
Polynomial complexity algorithm for Max-Cut problem

Mikhail Katkov

Howard Hughes Medical Institute;
Center for Neuroscience
New York University
New York, NY 10003
mikhail.katkov@gmail.com

Abstract

The standard NP-complete max-cut problem reduced to a binary quadratic program $\min x^T Q x$ s.t. $x^2 = 1$ is studied here. Without loss of generality matrix Q can be assumed positive semidefinite. It is shown that for some α a global minimum of sum of squares of polynomials $Q(\alpha, x) = \alpha x^T Q x + \sum_i (x_i^2 - 1)^2$, provides a solution to max-cut problem. In particular, (1) the global minimum of $Q(\alpha, x)$ can be found with any precision by polynomial complexity semi-definite program; (2) the sign of minimal point x^* is preserved; (3) for sufficiently small α the global minimum of $Q(\alpha, x)$ is the global minimum of binary quadratic program. The certificate of solution can be found from quadratic form representation of $Q(\alpha, x)$. The resulting algorithm solves arbitrary max-cut problem in polynomial time, therefore P=NP.

In memory of my Grandmother

1 Formulation of the problem

The standard NP-complete formulation of the maximum cut problem is considered here [Pol95]. In brief, let $G = (V, E)$ be finite, undirected, and loopless weighted graph with vertex set V and edge set $E = \{(i, j) : i, j \in V, i \neq j\}$ with weight function $w : E \rightarrow \mathbb{R}$. For any subset $S \subset V$ of vertexes, the set of edges with one end in S and the other end in $V \setminus S$ is called the *cut* ($\delta_G(S) := \{(i, j) | i \in S, j \notin S\}$) induced by S . The central problem of this paper is MAX-CUT PROBLEM: given a weighted graph (G, w) find

$$\max_{S \subset V} w(\delta_G(S)),$$

where

$$w(\delta_G(S)) = \sum_{i \in S, j \notin S} w_{i,j}.$$

2 Reduction to binary quadratic problem

Consider vector $\vec{x} = \{x_i | x_i \in \{-1, +1\}\} \in \mathbb{R}^n$ defined as follows.

$$\vec{x}(S) = \begin{cases} +1 & \text{for } i \in S \\ -1 & \text{for } i \notin S \end{cases},$$

and set $w_{i,j} = 0 \forall (i,j) \notin E$, then

$$\begin{aligned} w(\delta_G(S)) &= \frac{1}{8} \sum_{i=1}^n \sum_{j=1}^n w_{i,j} (x_i - x_j)^2 \\ &= \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n w_{i,j} - \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n w_{i,j} x_i x_j \\ &= W - \vec{x}^T Q \vec{x}, \end{aligned} \tag{1}$$

where

$$W = \sum_{i=1}^n \sum_{j=1}^n w_{i,j}$$

is a constant defined by the graph structure, Clearly, MAX-CUT PROBLEM is equivalent to the following quadratic problem with quadratic constraints

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & x_i^2 = 1. \end{aligned} \tag{2}$$

Lemma 2.1. *Let $x \in \mathbb{R}^n, c, d, f \in \mathbb{R}, Q \in \mathbb{R}^{n \times n}$ is symmetric matrix. The following problems have the same solutions x^* .*

(1)

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & x_i^2 = 1. \end{aligned}$$

(2)

$$\begin{aligned} \min \quad & x^T Q x + c \\ \text{s.t.} \quad & x_i^2 = 1. \end{aligned}$$

(3)

$$\begin{aligned} \min \quad & x^T Q x + dx^T x \\ \text{s.t.} \quad & x_i^2 = 1. \end{aligned}$$

(4)

$$\begin{aligned} \min \quad & (x^T Q x) f \\ \text{s.t.} \quad & x_i^2 = 1, \\ & f > 0 \end{aligned}$$

Proof. (1) \iff (2) \iff (4) obvious. (2) \iff (3) in the feasible set $x^T x = \sum_i x_i^2 \equiv n$ therefore for $c = dn$ (2) and (3) equivalent. \square

3 Sums of squares

3.1 Summary of [Par03]

Let bounded below polynomial $f(x_1, x_2, \dots, x_n)$ has even degree $2d$, $f^* = \min f(x), x \in \mathbb{R}^n$, X denote a column vector of all monomials in x_1, x_2, \dots, x_n of degree at most d . The length of the vector X equals a binomial coefficient

$$N = \binom{n+d}{d}.$$

Let L_f denote set of real symmetric matrices A such that $f(x) = X^T A X$. Assume that the constant monomial 1 is the first entry of X . Let E_{11} denote the matrix unit whose only nonzero entry is a one in the upper left corner.

Lemma 3.1 (Lemma 3.1. in [Par03]). *For any real number λ , the following two are equivalent:*

- *The polynomial $f(x) - \lambda$ is a sum of squares (sos) in $\mathbb{R}[x]$.*
- *There is a matrix $A \in L_f$ such that $A - \lambda E_{11}$ is positive semi-definite, that is, all eigenvalues of this symmetric matrix are non-negative reals.*

We write f^{sos} for the largest real number λ for which the two equivalent conditions are satisfied. We always have $f^* \geq f^{sos}$ [Par03, Sho87].

Theorem 3.2 (Theorem 3.2. in [Par03]). *Fix $\deg(f) = 2d$ and let the number of variables n vary. Then there exists a polynomial-time algorithm, based on semi-definite programming, for computing f^{sos} from f . The same statement holds if n is fixed and d varies.*

3.2 f^{sos} for sum of square polynomial

Lemma 3.3. *Let $f(x) = \sum_i p_i(x)^2$ be a sum of squares of some polynomials $p_i(x)$, $\lambda \in \mathbb{R}$. Then*

1. *If $f(x) - \lambda$ is a sum of squares then for any $\lambda' < \lambda$, $f(x) - \lambda'$ is a sum of squares.*
2. *$f^* = f^{sos}$.*

Proof. 1. Suppose $f(x) - \lambda$ is a sum of squares, then $f(x) - \lambda' = f(x) - \lambda + (\sqrt{\lambda - \lambda'})^2$ is a sum of squares.

2. Suppose $f^{sos} < f^*$ then for $\lambda = \frac{f^* + f^{sos}}{2} > 0$, $f(x) - f^{sos} + \lambda$ is sum of squares i.e. non-negative, but $\min f(x) - \lambda < 0$ - contradiction.

□

Corollary 3.4. $\min f(x) - f^{sos} = 0$, $A - f^{sos} E_{11}$ is singular, and $x^* | (f(x^*) = f^*) \in \ker(A - f^{sos} E_{11})$.

4 Approximation by quartic polynomial

Lemma 4.1. *Let $x \in \mathbb{R}^n, z \geq 0, z \in \mathbb{R}, \alpha \in \mathbb{R}, Q'$ positive semi-definite symmetric matrix*

$$Q(x) = \sum_i (x_i^2 - 1)^2 + \alpha x^T Q' x. \quad (3)$$

The following problem can be solved in polynomial time and space for any $\alpha > 0$.

$$\min Q(x) \quad (4)$$

Proof.

- (i) Since Q' is positive semi-definite by Cholesky decomposition $Q' = LL^T$, then polynomial $Q(x)$ is a sum of squares of polynomials $Q(x) = \sum_i (x_i^2 - 1)^2 + \alpha (L^T x)^T (L^T x)$, therefore by Theorem 3.2 f^{sos} can be found in polynomial time.

- (ii) By Lemma 3.3 $f^{sos} = \min Q(x)$.

□

Theorem 4.2. *Let Q' be positive semi-definite symmetric matrix, $Q(x)$ is defined as in (3), $X(\alpha) = \{x_k(\alpha) \in \mathbb{R}^n | x_k \text{ are local minima of } Q(x)\}$, then $\exists \alpha^* > 0$ that for all $0 \leq \alpha < \alpha^*$ signum $x_k(\alpha) = \text{signum } x_k(0)$, $\forall x_k(0) \in X(0)$.*

Proof.

1. Extrema of $Q(x)$ are defined by

$$\nabla_x \sum_i (x_i^2 - 1)^2 + \alpha x^T Q' x = 2\alpha Q' x + 4 \begin{bmatrix} (x_1^2 - 1)x_1 \\ \dots \\ (x_n^2 - 1)x_n \end{bmatrix} = \vec{0}, \quad (5)$$

2. Let $\alpha = 0$. Then, by direct inspection

1. $x_i = 0, \forall i$ is a local maximum.
2. $x_i^2 = 1, \forall i$ are local minima.
3. $x_i^2 = 1$, for some i , and $x_i = 0$ for the rest are saddle points.

3. Let $\vec{v} = Qx$. Then, $\vec{v}^T \vec{v} = (x^T Q Q x) \leq \lambda_{\max}^2 (x^T x), \forall x$, where λ_{\max} is the maximum eigenvalue of Q . Therefore, $\|\vec{v}\| \leq \lambda_{\max} \|x\|$. Therefore, $|v_i| \leq \lambda_{\max} \|x\|, \forall i$.

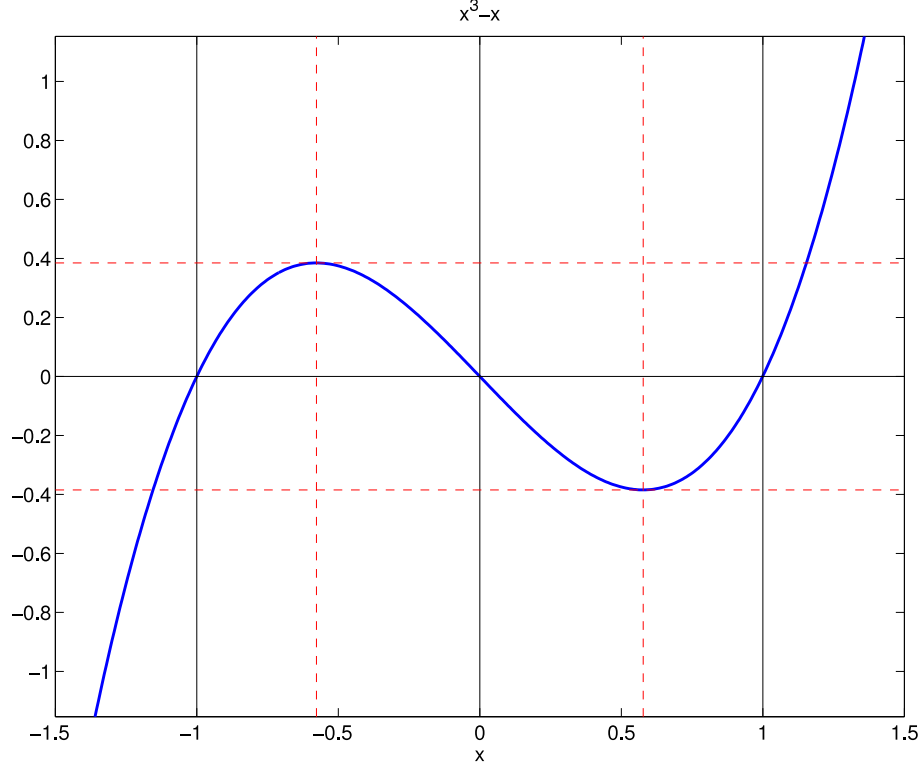


Figure 1: Graph of $x^3 - x$

There are 2 alternatives:

1.

$$\begin{aligned} x_i(0) = 1, \quad v_i < 0 & \text{ then } x_i(\alpha) > 1 \\ x_i(0) = -1, \quad v_i > 0 & \text{ then } x_i(\alpha) < -1 \end{aligned} \quad (6)$$

2.

$$\begin{aligned} x_i(0) = 1, \quad v_i > 0 & \text{ then } x_i(\alpha) < 1 \\ x_i(0) = -1, \quad v_i < 0 & \text{ then } x_i(\alpha) > -1 \end{aligned} \quad (7)$$

Suppose $x_i(0) = 1$, and if $\alpha|v_i| \leq \alpha\lambda_{\max} \|x(\alpha)\| \leq \alpha\lambda_{\max} \|x(0)\| \leq \alpha\sqrt{n}\lambda_{\max} < 2|p^*|$ then $x_i(\alpha) > \frac{1}{\sqrt{3}}$, i.e. the signum $x_i(\alpha) = \text{signum } x_i(0)$. The same argument holds for $x_i(0) = -1$.

Therefore,

$$\alpha^* = \frac{2|p^*|}{\sqrt{n}\lambda_{\max}}$$

□

4.1 Local minima of $Q(x)$

From (5) it follows that at extremal points \tilde{x}

$$\alpha \tilde{x}^T Q \tilde{x} = -2 \sum_i (\tilde{x}_i^2 - 1) \tilde{x}_i^2, \quad (8)$$

$$Q(\tilde{x}) = \sum_i (\tilde{x}_i^2 - 1)^2 - 2 \sum_i (\tilde{x}_i^2 - 1) \tilde{x}_i^2 \quad (9)$$

$$= \sum_i (\tilde{x}_i^2 - 1)(\tilde{x}_i^2 - 1) - 2(\tilde{x}_i^2 - 1)\tilde{x}_i^2 \quad (10)$$

$$= \sum_i (1 - \tilde{x}_i^2)(1 + \tilde{x}_i^2) \quad (11)$$

$$= \sum_i (1 - \tilde{x}_i^4) = n - \sum_i \tilde{x}_i^4 \quad (12)$$

Let $x_0^2 = 1$ one of the feasible point of problem (2), $\tilde{x} = x_0 - y$. Then,

$$Q(y) = n - \sum_i (x_{0,i} - y_i)^4 = n - \sum_i (x_{0,i}^4 - 4x_{0,i}^3 y_i + 6x_{0,i}^2 y_i^2 - 4x_{0,i} y_i^3 + y_i^4), \quad (13)$$

$$Q(y) = n - \sum_i (1 - 4x_{0,i} y_i + 6y_i^2 - 4x_{0,i} y_i^3 + y_i^4) \quad (14)$$

$$Q(y) = 4x_0^T y - 6y^T y + \sum_i (x_{0,i} y_i^3 - y_i^4). \quad (15)$$

From (5)

$$\begin{aligned} \alpha Q x + 2 \begin{bmatrix} (x_1^2 - 1)x_1 \\ \dots \\ (x_n^2 - 1)x_n \end{bmatrix} &= \alpha Q x - 2x + 2 \begin{bmatrix} x_1^3 \\ \dots \\ x_n^3 \end{bmatrix} \\ &= (\alpha Q - 2I) x + 2 \begin{bmatrix} x_1^3 \\ \dots \\ x_n^3 \end{bmatrix} = (\alpha Q - 2I) (x_0 - y) + 2 \begin{bmatrix} (x_{0,1} - y_1)^3 \\ \dots \\ (x_{0,n} - y_n)^3 \end{bmatrix} = \\ &(\alpha Q - 2I) x_0 - (\alpha Q - 2I) y + 2 \begin{bmatrix} x_{0,1}^3 - 3x_{0,1}^2 y_1 + 3x_{0,1} y_1^2 - y_1^3 \\ \dots \\ x_{0,n}^3 - 3x_{0,n}^2 y_n + 3x_{0,n} y_n^2 - y_n^3 \end{bmatrix} = \\ &(\alpha Q - 2I) x_0 - (\alpha Q - 2I) y + 2 \begin{bmatrix} x_{0,1} - 3y_1 + 3x_{0,1} y_1^2 - y_1^3 \\ \dots \\ x_{0,n} - 3y_n + 3x_{0,n} y_n^2 - y_n^3 \end{bmatrix} = \\ &\alpha Q x_0 - (\alpha Q + 4I) y + 2 \begin{bmatrix} 3x_{0,1} y_1^2 - y_1^3 \\ \dots \\ 3x_{0,n} y_n^2 - y_n^3 \end{bmatrix} = 0, \\ &y = (\alpha Q + 4I)^{-1} \left(\alpha Q x_0 + 2 \begin{bmatrix} 3x_{0,1} y_1^2 - y_1^3 \\ \dots \\ 3x_{0,n} y_n^2 - y_n^3 \end{bmatrix} \right). \quad (16) \end{aligned}$$

For $\alpha < \alpha^*$, (16) can be expanded in a Neumann series, and treated as a fix point equation. Since the right part of (16) is contracting operator for $\|y_n\| < 1$, one can apply fix point iterations, starting from $y_0 = 0$.

$$y = \frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{\alpha}{4} Q \right)^k \left(\alpha Q x_0 + 2 \begin{bmatrix} 3x_{0,1}y_1^2 - y_1^3 \\ \dots \\ 3x_{0,n}y_n^2 - y_n^3 \end{bmatrix} \right); \quad (17)$$

$$y_0 = 0; \quad (18)$$

$$y_1 = \sum_{k=1}^{\infty} \left(\frac{\alpha}{4} Q \right)^k x_0 = \frac{\alpha}{4} Q x_0 + \frac{\alpha^2}{16} Q^2 x_0 + o(\alpha^2) \quad (19)$$

$$y_2 = \sum_{k=1}^{\infty} \left(\frac{\alpha}{4} Q \right)^k x_0 + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\alpha}{4} Q \right)^k \left[3x_{0,i} \left[\sum_{k=1}^{\infty} \left(\frac{\alpha}{4} Q \right)^k x_0 \right]_i^2 - \left[\sum_{k=1}^{\infty} \left(\frac{\alpha}{4} Q \right)^k x_0 \right]_i^3 \right] \quad (20)$$

...

Keeping only quadratic terms in α

$$y = \frac{\alpha}{4} Q x_0 + \frac{\alpha^2}{16} Q^2 x_0 + \frac{1}{2} \left[3x_{0,i} \left[\frac{\alpha}{4} Q x_0 \right]_i^2 \right] + o(\alpha^2) \quad (21)$$

$$= \frac{\alpha}{4} Q x_0 + \frac{\alpha^2}{16} Q^2 x_0 + \frac{3\alpha^2}{32} [x_{0,i} [Q x_0]_i^2] + o(\alpha^2) \quad (22)$$

Substituting it back to (15), and keeping quadratic terms in α leads to

$$\begin{aligned} Q(\alpha; x_0) &= \alpha x_0^T Q x_0 + \frac{\alpha^2}{4} x_0^T Q^2 x_0 + \frac{3\alpha^2}{8} \sum_i x_{0,i}^2 [Q x_0]_i^2 - \frac{3\alpha^2}{8} x_0^T Q^2 x_0 + o(\alpha^2) \\ \frac{Q(\alpha; x_0)}{\alpha} &= x_0^T Q x_0 - \frac{\alpha}{8} x_0^T Q^2 x_0 + \frac{3\alpha}{8} \sum_i x_{0,i}^2 [Q x_0]_i^2 + o(\alpha). \end{aligned} \quad (23)$$

The minimum differences between cuts Δ is not greater than minimal nonzero difference in elements of matrix Q' . The minimum and maximum values of $x_0^T Q^2 x_0$ are $n\lambda_{min}^2$ and $n\lambda_{max}^2$. The same is for the term $\sum_i x_{0,i}^2 [Q x_0]_i^2$ (for eigenvectors aligned with x_0). Therefore, unique solution is obtained when

$$\Delta > \alpha n \left(\frac{\lambda_{max}^2}{2} - \frac{\lambda_{min}^2}{4} \right) + o(\alpha). \quad (24)$$

5 Certificate of Solution

Let $Y = \{1, \{x_i, i = 1..n\}, \{x_i x_j, i = 1..n, j = 1..n\}, z\}$ be a set of monomials. The polynomial $Q(x, z)$ can be represented as a quadratic form $Q(x, z) = \sum_{k,m} q_{k,m} y_k y_m$. This representation is not

unique. For instance, the coefficient in front of x_1^2 is $q_{(x_1), (x_1)} + 2 * q_{(x_1 x_1), 1} = \text{const}$. Therefore, for some constant t polynomial $Q(x, z) - t$, is non-negative, when matrix $q_{k,m}$ is positive semi-definite, and set of linear constraints on $q_{k,m}$ is satisfied [Par00]. The maximum value of t for which $Q(x, z) - t = 0$, and the set of constraints are satisfied provides the global minimum of the polynomial [Sho87], and the final matrix $[q_{k,m}]$ is singular. The structure of singular vectors provides the solution of the problem (2).

The global minima comes in pairs.. For instance if x^* is a global minimum $x' = -x^*$ is also a global minimum. On the other hand, $z(x') = z(x^*)$, and $x'_i x'_j = x_i^* x_j^*$. Therefore, the semi-definite problem has block-diagonal structure [Lof09]. In particular, if there is only one (up to a symmetry around origin) global optimum, than the final matrix $q_{k,m}$ have two zero eigenvalues, and two corresponding eigenvectors. One of them corresponds to monomials x_i , and another one to monomials $1, x_i x_j, z$. The vector composed of signum of the components of singular eigenvector corresponding to monomials x_i is a solution to problem (2). If there are more than one global minimum (more than one possible cut) there is a permutation symmetry, that lead to further split of the block corresponding to monomials x_i onto sub-blocks corresponding permutation symmetry and the rest of monomials, and the number of the singular vectors increases.

6 Complexity

The number of monomials in representation of $Q(x, z)$ is $n^2 + n + 2$. The number of variables in semi-definite program is limited by the size of matrix $[q_{m,k}] - (n^2 + n + 2)^2$. The interior-point algorithm in semi-definite program that guarantee worst case polynomial performance is second-order and require to compute Hessian with respect to all variables in $[q_{m,k}]$, therefore requires space is $(n^2 + n + 2)^4$, or $O(n^8)$. To compute constraints on values $q_{m,k}$ one need to compute Kronecker product $Q' \circ Q'$, i.e. $O(n^4)$ operations. In addition the polynomial complexity in time is required to solve semi-definite program, which depends on the particular algorithm, for details see [Boy04]. Overall, the complexity of the algorithm is polynomial.

Acknowledgments

The author would like to thank Eero P. Simoncelli for hosting during the work, and asking difficult questions.

References

- [Pol95] S. Poljak and Z. Tuza (1995). The Max-Cut problem: a survey. In: W. Cook, L. Lovasz and P. Seymour (Eds.), Special Year on Combinatorial Optimization, DIMACS Series in Discrete Mathematics and Computer Science. American Mathematical Society, Providence, Rhode Island.
- [Sho87] N. Z. Shor (1987) Class of global minimum bounds of polynomial functions. *Cybernetics and Systems Analysis* 23(6) pp. 731-734
- [Par00] P.A. Parrilo (2000) Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization. *Thesis* California Institute of Technology, Pasadena, California
- [Van96] L. Vandenberghe and S. Boyd (1996) Semidefinite Programming *SIAM Review* 38(1) pp. 49-95
- [Lof09] J. Löfberg. (2009) Pre- and post-processing sum-of-squares programs in practice. *IEEE Transactions on Automatic Control*, 54(5) pp. 1007-1011
- [Boy04] S. Boyd and L. Vandenberghe (2004) *Convex Optimization* Cambridge University Press
- [Par03] Pablo A. Parrilo, Bernd Sturmfels(2003) Algorithmic and quantitative real algebraic geometry, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, Vol. 60, pp. 83–99, AMS, 2003. ISBN: 0-8218-2863-0.