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# Polynomial complexity algorithm for Max-Cut problem

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## Abstract

The standard NP-complete max-cut problem reduced to a binary quadratic program  $\min x^T Qx$  s.t  $x^2 = 1$  is studied here. Without loss of generality matrix  $Q$  can be assumed positive semidefinite. It is shown that for some  $\alpha$  a global minimum of sum of squares of polynomials  $Q(\alpha, x) = \alpha x^T Qx + \sum_i (x_i^2 - 1)^2$ , provides a solution to max-cut problem. In particular, (1) the global minimum of  $Q(\alpha, x)$  can be found with any precision by polynomial complexity semi-definite program; (2) the sign of minimal point  $x^*$  is preserved; (3) for sufficiently small  $\alpha$  the global minimum of  $Q(\alpha, x)$  is the global minimum of binary quadratic program. The certificate of solution can be found from quadratic form representation of  $Q(\alpha, x)$ . The resulting algorithm solves arbitrary max-cut problem in polynomial time, therefore P=NP.

In memory of my Grandmother

## 1 Formulation of the problem

The standard NP-complete formulation of the maximum cut problem is considered here [Pol95]. In brief, let  $G = (V, E)$  be finite, undirected, and loopless weighted graph with vertex set  $V$  and edge set  $E = \{(i, j) : i, j \in V, i \neq j\}$  with weight function  $w : E \rightarrow \mathbb{R}$ . For any subset  $S \subset V$  of vertexes, the set of edges with one end in  $S$  and the other end in  $V \setminus S$  is called the *cut* ( $\delta_G(S) := \{(i, j) | i \in S, j \notin S\}$ ) induced by  $S$ . The central problem of this paper is MAX-CUT PROBLEM: given a weighted graph  $(G, w)$  find

$$\max_{S \subset V} w(\delta_G(S)),$$

where

$$w(\delta_G(S)) = \sum_{i \in S, j \notin S} w_{i,j}.$$

## 2 Reduction to binary quadratic problem

Consider vector  $\vec{x} = \{x_i | x_i \in \{-1, +1\}\} \in \mathbb{R}^n$  defined as follows.

$$\vec{x}(S) = \begin{cases} +1 & \text{for } i \in S \\ -1 & \text{for } i \notin S \end{cases},$$

and set  $w_{i,j} = 0 \forall (i,j) \notin E$ , then

$$\begin{aligned} w(\delta_G(S)) &= \frac{1}{8} \sum_{i=1}^n \sum_{j=1}^n w_{i,j} (x_i - x_j)^2 \\ &= \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n w_{i,j} - \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n w_{i,j} x_i x_j \\ &= W - \vec{x}^T Q \vec{x}, \end{aligned} \tag{1}$$

where

$$W = \sum_{i=1}^n \sum_{j=1}^n w_{i,j}$$

is a constant defined by the graph structure. Clearly, MAX-CUT PROBLEM is equivalent to the following quadratic problem with quadratic constraints

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t. } & x_i^2 = 1. \end{aligned} \tag{2}$$

**Lemma 2.1.** Let  $x \in \mathbb{R}^n, c, d, f \in \mathbb{R}, Q \in \mathbb{R}^{n \times n}$  is symmetric matrix. The following problems have the same solutions  $x^*$ .

(1)

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t. } & x_i^2 = 1. \end{aligned}$$

(2)

$$\begin{aligned} \min \quad & x^T Q x + c \\ \text{s.t. } & x_i^2 = 1. \end{aligned}$$

(3)

$$\begin{aligned} \min \quad & x^T Q x + d x^T x \\ \text{s.t. } & x_i^2 = 1. \end{aligned}$$

(4)

$$\begin{aligned} \min \quad & (x^T Q x) f \\ \text{s.t. } & x_i^2 = 1, \\ & f > 0 \end{aligned}$$

*Proof.* (1)  $\iff$  (2)  $\iff$  (4) obvious. (2)  $\iff$  (3) in the feasible set  $x^T x = \sum_i x_i^2 \equiv n$  therefore for  $c = dn$  (2) and (3) equivalent.  $\square$

### 3 Sums of squares

#### 3.1 Summary of [Par03]

Let bounded below polynomial  $f(x_1, x_2, \dots, x_n)$  has even degree  $2d$ ,  $f^* = \min f(x), x \in \mathbb{R}^n$ ,  $X$  denote a column vector of all monomials in  $x_1, x_2, \dots, x_n$  of degree at most  $d$ . The length of the vector  $X$  equals a binomial coefficient

$$N = \binom{n+d}{d}.$$

Let  $L_f$  denote set of real symmetric matrices  $A$  such that  $f(x) = X^T A X$ . Assume that the constant monomial 1 is the first entry of  $X$ . Let  $E_{11}$  denote the matrix unit whose only nonzero entry is a one in the upper left corner.

**Lemma 3.1** (Lemma 3.1. in [Par03]). *For any real number  $\lambda$ , the following two are equivalent:*

- *The polynomial  $f(x) - \lambda$  is a sum of squares (sos) in  $\mathbb{R}[x]$ .*
- *There is a matrix  $A \in L_f$  such that  $A - \lambda E_{11}$  is positive semi-definite, that is, all eigenvalues of this symmetric matrix are non-negative reals.*

We write  $f^{sos}$  for the largest real number  $\lambda$  for which the two equivalent conditions are satisfied. We always have  $f^* \geq f^{sos}$  [Par03, Sho87].

**Theorem 3.2** (Theorem 3.2. in [Par03]). *Fix  $\deg(f) = 2d$  and let the number of variables  $n$  vary. Then there exists a polynomial-time algorithm, based on semi-definite programming, for computing  $f^{sos}$  from  $f$ . The same statement holds if  $n$  is fixed and  $d$  varies.*

### 3.2 $f^{sos}$ for sum of square polynomial

**Lemma 3.3.** *Let  $f(x) = \sum_i p_i(x)^2$  be a sum of squares of some polynomials  $p_i(x)$ ,  $\lambda \in \mathbb{R}$ . Then*

1. *If  $f(x) - \lambda$  is a sum of squares then for any  $\lambda' < \lambda$ ,  $f(x) - \lambda'$  is a sum of squares.*
2.  *$f^* = f^{sos}$ .*

*Proof.* 1. Suppose  $f(x) - \lambda$  is a sum of squares, then  $f(x) - \lambda' = f(x) - \lambda + (\sqrt{\lambda - \lambda'})^2$  is a sum of squares.

2. Suppose  $f^{sos} < f^*$  than for  $\lambda = \frac{f^* - f^{sos}}{2} > 0$ ,  $f(x) - f^{sos} + \lambda$  is sum of squares i.e. non-negative, but  $\min f(x) - \lambda < 0$  - contradiction.

□

**Corollary 3.4.**  $\min f(x) - f^{sos} = 0$ ,  $A - f^{sos}E_{11}$  is singular, and  $x^* | (f(x^*) = f^*) \in \ker(A - f^{sos}E_{11})$ .

## 4 Approximation by quartic polynomial

**Lemma 4.1.** *Let  $x \in \mathbb{R}^n, z \geq 0, z \in \mathbb{R}, \alpha \in \mathbb{R}, Q'$  positive semi-definite symmetric matrix*

$$Q(x) = \sum_i (x_i^2 - 1)^2 + \alpha x^T Q' x. \quad (3)$$

*The following problem can be solved in polynomial time and space for any  $\alpha > 0$ .*

$$\min Q(x) \quad (4)$$

*Proof.*

(i) Since  $Q'$  is positive semi-definite by Cholesky decomposition  $Q' = LL^T$ , then polynomial  $Q(x)$  is a sum of squares of polynomials  $Q(x) = \sum_i (x_i^2 - 1)^2 + \alpha(L^T x)^T(L^T x)$ , therefore by Theorem 3.2  $f^{sos}$  can be found in polynomial time.

(ii) By Lemma 3.3  $f^{sos} = \min Q(x)$ . □

**Theorem 4.2.** *Let  $Q'$  be positive semi-definite symmetric matrix,  $Q(x)$  is defined as in (3),  $X(\alpha) = \{x_k(\alpha) \in \mathbb{R}^n | x_k \text{ are local minima of } Q(x)\}$ , then  $\exists \alpha^* > 0$  that for all  $0 \leq \alpha < \alpha^*$  signum  $x_k(\alpha) = \text{signum } x_k(0)$ ,  $\forall x_k(0) \in X(0)$ .*

*Proof.*

1. Extrema of  $Q(x)$  are defined by

$$\nabla_x \sum_i (x_i^2 - 1)^2 + \alpha x^T Q x = 2\alpha Q x + 4 \begin{bmatrix} (x_1^2 - 1)x_1 \\ \vdots \\ (x_n^2 - 1)x_n \end{bmatrix} = \vec{0}, \quad (5)$$

2. Let  $\alpha = 0$ . Then, by direct inspection

1.  $x_i = 0, \forall i$  is a local maximum.
2.  $x_i^2 = 1, \forall i$  are local minima.
3.  $x_i^2 = 1$ , for some  $i$ , and  $x_i = 0$  for the rest are saddle points.

3. Let  $\vec{v} = Qx$ . Then,  $\vec{v}^T \vec{v} = (x^T QQx) \leq \lambda_{\max}^2 (x^T x)$ ,  $\forall x$ , where  $\lambda_{\max}$  is the maximum eigenvalue of  $Q$ . Therefore,  $\|\vec{v}\| \leq \lambda_{\max} \|x\|$ . Therefore,  $|v_i| \leq \lambda_{\max} \|x\|, \forall i$ .

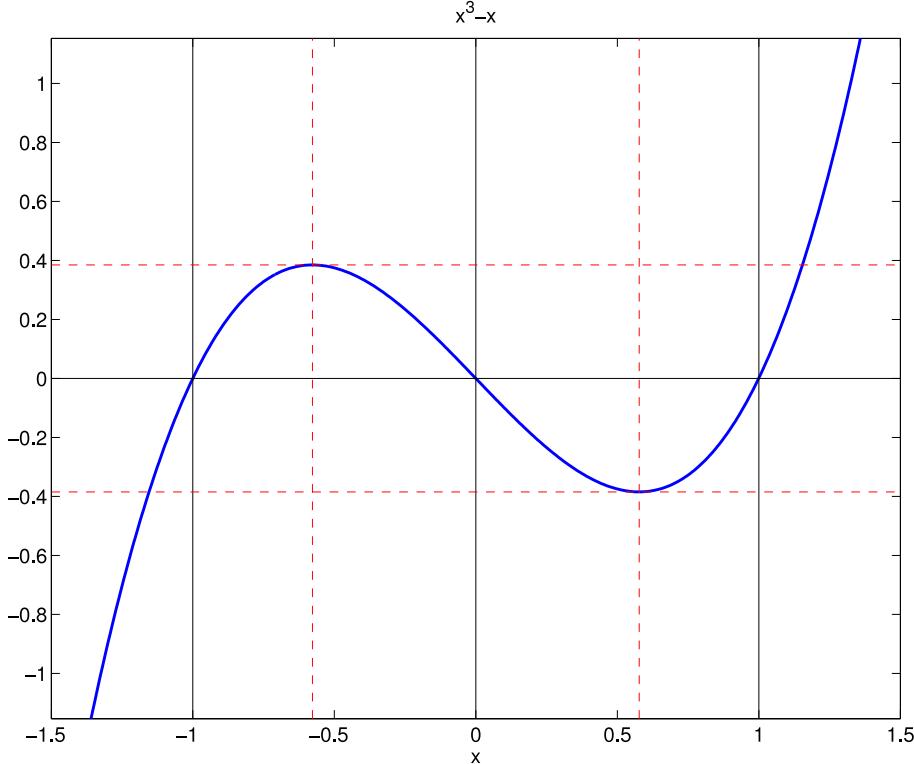


Figure 1: Graph of  $x^3 - x$

There are 2 alternatives:

1.

$$\begin{aligned} x_i(0) &= 1, & v_i < 0 &\text{ then } x_i(\alpha) > 1 \\ x_i(0) &= -1, & v_i > 0 &\text{ then } x_i(\alpha) < -1 \end{aligned} \quad (6)$$

2.

$$\begin{aligned} x_i(0) &= 1, & v_i > 0 &\text{ then } x_i(\alpha) < 1 \\ x_i(0) &= -1, & v_i < 0 &\text{ then } x_i(\alpha) > -1 \end{aligned} \quad (7)$$

Suppose  $x_i(0) = 1$ , and if  $\alpha|v_i| \leq \alpha\lambda_{\max} \|x(\alpha)\| \leq \alpha\lambda_{\max} \|x(0)\| \leq \alpha\sqrt{n}\lambda_{\max} < 2|p^*|$  then  $x_i(\alpha) > \frac{1}{\sqrt{3}}$ , i.e. the signum  $x_i(\alpha) = \text{signum } x_i(0)$ . The same argument holds for  $x_i(0) = -1$ .

Therefore,

$$\alpha^* = \frac{2|p^*|}{\sqrt{n}\lambda_{\max}}$$

□

#### 4.1 Local minima of $Q(x)$

From (5) it follows that at extremal points  $\tilde{x}$

$$\alpha \tilde{x}^T Q \tilde{x} = -2 \sum_i (\tilde{x}_i^2 - 1) \tilde{x}_i^2, \quad (8)$$

$$Q(\tilde{x}) = \sum_i (\tilde{x}_i^2 - 1)^2 - 2 \sum_i (\tilde{x}_i^2 - 1) \tilde{x}_i^2 \quad (9)$$

$$= \sum_i (\tilde{x}_i^2 - 1)(\tilde{x}_i^2 - 1) - 2(\tilde{x}_i^2 - 1) \tilde{x}_i^2 \quad (10)$$

$$= \sum_i (1 - \tilde{x}_i^2)(1 + \tilde{x}_i^2) \quad (11)$$

$$= \sum_i (1 - \tilde{x}_i^4) = n - \sum_i \tilde{x}_i^4 \quad (12)$$

Let  $x_0^2 = 1$  one of the feasible point of problem (2),  $\tilde{x} = x_0 - y$ . Then,

$$Q(y) = n - \sum_i (x_{0,i} - y_i)^4 = n - \sum_i (x_{0,i}^4 - 4x_{0,i}^3 y_i + 6x_{0,i}^2 y_i^2 - 4x_{0,i} y_i^3 + y_i^4), \quad (13)$$

$$Q(y) = n - \sum_i (1 - 4x_{0,i} y_i + 6y_i^2 - 4x_{0,i} y_i^3 + y_i^4) \quad (14)$$

$$Q(y) = 4x_0^T y - 6y^T y + \sum_i (x_{0,i} y_i^3 - y_i^4). \quad (15)$$

From (5)

$$\begin{aligned} \alpha Qx + 2 \begin{bmatrix} (x_1^2 - 1)x_1 \\ \dots \\ (x_n^2 - 1)x_n \end{bmatrix} &= \alpha Qx - 2x + 2 \begin{bmatrix} x_1^3 \\ \dots \\ x_n^3 \end{bmatrix} \\ &= (\alpha Q - 2I)x + 2 \begin{bmatrix} x_1^3 \\ \dots \\ x_n^3 \end{bmatrix} = (\alpha Q - 2I)(x_0 - y) + 2 \begin{bmatrix} (x_{0,1} - y_1)^3 \\ \dots \\ (x_{0,n} - y_n)^3 \end{bmatrix} = \\ &= (\alpha Q - 2I)x_0 - (\alpha Q - 2I)y + 2 \begin{bmatrix} x_{0,1}^3 - 3x_{0,1}^2 y_1 + 3x_{0,1} y_1^2 - y_1^3 \\ \dots \\ x_{0,n}^3 - 3x_{0,n}^2 y_n + 3x_{0,n} y_n^2 - y_n^3 \end{bmatrix} = \\ &= (\alpha Q - 2I)x_0 - (\alpha Q - 2I)y + 2 \begin{bmatrix} x_{0,1} - 3y_1 + 3x_{0,1} y_1^2 - y_1^3 \\ \dots \\ x_{0,n} - 3y_n + 3x_{0,n} y_n^2 - y_n^3 \end{bmatrix} = \\ &= \alpha Qx_0 - (\alpha Q + 4I)y + 2 \begin{bmatrix} 3x_{0,1} y_1^2 - y_1^3 \\ \dots \\ 3x_{0,n} y_n^2 - y_n^3 \end{bmatrix} = 0, \\ y &= (\alpha Q + 4I)^{-1} \left( \alpha Qx_0 + 2 \begin{bmatrix} 3x_{0,1} y_1^2 - y_1^3 \\ \dots \\ 3x_{0,n} y_n^2 - y_n^3 \end{bmatrix} \right). \end{aligned} \quad (16)$$

For  $\alpha < \alpha^*$ , (16) can be expanded in a Neumann series, and treated as a fix point equation. Since the right part of (16) is contracting operator for  $\|y_n\| < 1$ , one can apply fix point iterations, starting from  $y_0 = 0$ .

$$y = \frac{1}{4} \sum_{k=0}^{\infty} \left( \frac{\alpha}{4} Q \right)^k \begin{pmatrix} 3x_{0,1}y_1^2 - y_1^3 \\ \dots \\ 3x_{0,n}y_n^2 - y_n^3 \end{pmatrix}; \quad (17)$$

$$y_0 = 0; \quad (18)$$

$$y_1 = \sum_{k=1}^{\infty} \left( \frac{\alpha}{4} Q \right)^k x_0 = \frac{\alpha}{4} Q x_0 + \frac{\alpha^2}{16} Q^2 x_0 + o(\alpha^2) \quad (19)$$

$$y_2 = \sum_{k=1}^{\infty} \left( \frac{\alpha}{4} Q \right)^k x_0 + \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{\alpha}{4} Q \right)^k \left[ 3x_{0,i} \left[ \sum_{k=1}^{\infty} \left( \frac{\alpha}{4} Q \right)^k x_0 \right]_i^2 - \left[ \sum_{k=1}^{\infty} \left( \frac{\alpha}{4} Q \right)^k x_0 \right]_i^3 \right] \quad (20)$$

...

Keeping only quadratic terms in  $\alpha$

$$y = \frac{\alpha}{4} Q x_0 + \frac{\alpha^2}{16} Q^2 x_0 + \frac{1}{2} \left[ 3x_{0,i} \left[ \frac{\alpha}{4} Q x_0 \right]_i^2 \right] + o(\alpha^2) \quad (21)$$

$$= \frac{\alpha}{4} Q x_0 + \frac{\alpha^2}{16} Q^2 x_0 + \frac{3\alpha^2}{32} \left[ x_{0,i} [Q x_0]_i^2 \right] + o(\alpha^2) \quad (22)$$

Substituting it back to (15), and keeping quadratic terms in  $\alpha$  leads to

$$\begin{aligned} Q(\alpha; x_0) &= \alpha x_0^T Q x_0 + \frac{\alpha^2}{4} x_0^T Q^2 x_0 + \frac{3\alpha^2}{8} \sum_i x_{0,i}^2 [Q x_0]_i^2 - \frac{3\alpha^2}{8} x_0^T Q^2 x_0 + o(\alpha^2) \\ \frac{Q(\alpha; x_0)}{\alpha} &= x_0^T Q x_0 - \frac{\alpha}{8} x_0^T Q^2 x_0 + \frac{3\alpha}{8} \sum_i x_{0,i}^2 [Q x_0]_i^2 + o(\alpha). \end{aligned} \quad (23)$$

The minimum differences between cuts  $\Delta$  is not greater than minimal nonzero difference in elements of matrix  $Q'$ . The minimum and maximum values of  $x_0^T Q^2 x_0$  are  $n\lambda_{min}^2$  and  $n\lambda_{max}^2$ . The same is for the term  $\sum_i x_{0,i}^2 [Q x_0]_i^2$  (for eigenvectors aligned with  $x_0$ ). Therefore, unique solution is obtained when

$$\Delta > \alpha n \left( \frac{\lambda_{max}^2}{2} - \frac{\lambda_{min}^2}{4} \right) + o(\alpha). \quad (24)$$

## 5 Certificate of Solution

Let  $Y = \{1, \{x_i, i = 1..n\}, \{x_i x_j, i = 1..n, j = 1..n\}, z\}$  be a set of monomials. The polynomial  $Q(x, z)$  can be represented as a quadratic form  $Q(x, z) = \sum_{k,m} q_{k,m} y_k y_m$ . This representation is not

unique. For instance, the coefficient in front of  $x_1^2$  is  $q_{(x_1), (x_1)} + 2 * q_{(x_1 x_1), 1} = const$ . Therefore, for some constant  $t$  polynomial  $Q(x, z) - t$ , is non-negative, when matrix  $q_{k,m}$  is positive semi-definite, and set of linear constraints on  $q_{k,m}$  is satisfied [Par00]. The maximum value of  $t$  for which  $Q(x, z) - t = 0$ , and the set of constraints are satisfied provides the global minimum of the polynomial [Sho87], and the final matrix  $[q_{k,m}]$  is singular. The structure of singular vectors provides the solution of the problem (2).

The global minima comes in pairs.. For instance if  $x^*$  is a global minimum  $x' = -x^*$  is also a global minimum. On the other hand,  $z(x') = z(x^*)$ , and  $x'_i x'_j = x_i^* x_j^*$ . Therefore, the semi-definite problem has block-diagonal structure [Lof09]. In particular, if there is only one (up to a symmetry around origin) global optimum, than the final matrix  $q_{k,m}$  have two zero eigenvalues, and two corresponding eigenvectors. One of them corresponds to monomials  $x_i$ , and another one to monomials  $1, x_i x_j, z$ . The vector composed of signum of the components of singular eigenvector corresponding to monomials  $x_i$  is a solution to problem (2). If there are more than one global minimum (more than one possible cut) there is a permutation symmetry, that lead to further split of the block corresponding to monomials  $x_i$  onto sub-blocks corresponding permutation symmetry and the rest of monomials, and the number of the singular vectors increases.

## 6 Complexity

The number of monomials in representation of  $Q(x, z)$  is  $n^2 + n + 2$ . The number of variables in semi-definite program is limited by the size of matrix  $[q_{m,k}] - (n^2 + n + 2)^2$ . The interior-point algorithm in semi-definite program that guarantee worst case polynomial performance is second-order and require to compute Hessian with respect to all variables in  $[q_{m,k}]$ , therefore requires space is  $(n^2 + n + 2)^4$ , or  $O(n^8)$ . To compute constraints on values  $q_{m,k}$  one need to compute Kronecker product  $Q' \circ Q'$ , i.e.  $O(n^4)$  operations. In addition the polynomial complexity in time is required to solve semi-definite program, which depends on the particular algorithm, for details see [Boy04]. Overall, the complexity of the algorithm is polynomial.

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