

The Asymmetric Traveling Salesman Problem

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I. Introduction. Starting with $M(a)$, an $n \times n$ asymmetric cost matrix, Jonker and Volgenannt [1] transformed it into a $2n \times 2n$ symmetric cost matrix, $M(s)$, where $M(s)$ has unusual properties. One such property is that an optimal tour in $M(s)$ yields an optimal tour in $M(a)$. Modifying $M(s)$, we apply the modified Floyd-Warshall algorithm given in [2] to $M(s)$. Let T be a tour that is an upper bound for an optimal tour in $M(a)$. Due to the structure of $M(s)$, we either can always obtain an optimal tour in $M(s)$ that is derived from only one minimal positively-valued cycle in $\sigma_T^{-1}M^-$ whose value is less than $|T|$, (i.e., we don't have to link circuits), or else $T = T_{OPT}$. Thus, we can obtain an optimal tour in $M(a)$ in at most polynomial running time. If the proof of theorem 1 in section II is correct, since the asymmetric traveling salesman problem is NP-hard, P would equal NP.

II. A Theorem

Theorem 1.

Let

$$M(a) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

be an asymmetric cost matrix where $a_{ii} = \infty$, $i = 1, 2, \dots, n$. Using a modified version of the symmetric cost matrix, $M(s)$, obtained by Jonker and Volgenannt [1] as well as a result of Kleiman in [2] and the use of the modified F-W algorithm, we prove that we always can obtain an optimal solution to $M(a)$ in polynomial time.

Proof. If $M(a)$ contains a non-positive entry, let m be the smallest value of all the entries in $M(a)$. We then add $-m+1$ to each entry of $M(a)$. Thus, each of the entries in $M(a)$ now has a positive value. Jonker and Volgenannt [1] gave a method for transforming an $n \times n$ asymmetric cost matrix into a $2n \times 2n$ symmetric cost matrix such that an optimal tour in the latter yields an optimal tour in the former. Let M_∞ be an $n \times n$

matrix each of whose entries is ∞ . Furthermore, we change each diagonal entry of $M(a)$ into 0 to obtain the

matrix $M(a)_d$. Finally, we define $M(a)_d^T = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix}$. We then define the $2n \times 2n$ symmetric matrix

$\begin{pmatrix} M_\infty & M(a)_d \\ M(a)_d^T & M_\infty \end{pmatrix}$ as $M(s)$. We use any algorithm that yields an upper bound, say $T_{UPPERBOUND} = (t_1 \ t_2 \ \dots \ t_n)$, for an optimal tour in $M(a)$. In $M(s)$, $a_{i+n,i} = a_{n+i,i} = 0$, $i = 1, 2, \dots, n$. We now replace $T_{UPPERBOUND}$ by

$T = (t_1 \ t_{2+n} \ t_2 \ t_{3+n} \ t_3 \ \dots \ t_n \ t_{l+n})$ in $M(s)$. By construction, $|(t_{i+n} \ t_i)| = 0$, $i = 1, 2, \dots, n$ in $M(s)$. It follows that $\prod_{i=1}^{i=n} (t_{i+n} \ t_i) = \sigma_T$ where each 2-cycle $(t_{i+n} \ t_i)$ has a value of 0. We can always use a product of 2-cycles (edges) to obtain $\sigma_T^{-1} M(s)^-$. As mentioned earlier, the J-V paper proves that an optimal tour of $M(s)$ yields an optimal tour of $M(a)$. An acceptable path in $M(s)$ consists alternately of non-zero and zero arcs. We cannot link acceptable cycles of the kind found in $M(s)$ since by linking by deleting two arcs of form $(t_{i+n} \ t_i)$ or $(t_i \ t_{i+n})$, we obtain a circuit containing two consecutive *non-zero-valued* directed edges. Using the modified F-W algorithm, each cycle from a to b obtained in $M(s)$ has a value no greater than any other cycle from a to b . Thus, - using the modified F-W algorithm -, there is only one way that we could obtain T_{FWOPT} of $M(s)$: one minimal positively-valued acceptable cycle containing n arcs whose value is less than $|T|$ or – if one can't be found, $T_{UPPERBOUND} = T_{FWOPT} = T_{OPT}$. As proved in [2], this cycle always yields an *optimal* tour in $M(s)$ that yields an optimal tour in $M(a)$. We now show that the modified F-W algorithm when used for obtaining acceptable paths always obtains all acceptable paths in at most $O(n^4)$ running time. Since each such cycle is obtained using the modified F-W algorithm – together with an algorithm to insure that an acceptable path obtained stays acceptable which requires backtracking in only a smaller number of cases than otherwise – we can obtain a such a minimal positively-valued acceptable cycle containing n points of value less than $|T|$ (if it exists) in polynomial time. In particular, the Floyd-Warshall algorithm has $O(n^3)$ running time. Thus, even backtracking in every case, would raise the running time to at most $O(n^4)$.

III. The Construction of $\sigma_T^{-1}(M(s))$

$M(a)$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad a_{ii} = \infty, i = 1, 2, \dots, n .$$
 In J-V $M(a)$, $a_{ii} = -M'$ where M' is the largest value of a non-diagonal entry in $M(a)$. In $M(a)_d$, $a_{ii} = 0$. $M(a)_d$ is used in the construction of $M(s)$. In order that an optimal tour of $M(s)$ yields an optimal tour of $M(a)$, all arcs used in acceptable paths must belong to either $M(a)_d$ or $M(a)_d^T$. By applying σ_T^{-1} to the columns of $M(s)$, we obtain a matrix whose diagonal elements all have the value zero, while all other entries have a positive value. This is because $\sigma_T^{-1}(M(s)) = \sigma_T^{-1}(M(s))^+$. It follows that all acceptable paths contain only positive values, implying that all acceptable cycles have positive values.

REFERENCES

- [1] Jonker, R. and Volgenannt, T., Transforming asymmetric into symmetric traveling salesman problems, Oper. Res. Let., **2**, 161-163 (1983).
- [2] Kleiman, H., The symmetric traveling salesman problem, arXiv.org, math.CO/0509531.