Sets, relations, functions, and proofs

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Basic notions of (naïve) set theory; sets, elements, relations between and operations on sets; relations and their properties; functions and their properties. Examples of informal proofs: direct, indirect and counterexamples.

1 *Naïve set theory*

1.1 *Sets, elements, universe*

A *set* is a collection of entities. We use notation with curly braces "$\{\ldots\}$" to represent such a collection. If we have entities a and b, examples of sets are:

$$X = \{a\} \qquad\qquad Y = \{a, b\}$$

The entity a is an *element* of X and Y. We write this as $a \in X$ and $a \in Y$. The entity b is not an element of X. We write this as $b \notin X$.

A set is individuated by the elements it contains. This means that the order of representation of elements is irrelevant. For example, $\{a, b\} = \{b, a\}$. This also means that whenever any two sets (however obtained) contain the same elements, they are identical. In other words, for any two sets X and Y to be different, there has to be at least one element $x \in X$ such that $x \notin Y$ or some $y \in Y$ such that $y \notin X$.

It is possible for a set to have no element at all. This set is called the *empty set* and we refer to it with the symbol $\emptyset$.

Occasionally we might wish to specify the *universe* U of all entities which are under consideration.[1] Any specification of a set is then implicitly restricted to entities in U.

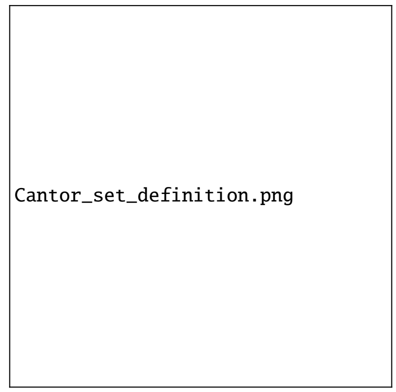

Figure 1: Passage first introducing the intuitive notion of a set from (the English translation of) Georg Cantor's *Beiträge zur Begründung der transfiniten Mengenlehre* from 1915.

[1] U need not be a set itself; it can be something bigger. But that is best left aside here. The whole concept of a universe might seem confusing at first sight. It is possible not to deeply understand what it is good for in the greater scheme of things, and still understand everything of current relevance about naïve set theory.

1.2 *Ways of describing or defining sets*

Three main methods for describing or defining sets exist:

1. by listing elements
2. by characteristic property
3. by recursive definition

The text above already gave examples for describing sets by *listing elements*. Sometimes we use notation "…" to indicate a range of elements when there is a clear intuitive ordering relation among them. Or we use "…" to abbreviate the obvious other members, even if there is no natural ordering. For example:

$X = \{2, 4, 6, 8, \dots\}$ [set of even integers bigger than 0]

$Y = \{2, 4, 6, 8, \dots, 20\}$ [set of even integers no bigger than 20]

$Z = \{\text{Russell}, \text{Wittgenstein}, \text{Frege}, \dots\}$[2] [set of authors to read]

[2] This would only be a good definition if the way to fill the "..." was absolutely clear.

To describe sets by a *characteristic property* of its elements, we might write:

$X = \{x \mid x \text{ is an even integer}\}$

$Y = \{x \mid x \text{ is an even integer no bigger than } 20\}$

$Z = \{x \mid x \text{ is a famous logician}\}$

To narrow down a reference set explicitly, we would write:[3]

[3] With a universe U in place, we should read a description like

$$X = \{x \mid \text{property of } x\}\ , \quad \text{as}$$
$$X = \{x \in U \mid \text{property of } x\}\ .$$

$X = \{x \in \{1, 2, 3, \dots\} \mid x \text{ is even}\}$

$Y = \{x \in \{1, 2, 3, \dots, 20\} \mid x \text{ is even}\}$

$Y = \{x \in \{y \mid y \text{ is a logician}\} \mid x \text{ is famous}\}$

To describe sets by *recursive definition*, we must:[4]

[4] Recursive definitions are useful because they allow for easier proofs and easier further definitions. This will become clear when we look at a recursive definition of the formulas of a logical language, to which we will then assign a meaning by exploiting the original recursive definition (keyword: "Tarski truth conditions").

(i) anchor the recursion

(ii) specify a recursion step

(iii) exclude elements untouched by anchor or recursive steps

An example is the following definition of natural numbers:

(i) 0 is a natural number

(ii) if n is a natural number, then so is $n + 1$

(iii) nothing else is a natural number

Another example is the definition of a simple formal language $\mathfrak{L}$. Unlike the previous example, we here define a set of symbols:[5]

[5] Examples of elements of $\mathfrak{L}$ by this definition are: "three", "twenty minus three", "twenty minus three plus four". Not an element of $\mathfrak{L}$ are "minus three", "plus one two", "one plus minus two".

1. words for all natural numbers are elements of $\mathfrak{L}$ (e.g., "one", "two", ...)
2. if $x, y \in \mathfrak{L}$, then so are the strings:

   "x plus y" "x minus y"

3. no string which is not constructible by this procedure is in $\mathfrak{L}$

1.3 *Important numerical sets to be familiar with*

Important sets to be familiar with are:[6]

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ [set of natural numbers]

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ [set of integers]

$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$ [set of rational numbers]

$\mathbb{R} = \{\pi, 0, e, \dots\}$ [set of real numbers]

[6] Notice the commonly used "double-stroke notation" for the capital letters used to refer to these special sets.

1.4 *Cardinality*

The number of elements in a set is called its *cardinality*. We write $|X|$ for the cardinality of X. The cardinality of X can be infinite. We then write $|X| = \infty$ and say that X is an infinite set.[7] If X is not an infinite set, it is called a *finite set*. Examples:

$|\{a\}| = 1$ $\quad$ $|\{a, b\}| = 2$

$|\emptyset| = 0$ $\quad$ $|\{2, 4, 6, 8 \dots\}| = \infty$

[7] Actually, infinite sets can have different cardinalities, so that writing $|X| = \infty$ could be misleading. For example, $|\mathbb{N}| = |Q| < |R|$. But this is not important for us at the moment.

1.5 *Relations between sets*

A set Y can contain another set X. Inversely, a set X can be an element of another set Y. We then write $X \in Y$. For example:

$X = \{a, b\}$

$Y = \{c, d, X\} = \{c, d, \{a, b\}\}$

It is important to note that $\{a, b\} \in Y$ but $a \notin Y$.

If all of the elements of X are also in Y, we say that X is a *subset* of Y, or that Y is a *superset* of X, and we write $X \subseteq Y$. If $X \subseteq Y$ and there is at least one element in Y which is not in X, we say that X is a *proper subset* of Y, or that Y is a *proper superset* of X, and we write $X \subset Y$. If X is not a (proper) subset of Y, we write $X \nsubseteq Y$ ($X \not\subset Y$). Some examples:

$\{a, b\} \subseteq \{a, b, c\}$ $\quad$ $\{a, b\} \subset \{a, b, c\}$

$\{a, b\} \nsubseteq \{a, c\}$ $\quad$ $\{a, b\} \not\subset \{a, b\}$

1.6 *Operations on sets*

Operations on sets take one or several sets as input and return another set. We consider here the power set operation and different kinds of logical operations.

The *power set* $\mathcal{P}(X)$ of X is the set of all subsets of X:

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$$

If X is finite, the cardinality of $\mathcal{P}(X)$ is $2^{|X|}$.[8] For example:

[8] This is because we decide $|X|$ times whether to include an element or not; so we collect all outcomes of $|X|$ binary decisions.

$$X = \{a, b\} \qquad |X| = 2$$

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \qquad |\mathcal{P}(X)| = 2^2 = 4$$

The following operations on sets correspond to logical operators (*and*, *or*, *not*). For any sets X and Y:

$$X \cap Y = \{z \mid z \in X \text{ and } z \in Y\} \qquad \text{[intersection]}$$

$$X \cup Y = \{z \mid z \in X \text{ or } z \in Y\} \qquad \text{[union]}$$

$$X \setminus Y = \{z \mid z \in X \text{ and } z \notin Y\} \qquad \text{[difference]}$$

$$\overline{X} = \{z \in U \mid z \notin X\} \qquad \text{[complement}^{9}\text{]}$$

Here are some examples:

$$X = \{a, b, c\} \qquad X \cup Y = \{a, b, c, d\}$$

$$Y = \{b, c, d\} \qquad X \setminus Y = \{a\}$$

$$X \cap Y = \{b, c\} \qquad Y \setminus X = \{d\}$$

A number of facts follows from the definitions so far. Some are shown in Figure 3. To conclusively show that something follows from a definition, we need the concept of a *proof*, the topic of the next section.

Figure 2: Venn diagrams of set operations.

[9] We need an explicit universe to interpret the complement operation.

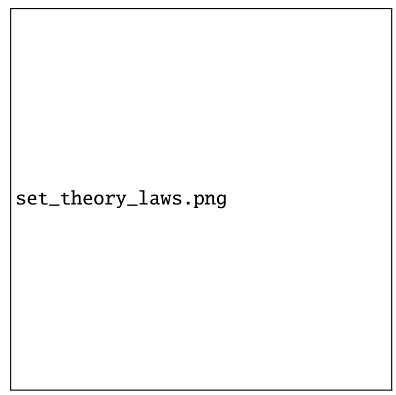

Figure 3: Some facts.

Exercise 1. Provide a natural language paraphrase for each of the following sets:

a. $A = \{5, 7, 9, 11, 13, \dots\}$

b. $B = \mathbb{N} \setminus A$

c. $C = \{c \in \mathbb{Z} \mid -2 \leq c \leq 2\}$

d. $D = \mathbb{N} \cap \overline{C}$

Exercise 2. Let's assume the following definitions:

$$X = \{a, b, c, d\}$$
$$Y = \{y \mid y \text{ is a vowel}\} = \{a, e, i, o, u\}$$
$$Z = \{z \mid z \text{ is an even natural number smaller than } 5\}$$

Write out the results of the following set operations:

a. $X \cap Y$

b. $X \cup Y$

c. $X \setminus Y$

d. $Y \setminus Z$

e. $Y \setminus Y$

f. $X \cap \overline{X}$

Exercise 3. For each of the following statements, determine whether it is true or false, using the sets X, Y and Z as defined above:

a. $X \subset Y$

b. $Y \in X$

c. $X \cap Y \subseteq X$

d. $|X| = |Y|$

e. $X \cup Z \subseteq X$

f. $X \cap Y \not\subset X$

2 Proofs

2.1 *Formal vs. informal proofs*

There are two general kinds of proofs. *Formal proofs* are rigid rule-based derivations operating on a formal language in a specific proof system.[10] *Informal proofs*, if done well, follow the structure of a formal proof but only describe the gist of it in more comprehensible language, usually a mix of natural language, specialized jargon and mathematical notation. When we speak of proofs from here on, think: informal proof.

[10] Formal proofs will be dealt with later in the context of a logic.

2.2 *Why proofs?*

Nothing can be known for certain, except mathematical-logical truth.[11] Proofs are the anchors of infallible, necessarily true knowledge. A proposition which has a valid proof must necessarily be true (in the system of logic and given the assumptions used to prove it). Therefore, proofs are the foundation of the only unshakable knowledge humankind is capable of.

[11] This is a strategically bold claim to clearly emphasize the significance of mathematical-logical knowledge. Please feel highly provoked and intrigued. Please push back, question and doubt!

2.3 *Proof strategies*

There are different kinds of proof strategies, which can be applied in different kinds of situations. Here we will look at the following four proof strategies:

(i) refutation by counterexample

(ii) direct proof

(iii) indirect proof

(iv) inductive proof

Refutation by counterexample. The perhaps easiest kind of proof is *refutation by counterexample*. It can be used to prove the falsety of a claim that some general law is true. Below is an example. The claim in Proposition 1 is a general statement about any two sets X and Y. But it is false. We can show that it is false by giving just a single instance of X and Y which refutes it.[12]

[12] We should not confuse mathematical statements with statistical generalizations, which might tolerate exceptions. It is irrelevant that there are other pairs of sets X and Y for which the claim would be true, such as: $X = \{a\}$, $Y = \{a, \{a\}\}$.

Proposition 1. The following claim is false: For any sets X and Y, if $X \in Y$, then all the elements of X are also elements of Y.

Proof. A counterexample to the claim in question is given by the following two sets:

$X = \{a, b\}$

$Y = \{c, d, X\} = \{c, d, \{a, b\}\}$

Although $X \in Y$ and $a \in X$, it is not true that $a \in Y$.[13] □

[13] To mark the end of a proof, we here use the symbol □. Another common end-of-proof notation is “QED”, short for *quod erat demonstrandum* (what was to be shown).

Here is a second example.

Proposition 2. The following claim is false: If $X \cup Y \neq \emptyset$, then $X \cap Y \neq \emptyset$.

Proof. A counterexample is $X = \{a\}$ and $Y = \{b\}$, because clearly $\{a\} \cup \{b\} = \{a, b\} \neq \emptyset$, but $\{a\} \cap \{b\} = \emptyset$. □

Direct proof. A direct proof of a statement proceeds by unraveling definitions and axioms until what needs to be shown is plain to see.

Proposition 3. For any X, $\emptyset \subseteq X$.

Proof. Consider an arbitrary set X. For a set Y be a subset of X, it is required that all elements of Y are also in X. Another way of putting this is that there cannot be a single element $y \in Y$ for which $y \notin X$. Since the empty set contains no elements at all, there cannot be any element in it, which is not also in X. □

If the statement to be proven is a conditional with *if* and *then*, then the direct proof may also use the content of the *if* part as part of its derivation.

Proposition 4. If $X \cap Y \neq \emptyset$, then $|X \cup Y| > 0$.

Proof. Suppose that $X \cap Y \neq \emptyset$. This means that there must be at least one element z that is in both X and Y. But then the number of elements that are in either X or Y must be at least one and so bigger than zero. □

Indirect proof. Direct proofs can sometimes be hard (even impossible), while a different strategy, namely an indirect proof is much easier. To indirectly prove a claim, we assume the logical opposite of what needs to be shown and derive from it a contradiction. This strategy is therefore also called *reductio ad absurdum* or *proof by refutation*. This is best demonstrated with a series of examples.

Let us start with an indirect proof for Proposition 3, for which we had a direct proof above already. The proposition is that: For any X, $\emptyset \subseteq X$.

Proof. Assume that there is an X for which $\emptyset \not\subseteq X$.[14] Then there must be an element in $\emptyset$ which is not in X. But there are no elements in $\emptyset$. So, we have a contradiction.[15] □

[14] Here is the *reductio* assumption. We simply assume the opposite of what we want to show.

[15] We derive a contradiction from the assumption that what needed to be shown is false. Hence, what needed to be shown must be true.

Here is another example.

Proposition 5. There can be at most one empty set.

Proof. Suppose that there are two empty sets.[16] Call them $\emptyset_1$ and $\emptyset_2$. Sets are individuated by the elements that they contain. So for $\emptyset_1$ and $\emptyset_2$ to be different, there needs to be an entity x such that $x \in \emptyset_1$ and $x \notin \emptyset_2$ or $x \in \emptyset_2$ and $x \notin \emptyset_1$. But since both $\emptyset_1$ and $\emptyset_2$ are empty, there cannot be such an entity x. Hence, there cannot be two empty sets. □

[16] This is the reductio assumption that the *opposite* of what we want to show is true.

And another example, this time for a conditional statement. Notice that now the *reductio* assumption is slightly more complicated.

Proposition 6. If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.

Proof. Let us assume that $X \subseteq Y$ and $Y \subseteq X$ and also that $X \neq Y$.[17] The latter means that there must be some element $x \in X$ such that $x \notin Y$ or some element $y \in Y$ such that $y \notin X$. If there is an $x \in X$ with $x \notin Y$, it cannot be that $X \subseteq Y$. If there is an $y \in Y$ with $y \notin X$, it cannot be that $Y \subseteq X$. This is a contradiction to our initial assumption. □

[17] We assume that the content of the *if*-statement is true and we also assume, towards deriving a contradiction, that the content of the *then*-statement is false, contrary to what the proposition says.

Inductive proof. There are also inductive proofs. These are more complicated, as they consist of three steps: the inductive base, the inductive assumption and the inductive step.

Inductive proofs are often useful in connection with recursive definitions. Let us consider a very simple example first. We use the following recursive definition of a set $\mathcal{F}$ of (flowery) strings:

1. **anchor:** the symbol "*" is part of $\mathcal{F}$
2. **step:** if $f \in \mathcal{F}$, then so is "(x)"
3. **exhaustion:** nothing else is in $\mathcal{F}$

Proposition 7. Each $f \in \mathcal{F}$ contains the exact same number of opening and closing parentheses.

Proof. The inductive proof is over the number n of opening parentheses.

Inductive base. Any element in $\mathcal{F}$ which has no opening parentheses is necessarily added to $\mathcal{F}$ by the recursive anchor. But since the recursive anchor introduces neither opening nor closing parentheses, the number of opening and closing parentheses for $n = 0$ is equal.

Inductive assumption. We assume that any string $f \in \mathcal{F}$ with $n = k - 1$ opening parentheses has the same number of opening and closing parentheses.

Inductive step. We now need to show that a string $f \in \mathcal{F}$ with $n = k$ opening parentheses also has $n = k$ closing parentheses. Indeed, the only way in which f can have $n > 0$ opening parentheses is by application of the recursive step. Therefore, f must be of the form $f =$ "(g)" where string $g \in \mathcal{F}$ has $k - 1$ opening parentheses. By inductive assumption, g has the same amount of opening and closing parentheses. But since $f =$ "(g)", and so exactly one parenthesis of each type is added to g, f must have an equal number of parentheses, too. □

Here is another, more difficult application of an inductive proof.

Proposition 8. The cardinality of the power set of finite set X is $|\mathcal{P}(X)| = 2^{|X|}$.

Proof. The inductive proof is over the cardinality of set X.

Inductive base. If $|X| = 0$, we know that $X = \emptyset$, so that $\mathcal{P}(X) = \{\emptyset\}$. The cardinality of $\{\emptyset\}$ is indeed $2^{|X|} = 2^0 = 1$.[18]

Inductive assumption. Assume that the claim is true for any set with cardinality of at most $n > 0$.

Inductive step. We need to show that the claim is true for any set X with $|X| = n$, given the inductive assumption. Let X be an arbitrary set with $|X| = n$. Let $x \in X$ and $Y = X \setminus \{x\}$. By inductive assumption, $|\mathcal{P}(Y)| = 2^{|Y|}$. The power set of X contains all sets in the power set of Y. It additionally also contains a version of each element in $\mathcal{P}(Y)$ that also contains x.[19] But that means that the cardinality of $\mathcal{P}(X)$ is $|\mathcal{P}(X)| = 2 \times |\mathcal{P}(Y)| = 2 \times 2^{n-1} = 2^n$. □

[18] Though strictly speaking unnecessary, we might also check the case of $n = 1$ just to be sure: If $|X| = 1$, we know that $X = \{x\}$, so that $\mathcal{P}(X) = \{\emptyset, \{x\}\}$, the cardinality of which is $2^{|X|} = 2$.

[19] This may be intuitively clear, but it might also be proven (as a so-called *lemma*). It is in this sense that these proofs are all informal: they do not spell out each and every piece of the derivation which may be plausible enough to be left out.

Exercise 4. Show that $X \cap \overline{X} = \emptyset$ for any set X.

Exercise 5. Previously, in Section 1, we defined a simple formal language $\mathfrak{L}$ recursively as the smallest set such that:

1. all words for natural numbers, i.e., "one", "two", ..., are in $\mathfrak{L}$
2. if $x, y \in \mathfrak{L}$, then so are the strings:

   "x plus y"   "x minus y"

Prove the following statement with an inductive proof strategy: No element of $\mathfrak{L}$ is a string that contains the work "bread."

Exercise 6. Using the set $\mathfrak{L}$ as constructed above, use an indirect proof to show that the length of elements of $\mathfrak{L}$ is unbounded (i.e., for any $x \in \mathfrak{L}$ there is a $y \in \mathfrak{L}$ such that y has strictly more words than x.)

3 Relations

While sets capture (what philosophers call) the *extensional meaning* of a property like "x is red", in terms of the set of all red objects (from a relevant universe), a relation captures the extensional meaning of expressions like "x is in love with y".

3.1 Tuples, Cartesian products

Recall that sets are individuated by their elements, but not by the way in which these elements are picked out or arranged. In particular $\{x, y\} = \{y, x\}$. An *ordered pair*, written as $\langle x, y\rangle$, is sensitive to ordering information, so that: $\langle x, y\rangle \neq \langle y, x\rangle$. We generalize the notion of an ordered pair to an *n-tuple*, written as $\langle x_1, x_2, \ldots, x_n\rangle$.[20,21] An n-tuple contains more information than the set of elements in that n-tuple. For example, we might be interested in the unordered set of cities that Hans visited last summer:

[20] We can allow for 1-tuples as well and think of them as just the element itself, i.e., $\langle a\rangle = a$.

[21] 3-tuples are also called triples; 4-tuples quadruples; 5-tuples quintuples ...

$$\{x \mid x \text{ is a city Hans visited last summer}\} = \{\text{London}, \text{Paris}, \text{Berlin}\}$$

or we might be interested in the cities that Hans visited in the order in which he actually visited them:

$$\langle \text{Berlin}, \text{London}, \text{Berlin}, \text{Paris}\rangle$$

The set does not give us information about the order, but also does not contain duplicates. The tuple does.

The *Cartesian product* of sets $X_1, X_2, \ldots, X_n$ is the set of all n-tuples $\langle x_1, x_2, \ldots, x_n\rangle$ such that x_i is an element from set X_i:[22]

[22] If we identify 1-tuples with the single element itself (see sidenote above), the Cartesian product of a single set X is X itself: $\{\langle x\rangle \mid x \in X\} = \{x \mid x \in X\} = X$.

$$X_1 \times X_2 \times \ldots \times X_n = \{\langle x_1, x_2, \ldots, x_n\rangle \mid x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}$$

The sets forming a Cartesian product need not be different from each other. Here are some examples for sets $X = \{a, b\}$ and $Y = \{c, d\}$:

$$X \times Y = \{\langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle b, d\rangle\}$$

$$X \times X = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle\}$$

$$X \times X \times Y = \{\langle a, a, c\rangle, \langle a, b, c\rangle, \langle b, a, c\rangle, \langle b, b, c\rangle, \\ \langle a, a, d\rangle, \langle a, b, d\rangle, \langle b, a, d\rangle, \langle b, b, d\rangle\}$$

The n-place Cartesian product with the same set X can also be written as X^n:[23]

[23] For example the set $\mathbb{R}^3$ is a three dimensional vector space. Numerical tuples, like the elements of $\mathbb{R}^3$ are *vectors*.

$$\underbrace{X \times X \times \cdots \times X}_{n \text{ times}} = X^n$$

3.2 *Relations*

An *n-place relation* R is a set of n-tuples $R \subseteq X_1 \times \cdots \times X_n$.[24,25] For example, consider the set of people $P = \{j, m, s\}$ of John, Mary and Sue and the binary relation $L \subseteq P \times P$ which encodes who loves whom:

[24] Instead of "n-place" we might also say "n-ary" and speak of the *arity* of a relation.

[25] A 1-place relation on set X is just a subset of X; a 2-place relation is called binary relation; a 3-place relation is called ternary relation; ...

$$L = \{\langle x, y\rangle \in P \times P \mid x \text{ loves } y\}$$

Suppose we live in a world in which these are the facts:

$$L = \{\langle j, j\rangle, \langle j, s\rangle, \langle m, s\rangle, \langle s, m\rangle\} \subset P \times P$$

We can visualize relation L in a diagram, like shown in Figure 4, where we draw elements as dots (possibly with labels) and where we draw an arrow from element x to element y whenever $\langle x, y\rangle$ is part of the relation.

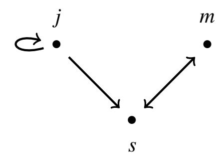

Figure 4: Diagram of relation L.

Another example is the binary relation "n_1 is the predecessor of n_2" on the set $\mathbb{N}$ of natural numbers. The predecessor relation $P \subseteq \mathbb{N} \times \mathbb{N}$, shown in Figure 5, is defined as:

$$P = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \ldots\}$$

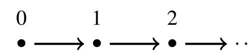

Figure 5: The predecessor relation on $\mathbb{N}$.

For a binary relation $R \subseteq X \times Y$, there are several shortcut notations to express the same content as when we write "$\langle x, y\rangle \in R$":

prefix notation: Rxy

infix notation: xRy

postfix notation: xyR

From here on we will predominantly use prefix notation, except for known mathematical relations like $\leq$ or $=$.

If $R \subseteq X \times Y$ is a binary relation, the *domain* of R is

$$\mathrm{dom}(R) = \{x \in X \mid \text{there is some } y \in Y \text{ with } Rxy\}$$

The *range* of R is

$$\mathrm{range}(R) = \{y \in Y \mid \text{there is some } x \in X \text{ with } Rxy\}$$

The *negation* of n-place relation $R \subseteq X_1 \times \cdots \times X_n$ is

$$\overline{R} = (X_1 \times \cdots \times X_n) \setminus R$$

The *convervse* of n-place relation $R \subseteq X_1 \times \cdots \times X_n$ is

$$R^{-1} = \{\langle y, x\rangle \mid Rxy\}$$

Claim 9. The following is false: If $R \subseteq X \times X$, then $\mathrm{dom}(R) = \mathrm{range}(R)$.

Proof. A counterexample is $L' = L \setminus \{\langle j, j\rangle\}$, with L being the "love relation" introduced above in Figure 4. Notice that: $L = \{\langle j, j\rangle, \langle j, s\rangle, \langle m, s\rangle, \langle s, m\rangle\}$ and so $L' = \{\langle j, s\rangle, \langle m, s\rangle, \langle s, m\rangle\}$. According to L' everybody loves someone so $\mathrm{dom}(L') = \{j, m, s\}$, but nobody loves John, so $j \notin \mathrm{range}(L') = \{m, s\}$. □

3.3 Properties of binary relations

Binary relation $R \subseteq X \times X$ is[26]

reflexive iff Rxx for all $x \in X$

irreflexive iff Rxx for no $x \in X$

symmetric iff for all $x, y \in X$ if Rxy then also Ryx

asymmetric iff for no $x, y \in X$ both Rxy and Ryx

anti-symmetric iff for all $x, y \in X$ if Rxy and Ryx, then $x = y$

transitive iff for all $x, y \in X$ if Rxy and Ryz, then also Rxz

intransitive iff for all $x, y \in X$ if Rxy and Ryz, then not Rxz

connected iff for all $x, y \in X$ either Rxy or Ryx or $x = y$

[26] In definitions like these, we often write "iff" which can be read as "if and only if" or "exactly if".

The relation L in Figure 4 does not have any of the properties above. For example, it is not reflexive because there is an element, namely m, for which Lmm is false. It is also not irreflexive because there is an element, namely j, for which Ljj is true. It is not transitive, because although Ljs and Lsm it is not the case that Ljm.

The relation "n_1 is the predecessor of n_2" from Figure 5 is irreflexive, asymmetric and intransitive. It is intransitive, because whenever x is the predecessor of y we have $x + 1 = y$, and whenever y is the predecessor of z we have $y + 1 = z$. But then $x + 2 = z$, so x is not the predecessor of z.

Proposition 10. If $R \subseteq X \times X$ is reflexive, then $\text{dom}(R) = \text{range}(R)$.

Proof. Let $R \subseteq X \times X$ be reflexive and assume towards contradiction that $\text{dom}(R) \neq \text{range}(R)$. The latter means that there is either an $x \in \text{dom}(R)$ with $x \notin \text{range}(R)$, or that there is $x \notin \text{dom}(R)$ with $x \in \text{range}(R)$. But if we take an arbitrary $x \in X$, then by reflexivity Rxx. So $x \in \text{dom}(R)$ and $x \in \text{range}(R)$, which contradicts our assumption. □

Proposition 11. If $R \subseteq X \times X$ is asymmetric, it is also irreflexive.

Proof. If $R \subseteq X \times X$ is not irreflexive, then there is at least one $x^* \in X$ such that Rx^*x^*. But then there is also a pair $x, y \in X$ (namely with $x = x^*$ and $y = x^*$) such that Rxy and Ryx. So R is not asymmetric.[27] □

[27] This is a *proof by contraposition*. To show that "if A, then B" we show that "if not B, then not A". This is justified because $p \leftarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent in propositional logic (as we will learn later).

Binary relation $R \subseteq X \times X$ is an *equivalence relation* iff R is reflexive, symmetric and transitive. Equivalence relations are interesting because they cluster elements by some criterion of sameness. Whence also the name. Given appropriate domains, the following are examples of equivalence relations:

... and ... have the same shoe size

... and ... are born in the same year

… and … have the same color (see Figure 6)

… and … have the same cardinality

equivalence_relation.png

Figure 6: Example of an equivalence relation based on property "… and … have the same color".

Based on different properties of relations, we can also define various notions of "ordering". Binary relation $R \subseteq X \times X$ is a

partial weak order iff R is reflexive, anti-symmetric and transitive
[example: relation "$\subseteq$" on $\mathcal{P}(Y)$]

partial strict order iff R is irreflexive, asymmetric and transitive
[example: relation "$\subset$" on $\mathcal{P}(Y)$]

linear weak order iff R is a partial weak order and connected
[example: relation "$\leq$" on $\mathbb{N}$]

linear strict order iff R is a partial strict order and connected
[example: relation "$<$" on $\mathbb{N}$]

Exercise 7. Which proof strategy is used in the proof of Proposition 10?

Exercise 8. Which properties do the following binary relations, expressed here in natural language, on the set of all human beings have?

a. x is taller than y

b. x is the same person as y

c. x is the father of y

d. x has the same first name as y

Exercise 9. Consider the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers and the binary relation $R \subseteq \mathbb{N} \times \mathbb{N}$ defined as follows:

$$R = \left\{\langle x, y\rangle \mid x, y \in \mathbb{N} \text{ and } y = x^2\right\}$$

Which properties does R have? Is R a partial weak/strict order? Is it a linear weak/strict order? Is it an equivalence relation?

4 *Functions*

Intuitively, a function maps each element from set X to exactly one element from some set Y (where $X = Y$ is a possibility). Functions capture uniquely referring expressions such as "the head of state x" or "the first name of x" or "the height of x" (see Figure 7).

Formally, a function $f : X \to Y$ is a relation $f \subseteq X \times Y$ such that for every $x \in X$ there is a unique $y \in Y$ with $\langle x, y \rangle \in f$. We write $f(x)$ for the unique $y \in Y$ with $\langle x, y \rangle \in f$. Alternative notation is $f : x \mapsto f(x)$. Examples of functions $f : \mathbb{N} \to \mathbb{N}$ in alternative notation styles are:

$f(x) = x + 1$ alt.: $x \mapsto x + 1$ [the successor function]

$f(x) = x$ alt.: $x \mapsto x$ [the identity function]

$f(x) = x^2$ alt.: $x \mapsto x^2$

The construction $f : x \mapsto$ "the son of x" is not necessarily a function. We might deal with a domain[28] X where some person has no son, or where some person has more than one son.

A function $f : X \to Y$ is

injective iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$

surjective iff for each $y \in Y$ there is an $x \in X$ with $f(x) = y$

bijective iff f is injective and a surjective

Alternatively, we may say that f is an *injection*, *surjection* or a *bijection*. If $f : X \to Y$ is a bijection then there exists a bijection $f^{-1} : Y \to X$ called the *inverse (function) of f* s.t. $f^{-1}(y) = x$ iff $f(x) = y$.

function_example.png

Figure 7: Example of function corresponding to description "the height of x".

[28] Since a function is a special kind of relation, all relevant terminology defined for relations (e.g., *domain*, *range*, *inverse*, ...) applies.

Exercise 10. Which of the following natural language expressions is such that its meaning could be captured by a function $f\colon X \to Y$ (rather than just a relation which is *not* a function)?

a. x admires y

b. x is the father of y

c. x is the same person as y

d. x self-identifies as gender y

Exercise 11. Look at the following relation $R \subseteq \{a, b, c, d\} \times \{u, v, w, x, y, z\}$:

$$R = \{\langle a, v\rangle, \langle b, z\rangle, \langle c, z\rangle, \langle a, w\rangle\}$$

Is this a function? If so, is it an injection, surjection or a bijection? If not, can you make a minimal change so that it is a function (e.g., adding or subtracting some element in R)? Then, after any change, is it now an injection, surjection or a bijection?

Exercise 12. Consider the claim that any function f which is not an injection is a surjection. Is this true or false? Whatever you think it is, can you prove it?

5 *Revisiting cardinality*

Now that we have increased our mathematical toolkit we can go back to the first section and reexamine some of the terms we've brushed against back in section 1.

5.1 *Finite sets*

A set A is called finite if there exists a bijection $f : \{0, 1, ..., n-1\} \to A$ for some $n \in \mathbb{N}$. We call n the cardinality of A denoted by $|A|$.

Proposition 12. The cardinality is uniquely defined for all finite sets.

Proof. TODO! □

Proposition 13. If A and B are finite then $|A| \leq |B|$ iff there exists an injection $f : A \to B$.

Proof. TODO! □

Proposition 14. If A and B are finite then $|A| = |B|$ iff there exists a bijection $f : A \to B$.

Proof. TODO! □

5.2 *Infinite sets*

A set that is not finite is called *infinite*. We now generalise cardinalities to infinite sets.

Properties of cardinality. For any two sets A, B; $|A| \leq |B|$ iff there exists an injection $f : A \to B$, and $|A| = |B|$ iff f is bijective. Note that we've proven these properties for finite sets already in the previous section, but for infinite sets they are axiomatic, we simply assert them to be true. [29]

We say that the cardinality of some set A is *strictly less* than those of the set B, denoted $|A| < |B|$, iff $|A| \leq |B|$ and $|A| \neq |B|$. Let's see if infinite sets behave like we would expect them to in relation to finite ones:

Proposition 15. If A is finite and B is infinite, then $|A| < |B|$.

Proof. TODO! □

Countable sets. A set A is called *countable* iff it is either finite or infinite with $|A| = |\mathbb{N}|$. An infinite countable set is called *countably infinite*. Sets which are not countable are called *uncountable*.

Proposition 16. There exists at least one uncountable set.

Proof. TODO! □

[29] If you're wondering why somenone would define a size without an explicit relation to some numberic value, you may be suprised to learn that that's what we've been doing all along. Set-theoretically, a natural number is just a set of it's predecessors, so

$$
\begin{aligned}
0 &= \emptyset \\
1 &= \{0\} = \{\emptyset\} \\
2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\
&\vdots \\
n &= \{0, 1, ..., n-1\}
\end{aligned}
$$

The natural numbers are just an arbitrary collection of sets with an intuitive interpretation that we choose to compare other sets against.