increments by x_i leads to selecting a subset of a MinSet sequence, such a subset of that sequence will also be a valid subset in $VMPSet(m_i, m_{n-1})$. Hence (4.13) of Lemma 4.18 is always satisfied.

□

5.6 Algorithm: Increment a MinSet

Increment of a MinSet, $\text{MinSet}(m_{i+1}, m_s)$ by an adjacent mdag, $\text{mdag} < x_i >$ would involve same kind of operations as in the case of $\text{VMPSet}(m_{i+1}, m_s)$. Unless the R -edge from x_i is incident at the two distinguished nodes x_{i+1} or at x_{i+2} , the multiplication would cover only a subset of VMPs in $\text{MinSet}(m_{i+1}, m_s)$. This subset is determined by the node partition t in which the node x_t in $\text{MinSet}(m_{i+1}, m_s)$ lies while $x_i R x_t$ holds true.

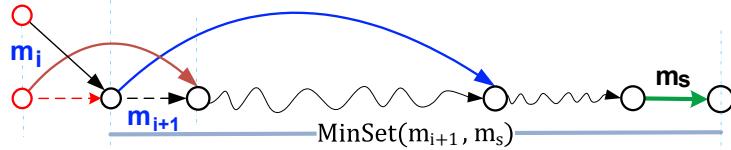


Figure 21: Special Case: Incrementing a MinSet at a Common Node x_{i+1}

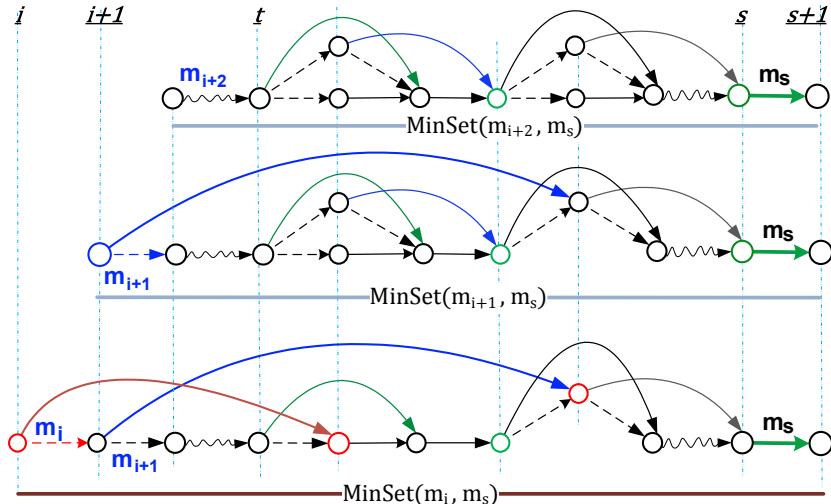


Figure 22: Two Successive Increments at x_{i+2} and x_{i+1}

The Step 0 in the algorithm 5.3 covers special cases, where the multiplying mdag multiplies all the VMPs in the MinSet using a common node (Figure 21). The Steps 1-2 show a general case (Figure 22) where the mdag can multiply only a subset of the VMPs from the original MinSet. This involves effectively re-constructing the whole MinSet by the revised list of the multiplying mdags in each node partition.

Algorithm 5.3 $incrMinSet(mdag\langle x_i \rangle, MinSet(m_{i+1}, m_s))$

Input: $m_i = mdag(x_i, x_{i+1}, x_{t+1})$, $m_{i+1} = mdag(x_{i+1}, x_{i+2}, x_{t+r})$, $r \geq 1$; $MinSet(m_{i+1}, m_s)$
Output: $MinSet(m_i, m_s)$ or $\{MinSet(m_i, m_{i+1}), MinSet(m_{i+1}, m_s)\}$

Step 0: Initialization and Special Cases

```
1: let  $x_i R x_{t+1}$ ; given  $m_{i+1} = mdag(x_{i+1}, x_{i+2}, x_{t+1})$ ;  
2: if  $ER(x_{i+2}) \neq \emptyset$  then  
3:   return  $\{MinSet(m_i, m_{i+1}), MinSet(m_{i+1}, m_s)\}$ ;  
4: end if  
5: if  $(x_i R x_{i+1} \text{ OR } x_i R x_{i+2})$  then  
6:   return  $MinSet(m_i, m_s) = m_i \cdot MinSet(m_{i+1}, m_s)$ ;  
7: end if
```

Step 1: Determine the candidate mdags in each node partition of $MinSet(m_{i+1}, m_s)$;

assumption: $i < t \leq s$;

```
1: for all  $nodePartition j \in [i + 1 \dots s - 1]$  do  
2:    $mdagList[j] = \{mdag\langle x_j \rangle\}$ ;  
3: end for  
4: remove each  $mdag(x_t, x_{t+1}, x_r)$  from  $mdagList[t]$  where  $x_i \not R x_{t+1}$ ;
```

Step 2: Rebuild the whole $MinSet$ with the updated mdags

(sequentially Increment the vmpSets by the mdag List)

```
1:  $vmpSetList = \{mdag\langle x_{s-1} \rangle \cdot m_s\}$ ;  
2: for all  $nodePartition j = s - 2$  downto  $i$  do  
3:   for all  $mdag \in mdagList[j]$  do  
4:      $newList = \emptyset$ ;  
5:     for all  $vmpSet \in vmpSetList$  and adjacent to  $mdag$  do  
6:       update: add  $mdag \cdot vmpSet$  to  $newList$ ;  
7:     end for  
8:      $vmpSetList \leftarrow newList$ ;  
9:   end for  
10:  if  $(|mdagList[j]| = 1)$  then  
11:     $vmpSetList \leftarrow mergeMinSet(vmpSetList)$ ;  
12:  end if  
13: end for  
14: output  $vmpSetList = MinSet(m_i, m_s)$ ; //
```

End.

Lemma 5.5. *The time complexity of Algorithm 5.3, $\text{incrMinset}()$, is $O(n^{12} \log n)$.*

Proof. The dominant time comes from Step 2. Each of the FOR loops at lines 3 and 5 are iterated $O(n^5)$ times and $O(n^3)$ times, determined by the bound $O(n^5)$ on the cardinality of the mdags in any node partition in $\text{MinSet}(m_{i+1}, m_s)$.

The merge operation at line Step (2:11) takes at the most $O(n^{11} \log n)$ steps, and it dominates in Step 2.

The FOR loop at line Step(2:2) can be iterated $O(n)$ times.

Therefore, the time complexity of the above algorithm is $O(n * n^{11} \log n) = O(n^{12} \log n)$. \square

5.6.1 Joining two Adjacent MinSets

The following algorithm 5.4 is essentially an iterative increment of the second MinSet, $\text{MinSet}(m_t, m_s)$, by the available adjacent mdags in the successive partitions of the first MinSet, $\text{MinSet}(m_i, m_t)$.

At each iteration, a new set of VMP sets is created and which could also be merged into one VMP set.

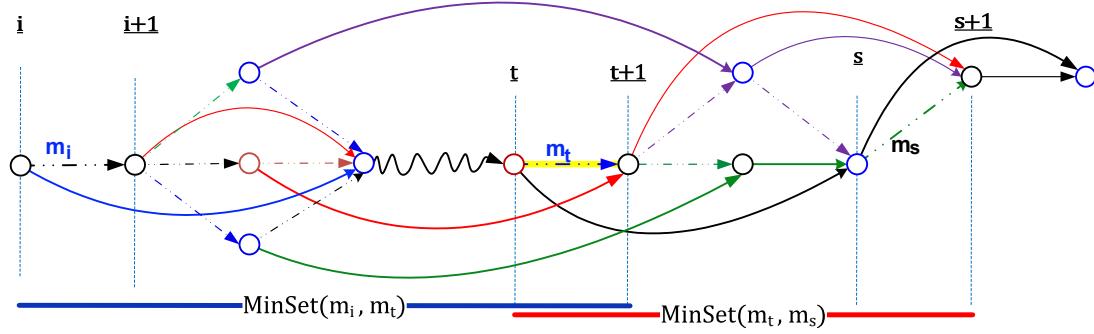


Figure 23: Joining two Adjacent MinSets

Algorithm 5.4 $joinMinSet(MinSet(m_i, m_t), MinSet(m_t, m_s))$

Input: $MinSet(m_i, m_t)$, $MinSet(m_t, m_s)$;
Output: $MinSet(m_i, m_t, m_s)$;

Compute the Product $MinSet(m_i, m_t) \cdot MinSet(m_t, m_s)$

```

1: for all nodePartition  $j \in [i \dots t]$  do
2:    $partition[j] = \{x_j | x_j \in MinSet(m_i, m_t)\}$ ;
3: end for
4:  $vmpSetList = \{MinSet(m_t, m_s)\}$ ;
5: for all nodePartition  $j = t - 1$  downto  $i$  do
6:   for all  $mdag\langle x_j \rangle$  in  $partition[j]$  do
7:      $newList = \emptyset$ ;
8:     for all  $vmpSet \in vmpSetList$  and adjacent to  $mdag\langle x_j \rangle$  do
9:       add  $incrMinSet(mdag\langle x_j \rangle, vmpSet)$  to  $newList$ ;
10:    end for
11:   end for
12:    $vmpSetList \leftarrow newList$ ;
13:   if ( $|partition[j]| = 1$ ) then
14:      $mergeMinSet(vmpSetList)$ ;
15:   end if
16: end for
17: output  $vmpSetList$ ;
```

Lemma 5.6. *The time complexity of $joinMinSet()$ by Algorithm 5.4 is $O(n^{21} \log n)$.*

Proof.

In the above Algorithm 5.4,

the FOR loop on line 5 is iterated $O(n)$ times,

the FOR loop on line 6 is iterated $O(n^5)$ times, determined by the cardinality of $\{mdag < x_j >\}$,
the FOR loop on line 8 is iterated $O(n^3)$ times, determined by the subset of $vmpSetList$ adjacent
to each $mdag$, and

the time complexity of $incrMinSet()$ at line 9 is $O(n^{12} \log n)$.

The merge operation at line 11 can be done in time $O(n^{11} \log n)$.

Clearly the time between the lines 6-11 dominates and which is:

$$O(n^5 * n^3 * n^{12} \log n) = O(n^{20} \log n).$$

Therefore, the total time over $O(n)$ iterations between the lines 2-13 is $O(n^{21} \log n)$.

□

5.6.2 Updating the MinSet Prefixes

Joining two MinSets by an R -edge can become overly intricate when the R -edge from the incrementing node $x_i \in g(i)$ at step (a) of the algorithm $incrementMS()$ is incident at MinSets that are not adjacent to x_i . The algorithm $incrementMS()$ defers that "join" until an incrementing node x_i which will join the MinSets adjacent to it is found.

The following Figure 24 extends the previous Figure 20 (Incrementing & Joining Adjacent MinSets) to capture the basic behavior of this algorithm. It shows a chain of joining operations in a MinSet sequence induced by the R -edges stemming from the prefix MinSet, $\text{MinSet}(m_i, m_t)$.

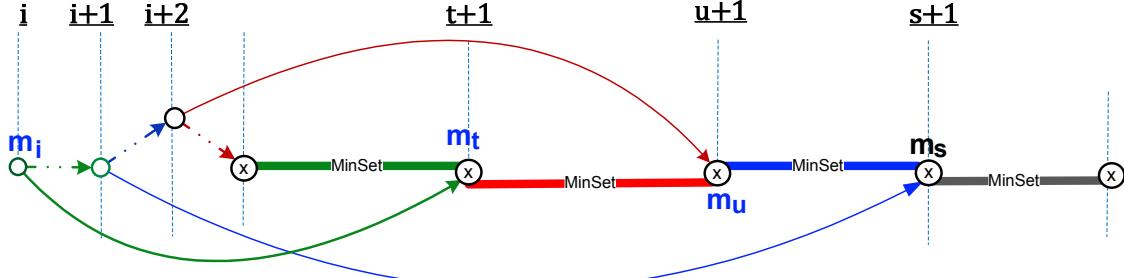


Figure 24: Joining a Chain of MinSet Sequences by the Incrementing mdags

Algorithm 5.5 $\text{updateSequence}(\text{MinSet}(m_i, m_t), \text{GMS } (i, n))$

Input: the prefix $\text{MinSet}(m_i, m_t)$, $\text{GMS } (i, n)$;

Output: updated $\text{GMS } (i, n)$ with incremented MinSet prefixes in each sequence;

Step (a): Find all the free jump edges in each partition in the Prefix $\text{MinSet}(m_i, m_t)$

- 1: $\text{freeRList} := \{(x_j, x_k) \in \text{MinSet}(m_i, m_t) \mid k \geq t; x_j R x_k\}; // |\text{freeRList}| = O(n^4);$
- 2: sort freeRList in ascending k ;

Step (b): Join the Sub-Sequences induced by the R -edges in freeRList

- 1: $mtSet := \{m_t\}; prefixSet := \{\text{MinSet}(m_i, m_t)\};$
 - 2: **while** $\text{freeRList} \neq \emptyset$ **do**
 - 3: get next free jump edge $(x_j, x_k) \in \text{freeRList};$
 - 4: **while** $mtSet \neq \emptyset$ **do**
 - 5: remove next mt from $mtSet$;
 - 6: find next $prefix = \text{MinSet}(m_i, m_t) \in prefixSet;$
 - 7: **if** $(mdag < x_k > = mt)$ **then**
 - 8: **for all** $minset = \text{MinSet}(mt, m_u) \in VMPSet(mt, m_{n-1})$, where $m_u = mdag < x_u >$ **do**
 - 9: $prefix := \text{joinMinSet}(prefix, \text{MinSet}(mt, m_u));$
 - 10: add $mt := m_u$ to $mtSet$;
 - 11: add $prefix = \text{MinSet}(m_i, m_t)$ to $prefixSet$;
 - 12: recompute freeRList using Step(a);
 - 13: **end for**
 - 14: **end if**
 - 15: **end while**
 - 16: **end while**
 - 17: replace all prefix MinSets in $\text{GMS } (i, n)$ with $prefixSet$;
-

Claim 5.7. *The time complexity of $\text{updateSequence}()$ in Algorithm 5.5 is $O(n^{36} \log n)$.*

Proof. It should be easy to see that the time in Step(b) dominates.

The loop at line b(2) is iterated $O(n^4)$ times.

The While loop at lines b(4) is iterated $O(n^5)$ times.

The For loop at line b(8) is iterated $O(n^6)$ times, determined by the number of mdags to be searched for $ER \neq \emptyset$.

The time complexity at line b(9) of $\text{joinMinSet}()$ is $O(n^{21} \log n)$ and dominates all other operations even outside the For loops.

Therefore, the time complexity of $\text{updateSequence}()$ is,

$$T(\text{Step}(b)) = O(n^4 * n^5 * n^6 * n^{21} \log n) = O(n^{36} \log n).$$

□

5.6.3 The Merge Operation

The following are high level algorithms for the merge operation used in the above algorithms, 5.2 and 5.4.

Merging two MinSets

This is called by $\text{mergeMinSets}()$ to merge all the MinSets in a given list.

Algorithm 5.6 $\text{merge2MinSets}(\text{minSet1}, \text{minSet2})$

Input: $\text{minSet1} = \text{MinSet}(m_r, m_s)$, $\text{minSet2} = \text{MinSet}(m_r, m_s)$;

Output: $\text{MinSet}(m_r, m_s)$;

- 1: *union & merge* each node partition pair for the MinSets, minSet1 and minSet2 ;
 - 2: *add* the counts if the MinSets can be merged:

$$\text{MinSet}(m_r, m_s).\text{Count} := \text{minSet1.Count} + \text{minSet2.Count};$$
 - 3: **return** $\text{MinSet}(m_r, m_s)$;
-

Claim 5.8. *The time complexity of $\text{merge2MinSets}()$ in Algorithm 5.6 is $O(n^6 \log n)$.*

Proof. This merge operation is essentially a union operation of the $O(n^5)$ distinct mdags in each node partition. Assuming that all the node partitions have been pre-sorted, the search for any mdag in a partition with $O(n^5)$ mdags can be done in $O(\log n)$ time. Thus the union operation in each node partition requires $O(n^5 \log n)$ time.

Clearly, the add operation at line 2 is not dominating, taking only $O(n^3)$ time.

□

5.6.4 Merge all MinSets

The following algorithm merges all the MinSets, $\text{MinSet}(m_i, m_j)$, given in a list of MinSets.

Algorithm 5.7 *mergeMinSets(MinSetList)*

Input: A list of $\text{MinSet}(m_r, m_s)$ in MinSetList ;

Output: $\{\text{MinSet}(m_r, m_s)\}$;

```
1: mergeSet := MinSetList[1]; l := length(MinSetList);
2: for j := 2 to l do
3:   mergeSet := merge2MinSet(mergeSet, MinSetList[j]);
4: end for
5: return mergeSet;
```

Claim 5.9. *The time complexity of Algorithm 5.7 is $O(mn^6 \log n)$, where m is the size of MinSetList .*

6 Conclusions

6.1 Collapse of the Polynomial Hierarchy

We can re-state Lemma 5.2 as the following Theorem in terms of the class **FP** which is defined as the class of functions $f : \{0, 1\}^* \rightarrow \mathbb{N}$ computable in polynomial time on a deterministic model of computation such as a deterministic Turing machine or a RAM.

Theorem 6.1. *The counting problem for perfect matching is in **FP**, and therefore, $\#\mathbf{P} = \mathbf{FP}$ and $\mathbf{NP} = \mathbf{P}$.*

Based on the fact that every $\#P$ -complete problem is also NP -hard, it follows that $\mathbf{NP} \subseteq \mathbf{P}^{\#\mathbf{P}}$. And therefore, the above Theorem implies that polynomial hierarchy **PH** collapses into **P**. Needless to say that the main Theorem of Toda [Tod89], which states that the class $\#\mathbf{P}$ contains PH , is a re-confirmation of **PH** collapsing into **P**.

6.2 A Characterization of P-time Enumeration

The forgoing enumeration technique gives rise to the following conjecture on a characterization of the polynomial time enumeration:

A sufficient condition for the existence of a P-time algorithm for any enumeration problem is the existence of a partition hierarchy of the exponentially decreasing solution spaces, where each partition is polynomially bounded and the disjoint subsets in each partition are P-time enumerable for each $n \geq 1$, n being a problem size parameter.

Although we may have an existential proof for a sufficient condition for the P-time enumeration, a more fundamental question is if this condition is also necessary.

An attempt to prove that necessary condition was made in an unpublished paper [Asl92]. The basic logic behind the proof was that any deterministic search must cover the entire solution

space of the search problem, and hence must also be able to count all the solutions in essentially the same time bound. This logic lead to the enumeration model conjectured in this paper.

A simultaneous polynomial bound on the depth as well as on the width (partition size) of the partition hierarchy creates a non-trivial relationship between the partitions at any two consecutive levels. This is simply because an exponentially large set can never be reduced to a constant size in polynomially many steps by the subset operations.

This is an area of algorithm design which has not received much attention so far. Some thoughts along this line have been covered in Appendix A.9.

References

- [Asl92] Javaid Aslam, *An Information Theoretic Model of Parallel Search: Counting is NC-reducible to Search* (*Unpublished work, Computer Science Dept., SUNY Buffalo, NY*), 1992.
- [But91] Gregory Butler, *Fundamental Algorithms for Permutation Groups*, Lecture Notes in Computer Science, vol. 559, Springer-Verlag, 1991.
- [Edm65] Jack Edmonds, *Paths, Trees, and Flowers*, Canadian Journal of Mathematics **17** (1965), 449–467.
- [Ege31] E. Egervary, *On Combinatorial Properties of Matrices (in Hungarian)*, Math. Lapok 38, 1931, pp. 16–28.
- [GJ79] M. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman & Co., SanFrancisco, CA, 1979.
- [Hof82] C. M. Hoffmann, *Group-theoretic Algorithms and Graph Isomorphism*, Lecture Notes in Computer Science, vol. 136, ch. II, Springer-Verlag, Berlin, 1982.
- [Jer94] Mark Jerrum, *The Computational Complexity of Counting*, Proceedings of the International Congress of Mathematicians, Birkhauser Verlag, 1994, pp. 1407–1416.
- [Kuh55] H. W. Kuhn, *The Hungarian Method for the Assignment Problem*, Naval Research Logistics Quarterly 2, 1955, pp. 83–97.
- [Tod89] S. Toda, *On the Computational Power of PP and $\bigoplus \mathbf{P}$* , Proc. 30th IEEE Symp. on Foundations of Computer Science, 1989, pp. 514–519.
- [Val79a] L. G. Valiant, *The Complexity of Computing the Permanent*, Theoretical Computer Science **8** (1979), 189–201.
- [Val79b] ———, *The Complexity of Enumeration and Reliability Problems*, SIAM J. Computing **8(3)** (1979), 410—421.

Appendix A

A.1 Permutation Multiplication- Proof of Theorem 3.1

Proof.

Let $\psi = (j, k)$ be a transposition in S_n . Note that ψ need not be realized by BG' , however, we will show that there are two unique edges in BG' that represent ψ , and depend on $E(\pi)$ whenever $\pi\psi$ is realized by BG' .

Let $i, t \in \Omega$ be the two points mapped by π such that $i^\pi = j$, and $t^\pi = k$. Thus $E(\pi)$ covers the edges v_iw_j and v_tw_k in BG' .

$$\underline{\pi\psi \in M(BG')} \implies \text{a cycle of length 4}$$

If the product $\pi\psi$ is realized by BG' , then we must have:

$$\begin{aligned} i^{\pi\psi} &= j^\psi = k, \quad \text{and} \\ t^{\pi\psi} &= k^\psi = j. \end{aligned}$$

That is, the existence of the edges in $E(\pi\psi)$ dictates that BG' contain the edges v_iw_j and v_tw_k at the vertex $v_i \in V$, and v_tw_j and v_iw_k at the vertex $v_t \in V$. And hence, BG' has a cycle $v_iw_jv_tw_kv_i$ of length 4.

$$\underline{\text{A cycle of length 4} \implies \pi\psi \in M(BG')}$$

Let $C = v_iw_jv_tw_k$ be a cycle of length 4 in BG' where π is such that $i^\pi = j$ and $t^\pi = k$, and thus π covers v_iw_j and v_tw_k .

The new permutation $\pi_1 = \pi\psi$ can be realized by swapping the alternate edges of C such that π_1 differs from π only in two positions, viz., $i^{\pi_1} = k$ and $t^{\pi_1} = j$, corresponding to the edges v_iw_k and v_tw_j .

Now we show how ψ is encoded by the two alternate edges of C .

Since $\psi = \pi^{-1}\pi_1$, we have

$$\begin{aligned} j^\psi &= j^{\pi^{-1}\pi_1} = i^{\pi_1} = k, \quad \text{and} \\ k^\psi &= k^{\pi^{-1}\pi_1} = t^{\pi_1} = j. \end{aligned}$$

Therefore, $\psi = (j, k)$ is represented by the alternate edges, v_iw_k and v_tw_j in C which effectively realizes $\pi\psi$. Clearly, the edges in C representing ψ depend on π by the mapping $t^\pi = k$. □

The following Corollary of Theorem 3.1 generalizes the multiplier ψ to any permutation cycle not necessarily a transposition.

Corollary A.1. *Let $\pi \in S_n$ is realized by a bipartite graph BG' on $2n$ nodes. If $\psi \in S_n$ is a permutation cycle of length $r \leq n$, then $\pi\psi$ is realized by a bipartite graph BG' iff there exists a graph cycle of length $2r$ in BG' such that the alternate edges in the cycle are covered by π and $\pi\psi$.*

Proof. The result can be easily proved by an inductive application of the above Theorem. □

A.2 Proof of Corollary 3.2

Proof. Recall that BG_i is a subgraph of the complete bipartite graph $BG = K_{n,n}$ induced by the subgroup $G^{(i)}$. That is, $\forall j \in \{1, 2, \dots, i\}$, and for each $E(\pi)$ in BG_i , $j^\pi = j$. Following Theorem 3.1 we can identify the cycle responsible for realizing the multiplication $\pi\psi$, and see how ψ depends on $\pi \in G^{(i)}$.

$$\begin{aligned} \pi \text{ and } \pi\psi \text{ is realized by } BG_{i-1} &\iff i^{\pi\psi} = i^\psi = k \text{ and } t^{\pi\psi} = k^\psi = i \\ &\iff \text{edges } v_i w_k, v_t w_i \in BG_{i-1}, \text{ where } BG_0 = BG. \end{aligned}$$

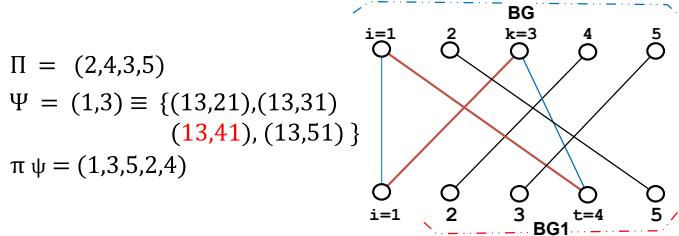


Figure 25: Multiplication by a Coset Representative $\psi = (1, 3)$

Clearly, the point k is fixed by ψ for a given i , and t is then fixed by π . Therefore, each (ψ, π) pair uniquely defines the edge pair $a_i(\psi, \pi) = (v_i w_k, v_t w_i)$. Also, it is easy to see that the only edge pair that can form a cycle of length four with the edge pair $(v_i w_i, v_t w_k)$ is $(v_i w_k, v_t w_i)$, giving the cycle $(v_i w_k v_t w_i)$.

□

Remark A.2. One should note the analogy of forming the product $\pi\psi$ with the augmenting path concept in constructing a perfect matching [Kuh55, Ege31]. The cycle (v_i, w_k, v_t, w_i) [Figure 25], which is used to multiply π and ψ , always contains the augmenting path (v_i, w_k, v_t, w_i) corresponding to the matched edge $v_t w_k$ in $E(\pi)$.

A.3 Permutation Multiplication Defined by an R-Cycle

The following Lemma shows how does an R -cycle compose a sequence of coset representatives. It is an extension of Corollary 3.2.

Lemma A.3. Let C_{ab} be an R -cycle, defining aRb , in a bipartite graph $K_{n,n}$, where $a \in g(i)$ and $b \in g(j)$, $1 \leq i < j \leq n$, and $x_{i_r} \in g(i_r)$, $1 \leq r \leq j-i$, are all the edge pairs covered by C_{ab} such that $i = i_1 < i_2 < \dots < i_{r-1} < i_r < i_{r+1} = j$. Also let $\pi(b) \in G^{(j-1)}$ be a permutation realized by the bipartite graph BG_{j-1} . Then C_{ab} represents a composition of the coset representatives leading to the permutation π_a given by

$$\pi_a = \psi(x_{i_r})\psi(x_{i_{r-1}}) \cdots \psi(x_{i_2})\psi(x_{i_1}), \text{ where } \psi(x_{i_r}) = \psi_{i_r} \in U_{i_r}, \quad (\text{A.1})$$

such that $\pi(b)\pi_a \in G^{(i-1)}$ covers a and other alternate edges in C_{ab} .

Proof. The proof is by induction on r , using the arguments in the proof of Corollary 3.2. □

The following Lemma provides a group theoretic semantics for the relation R . It correlates the permutation multiplication in $\text{UK}_{n,n}$ and the relation R in $\Gamma(n)$.

Lemma A.4. *Let $a \in g(i)$, $b \in g(j)$ be the edge pairs at the nodes i and j respectively in $BG = K_{n,n}$, such that $G^{(j)} < G^{(i)}$, $1 \leq i < j \leq n$. Let aRb be realized by the transitivity over the intermediate nodes such that $\forall k$, $j > k \geq i$, $\exists x_k \in g(k)$, $x_{k+1} \in g(k+1)$ and x_kRx_{k+1} . Then aRb represents a permutation*

$$\pi_a = \psi(x_{j-1})\psi(x_{j-2}) \cdots \psi(x_{i-1})\psi(x_i) \quad (\text{A.2})$$

where $\psi(x_r) = \psi_r \in U_r$, $i \leq r \leq j-1$, such that the product $\pi(b)\pi_a$ is realized by BG_{i-1} and that it covers a , b and other alternate edges of the R -cycle(s) defined by aRb .

Proof. The proof is essentially by induction on the number of R -cycles in the transitive chain aRb . When there is exactly one R -cycle defined by aRb , the result follows directly from the above Lemma A.3.

Whenever there are one or more ID nodes between i and j , we have two or more disjoint R -cycles such that each cycle represents a permutation given by Lemma A.3. □

A.4 More Properties of the Generating Graph

We now present few basic properties and attributes of the generating graph.

The R -in (out)degree of a node $x \in \Gamma$ is defined as the number of R -edges incident (going out) on (from) x . The S - in (out) degree of a node $x \in \Gamma$ is defined analogously.

Property A.5. *In every generating graph $\Gamma(n)$, $\forall i < n$ and $\forall x_i \in g(i)$, $\exists j \leq n$ and $x_j \in g(j)$ such that x_iRx_j . Similarly, the reverse result is also true— for all $x_j \in g(j)$ and $\forall i < j$ there exists $x_i \in g(i)$, such that x_iRx_j .*

Proof. The result is due to the completeness of the bipartite graph.

For all $x_i = (v_iw_k, v_jw_i) \in g(i)$, $1 \leq i < j, k \leq n$, there exist edges, v_jw_k and v_iw_i in BG , such that they form an R -cycle of length 4 with x_i covering the edge v_jw_k . Therefore, we will always have either x_iRx_j or x_iRx_k . □

Property A.6. *In every generating graph $\Gamma(n)$, $\forall (i, j)$, $1 \leq i < j \leq n$, $\exists x_i \in g(i)$ and $x_j \in g(j)$, such that x_iRx_j*

Proof. Simply note that the edges needed for forming a cycle of length four with x_i and one of the edges in x_j are always available in $K_{n,n}$. □

Property A.7. Let i and $j > i$ be any two node partitions in $\Gamma(n)$. Then $\forall x_i \in g(i)$, $x_i Rx_j \implies \exists y_j \in g(j)$ such that x_i and y_j are disjoint, and $x_i Rx_j$ is false. Similarly x_i and y_j being disjoint, and $x_i Rx_j$ being false implies $\exists y_j \in g(j)$ such that $x_i Ry_j$.

Proof. One should note that the condition for two edge pairs in $K_{n,n}$ being related by R is mutually exclusive to the condition for the corresponding nodes in $\Gamma(n)$ being disjoint. In one case, when $x_i Ry_j$ is true, the node pairs at j overlap with the vertex of one of the edges in the edge pair x_i in BG , and in the other case, $x_i Rx_j$ being false, j must be disjoint with the vertices at the node pairs covered by x_i . □

The following Property is essentially a complement of Property A.7.

Property A.8. For all (i, j) , $1 \leq i < j < n$, and $\forall x_i \in g(i)$, if $\exists x_k \in g(k)$, $n \geq k > j$, such that $x_i Rx_k$, then $\exists x_j \in g(j)$ such that x_i and x_j are disjoint.

Proof. An instance of this property can best be understood by looking at the layout of the edge pairs, x_i, x_j and x_k in $K_{n,n}$. The relation $x_i Rx_k$ directly implies that the edge pairs in all the partitions in $\{t \mid i < t < k\}$ have at least one edge pair x_t available such that a perfect matching can be formed. This must be true since we have a complete bipartite graph. And hence x_t must be disjoint to x_i (although not necessarily to x_k). □

Property A.9. All the R -edges coming from a given node in $\Gamma(n)$ go to the same node partition. Thus either all R -edges coming from a node are direct edges, or all are jump edges.

A.5 Permutation Represented by an R -Path

The following is a direct Corollary of Theorem A.4, noting that the product $\pi(b)\pi_a$ is realized by BG_{i-1} . It provides a group theoretic semantics to an R -path in $\Gamma(n)$.

Corollary A.10. Let $p = x_i x_{i+1} \cdots x_{j-1} x_j$, $1 \leq i < j \leq n$, be an R -path in $\Gamma(n)$ defined by $x_i Rx_j$, where $x_i \in g(i)$, and let $\psi(x_k)$ be the transposition defined by the edge-pair x_k . Then p defines a permutation cycle π_p given by the product of the transpositions

$$\pi_p = \psi(x_j)\psi(x_{j-1}) \cdots \psi(x_{i+1})\psi(x_i), \quad (\text{A.3})$$

such that π_p covers x_i, x_j and other alternate edges of the R -cycle(s) defined by $x_i Rx_j$.

The above Corollary A.10 effectively describes how larger permutation cycles are composed by the R -paths which eventually lead to a perfect matching whenever that R -path covers all the n node partitions in $\Gamma(n)$.

A.6 More VMP Properties

Property A.11. A VMP, $p = x_i x_{i+1} \cdots x_{t-1} x_t$ in $\Gamma(n)$, is a complete VMP if it satisfies any one of the following conditions:

1. p is an R -path with no jump edges.
2. The path $p = x_i p'$ obtained by incrementing a CVMP, $p' = x_{i+1} x_{i+2} \cdots x_t$, using a valid mdag, $mdag(x_i, x_{i+1}, x_t)$, $x_t \in p'$, or by an R -edge $x_i x_{i+1}$.
3. $p = p_1 p_2$, where p_1 and p_2 are CVMPs.

Proof. The proof of the above three properties is as follows.

1. p is an R -path: Obvious.

2. $p = x_i p'$ is a CVMP:

Clearly, the new path p is a VMP by virtue of the valid mdag, $mdag(x_i, x_{i+1}, x_t)$, and this mdag is covered by p .

3. $p = p_1 p_2$:

Simply note that the concatenation behavior of two or more CVMPs is exactly same as that of the R -edges— except that in CVMPs there may be two R -edges meeting at the starting node of p_2 .

□

A.7 The Permutation Represented by a CVMP

The following Lemma provides a group theoretic semantics of a CVMP, showing how a CVMP represents a product of coset representatives that would multiply any element of the associated subgroup. Further, it shows how that product is represented by a set of matched edges.

Let $E'(\pi)$ represent a subset of the matched edges in $E(\pi)$.

Lemma A.12. Every CVMP, $p = x_i x_{i+1} \cdots x_{j-1} x_j$ in $\Gamma(n)$, represents a permutation $\pi \in G^{(i-1)}$, and a matching $E'(\pi) \subseteq E(\pi)$ (on the nodes $i, i+1, \dots, j$ in $K_{n,n}$) given by

$$\pi = \psi(x_j)\psi(x_{j-1}) \cdots \psi(x_{i-1})\psi(x_i) \quad (\text{A.4})$$

where $1 \leq i < j \leq n$, and $x_i \in g(i)$.

Note. It is implicit that whenever $j < n$, $\exists x_k$ such that $x_j Rx_k$, where $j < k \leq n$. Therefore, by Theorem A.4, π would multiply all the permutations $\pi'(x_k) \in M(BG_{k-1})$, to give rise to $\pi'(x_k)\pi \in M(BG_{i-1})$.

Proof. The proof is by induction on the length, $l = |p|$ of the CVMP, p . For notational convenience we can assume each edge pair x_i to be a set of two edges.

Basis

For $l = 1$ the CVMP is an R -edge, $x_i x_{i+1}$, which represents the permutation, $\pi = \psi(x_{i+1})\psi(x_i)$ (Corollary A.10).

For $l = 2$ the CVMP is either an R -path of length 2, or an mdag, $mdag(x_i, x_{i+1}, x_{i+2})$, which represents $\pi = \psi(x_{i+2})\psi(x_{i+1})\psi(x_i)$.

Induction

Let (A.4) be true for all p , $2 \leq |p| \leq l < n - 1$, that is, we have a CVMP, p , of length $j - i$ that realizes the permutation π and a matching $E'(\pi)$. Let the new CVMP of length $j - i + 1$ be $x_{i-1}p$, $x_{i-1} \in g(i - 1)$, and let $x_t \in p$ be such that $x_{i-1}Rx_t$. It will suffice to show that the new CVMP realizes the permutation $\pi\psi(x_{i-1}) \in G^{(i-2)}$.

Note: We assume that x_{i-1} is not an ID node, i.e., $x_{i-1} \neq id_{i-1}$, otherwise the result would be trivially true.

Since the new CVMP p' of length $l + 1$ is derived from Property A.11(2), there is an mdag, $mdag(x_{i-1}, x_i, x_t)$, or an R -edge $x_{i-1}x_i$, such that $\psi(x_{i-1}) = (i - 1, k)$, and $k^\pi = t$. Therefore, by Corollary 3.2, the cycle defined by $x_{i-1}Rx_t$ realizes the product $\pi\psi(x_{i-1}) \in G^{(i-2)}$.

□

The Matching Represented by a CVMP

Lemma A.13. Every CVMP (m_i, m_j) , $p = x_i x_{i+1} \cdots x_{j-1} x_j$ in $\Gamma(n)$, represents a matching $E'(\pi) \subseteq E(\pi)$ (on the nodes $i, i + 1, \dots, j$ in $K_{n,n}$) given by

$$E'(\pi) = \{e \mid e \in x_i \in p\} - \{SE(x_s x_t) \mid x_s, x_t \in p\}, \quad (\text{A.5})$$

where $\pi \in G^{(i-1)} < S_n$, $1 \leq i < j \leq n$, and $x_i \in g(i)$.

Note. It is implicit that whenever $j < n$, $\exists x_k$ such that $x_j Rx_k$, where $j < k \leq n$. Therefore, by Theorem A.4, π would multiply all the permutations $\pi'(x_k) \in M(BG_{k-1})$, to give rise to $\pi'(x_k)\pi \in M(BG_{i-1})$.

Proof. The proof is by induction on the length, $l = |p|$ of the CVMP, p . For notational convenience we can assume each edge pair x_i to be a set of two edges.

Basis

For $l = 1$ the CVMP is an R -edge, $x_i x_{i+1}$, which represents the permutation, $\pi = \psi(x_{i+1})\psi(x_i)$, and a matching $E'(\pi) = x_i \cup x_{i+1} - \{SE(x_i x_{i+1})\}$. For $l = 2$ the CVMP is either an R -path of length 2, or an mdag, $mdag(x_i, x_{i+1}, x_{i+2})$, which represents $\pi = \psi(x_{i+2})\psi(x_{i+1})\psi(x_i)$. The matched edges can be deduced from the $SE(e)$ of associated R -edge e . That is, we have either

$$E'(\pi) = x_1 \cup x_2 \cup x_3 - \{SE(x_1 x_2), SE(x_2 x_3)\},$$

or

$$E'(\pi) = x_1 \cup x_2 \cup x_3 - \{SE(x_1 x_3), SE(x_2 x_3)\}.$$

Induction

Let (A.5) be true for all p , $2 \leq |p| \leq l < n - 1$, that is, we have a CVMP, p , of length $j - i$ that realizes the matching $E'(\pi)$. Let the new CVMP of length $j - i + 1$ be $x_{i-1}p$, $x_{i-1} \in g(i - 1)$, and let $x_t \in p$ be such that $x_{i-1}Rx_t$. It will suffice to show that the new CVMP realizes the permutation $\pi\psi(x_{i-1}) \in G^{(i-2)}$, and the new matching $E'(\pi\psi(x_{i-1})) = E'(\pi) \cup x_{i-1} - \{SE(x_{i-1}x_t)\}$.

Note: We assume that x_{i-1} is not an ID node, i.e., $x_{i-1} \neq id_{i-1}$, otherwise the result would be trivially true.

Since the new CVMP p' of length $l + 1$ is derived from Property A.11(2), there is an mdag, $mdag(x_{i-1}, x_i, x_t)$, or an R -edge $x_{i-1}x_i$, such that $\psi(x_{i-1}) = (i - 1, k)$, and $k^\pi = t$. Therefore, by Corollary 3.2, the cycle defined by $x_{i-1}Rx_t$ realizes the product $\pi\psi(x_{i-1}) \in G^{(i-2)}$.

The addition of the new node x_{i-1} to p adds the corresponding edge pair x_{i-1} in the bipartite graph to the matched edges. Moreover, the new R -edge $x_{i-1}x_t$ in p' will remove the edge $SE(x_{i-1}x_t)$ from the set $E'(\pi) \cup x_{i-1}$. Therefore,

$$E'(\pi\psi(x_{i-1})) = \{e \mid e \in x_i \in p'\} - \{SE(x_jx_k) \mid x_j, x_k \in p'\}.$$

□

A.8 A Partitioning Scheme for CVMP Sets

Claim A.14. For each $(\pi, \psi) \in G^{(i)} \times U_i$, there exists a unique pair $(p, x_i) \in prodVMPSet(m_{i+1}, m_t, m_{n-1}) \times g(i)$, where x_iRx_{t+1} , and $x_{t+1} \in m_t$, such that the product $\pi\psi$ is uniquely realized by $x_i \cdot p \in x_i \cdot prodVMPSet(m_{i+1}, m_t, m_{n-1})$.

Proof. The result follows from the existence of a unique R -edge incident from x_i to p whenever the associated permutation product $\pi\psi$ is realized by $x_i \cdot p$. [Corollary 3.2].

Note that there exists a mapping

$f : G^{(i)} \times U_i \rightarrow prodVMPSet(m_{i+1}, m_t, m_{n-1}) \times g(i)$, such that
 $\forall (\pi, \psi) \in G^{(i)} \times U_i$, the product $x_i \cdot p \in x_i \cdot prodVMPSet(m_{i+1}, m_t, m_{n-1})$ is realized by a unique R -edge such that x_iRx_{t+1} , where $x_{t+1} \in m_t$, $\psi(x_i) = \psi$ and $\pi(p) = \pi$.

□

Let $x_i \cdot prodVMPSet(m_{i+1}, m_t, m_{n-1})$ denote all the allowed multiplication of the CVMPs in $prodVMPSet(m_{i+1}, m_t, m_{n-1})$ by $x_i \in g(i)$, with the resulting CVMPs of length $n - i$ between the nodes x_i and x_n . Also let $m_i = mdag < x_i >$.

Lemma A.15. Given all the partitions, $\{prodVMPSet(m_{i+1}, m_t, m_{n-1})\}$, where $i \leq t \leq n - 1$, of $\{CVMPSet(m_{i+1}, m_{n-1})\}$, the next larger CVMP sets, can be computed as follows:

$$CVMPSet(m_i, m_{n-1}) = \biguplus_{m_t} \left\{ (x_i \cdot prodVMPSet(m_{i+1}, m_t, m_{n-1})) \mid (x_iRx_{t+1} \text{ and } x_iSx_{i+1}) \text{ or } x_iRx_{i+1}, x_i \in g(i), i \leq t \leq n - 1 \right\} \quad (A.6)$$

where (m_i, m_{n-1}) covers $g(i) \times g(n - 1)$, and (m_{i+1}, m_t, m_{n-1}) covers $g(i + 1) \times g(t) \times g(n - 1)$.

Proof.

Follows from Lemma 3.5 and Claim A.14.

□

Lemma A.16. All the $CVMPSet(m_i, m_{n-1})$, $1 \leq i \leq n-1$, in (A.6), can be constructed from the subsets, $prodVMPSet(m_{i+1}, m_t, m_{n-1})$, in polynomial time where each $CVMPSet(m_i, m_{n-1})$ uses only $O(n^6)$ $prodVMPSet(m_{i+1}, m_t, m_{n-1})$.

Proof. The proof on the size of these disjoint sets follows from (4.6), noting that

$$|\{prodVMPSet(m_{i+1}, m_t, m_{n-1})\}| = O(n^6) \text{ for each pair } (m_{i+1}, m_{n-1}).$$

The bound on the time follows from the polynomial bound on the "increment $CVMPSet()$ ".

□

A.9 A Characterization of Polynomial Time Enumeration

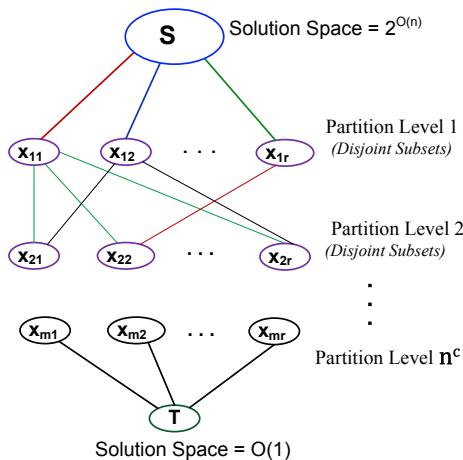


Figure 26: Partitions of the Solution Space of a Search Problem

A Sufficient Condition for P-time Enumeration.

Conjecture 1. An enumeration problem χ is in **FP** if there exists a hierarchy of the partitions of the solution space of χ such that

1. Each partition at level i in the hierarchy is a polynomially bounded partition of the solution space of a subproblem represented by with mutually disjoint subsets.
2. All the disjoint subsets at each level can be represented by a unique set of attributes– that is, the partitioning is not recursive but represented by the generators of polynomial size.
3. The solution space at each level in the hierarchy decreases by a factor c , $c > 1$.

A Necessary Condition for P-time Enumeration.

While the conditions listed under Conjecture 1 could be somewhat over restrictive, they must still hold true, in a perhaps more abstract form, in order to allow a P-time enumeration.

Example: Directed s – t Paths in an n-partite graph

Following is an example of an enumeration problem which meets the above characterization.

Consider an n -partite directed acyclic graph [Fig 27], $G = (V, E)$, where $V = Vs \cup V_1 \cup V_2 \dots \cup V_n \cup Vt$, and $E = \biguplus E_i$, where $E_i \subseteq V_i \times V_{i+1}$.

Further, let $x_i \in V_i = \{(i, 1), (i, 2), (i, 3), \dots, (i, r) \mid r \leq n^{O(1)}\}$.

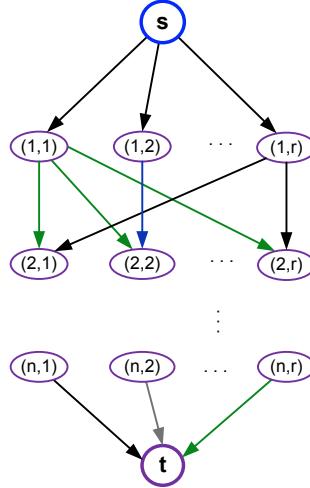


Figure 27: An n -partite Graph

Let $s \in Vs$, and $t \in Vt$.

Let $P(x_i)$ define the set of all paths between the node pair (x_i, t) in G , i.e.,

$$P(x_i) \stackrel{\text{def}}{=} \{x_i x_{i+1} \dots x_n \mid x_r \in V_r\}.$$

Since each path $p \in P(x_i)$ covers exactly one distinct node at each level i , $P(x_i)$ can be written as:

$$|P(x_i)| = \sum_{(x_i, x_{i+1}) \in E_i} |P(x_{i+1})|.$$

Note that all $P(x_i)$ are disjoint at any level i , and hence, $P(x_i)$ is an equivalence class.

The polynomial bound of $O(|V|^3)$, for enumerating $P(s)$ can certainly be achieved by a transitive closure of the n -partite graph, assuming the each edge $(x_i, x_{i+1}) \in E_i$ can be found in $O(1)$ time. In fact, an optimal bound of $O(|E|)$ can be determined by

$$T(P(x_i)) = O(|E_i|) + T(P(x_{i+1})).$$

Conjecture 2. *The solution space of every enumeration problem of size n is a subset of a universe which is a group isomorphic to a symmetric group of degree $n^{O(1)}$. This solution space is an equivalence class determined by the problem instance.*

A.10 The Equivalence Classes in the Partition Hierarchy for Perfect Matchings

Extending the original partitioning hierarchy means we have additional equivalence classes implied by the disjoint partitions at each partition level. These additional classes are described below.

The Class CVMPSet

Consider the following relation \equiv over the set, $\{CVMPSet(m_1, m_{n-1})\}$:

For each $p, q \in \{CVMPSet(m_1, m_{n-1})\}$,

$p \equiv q \iff \exists m_1 = mdag < x_1 >$ and $(p', q') \in \{CVMPSet(m_2, m_{n-1})\}$,

such that

$$m_1 \cdot p' = p \text{ and } m_1 \cdot q' = q.$$

Then the relation \equiv is an equivalence relation giving $CVMPSet(m_1, m_{n-1})$ as an equivalence class.

Other equivalence classes are:

- $prodVMPSet(m_i, m_t, m_{n-1})$ induced by the mdag pair (m_i, m_t) .
- the subset of MinSet sequences, $CMS_{in}(r)$ in $CVMPSet(m_i, m_{n-1})$, induced by the MinSet, $MinSet(m_i, m_t)$ such that $ER(x_{t+1}) \neq \emptyset$, where $x_{t+1} \in m_t$.