

# Understanding SAT is in P

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**Abstract.** We introduce the idea of an understanding with respect to a set of clauses as a satisfying truth assignment explained by the contexts of the literals in the clauses. Following this idea, we present a mechanical process that obtains, if it exists, an understanding with respect to a 3SAT problem instance based on the contexts of each literal in the instance, otherwise it determines that none exists. We demonstrate that our process is correct and efficient in solving 3SAT.

Satisfiability (SAT, for short) is regarded as one of the most fundamental computational problems. In the 1970's, when the class NP of problems was first defined [1, 2, 3], both SAT and its special case 3SAT were among the first problems shown to be NP-complete in [1]. This highlighted the importance of 3SAT and SAT since, by the definition of NP-completeness, if a polynomial-time algorithm exists for 3SAT or SAT then all NP-complete problems, and indeed all problems in NP, can be solved efficiently. To date, however, no polynomial-time algorithm has been found for any of the NP-complete problems, which among other reasons has led to the widely accepted belief that no such algorithm exists [4, 5].

Although no polynomial-time algorithm has been found for SAT or 3SAT, remarkable improvements in terms of efficiency have been achieved throughout the years. This has been accentuated recently (starting in the 1990's) in the form of so called SAT solvers, which are practical procedures for SAT able to handle large instances considerably fast. Modern SAT solvers can be divided into two groups: *a*) complete solvers, mainly based on the backtracking search procedure of Davis-Putnam-Logemann-Loveland (DPLL) algorithm [6, 7] and Conflict-Driven Clause Learning (CDCL) algorithm [8], which are meant to always provide the correct solution given enough time; and *b*) incomplete solvers, mostly based on stochastic local search [9], which at the expense of statistically minimal errors seek to produce a quick answer.

As mentioned above, in spite of great advances, until now no algorithm has been proposed to solve SAT or 3SAT in polynomial time. In this paper we propose an algorithm that achieves this for 3SAT. We introduce the idea of an understanding with respect to a set of clauses as a satisfying truth assignment explained by the contexts of the literals in the clauses, where the key point is the use of contexts which allow to construct the assignment without searching (locally or systematically) the space of potential solutions. Following this idea, our algorithm obtains, if it exists, an understanding with respect to

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a 3SAT problem instance based on the contexts of each literal in the instance, otherwise it determines that none exists.

The outline of the paper is as follows. In § 0 we recall definitions on 3SAT. In § 1 we present the idea introduced in this paper, including definitions, lemmas, and algorithms that lead to our main algorithm. Finally, in § 2 we present the analysis of our main algorithm in terms of correctness and asymptotic time complexity.

## 0. Preliminaries.

We recall some definitions from [10].

Let  $X = \{x_1, x_2, \dots, x_m\}$  be a set of Boolean variables. A *truth assignment* for  $X$  is a function  $\alpha: X \rightarrow \{0, 1\}$ . If  $\alpha(x) = 1$  we say that  $x$  is “true” under  $\alpha$ ; if  $\alpha(x) = 0$  we say that  $x$  is “false”. If  $x$  is a variable in  $X$ , then  $x$  and  $\bar{x}$  are *literals* over  $X$ . We say that  $\bar{x}$  is the *negation* of  $x$  and  $x$  is the negation of  $\bar{x}$ . The literal  $x$  is true under  $\alpha$  if and only if the variable  $x$  is true under  $\alpha$ ; the literal  $\bar{x}$  is true if and only if the variable  $x$  is false.

A *clause* over  $X$  is a set of literals over  $X$ , such as  $\{x_1, \bar{x}_3, x_8\}$ . It represents the disjunctions of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. The clause above will be satisfied by  $\alpha$  unless  $\alpha(x_1) = 0$ ,  $\alpha(x_3) = 1$ , and  $\alpha(x_8) = 0$ . A collection  $\Phi$  of clauses over  $X$  is *satisfiable* if and only if there exists some truth assignment for  $X$  that simultaneously satisfies all clauses in  $\Phi$ . Such a truth assignment is called a *satisfying truth assignment* for  $\Phi$ .

The *3-satisfiability* (3SAT) problem is specified as follows:

Given a collection  $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_m\}$  of clauses on a finite set  $X$  of variables such that  $|\varphi_i| = 3$  for  $1 \leq i \leq m$ . Is there a truth assignment for  $X$  that satisfies all clauses in  $\Phi$ ?

**1. Idea.** Let  $\Phi$  be an instance of the 3SAT problem on a finite set  $X$  of variables, and let  $L$  be the set of all literals over  $X$ . An *understanding* for  $L$  is a function  $\tilde{u}: L \rightarrow \{t, f, \varepsilon\}$ . For any literal  $\lambda$  in  $L$ , if  $\tilde{u}(\lambda) = t$  we say that  $\lambda$  is “true” under  $\tilde{u}$ ; if  $\tilde{u}(\lambda) = f$  we say that  $\lambda$  is “false”; and if  $\tilde{u}(\lambda) = \varepsilon$  we say that  $\lambda$  is “free”.

The literal  $x$  is true and its negation  $\bar{x}$  is false under  $\tilde{u}$  if and only if the variable  $x$  is true under  $\alpha$ ; the literal  $\bar{x}$  is true and its negation  $x$  is false under  $\tilde{u}$  if and only if the variable  $x$  is false under  $\alpha$ ; the literal  $x$  is free and its negation  $\bar{x}$  is free under  $\tilde{u}$  if and only if the variable  $x$  is unassigned.

Let  $\varphi: \{l_1, l_2, l_3\}$  be a clause in  $\Phi$ , where  $l_1$ ,  $l_2$ , and  $l_3$  are all distinct literals. We assume that  $\varphi$  is satisfied. If we focus on one of the literals in  $\varphi$ , say  $l_1$ , we say that the *context* of  $l_1$  in  $\varphi$  is the set of literals that appear in  $\varphi$  that are different from  $l_1$ , i.e.,  $\{l_2, l_3\}$ . We call *concept* to a context in which its literals are interpreted according to a particular understanding. Thus, the concept of  $l_1$  in  $\varphi$ , interpreted according to some understanding  $\tilde{u}$ , is denoted as  $\mathcal{C}: \{\tilde{u}(l_2), \tilde{u}(l_3)\}$ .

Based on the elements in the codomain of a function of understanding, the possible combinations of elements, under some understanding  $\tilde{u}$ , in some concept  $\mathcal{C}$  are:

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|---------------------------|---|
| i) both literals free;    | iv) one literal free and the other true;  |
| ii) both literals true;   | v) one literal true and the other false;  |
| iii) both literals false; | vi) one literal free and the other false. |

We say that a concept as in (i), (iii), or (vi) is of type  $\mathcal{C}^+$  and a concept as in (ii), (iv), or (v) is of type  $\mathcal{C}^*$ .

Let  $\phi$  be a set of clauses, subset of  $\Phi$ , that are assumed to be satisfied. Let  $\tilde{\mathcal{C}}$  be the set of all concepts of literals in clauses of  $\phi$  interpreted according to an understanding  $\tilde{u}$ . Let  $\lambda$  be any literal that appears in one or more clauses in  $\phi$  and let  $\neg\lambda$  be its negation. Further, let  $\tilde{\mathcal{C}}[\lambda]$  be the set of concepts of  $\lambda$  in  $\phi$  and let  $\tilde{\mathcal{C}}[\lambda]^-$  be the set of concepts of type  $\mathcal{C}^+$  in  $\tilde{\mathcal{C}}[\neg\lambda]$ . It should be clear that  $\tilde{\mathcal{C}}[\lambda]$  is a subset of  $\tilde{\mathcal{C}}$  and  $\tilde{\mathcal{C}}[\lambda]^-$  is a subset of  $\tilde{\mathcal{C}}[\neg\lambda]$ . We say that a set of concepts is of type  $\tilde{\mathcal{C}}^*$  if all its elements are of type  $\mathcal{C}^*$ ; and a set of concepts is of type  $\tilde{\mathcal{C}}^+$  if at least one of its elements is of type  $\mathcal{C}^+$ .

We define the understanding  $\tilde{u}$  of  $\lambda$  with respect to the set  $\phi$  as follows:

$$\tilde{u}(\lambda) = \begin{cases} \varepsilon, & \text{if } \tilde{\mathcal{C}}[\lambda] \text{ is empty or } (\tilde{\mathcal{C}}[\lambda]^- \text{ is empty and } \tilde{\mathcal{C}}[\lambda] \text{ is of type } \tilde{\mathcal{C}}^*); \\ t, & \text{if } \tilde{\mathcal{C}}[\lambda] \text{ is of type } \tilde{\mathcal{C}}^+ \text{ and } \tilde{\mathcal{C}}[\lambda]^- \text{ is empty}; \\ f, & \text{if } \tilde{\mathcal{C}}[\lambda]^- \text{ is not empty and } \tilde{\mathcal{C}}[\lambda] \text{ is not of type } \tilde{\mathcal{C}}^+. \end{cases}$$

It should be clear that the definition of understanding above leaves one possible case out of consideration, that is, if  $\tilde{\mathcal{C}}[\lambda]$  is of type  $\tilde{\mathcal{C}}^+$  and  $\tilde{\mathcal{C}}[\lambda]^-$  is not empty. This is because in such a case, the understanding  $\tilde{u}$  is considered *undefined*.

**Lemma A.** An understanding  $\tilde{u}$  is defined with respect to a set  $\phi$  of clauses if and only if  $\tilde{u}$  is equivalent to a satisfying truth assignment for  $\phi$ .

*Proof.* We assume first that the understanding  $\tilde{u}$  is equivalent to a satisfying truth assignment  $\alpha$  for  $\phi$ . It follows that for each clause in  $\phi$  there is at least one literal that is true under  $\alpha$ . Then, based on the equivalence between truth assignments and understandings, defined at the beginning of this section, we have that for each clause in  $\phi$  there is at least one literal that is true under  $\tilde{u}$  as well. Therefore  $\tilde{u}$  is defined for at least one literal of each of the clauses in  $\phi$ , and since it is defined to be true, the rest of literals of each clause must be defined to be free under  $\tilde{u}$ . Thus, we have that  $\tilde{u}$  is defined for all  $\phi$ .

For the converse, we show the contrapositive. Assume we have a truth assignment for all variables in  $\phi$  which is not a satisfying truth assignment for  $\phi$ , call it  $\bar{\alpha}$ . This means that in at least one clause of  $\phi$  all literals are assigned to false under  $\bar{\alpha}$ . Let  $\phi'$  be the set of clauses, subset of  $\phi$ , satisfied by  $\bar{\alpha}$  and let  $\varphi$  be a clause in  $\phi$  that is not satisfied by  $\bar{\alpha}$  (i.e.,  $\varphi \notin \phi'$ ). From our assumption it follows that  $\tilde{u}$  is defined with respect to  $\phi'$  and

all literals in  $\varphi$  are false under  $\tilde{u}$ . This implies that  $\tilde{\mathcal{C}}[\lambda]^-$  is not empty for any literal  $\lambda$  in  $\varphi$ . Furthermore, the concepts that can be defined of literals from  $\varphi$  are all of type  $\mathcal{C}^+$ . Therefore,  $\tilde{u}$  is not defined for  $\phi$ .  $\square$

Following Lemma A, for any set  $\phi$  of clauses it might be the case that one or more understandings can be defined, or that no understanding can be defined at all. We say that two understandings are *equivalent* if both are defined with respect to the same set of clauses.

As stated in the introduction, one of the key points of the idea herein presented is that based on an understanding defined with respect to a set  $\phi$  of clauses and its corresponding set  $\tilde{\mathcal{C}}$  of concepts, it is possible to define all understandings that exists with respect to  $\phi$ . In what follows we present two lemmas (Lemma G and Lemma D) and two algorithms (Algorithm G and Algorithm D) that establish some truths and processes related to the existence of equivalent understandings, which are relevant for our main algorithm (Algorithm  $\tilde{U}$ ), presented at the end of this section.

**Lemma G.** Let  $\tilde{u}$  be an understanding defined with respect to a set  $\phi$  of clauses, where a literal  $\lambda$  in  $\phi$  is free under  $\tilde{u}$ . Let also  $\phi_\lambda$  be a subset of  $\phi$  that contains exclusively all clauses from  $\phi$  where  $\lambda$  or  $\neg\lambda$  appear. If there does not exist an understanding  $\tilde{u}_\lambda$  defined with respect to  $\phi_\lambda$  such that  $\lambda$  is true under  $\tilde{u}_\lambda$ , then there exists no understanding defined with respect to  $\phi$  under which  $\lambda$  is true.

*Proof.* We let  $\tilde{\mathcal{C}}$  be the set of concepts of  $\phi$  from which  $\tilde{u}$  is defined. Let also  $\tilde{\mathcal{C}}'$  be the set of concepts that contains only all concepts from  $\tilde{\mathcal{C}}$  that correspond to clauses in  $\phi_\lambda$  (the interpretation of  $\tilde{\mathcal{C}}'$  we leave it to be  $\tilde{u}$  as in  $\tilde{\mathcal{C}}$ ), i.e.,  $\tilde{\mathcal{C}}'$  is a subset of  $\tilde{\mathcal{C}}$ .

Since we have from  $\tilde{u}$  that neither  $\lambda$  nor  $\neg\lambda$  were needed for clauses in  $\phi_\lambda$  to be satisfied (since they are free under  $\tilde{u}$ ), it follows that an understanding  $\tilde{u}_\lambda$  under which  $\lambda$  is true can be defined provided that both of the following conditions are met for at least one concept  $\mathcal{C}$  in  $\tilde{\mathcal{C}}'[\lambda]$ :

- a) It is not the case that both of the literals in the definition of  $\mathcal{C}$  are part of the definition of a concept in  $\tilde{\mathcal{C}}'[\neg\lambda]$ .
- b) It is not the case that of the two literals,  $l_1$  and  $l_2$ , that define  $\mathcal{C}$ , there is a concept  $\mathcal{C}_1$  in  $\tilde{\mathcal{C}}'[\neg\lambda]$  defined by  $l_1$  and a literal  $l_x$ , and there is a concept  $\mathcal{C}_2$  in  $\tilde{\mathcal{C}}'[\neg\lambda]$  defined by  $l_2$  and the negation of  $l_x$  (i.e.,  $\neg l_x$ ).

From above it follows that the understanding  $\tilde{u}_\lambda$  cannot exist if no concept in  $\tilde{\mathcal{C}}'[\lambda]$  meet the conditions stated. And if that is the case, it follows that there exists no understanding defined with respect to  $\phi$  under which  $\lambda$  is true, because  $\phi_\lambda$  contains all clauses where  $\lambda$  appears, and thus there cannot be any concept for  $\lambda$  with respect to  $\phi$  that is not already in  $\tilde{\mathcal{C}}'[\lambda]$ .  $\square$

**Algorithm G.** (*Verify if there exists an understanding defined with respect to a set of clauses that contains exclusively clauses where literals  $\lambda$  or  $\neg\lambda$  appear, such that  $\lambda$  is true under such understanding*). Given a literal  $\lambda$  in a set  $\phi$  of clauses, and an understanding  $\tilde{u}$  and set  $\tilde{\mathcal{C}}$  of concepts defined with respect to  $\phi$ , such that  $\lambda$  is free under  $\tilde{u}$ . Let  $\phi_\lambda$  be a subset of  $\phi$  that contains exclusively all clauses of  $\phi$  where  $\lambda$  or  $\neg\lambda$  appear. Verify if there exists an understanding  $\tilde{u}_\lambda$  defined with respect to  $\phi_\lambda$ , such that  $\lambda$  is true under  $\tilde{u}_\lambda$ .

**G1.** Set  $\tilde{u}' \leftarrow \tilde{u}$  and  $\tilde{\mathcal{C}}' \leftarrow \tilde{\mathcal{C}}$ . Set  $\tilde{u}(\lambda)' \leftarrow t$  (this is our assumption) and  $\tilde{u}(\neg\lambda)' \leftarrow f$ .

**G2.** Consider a concept  $\mathcal{C}$  in  $\tilde{\mathcal{C}}'[\lambda]$ , not yet considered. If all concepts in  $\tilde{\mathcal{C}}'[\lambda]$  have been considered, the algorithm terminates unsuccessfully; output **False**.

**G3.** Let  $l_1$  and  $l_2$  be the literals in concept  $\mathcal{C}$ . Set both  $l_1$  and  $l_2$  to *not true* (i.e., either  $\varepsilon$  or  $f$ ) under  $\tilde{u}'$  (following our assumption). If  $\langle \text{Compute } \tilde{u}' \rangle$  causes no contradiction (in  $\tilde{u}'$ ), the algorithm terminates successfully; output **True**. Otherwise, go back to G2. ■

The operation  $\langle \text{Compute } \tilde{u} \rangle$  used in Algorithm G above and, later, in Algorithm D and Algorithm  $\tilde{U}$ , is defined next.

$\langle \text{Compute } \tilde{u} \rangle =$

Compute  $\tilde{u}$  for each literal  $\lambda$  and its negation for which the type of  $\tilde{\mathcal{C}}[\lambda]$  has changed, until there is no change of type on any subset of concepts of  $\tilde{\mathcal{C}}$ .

**Lemma D.** Let  $\tilde{u}$  be an understanding defined with respect to a set  $\phi$  of clauses, and let  $\lambda$  be a literal in  $\phi$  that is false under  $\tilde{u}$ . Let also  $\mathcal{H}$  be a given set of literals (considered empty, if not given). Considering  $\mathcal{H}$ , there exists an understanding  $\tilde{u}'$  equivalent to  $\tilde{u}$ , such that  $\lambda$  is free under  $\tilde{u}'$ , if and only if, under understanding  $\tilde{u}$ , there is at least one literal  $l$  (not in  $\mathcal{H}$ ) in each of the concepts in  $\tilde{\mathcal{C}}[\lambda]^-$ , for which the following two conditions are true:

- $d_1$ . If  $l$  is false under  $\tilde{u}$  then, considering  $\mathcal{H}'$ , defined as  $\mathcal{H}' \leftarrow \mathcal{H} + \lambda$ , there exists an understanding  $\tilde{u}''$  equivalent to  $\tilde{u}$ , such that  $l$  is free under  $\tilde{u}''$ .
- $d_2$ . There exists an understanding, defined with respect to a subset of  $\phi$  that contains exclusively all clauses from  $\phi$  where  $l$  or  $\neg l$  appear, under which  $l$  is true.

(Any literal  $l$  that is in  $\mathcal{H}$  is skipped to avoid circular arguments. It should be clear that if  $\mathcal{H}$  is empty, the existence of  $\tilde{u}'$  is valid in general; otherwise, the existence or nonexistence of  $\tilde{u}'$  is only valid for the case in which the elements in  $\mathcal{H}$  are fixed to be false.)

*Proof.* Let us first assume that  $d_1$  and  $d_2$  are true for at least one literal  $l$  (not in  $\mathcal{H}$ ) in each of the concepts in  $\tilde{\mathcal{C}}[\lambda]^-$ . Condition  $d_1$  is true if either  $l$  is free under  $\tilde{u}$  or if  $l$  is false

under  $\tilde{u}$  and there exists an understanding  $\tilde{u}''$  defined with respect to  $\phi$ , such that  $l$  is free under  $\tilde{u}''$ . Thus, in both cases we have that there exists an understanding defined with respect to  $\phi$  under which  $l$  is free.

On the other hand, condition  $d_2$  states that there exists an understanding defined with respect to a set of clauses  $\phi_l$  under which  $l$  is true, where  $\phi_l$  is a subset of  $\phi$  that contains exclusively all clauses from  $\phi$  where  $l$  or  $\neg l$  appear. In order for this to be valid not only for  $\phi_l$  but for the whole  $\phi$  we need an understanding under which, for at least one concept in the set of concepts of  $l$ , both literals that define it are not true. One such concept is the one that is obtained from the same clause as the concept  $C$  in  $\tilde{\mathcal{C}}[\lambda]^-$ . Thus, while  $C$  is defined as  $\{\tilde{u}(l), \tilde{u}(l_x)\}$ , the corresponding concept in  $\tilde{\mathcal{C}}[l]$  is defined as  $\{\tilde{u}(\neg\lambda), \tilde{u}(l_x)\}$ . We have then that, under  $\tilde{u}$ ,  $\neg\lambda$  is true,  $\lambda$  is false, and  $l_x$  is not true (since it is part of a concept in  $\tilde{\mathcal{C}}[\lambda]^-$ ).

From above it follows that if we define an understanding  $\tilde{u}'$  and related set  $\tilde{\mathcal{C}}'$  of concepts initially as a copy of  $\tilde{u}$  and  $\tilde{\mathcal{C}}$  respectively, and we impose, under  $\tilde{u}'$ , at least one literal  $l$  on each concept in  $\tilde{\mathcal{C}}'[\lambda]^-$  to be true, then the set  $\tilde{\mathcal{C}}'[\lambda]^-$  is empty, with which we have  $\lambda$  free under  $\tilde{u}'$

For the converse, we show the contrapositive. Thus, we assume that the conditions  $d_1$  and  $d_2$  are both, or at least one of them, not true for at least the two literals  $l_x$  and  $l_y$  that define one of the concepts in  $\tilde{\mathcal{C}}[\lambda]^-$ .

Let us first assume that  $d_1$  is not true. That is,  $l_x$  and  $l_y$  are false under  $\tilde{u}$  and there exists no understanding  $\tilde{u}''$  defined with respect to  $\phi$  under which  $l_x$  or  $l_y$  is free. For  $\lambda$  to be free it is necessary that the set of concepts of  $\lambda$  is of type  $\tilde{\mathcal{C}}^*$ . However, based on our assumption one of the concepts of  $\lambda$  is defined by two literals ( $l_x$  and  $l_y$ ) that are false under  $\tilde{u}$  and no understanding exists, defined with respect to  $\phi$ , under which at least of one these literals is free. This means that  $\neg l_x$  and  $\neg l_y$  must be true under any understanding defined with respect to  $\phi$ . Therefore, the set of concepts of  $\lambda$  is of type  $\tilde{\mathcal{C}}^*$  under any understanding defined with respect to  $\phi$ .

Finally, we assume that  $d_2$  is not true. Let  $\phi_x$  be a subset of clauses that contains exclusively all clauses from  $\phi$  where  $l_x$  or  $\neg l_x$  appear, and let  $\phi_y$  be a subset of clauses that contains exclusively all clauses from  $\phi$  where  $l_y$  or  $\neg l_y$  appear. We assume that there exists no understanding defined with respect to  $\phi_x$  under which  $l_x$  is true and there exists no understanding defined with respect to  $\phi_y$  under which  $l_y$  is true. Based on Lemma G, this assumption implies that there exists no understanding defined with respect to  $\phi$ , such that at least one of  $l_x$  or  $l_y$  is true. Therefore, under any understanding defined with respect to  $\phi$  the set of concepts of  $\neg\lambda$  is of type  $\tilde{\mathcal{C}}^+$  and thus  $\lambda$  is false.  $\square$

**Algorithm D.** (*Define an understanding  $\tilde{u}'$ , equivalent to a given understanding  $\tilde{u}$  under which a literal  $\lambda$  is false, such that  $\lambda$  is free under  $\tilde{u}'$* ). Given an understanding  $\tilde{u}$  and a set  $\tilde{\mathcal{C}}$

of concepts defined with respect to a set  $\phi$  of clauses, a literal  $\lambda$  that is false under  $\tilde{u}$ , and a set  $\mathcal{H}$  of literals (considered empty, if not given). Define, if possible, an understanding  $\tilde{u}'$  and a set  $\tilde{\mathcal{C}}'$  of concepts equivalent to  $\tilde{u}$  and  $\tilde{\mathcal{C}}$ , such that  $\lambda$  is free under  $\tilde{u}'$ .

**D0.** Set  $\tilde{u}' \leftarrow \tilde{u}$  and  $\tilde{\mathcal{C}}' \leftarrow \tilde{\mathcal{C}}$ .

**D1.** Consider a concept  $\mathcal{C}$  in  $\tilde{\mathcal{C}}'[\lambda]^-$ , not yet considered. If all concepts in  $\tilde{\mathcal{C}}'[\lambda]^-$  have been considered, the algorithm terminates successfully; output  $\tilde{u}'$  and  $\tilde{\mathcal{C}}'$ .

**D2.** Consider an element  $\tilde{u}'(l)$  in  $\mathcal{C}$ , not yet considered. If all elements of concept  $\mathcal{C}$  have been considered, the algorithm terminates unsuccessfully; output ‘**there is no such understanding**’.

**D3.** If  $l$  is in  $\mathcal{H}$ , go back to D2.

**D4.** If  $l$  is false under  $\tilde{u}'$ , then set  $\mathcal{H}' \leftarrow \mathcal{H} + \lambda$  and, based on  $\tilde{u}'$  and  $\tilde{\mathcal{C}}'$  and considering  $\mathcal{H}'$ , define if possible an understanding  $\tilde{u}''$  and a set  $\tilde{\mathcal{C}}''$  of concepts equivalent to  $\tilde{u}'$  and  $\tilde{\mathcal{C}}'$ , such that  $l$  is free under  $\tilde{u}''$  (this is done by Algorithm D). If no such understanding exists, go back to D2.

**D5.** If there does not exist an understanding  $\tilde{u}_l$  defined with respect to a subset of  $\phi$  that contains exclusively all clauses from  $\phi$  where  $l$  or  $\neg l$  appear, such that  $l$  is true under  $\tilde{u}_l$  (checked by Algorithm G), go back to D2. Otherwise, if an understanding  $\tilde{u}''$  was defined in D4 for  $l$ , set  $\tilde{u}' \leftarrow \tilde{u}''$  and  $\tilde{\mathcal{C}}' \leftarrow \tilde{\mathcal{C}}''$ , and, irrespectively of that, set  $\tilde{u}'(l) \leftarrow t$  and  $\tilde{u}'(\neg l) \leftarrow f$ ,  $\langle \text{Compute } \tilde{u}' \rangle$ , and go back to D1. ■

Next, we present our main algorithm which defines for any given 3SAT problem instance  $\Phi$  an understanding with respect to  $\phi$ , if one exists, or it determines that none exists.

**Algorithm  $\tilde{U}$ .** (*Define an understanding with respect to a 3SAT problem instance*). Given a 3SAT problem instance  $\Phi$ , define if possible an understanding with respect to  $\Phi$ .

**$\tilde{U}0$ .** Let  $\phi$  be an empty set of clauses and let  $\tilde{u}$  be an understanding defined with respect to  $\phi$  and  $\tilde{\mathcal{C}}$  be an empty set of concepts interpreted according to  $\tilde{u}$ .

**$\tilde{U}1$ .** Consider a clause  $\varphi$  that is in  $\Phi$  but not in  $\phi$ . Assume that  $\varphi$  is satisfied. If all clauses in  $\Phi$  are in  $\phi$ , the algorithm terminates successfully;  $\tilde{u}$  is the answer.

**$\tilde{U}2$ .** If all literals in  $\varphi$  are false under  $\tilde{u}$ , define if possible an understanding  $\tilde{u}'$  and a set  $\tilde{\mathcal{C}}'$  of concepts equivalent to  $\tilde{u}$  and  $\tilde{\mathcal{C}}$ , such that at least one literal in  $\varphi$  is free under  $\tilde{u}'$  (this is done by applying Algorithm D over each of the literals in  $\varphi$  until  $\tilde{u}'$  is successfully defined for one of them or all have been processed without success). If no understanding  $\tilde{u}'$  exists for any of the literals, the algorithm terminates

unsuccessfully; output ‘there exists no understanding with respect to  $\Phi$ ’. Otherwise, set  $\tilde{u} \leftarrow \tilde{u}'$  and  $\tilde{\mathcal{C}} \leftarrow \tilde{\mathcal{C}}'$ .

**U3.** Consider a literal  $\lambda$  in  $\varphi$ , not yet considered, taking first literals that are not false under  $\tilde{u}$ . If all literals in  $\varphi$  have been considered, go back to **U1**.

**U4.** Add the concept of  $\lambda$  in  $\varphi$  to the set  $\tilde{\mathcal{C}}$ ,  $\langle \text{Compute } \tilde{u} \rangle$ , and add  $\varphi$  to the set  $\phi$ . ■

Algorithm  $\tilde{U}$  follows straightforward from the definitions and lemmas stated previously. One detail, however, is the order in which literals in  $\varphi$  are considered in **U3**, taking first literals that are not false under  $\tilde{u}$ . This is meant to avoid getting an undefined understanding form a clause  $\varphi$  where some literals are free, others are false, but none are true under  $\tilde{u}$ . Clearly, if the false literals are considered first we will get an undefined understanding. However, by taking literals that are not false first we ensure that for one of them its concept in  $\varphi$  will be of type  $\mathcal{C}^+$ , and for any literal that is false its concept in  $\varphi$  will be of type  $\mathcal{C}^*$ .

## 2. Analysis.

In this section we present the analysis of our main algorithm (Algorithm  $\tilde{U}$ ) in terms of correctness (§ 2.1) and asymptotic time complexity (§ 2.2).

### 2.1 Correctness.

**Theorem 1.** Algorithm  $\tilde{U}$  terminates successfully if and only if  $\Phi$  is satisfiable.

We prove Theorem 1 through a sequence of lemmas.

**Lemma 1.** If  $\Phi$  is satisfiable, Algorithm  $\tilde{U}$  terminates successfully.

*Proof.* We assume initially that  $\Phi$  is satisfiable.

The proof is by induction, where the induction hypothesis is that there exists an understanding defined with respect to a set  $\phi$  of clauses, subset of  $\Phi$ .

The base case is for  $|\phi| = 1$ . In this case an understanding  $\tilde{u}$  is always defined with respect to  $\phi$ , with the first literal considered in **U3** made true in **U4**, since its concept is of type  $\mathcal{C}^+$  (due to literals in that concept being initially free), and then the type of the concept of the other two literals is  $\mathcal{C}^*$ , making them free under  $\tilde{u}$ .

For the induction step, we wish to show that there exists an understanding defined with respect to  $\phi + \varphi$ , where  $\phi$  is a subset of  $\Phi$  and  $\varphi$  is a clause in  $\Phi$  but not in  $\phi$ . In all but one case there exists an understanding defined with respect to  $\phi + \varphi$ . Such case happens if all literals in  $\varphi$  are false under all understandings that can be defined with respect to  $\phi$ . Based on the induction hypothesis, there exists an understanding defined

with respect to  $\phi$ . And, from our initial assumption (i.e.,  $\Phi$  is satisfiable) we have that there exists a truth assignment which satisfies all clauses, thus there is no clause in  $\Phi$  with all its literals assigned to false. Therefore, the call to Algorithm D in  $\tilde{U}2$ , defines successfully an understanding with respect to  $\phi$  under which one literal  $\lambda$  in  $\phi$  is free. Then, in  $\tilde{U}3$ ,  $\lambda$  is considered first since the other two literals in  $\varphi$  are false. And finally in  $\tilde{U}4$ , understanding  $\tilde{u}$  is defined with respect to  $\phi + \varphi$ .  $\square$

**Lemma 2.** If Algorithm  $\tilde{U}$  terminates successfully then  $\Phi$  is satisfiable.

*Proof.* We show the contrapositive. Thus, we assume that  $\Phi$  is not satisfiable. Based on Lemma A we use the equivalence. That is, our assumption is that there exists no understanding defined with respect to  $\Phi$ .

For our assumption to be true it is necessary that for at least one clause  $\varphi$ , in  $\Phi$ , all its literals are false under all understandings that can be defined with respect to a set  $\phi$ , subset of  $\Phi$ , that does not include  $\varphi$ . In such a case, Algorithm  $\tilde{U}$  executes  $\tilde{U}2$ , where Algorithm D is executed to try to define an understanding with respect to  $\phi$  under which one of the literals in  $\varphi$  is free. However, due to our assumption Algorithm D fails. Consequently, Algorithm  $\tilde{U}$  terminates unsuccessfully.  $\square$

This concludes the proof of Theorem 1.

## 2.2 Time Complexity.

**Theorem 2.** For any given 3SAT problem instance  $\Phi$ , Algorithm  $\tilde{U}$  terminates in polynomial time.

*Proof.* We analyze the algorithm complexity in two parts.

The first part is concerned with  $\tilde{U}1$ ,  $\tilde{U}3$ , and  $\tilde{U}4$ . These steps perform a constant number of operations on the number of literals in  $\varphi$ , except for  $\langle \text{Compute } \tilde{u} \rangle$  which, in case the type of the set of concepts of  $\lambda$  has changed, it has to recompute  $\tilde{u}$  for  $\lambda$  and its negation and check if it is necessary to recompute  $\tilde{u}$  for any other literal for which  $\lambda$  is part of the definition of its set of concepts. In the worst case this process goes through all concepts that have been defined with respect to  $\phi$ . That is, at most three times the number of clauses in  $\phi$ . Thus, if we consider that at every iteration Algorithm  $\tilde{U}$  should go through this worst case (until all clauses in  $\Phi$  are processed), we get roughly an arithmetic series as the number of operations performed.

The second part is concerned with  $\tilde{U}2$ , where assuming that all literals in  $\varphi$  are false, Algorithm D will be executed for each literal  $\lambda$  in  $\varphi$  until it defines an understanding under which  $\lambda$  is free. The number of iterations in Algorithm D depends on the number of concepts in  $\tilde{\mathcal{C}}[\lambda]^-$ , for the literal  $\lambda$  for which Algorithm D is meant to define an understanding.

We recall that  $\tilde{\mathcal{C}}[\lambda]^-$  is the set of concepts of type  $\mathcal{C}^+$  of  $\neg\lambda$ . Since by definition there can be only one concept of type  $\mathcal{C}^+$  defined from each clause, we have that the number of concepts of type  $\mathcal{C}^+$  in the set of concepts defined with respect to  $\phi$  is at most equal to the number of clauses in  $\phi$ . Thus, the maximum number of iterations of Algorithm D overall (including its recursive call in D4 for some literals in concepts in  $\tilde{\mathcal{C}}[\lambda]^-$ ) is bounded by the total number of clauses in  $\phi$  (times some constant). Therefore, if we consider that at every iteration Algorithm  $\tilde{U}$  should execute Algorithm D in  $\tilde{U}2$  over each of the literals of clause  $\varphi$ , we have in the worst case roughly an arithmetic series as the total number of operations.

In both parts above we have an upper bound of approximately  $O(m^2)$ , where  $m$  is the number of clauses in  $\Phi$ . Therefore, Algorithm  $\tilde{U}$  terminates in polynomial time.  $\square$

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