

On the existence of polynomial-time algorithms to the subset sum problem

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Abstract. This paper proves that there does not exist a polynomial-time algorithm to the the subset sum problem. As this problem is in NP , the result implies that the class P of problems admitting polynomial-time algorithms does not equal the class NP of problems admitting nondeterministic polynomial-time algorithms.

Key words: computational complexity, polynomial-time, algorithm, knapsack problem.

1 Introduction

Let \mathbb{N} and \mathbb{R} indicate natural and real numbers respectively.

Definition 1. A knapsack is a pair of the form $(j, (d_1, \dots, d_n))$ where $j, n \in \mathbb{N}$, $j, n > 0$ and $d_k \in \mathbb{N}$, $d_k > 0$ for $1 \leq k \leq n$.

The knapsack problem means the following: given a knapsack $(j, (d_1, \dots, d_n))$ determine if there exist binary numbers $c_k \in \{0, 1\}$, $1 \leq k \leq n$, such that

$$j = \sum_{k=1}^n c_k d_k.$$

Let $B, \alpha \in \mathbb{R}$, $B \geq 1$, $\alpha \geq 0$ be fixed numbers. An algorithm A is called polynomial-time algorithm to the knapsack problem if there exist numbers $C, \beta \in$

IR that depend on B and α but not on n such that the following condition is true:

For any sequence of knapsacks of the form

$$((j_n, (d_{1,n}, \dots, d_{n,n})))_{n \geq 1}$$

satisfying

$$\log_2 j_n < Bn^\alpha, \log_2 d_{k,n} < Bn^\alpha, (1 \leq k \leq n), (n \geq 1) \quad (1.1)$$

the number N_n of elementary operations that the algorithm A needs to produce an answer *yes* or *no* to the question if there exists binary numbers $c_{k,n} \in \{0, 1\}$, $1 \leq k \leq n$, such that

$$j_n = \sum_{k=1}^n c_{k,n} d_{k,n} \quad (1.2)$$

satisfies $N_n < Cn^\beta$ for all $n \geq 1$.

The problem that has been described is used in the Merkle-Hellman knapsack cryptosystem and today it is commonly known as the knapsack problem. The name Subset sum problem is used for it in [2] p. 301, while the name Knapsack problem is reserved for a more general problem involving selecting objects with weights and profits. The name *knapsack* is more convenient than *subset sum* and it is often used in this paper.

In the definition of a polynomial-time algorithm for the knapsack problem we have included an upper bound on j_n and on each $d_{k,n}$, $1 \leq k \leq n$. Such bounds are necessary for the following two reasons (i) and (ii).

(i) The number m of bits in the binary representation of j_n satisfies $m \leq \log_2 j_n < m+1$. Thus, if $\log_2 j_n$ grows faster than any polynomial as a function of n then so does the length of j_n in the binary representation. It is necessary to verify that (1.2) is satisfied. It requires making some operations (like compare, copy, read, add, subtract, multiply, divide, modulus) that act on a representation of j_n on some base number. We may assume that the number base is 2 as changing a

number base does not change the character of the algorithm from polynomial-time to non-polynomial-time. Any operations that require all bits of j_n must require more than a polynomial number of elementary operations from any algorithm A if the number of bits in j_n grows faster than any polynomial. Similar comments apply to $d_{k,n}$.

(ii) If j_n has an upper bound independent of n , then there exist a polynomial-time algorithm solving the knapsack problem. The Annex gives one such algorithm in Lemma A2. The algorithm in Lemma A2 calculates an exponentially growing number of combinations of c_k in the same polynomial time run.

Because of (i) and (ii) j_n must grow polynomially with n . We can select j_n as growing linearly as in (1.1). It gives an NP-complete knapsack problem.

Remark 1. Lemma A2 in the Annex solves all possible values of $j_n < Bn^\alpha$ with the same polynomial time run of Algorithm A0 because j_n is not used in A0 before checking the final result $b_{n,k}$. Let us consider the case when j_n is not limited from above by a polynomial of n . Lemma A1 runs in polynomial time even if the upper bound for j_n grows faster than a polynomial of n but it does not produce results that can tell if there exists a solution for a particular value j_n . A polynomial-time test, such as taking a modulus in (A1), maps the superpolynomial set of possible values of $j = \sum_{k=1}^n c_{k,n} d_{k,n}$ into a polynomial number of classes. In (A1) the classes are all sums j with the same moduli by r_n . At least one such a class corresponds to an superpolynomial number of values j . In order to check if any value j in the class equals j_n the algorithm should in some way check all of the values j in the class, but if the algorithm at the same run checks all values of j then it should in some way loop over a superpolynomial set which is not possible for a polynomial time algorithm. In general, we can say that a single polynomial time run of an algorithm cannot solve all values of j_n that are below a superpolynomial upper bound because the algorithm can only produce a polynomial number of results and there exist a superpolynomial number of possible values j_n . A polynomial time algorithm that solves the subset

sum problem for any value j_n below a superpolynomial upper bound must limit search and there must be values j_n that are solved with different runs of the algorithm.

Remark 2. An algorithm is a finite set of rules that at every step tell what to do next. We can implement an algorithm as a computer program in a second generation language on a von Neumann machine and a polynomial time algorithm can be implemented in this way so that it requires time and memory that grow polynomially with respect to the problem dimension. In the case when the smallest upper bound of j_n in Remark 1 grows exponentially a program in a second generation computer language implementing a polynomial time algorithm needs to limit search by branching instructions, or by acting differently on different data (like in add, subtract and compare instructions). Thus, we can find values of j_n such that the algorithm uses different branches, or acts differently on data, in solving the subset sum problem.

2 The inequality (2.6) means non-polynomial time

It is not possible to select a fixed sequence of specific subset sum problems and show that no algorithm can solve this specific sequence of problems in polynomial time. This is so because we can create an algorithm that treats these specific problems in a particular way and can solve that specific sequence of problem in a fast way. Instead, we must first select the algorithm and pose that selected algorithm a sequence of subset sum problems that are particularly hard for that specific algorithm. As the algorithm can be any possible algorithm, the sequence of problems can only be defined by using some suitable definition of a difficult problem to the selected algorithm and we cannot give any numerical values for all of the numbers $c_{k,n}$ in (1.2). The selection will be done by using the following definition of the computation time of a subset sum problem.

For convenience, let us select n to be of the form $n = 2^{i+2}$ for some $i > 0$. This simplifies expressions since it is not necessary to truncate numbers to integers.

Definition 2.

We define a function $f(n)$ that describes (in a certain sense) the worst computation time for a selected algorithm.

Let the worst in the median n -tuple as be defined as follows. Let

$$h(d_{1,n}, \dots, d_{n,n}, j_n)$$

be the computation time for deciding if the knapsack

$$(j_n, (d_{1,n}, \dots, d_{n,n}))$$

has a solution or not. Let

$$\text{Median}_{j_n} h(d_{1,n}, \dots, d_{n,n}, j_n) \quad (2.1)$$

be the median computation time where j_n ranges over numbers

$$j_n \in \{C + 1, \dots, 2^{n+1} - 1\} \quad (2.2)$$

satisfying the two conditions

$$j_{n,l} = j_n - C \left\lfloor \frac{j_n}{C} \right\rfloor > 2^{\frac{n}{4}+2} \quad (2.3)$$

where $C = 2^{\frac{n}{2}+1}$, and that there is no solution to the knapsack $(j_n, (d_{1,n}, \dots, d_{n,n}))$. That is, $j_{n,l}$ are the lower half bits of j_n . The values of j_n are computed separately in calculation of the median, i.e., no partial results from previously computed values of j_n are used.

Let $(d_{1,n}, \dots, d_{n,n})$ range over all knapsack sequences with

$$\lceil \log_2 \sum_{k=1}^n d_{k,n} \rceil = n$$

and $d_{k,n} \leq \frac{2^n - 1}{n}$. Because of this requirement at most every second value of j_n in (2.2) is a solution to the knapsack, i.e., there are 2^n combinations of $(c_{1,n}, \dots, c_{n,n})$ mapped to numbers from zero to $2^{n+1} - 1$. The worst in the median tuple for n is an n -tuple $(d_{1,n}, \dots, d_{n,n})$ (possibly not unique) that maximizes the median computation time (2.1).

Let this maximal median computation time be denoted by $f(n)$. Thus

$$f(n) = \max_{d_{1,n}, \dots, d_{n,n}} \text{Median}_{j_n} h(d_{1,n}, \dots, d_{n,n}, j_n). \quad (2.4)$$

We use the median in Definition 2 instead of the worst case or the worst in the average case because we need $\frac{n}{2}$ almost as long computations as the worst in (2.6). In the worst and in the worst in the average, a very slow computation of one value j_n can be the reason for the long computation time. By using the median we can find many values j_n giving almost the median computational time because the distribution of the computational time for j_n becomes almost normally distributed when n grows due to the law of large numbers. We include only unsuccessful cases of j_n in the computation of the median because this choice implies that a more complicated knapsack problem (i.e., more cases to check) gives a longer computation time. If there are more cases to choose, there are more successful cases. Therefore the time for finding a solution decreases if there are more cases to check.

Lemma 1. *Let m be fixed and n be a power of m . If $f(n)$ satisfies the inequality*

$$\frac{n}{m} f\left(\frac{n}{m}\right) < f(n) \quad (2.5)$$

then $f(n)$ does not grow polynomially with n .

Proof. Iterating we get

$$\frac{n}{m} \frac{n}{m^2} f\left(\frac{n}{m^2}\right) < f(n)$$

and iterating up to k yields

$$\frac{n^k}{m^{\sum_{i=1}^k i}} f\left(\frac{n}{m^k}\right) < f(n)$$

i.e.,

$$e^{k \ln n - \frac{1}{2} k^2 \ln m - \frac{k}{2} \ln m} f\left(\frac{n}{m^k}\right) < f(n).$$

Setting $k = \frac{\ln n}{\ln m}$ (i.e., $1 = \frac{n}{m^k}$) gives

$$(n^{\ln n})^{\frac{1}{2 \ln m}} n^{-\frac{1}{2}} f(1) < f(n).$$

If m is any fixed number we see that $f(n)$ satisfying (2.5) is not bounded by a polynomial function of n . \square

Lemma 2. *Let n be a power of 2. If $f(n) = f_1(n) + f_2(n)$ where $f_1(n)$ is a polynomial function of n and $f_2(n)$ satisfies the inequality*

$$\frac{n}{2} f_2\left(\frac{n}{2}\right) < f_2(n) \quad (2.6)$$

then $f(n)$ does not grow polynomially with n .

Proof. If $f(n)$ is a polynomial function of n and since $f_1(n)$ is a polynomial function of n by assumption, it follows that $f_2(n)$ must also be a polynomial function of n . By Lemma 1, $f_2(n)$ is not a polynomial function of n , thus neither is $f(n)$. \square

3 Construction of a special subset sum problem

In this section we will define a special subset sum problem K_{1,j_n} in Definition 3 and show that it can only be solved by solving $n_1 = n/2$ subknapsacks $(j'_i, (d_{1,n}, \dots, d_{n_1,n}))$ with different values of j'_i . We will use the denotation $n_1 = n/2$ throughout this article for brevity.

Definition 3. Construction of K_{1,j_n} . We first make a knapsack where the only solutions must satisfy the condition that exactly one c_k must be 1 and the others must be zero for $k = n_1 + 1$ to $k = n$. Let us construct the values $d_{k,n}$, $k = n_1 + 1, \dots, n$ of K_{1,j_n} for a given j_n . Let $C = 2^{\frac{n}{2}+1}$ and

$$j_{n,h} = C \left\lfloor \frac{j_n}{C} \right\rfloor, \quad j_{n,l} = j_n - j_{n,h} \quad (3.1)$$

be the high and low bit parts of j_n . Because of (2.2), $j_{n,h} \neq 0$. Let

$$d_{n_1+k,n} = j_{n,h} + a_k \quad (3.2)$$

where $0 < a_k < \min\{j_{n,l}, \frac{2^{n_1}-1}{n_1}\}$ are distinct integers and there exists no solution to the knapsack problem for the knapsack

$$(j'_i, (d_{1,n}, \dots, d_{n_1,n}))$$

where

$$j'_i = j_{n,l} - a_i. \quad (3.3)$$

Let us also require that the computation time for j'_i is at least as long as the median computation time $f(n_1)$ for $(j, (d_{1,n}, \dots, d_{n_1,n}))$. We can select j'_i filling this condition because half of the values j are above the median. Notice that we compute the median only over values j that do not give a solution to the knapsack. We will also assume that the j'_i are in the set corresponding to (2.2)-(2.3) for $f(n_1)$, i.e.,

$$j'_i \in \{C' + 1, \dots, 2^{n_1+1} - 1\} \quad (3.4)$$

satisfying the condition

$$j'_i - C' \left\lfloor \frac{j'_i}{C'} \right\rfloor > 2^{\frac{n}{8}+2} \quad (3.5)$$

where $C' = 2^{\frac{n_1}{2}+1}$. We may assume so because there are enough values from which to choose j'_i .

In (3.2) we select the numbers a_k in such a way that the $d_{n_1+k,n}$ satisfy the size condition $d_{n_1+k,n} \leq \frac{2^n-1}{n}$. Because of the bound (2.3) we have an exponential number of choices for a_i . It is possible to find numbers j'_i such that there is no solution since only for about half of the values of j there exists a solution for $(j, (d_{1,n}, \dots, d_{n_1,n}))$. If $j_{n,l}$ is too small and we cannot find values j'_i , we take a carry from $j_{n,h}$ in (3.3) and reselect a_k . Because of the lower bound on j in (2.2), $j_{n,h}$ is not zero and we can take the carry. Then $j_{n,h}$ is decreased by the carry.

Exactly one c_k must be 1 and the others must be zero for $k = n_1 + 1$ to $k = n$. There cannot be more values $c_k = 1$ for $k > n_1$ because then the higher bits of j_n are not matched. The unknown algorithm can try also other combinations but these are the only possible combinations and the algorithm must also try them (i.e., check these cases in some way unknown to us). The sum of the numbers $d_{k,n}$, $k \leq \frac{n}{2}$ is less than $2^{\frac{n}{2}+1} - 1$. Adding one c_k can give a carry and there may not be a solution to the knapsack because the high bits of j_n do not match but this is not an issue since we do not want solutions. We select the n -tuple so that there are no solutions to the knapsack already because the lower bits do not match.

Lemma 3. *The algorithm cannot stop to finding a solution because for every j_n none of the $\frac{n}{2}$ values of j'_i solve the knapsack problem. Every value j'_i gives at least as long computation as the median computation time $f(n_1)$.*

Proof. We have selected K_{1,j_n} such that $(j'_i, (d_{1,n}, \dots, d_{n_1,n}))$ has no solution for any j'_i . Thus the algorithm cannot stop because it finds a solution. By construction the values j'_i give at least as long computation time as the median for the tuple at $k = 1, \dots, n_1$. Since that tuple is the worst in the median tuple for n_1 , the computation time for each j'_i is at least $f(n_1)$. \square

Lemma 4. *There is no way to discard any values j'_i without checking if they solve the subknapsack from $k = 1$ to $k = n_1$. Any case of using the values of $d_{k,n}$ in order to get the result is considered checking.*

Proof. We can select any a_k in such a way that there either exists a solution or does not exist. Knowledge from other $c_{i,n}$ ($i \neq n_1+k$) cannot give any information on how this a_k was selected. Thus, the existence of a solution must be checked using the value $d_{n_1+k,n}$. \square

Lemma 5. *Several values of j'_i cannot be evaluated on the same run. The median computation time of K_{1,j_n} is at least*

$$f_1(n_1) + n_1 f_2(n_1)$$

where $f(n) = f_1(n) + f_2(n)$ is a lower bound for the computation time of one j'_i and $f_1(n)$ is a polynomial function of n , the shared part of the computation time of all j'_i .

Proof. As explained in Remark 1, a polynomial time algorithm cannot solve all values of j'_i at the same run because it would require an exponential amount of memory. As explained in Remark 2, we can assume that the algorithm is implemented in a second generation computer language on a von Neumann machine and its code has branching instructions, or it acts differently on different data in an instruction (like add depends on the data), which has the same effect as a branching instruction: for a different j_n there is needed a different run. These branching instructions define a branching tree describing the execution of the algorithm for any input data. The tree is fixed when the algorithm is selected. At each branching point the input data is divided into a finite number of classes. Because this division is fixed, we can always find two values j'_i which are not executed by the same polynomial time run. After finding two, we can continue to find three values j'_i which all are executed by different polynomial time runs of the algorithm. This

can be extended to $\frac{n}{2}$ values j'_i : we can select j'_i in such a way that no two values j'_i are computed in the same run. The runs for different values j'_i can have parts that are shared, as long as the shared parts are computed in polynomial time. This is necessarily the case for practical algorithms: the runs must share at least the beginning of the code before branch instructions are reached and this shared part must take only polynomial time for the algorithm to make any sense. The shared part of the computation time can be described by a polynomial function $f_1(n)$ and a lower bound for the nonshared computation time can be denoted by a function $f_2(n)$. \square

4 Proving the inequality (2.6)

Let the algorithm be chosen. We selected a tuple K_{1,j_n} for a chosen j_n and showed in Lemma 5 that the computation time for the set of K_{1,j_n} for the single value j_n is at least as high as the left hand side of (2.6). We have obtained the left side of the inequality (2.6) for an arbitrarily chosen algorithm solving the knapsack problem. However, the set of K_{1,j_n} is a (reasonably) hard problem only for the chosen value j_n . Let us call this j_n with the name j_{n0} . In the right side of (2.6) the number j_n must range over all values and we calculate the median computation time over those values of j_n where there is no solution. In $K_{1,j_{n0}}$ it is very fast to conclude that most values for j_n do not have a solution: it is usually enough just to check the bits of j_n in the most significant half of the number. If they do not match the most significant bits of j_{n0} , then there is no solution.

We want to change the knapsack problem $K_{1,j_{n0}}$ to another knapsack problem K_2 (the problem K_2 will be defined later in Definition 5) where j_n can range over all numbers and for many values of j_n there is no solution and the knapsack problem is difficult. The knapsack problem K_2 has at most as long median computation time as the worst in the median tuple for n because the worst is the worst.

We will do the change in two steps. First we change $K_{1,j_{n_0}}$ to $K_{3,j_{n_0}}$ where the bits in the lower half of j_n can vary. In the second step we change $K_{3,j_{n_0}}$ to K_2 where also the upper half bits of j_n can vary. What we have to show is that the computation time of the set $K_{1,j_{n_0}}$ with a single $j_n = j_{n_0}$ is not larger than the median computation time for $K_{3,j_{n_0}}$ when j_n can have any lower half bits. In $K_{3,j_{n_0}}$ only one $d_{j,n}$, the one with $j = n$, has the most significant bits of j_{n_0} . Therefore $c_{n,n}$ must be one in order to have a possibility of finding a solution for j_n that has the high bits of j_{n_0} . We put some numbers to $d_{j,n}$ for $j = n_1 + 1, \dots, n - 1$. These numbers have zero high bits. There are more combinations that can give a solution in $K_{3,j_{n_0}}$ than in $K_{1,j_{n_0}}$, thus it is easier (and faster) to find a solution, provided that there is a solution for a chosen j_n . The trick here is that in the calculation of the median computation time we take only those j_n where there is no solution. Then the fact that there are more possible combinations only makes it harder to conclude that there is no solution. We conclude in Lemma 6 that the median computation time for $K_{3,j_{n_0}}$ when the lower half bits of j_n vary is larger than the computation time of $K_{1,j_{n_0}}$.

Next we have to show that K_2 gives a larger median computation time when j_n varies over all numbers than $K_{3,j_{n_0}}$ when the bits of the lower half of j_n vary. It is a similar situation here: there are more combinations in K_2 that can give a solution for a given j_n , but only those j_n that give no solution are counted in the median computation time. Therefore adding complexity makes the median computation time longer. In K_2 we replace $d_{n,n}$ of $K_{3,j_{n_0}}$ by a difficult knapsack problem in the upper half bits. As this difficult knapsack problem in the upper half has n numbers $d_{j,n}$ and the bit length of each $d_{j,n}$ is only $n/2$, there usually always are solutions to the upper half knapsack problem. Looking at the upper half knapsack problem does not help in finding values j_n that give no solution to the knapsack problem K_2 . Because of this, the knapsack problem K_2 is not any easier than the knapsack problem $K_{3,j_{n_0}}$.

Figure 1 shows the main idea.

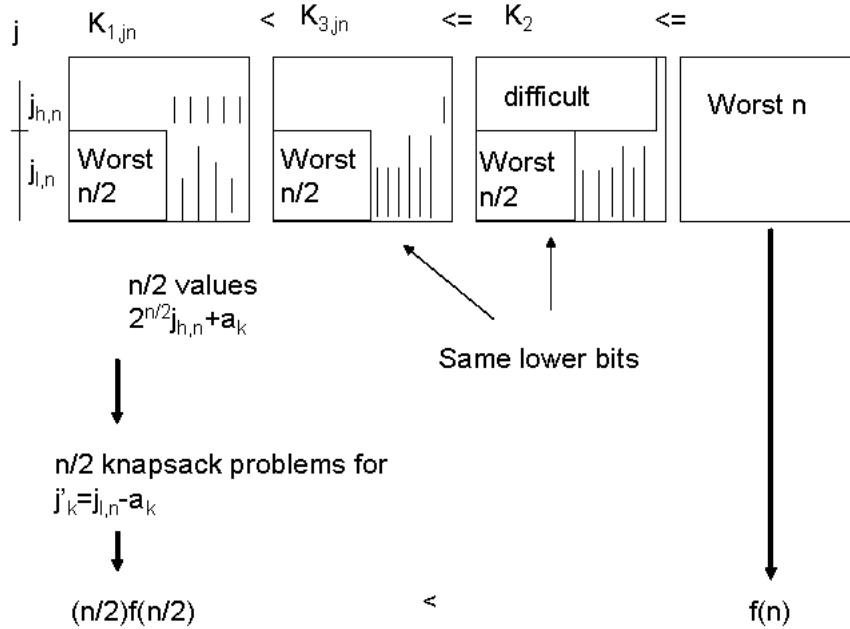


Fig. 1. The idea of the proof.

In Figure 1 the set $K_{1,j_{n0}}$ has the worst in the median n_1 -tuple in the left side and the right side has numbers from which it is necessary to select exactly one in order to satisfy the high bits of j_{n0} . This yields n_1 separate subset sum problems and we get the computation time corresponding to the left side of (2.6). The set $K_{3,j_{n0}}$ has only one element which has high order bits and it must always be selected in order to satisfy the high bits of j_n . Here the bits of the upper half of j_n are the same as in j_{n0} . There is the same worst in the median n_1 -tuple and the remaining $n_1 - 1$ elements can be assigned in any way yielding of the order n^2 knapsack problems. It is easier to find a solution than in $K_{1,j_{n0}}$, but it is harder to conclude that there are no solutions. Lemma 6 shows that the time of solving K_{1,j_n} is not higher than the median computation time for K_{3,j_n} for almost any j_n that does not yield a solution.

The n -tuple K_2 has some difficult upper half knapsack problem which has to be satisfied with the same values c_k as the lower half knapsack. It is not of

any use to check if the upper half knapsack half has a solution when trying to show that there is no solution to the whole knapsack since there almost always are many solutions to the upper half knapsack problem. The algorithm must look at all bits. As finding a solution in K_2 requires looking at both the upper and lower half bits, it should be more difficult to conclude that there are no solutions. We will show that at least it is not faster. Finally, the inequality from K_2 to the worst in the median n -tuple is obtained directly by the definition of what the worst means.

Definition 4. Construction of $K_{3,j_{n0}}$. Let j_n be given and let us define a n -tuple $K_{3,j_{n0}}$ as an n -tuple with elements $(d_{1,n,3}, \dots, d_{n,n,3})$ by specifying the elements

$$\begin{aligned} d_{k,n,3} &= d_{k,n} \quad (k = 1, \dots, \frac{n}{2}) \\ d_{k,n,3} &= e_1 \quad (k = \frac{n}{2} + 1, \dots, \frac{3n}{4}) \\ d_{k,n,3} &= e_2 \quad (k = \frac{3n}{4} + 1, \dots, n - 1) \\ d_{n,n,3} &= j_{n0,h}. \end{aligned} \tag{4.1}$$

We select two nonnegative integers $e_i \leq \frac{2^{n_1}-1}{n_1}$, $i = 1, 2$. The selected e_1 and e_2 are so small that if $c_n = 0$ the higher bits of j_n are not matched because there is no carry. That is, the worst in the median knapsack for $n_1 = n/2$ is still in the left side. The high bits of j_{n0} are in $d_{n,n,3}$. We choose some numbers to the elements $d_{k,n,3}$ for $k = n_1 + 1, \dots, n - 1$.

This n -tuple has a simple upper half tuple. The sum of the numbers $d_{k,n,3}$, $k \leq \frac{n}{2}$ is less than $2^{\frac{n}{2}+1} - 1$. It is always necessary to set $c_n = 1$ and this satisfies the upper half bits of j_n when j_n ranges over numbers that have the same upper half bits as j_{n0} .

Definition 5. Construction of K_2 . We will define K_2 as an n -tuple with elements $(d_{1,n,2}, \dots, d_{n,n,2})$. Let us remember that the n -tuple $(d_{1,n}, \dots, d_{\frac{n}{2},n})$ is the worst in the median tuple for $\frac{n}{2}$. Let $(d_{0,1}, \dots, d_{0,n})$ be an n -tuple where each

$d_{0,k} \leq \frac{2^{n_1}-1}{n_1}$. We define

$$d_{k,n,2} = Cd_{k,n,2} + d_{k,n} \quad (4.2)$$

for $k = 1, \dots, n_1$. The numbers e_1 and e_2 are as in $K_{3,j_{n_0}}$ and we define the elements of K_2 for $k = n_1 + 1$ to $k = n$ as

$$\begin{aligned} d'_k &= Cd_{0,k} + e_1 \quad (k = \frac{n}{2} + 1, \dots, \frac{3n}{4}) \\ d'_k &= Cd_{0,k} + e_2 \quad (k = \frac{3n}{4} + 1, \dots, n - 1) \\ d'_n &= Cd_{0,n}. \end{aligned} \quad (4.3)$$

Thus, K_2 has the same lower half tuple elements as K_{3,j_n} and in the upper half there is the n -tuple $(d_{0,1}, \dots, d_{0,n})$. In this definition we do not specify the n -tuple $(d_{0,1}, \dots, d_{0,n})$, but it will be chosen as a sufficiently difficult n -tuple.

In K_{3,j_n} our chosen algorithm may fast find a solution and stop for any j_n , but we are only interested at such j_n that give no solution. The tuple K_2 can be split into two n -tuples: the lower half tuple with elements smaller than C and the upper half tuple that has the higher bit parts. In K_2 the algorithm usually does not stop to a solution of the lower half tuple since the upper half tuple is usually not satisfied by c_k that satisfy the lower half knapsack.

Lemma 6. *The time for the chosen algorithm to solve $K_{1,j_{n_0}}$ is not larger than the median computation time for the algorithm for solving $K_{3,j_{n_0}}$ when j_n ranges over all values where $j_{n,h} = j_{n_0,h}$.*

Proof. In $K_{3,j_{n_0}}$ the indices $k > n_1$ give $\frac{(n+4)n}{16}$ values of j for a knapsack problem in the indices $k \leq n$. Let us name these values j'_i where $i, i = 1, \dots, \frac{(n+4)n}{16}$.

In the indices $k = 1, \dots, n_1$ there is the worst in the median n_1 -tuple. The values j'_i that we get are a sample of all possible values j_{n_1} for the knapsack problem for this worst in the median n_1 -tuple.

Half of all possible values of j_{n_1} yield a longer computation time than $f(n_1)$ in the worst in the median knapsack problem for n_1 because $f(n_1)$ is the median computation time. If the values of j'_i that we get are a representative sample of all j_{n_1} , then about half of the values of j'_i that do not give a solution yield a longer computation time than $f(n_1)$.

We can select e_1 and e_2 from an exponential set of numbers. Therefore we can assume that the numbers j'_i are sufficiently well randomly distributed over the possible range of the numbers j_{n_1} for the knapsack problem for $n_1 = n/2$ and they are a representative sample of all numbers j_{n_1} .

Also, because the numbers j'_i are sufficiently randomly distributed over all possible values of j_{n_1} we may assume that about half of the values j'_i are on the range (3.4).

There are more values j'_i to check in $K_{3,j_{n_0}}$ than the $n/2$ in $K_{1,j_{n_0}}$. If there is no solution for some j_n , then it is necessary to check all j'_i before the algorithm can conclude that there are no solutions. Therefore the computation time of the chose algorithm to solve $K_{1,j_{n_0}}$ is not longer than the median computation time for the algorithm to compute $K_{3,j_{n_0}}$ when j_n ranges over all numbers that have $j_{n,h} = j_{n0,h}$. \square

The median computation time in (2.1) is calculated over the *no* instances only. Thus, *yes* instances are ignored. It is sufficient that there are at least some *no* instances so that (2.1) can be calculated. We give an argument that estimates the number of solutions to the knapsack problem (j_n, K_2) . The argument makes use of averages but it is quite sufficient for showing that there are some *no* instances for computation of (2.1) if the upper bits of K_2 are selected in a suitable way, indeed a random selection of these bits is likely to yield many *no* instances.

Lemma 7. *There are in average $2^{\frac{n}{2}}$ solutions possible choices of (c_1, \dots, c_n) that give the same sum $\sum_{k=1}^n c_k d_{o,k}$.*

Proof. The number of combinations of c_k is 2^n and the sum $\sum_{k=1}^n d_{o,k}$ is at most $2^{\frac{n}{2}}$. There are fewer combinations that yield very small or large sums and most sums are in the middle ranges. \square

Lemma 11. We can select the numbers $d_{o,k}$ in such a way that there are in average about $2^{\frac{n}{4}}$ solutions possible choices of (c_1, \dots, c_{n_1}) that give the same sum $\sum_{k=1}^{n_1} c_k d_{o,k}$.

Proof. Most random selections of the numbers $d_{o,k}$ give this result. There are fewer combinations that yield very small or large sums and most sums are in the middle ranges. \square

Lemma 12. The lower half tuple in the indices $k = n_1 + 1, \dots, n$ has only $\frac{n+4}{4}$ possible values j .

Proof. These numbers are

$$j = \sum_{k=n_1+1}^n c_k (d_{k,n,2} - Cd_{0,k}) = k_1 e_1 + k_2 e_2 \quad (4.4)$$

where $0 \leq k_1 \leq \frac{n}{4}$ and $0 \leq k_2 \leq \frac{n}{4} - 1$. \square

The elements in the worst in the median tuple for n_1 satisfy $d_{k,n} \leq \frac{2^{n_1}-1}{n_1}$ because we only consider such values of $d_{k,n}$ when finding the worst in the median tuple for n_1 . Also $e_i \leq \frac{2^{n_1}-1}{n_1}$. Thus, there is no carry from the lower half tuple to the upper half tuple.

Lemma 13. It is possible to compute the median (2.1) for K_2 .

Proof. Let us assume that the values c_k are fixed for the indices $k > n_1 + 1$. This fixes some value j that must be obtained from the knapsack in the indices $k = 1, \dots, n_1$ as the subset sum. By Lemma 12 there are only $\frac{n+4}{4}$ possible values j . The upper half tuple yields about $2^{\frac{n}{4}}$ possible solutions for a given j in the indices $k = 1, \dots, n_1$ by Lemma 11. The worst in the median tuple in the

lower half tuple has $\frac{n}{2}$ elements, thus $2^{\frac{n}{2}}$ possible numbers can be constructed as sums $\sum_{k=1}^{n_1} c_k d'_k$ in the lower half tuple. The set of the about $2^{\frac{n}{4}}$ possible solutions of the upper half tuple for a randomly selected j is a small subset of all possible combinations of c_k in the lower half tuple in the indices $k = 1, \dots, n_1$. The probability that any of the possible solutions from the upper half tuple is a solution of the lower half tuple is only on the range of $\frac{(n+4)n}{16} 2^{-\frac{n}{4}}$. The events of selecting the upper half tuple, the lower half tuple, and the value j can all be considered independent events. There are only a polynomial number of sums (4.4), thus when j_n is selected, there are only a polynomial number of possible values for the lower half of j in $(j, (d'_1, \dots, d'_{n_1}))$. For a randomly selected j_n there are then only a polynomial number of c_k , $k \leq n_1$, that satisfy the lower half bits of j_n . The choice of c_k , $k \leq n_1$, fixes the upper half of j . We are left with an upper half knapsack problem for the indices $k = n_1 + 1, \dots, n$. In this knapsack problem the elements have the size about 2^{n_1} and there are n_1 elements. Thus, for a randomly selected j_n we expect about one solution. The solution is constrained by the demand that the lower half bits give j , i.e., not all combinations are possible. We conclude that we get at least some *no* instances for computation of (2.1) for some choice of $(d_{0,1}, \dots, d_{0,n})$. \square

Lemma 14. *The time for the chosen algorithm to solve $K_{3,j_{n_0}}$ when j_n ranges over numbers satisfying $j_{n,j} = j_{n,0,h}$ is not larger than the median computation time for the algorithm for solving K_2 when j_n ranges over all values of j_n .*

Proof. In $K_{3,j_{n_0}}$ the upper bits are easily satisfied by selecting $c_{n,n} = 1$. In order to find a solution to the subset sum problem for K_2 the algorithm must find a common solution to two knapsacks, i.e., both the upper bits and the lower bits knapsacks in K_2 must be solved with the same numbers (c_1, \dots, c_n) . We may choose any difficult knapsack $(d_{0,1}, \dots, d_{0,n})$ to the upper bits of K_2 .

The algorithm cannot conclude that there are no solutions to the whole knapsack problem because there are no solutions to the upper half knapsack problem.

This is so since there almost always are many solutions to the upper half knapsack problem for any value of j : the upper half knapsack problem has n elements of the bit length at most $n/2$. This means that there are 2^n possible combinations of c_k and they are mapped to $2^{n/2}$ different numbers j . Each number j is likely to come from many combinations of c_k since in average $2^{n/2}$ combinations give the same j .

It is also not possible to the algorithm to check that none of the solutions to the upper half knapsack problem give a solution to the lower half knapsack problem. This is so because there are exponentially many (i.e., $2^{n/2}$) solutions to the upper half knapsack problem. They cannot be checked in a polynomial time.

Because of these two reasons the median computation time of $K_{3,j_{n0}}$ when j_n ranges over all j_n that has the same high bits as j_{n0} cannot be higher than the median computation time for K_2 where j_n ranges over all numbers. In the computation of the median time we only take cases of j_n where there is no solution and a more complicated n -tuple must give a longer time for concluding that there are no solutions. \square

Lemma 15. *The inequality (2.6) holds for the chosen algorithm.*

Proof. By Lemma 6 the median computation time for $K_{3,j_{n0}}$ when the median is taken over the set of j_n having $j_{n,h} = j_{n0,h}$ is at least as high as the time to solve $K_{1,j_{n0}}$. By Lemma 13 we can calculate the median of computation times over cases when there is no solution for K_2 . By Lemma 14 the median computation time for K_2 when j_n ranges over all values is not smaller than the median computation time for $K_{3,j_{n0}}$ when the median is computed over the set j_n where $j_{n,h} = j_{n0,h}$. As K_2 is a fixed n -tuple it follows from the definition of the worst in the median tuple that K_2 has at most as long median computation time as the worst in the median tuple for n , i.e., $f(n)$. Thus the inequality (2.6) holds. \square

Theorem 1. *Let an algorithm for the knapsack problem be selected. There exist numbers $B, \alpha \in \mathbb{R}$, $B \geq 1$, $\alpha \geq 0$ and a sequence*

$$((j_n, (d_{1,n}, \dots, d_{n,n})))_{n \geq 1}$$

of knapsacks satisfying

$$\log_2 j_n < Bn^\alpha, \log_2 d_{k,n} < Bn^\alpha, (1 \leq k \leq n), (n \geq 1)$$

such that the algorithm cannot determine in polynomial time if there exist binary numbers $c_{k,n}$, $1 \leq k \leq n$, satisfying

$$j_n = \sum_{k=1}^n c_{k,n} d_{k,n}.$$

Proof. The idea of this proof is to compare the computation time of the worst (in some sense) knapsack of size n to the computation time of (in the same sense) worst knapsack of $\frac{n}{2}$. The computation time was defined in (2.4) and denoted by $f(n)$. By Lemma 15 the inequality (2.6) holds for an arbitrary chosen algorithm. By Lemma 2 the arbitrarily chosen algorithm is not a polynomial time algorithm.

□

Theorem 2. \mathbf{P} does not equal \mathbf{NP} .

Proof. The knapsack problem is well known to be in \mathbf{NP} . □

References

1. S. Cook, The P versus NP problem. *available on-line at* www.claymath.org.
2. D. L. Kreher and D. R. Stinson, Combinatorial algorithms, generation, enumeration, and search, CRC Press, Boca Raton, 1999.

5 Annex

Lemma A1. Let $B \geq 1$, $\alpha \geq 0$ and $\gamma \geq 0$ be selected. Let $r_n > 0$ and j_n be integers satisfying

$$r_n < n^\gamma, \quad \log_2 j_n < Bn^\alpha \quad (n \geq 1).$$

There exist numbers $C, \beta \in \mathbb{R}, C \geq 1, \beta \geq 0$ and an algorithm that given any sequence of knapsacks

$$((j_n, (d_{1,n}, \dots, d_{n,n})))_{n \geq 1}$$

can determine for each n if there exist binary numbers $c_{k,n}$, $1 \leq k \leq n$, such that

$$j_n \equiv \sum_{k=1}^n c_{k,n} d_{k,n} \pmod{r_n}. \quad (A1)$$

The number N_n of elementary operations needed by the algorithm satisfies $N_n < Cn^\beta$ for every $n > 1$.

Proof. The bound on the logarithm of j_n guarantees that modular arithmetic operations on $d_{k,n}$ can be made in polynomial time since we can assume that $d_{k,n} \leq j_n$. We can find the numbers $c_{k,n}$ by computing numbers $s_{k,j,n}$ from the recursion equations for k

$$s_{k,j,n} = s_{k-1,j,n} + s_{k-1,(j-d_{k,n}) \pmod{r_n},n} \quad (A2)$$

$$s_{0,j,n} = \delta_{j=0},$$

where the index j ranges from 0 to $r_n - 1$ and is calculated modulo r_n . The index n is fixed and only indicates that the numbers are for the n^{th} knapsack. Here δ_x is an indicator function: $\delta_x = 1$ if the statement x (i.e., j equals 0 in (A2)) is

true and $\delta_x = 0$ if x is false. Let

$$G_{k,n}(x) = \sum_{j=0}^{r_n-1} s_{k,j,n} x^j,$$

where $|x| < 1$. From (A2) follows

$$\sum_{j=0}^{r_n-1} s_{k,j,n} x^j = \sum_{j=0}^{r_n-1} s_{k-1,j,n} x^j + \sum_{j=0}^{r_n-1} s_{k-1,(j-d_{k,n})(\bmod r_n),n} x^j.$$

Changing summation to $j' = j - d_{k,n}$ yields

$$G_{k,n}(x) = G_{k-1,n}(x) + \sum_{j'=-d_{k,n}}^{r_n-1-d_{k,n}} s_{k-1,j'(\bmod r_n),n} x^{j'+d_{k,n}}.$$

Changing the order of summation of j' shows that

$$G_{k,n}(x) = G_{k-1,n}(x) + x^{d_{k,n}} \sum_{j'=0}^{r_n-1} s_{k-1,j',n} x^{j'}. \quad (A3)$$

Simplifying (A3) gives

$$G_{k,n}(x) = G_{k-1,n}(x) + x^{d_{k,n}} G_{k-1,n}(x).$$

As $G_{0,n}(x) = s_{0,0,n} = 1$, we get

$$G_{n,n}(x) = \prod_{k=1}^n (1 + x^{d_{k,n}}).$$

Expanding the product shows that $s_{k,j,n} \neq 0$ if and only if there exist binary numbers c_m , $c_m \in \{0, 1\}$, $1 \leq m \leq n$, satisfying

$$j \equiv \sum_{m=1}^n c_m d_{m,n} \pmod{r_n}.$$

For $j = j_n$ and $k = n$ we get the knapsack problem. This means that we can solve the knapsack problem by computing all $s_{k,j,n}$ from (A2). We do not actually need the numbers $s_{k,j,n}$ but only the information if $s_{k,j,n} \neq 0$. Therefore we will not compute the terms $s_{k,j,n}$ directly but calculate binary numbers $b_{j,k} \in \{0, 1\}$ by Algorithm A0 below. The number $b_{k,j}$ calculated by A0 is zero if and only if the number $s_{k,j,n} = 0$ is zero.

Algorithm A0:

```

Loop from  $k = 0$  to  $k = n$  with the step  $k := k + 1$  do {
    Loop from  $j = 0$  to  $j = r_n - 1$  with the step  $j := j + 1$  do
         $b_{j,k} := 0$ 
    }
     $b_{0,0} := 1$ 
    Loop from  $k = 1$  to  $k = n$  with the step  $k := k + 1$  do {
         $M := \min\{r_n - 1, \sum_{m=1}^k d_{m,n}\}$ 
        Loop from  $j = 0$  to  $j = M$  with the step  $j := j + 1$  do {
            If  $(b_{k-1,j} = 0 \text{ and } b_{k-1,(j-d_{k,n}) \pmod{r_n}} = 0)$  do  $b_{j,k} := 0$ 
            else do  $b_{j,k} := 1$ 
        }
    }
    If  $b_{n,j_n} = 1$  do  $result := TRUE$  else do  $result := FALSE$ 
}

```

Algorithm A0 loops from $k = 0$ to $k = n$ and from $j = 0$ to $j = r_n - 1 < n^\gamma$.

Thus A0 needs a polynomial number of elementary operations as a function of n in order to give the result $TRUE$ or $FALSE$ to the existence of a solution to (A1). \square

Lemma A2. *Let $B, \alpha \in \mathbb{R}$, $B \geq 1$, $\alpha \geq 0$ be fixed. There exist numbers $C, \beta \in \mathbb{R}$, $C \geq 1$, $\beta \geq 0$ and an algorithm that for any sequence*

$$((j_n, (d_{1,n}, \dots, d_{n,n})))_{n \geq 1}$$

of knapsacks satisfying

$$j_n \leq Bn^\alpha, \quad d_{k,n} \leq j_n \quad (1 \leq k \leq n),$$

can determine if there exist binary numbers $c_{k,n}$, $1 \leq k \leq n$, such that

$$j_n = \sum_{k=1}^n c_{k,n} d_{k,n}.$$

The number N_n of elementary operations needed by the algorithm satisfies $N_n < Cn^\beta$ for every $n > 1$.

Proof. The result follows directly from Lemma A1 by selecting $r_n = \sum_{k=1}^n d_{k,n} \leq nj_n$. \blacksquare