第七、八周作业参考答案

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习题 1

狄拉克方程的平面波解如下：

$$u^s(\vec{p})=\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix},\quad v^s(\vec{p})=\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ -\sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}\tag{1}$$

请证明如下关系：

$$\begin{gathered}u^r(\vec{p})^\dagger\cdot u^s(\vec{p})=2\omega_p\delta^{rs},\qquad v^r(\vec{p})^\dagger\cdot v^s(\vec{p})=2\omega_p\delta^{rs}\\ \bar{u}^r(\vec{p})\cdot u^s(\vec{p})=2m\delta^{rs},\qquad \bar{v}^r(\vec{p})\cdot v^s(\vec{p})=-2m\delta^{rs}\\ \bar{u}^s(\vec{p})\cdot v^r(\vec{p})=0,\qquad u^r(\vec{p})^\dagger\cdot v^s(-\vec{p})=0\\ \textstyle\sum_{s=1}^2 u^s(\vec{p})\bar{u}^s(\vec{p})=\not{p}+m,\quad \sum_{s=1}^2 v^s(\vec{p})\bar{v}^s(\vec{p})=\not{p}-m\end{gathered}\tag{2}$$

解：

$$u^r(\boldsymbol{p})^\dagger u^s(\boldsymbol{p})=\begin{pmatrix}\xi^{r\dagger}\sqrt{p\cdot\sigma},\xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}=\xi^{r\dagger}p\cdot\sigma\xi^s+\xi^{r\dagger}p\cdot\bar{\sigma}\xi^s=2\omega_p\delta^{rs}\tag{3}$$

$$v^r(\boldsymbol{p})^\dagger v^s(\boldsymbol{p})=\begin{pmatrix}\xi^{r\dagger}\sqrt{p\cdot\sigma},-\xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ -\sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}=\xi^{r\dagger}p\cdot\sigma\xi^s+\xi^{r\dagger}p\cdot\bar{\sigma}\xi^s=2\omega_p\delta^{rs}\tag{4}$$

$$\begin{aligned}\bar{u}^r(\boldsymbol{p})u^s(\boldsymbol{p})&=\begin{pmatrix}\xi^{r\dagger}\sqrt{p\cdot\sigma},\xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\gamma^0\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}\\ &=\begin{pmatrix}\xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}},\xi^{r\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}=\xi^{r\dagger}(\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})}+\sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)})\xi^s\\ &=2m\delta^{rs}\end{aligned}\tag{5}$$

$$\bar{v}^r(\boldsymbol{p})v^s(\boldsymbol{p}) = \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma}, -\xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix}\gamma^0 \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}$$
$$= \begin{pmatrix} -\xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{r\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} = -\xi^{r\dagger}(\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} + \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)})\xi^s$$
$$= -2m\delta^{rs} \tag{6}$$

$$\bar{u}^r(\boldsymbol{p})v^s(\boldsymbol{p}) = \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix}\gamma^0 \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}$$
$$= \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{r\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} = \xi^{r\dagger}(\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} + \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)})\xi^s$$
$$= 2m\delta^{rs} \tag{7}$$

$$\bar{u}^r(\boldsymbol{p})v^s(\boldsymbol{p}) = \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix}\gamma^0 \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}$$
$$= \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{r\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} = \xi^{r\dagger}(\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} - \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)})\xi^s$$
$$= 0 \tag{8}$$

$$u^r(\boldsymbol{p})^\dagger v^s(-\boldsymbol{p}) = \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\xi^s \\ -\sqrt{p\cdot\sigma}\xi^s \end{pmatrix} = \xi^{r\dagger}(\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} - \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)})\xi^s$$
$$= 0 \tag{9}$$

$$\begin{aligned} \sum_s u^s(\boldsymbol{p})\bar{u}^s(\boldsymbol{p}) &= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \\ &= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = \not{p} + m \end{aligned} \tag{10}$$

$$
\begin{aligned}
\sum_s v^s(\boldsymbol{p})\bar{v}^s(\boldsymbol{p}) &= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \begin{pmatrix} -\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \\
&= \sum_s \begin{pmatrix} -\sqrt{p\cdot\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & -\sqrt{p\cdot\bar{\sigma}}\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & -\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} = \not{p} - m
\end{aligned}
\tag{11}
$$

习题 2

计算如下场算符分别在 P,T,C 作用下的变换:

$$
\begin{aligned}
&i\bar{\psi}\gamma^5\psi \\
&\bar{\psi}\gamma^\mu\psi \\
&\bar{\psi}\gamma^\mu\gamma^5\psi
\end{aligned}
\tag{12}
$$

解:

$$
\begin{aligned}
&Pi\bar{\psi}\gamma^5\psi P^{-1} = i\bar{\psi}(t,-\vec{x})\gamma^0\gamma^5\gamma^0\psi(t,-\vec{x}) = -i\bar{\psi}\gamma^5\psi(t,-\vec{x}) \\
&Ti\bar{\psi}\gamma^5\psi T^{-1} = -i\left(\gamma^1\gamma^3\psi(-t,\vec{x})\right)^\dagger\gamma^{0*}\gamma^{5*}\gamma^1\gamma^3\psi(-t,\vec{x}) = -i\psi^\dagger\gamma^0\gamma^5\psi(-t,\vec{x}) = -i\bar{\psi}\gamma^5\psi(-t,\vec{x}) \\
&Ci\bar{\psi}\gamma^5\psi C^{-1} = i\left(-i\gamma^0\gamma^2\psi\right)^{\mathrm{T}}\gamma^5\left(-\bar{\psi}i\gamma^0\gamma^2\right)^{\mathrm{T}} = -i\left(\bar{\psi}i\gamma^0\gamma^2\gamma^{5\mathrm{T}}i\gamma^0\gamma^2\psi\right)^{\mathrm{T}} = \bar{\psi}i\gamma^5\psi
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&P\bar{\psi}\gamma^\mu\psi P^{-1} = \left(\gamma^0\psi\right)^\dagger\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec{x}) = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec{x}) = \begin{cases} \bar{\psi}\gamma^0\psi(t,-\vec{x}) \\ -\bar{\psi}\gamma^i\psi(t,-\vec{x}) \end{cases} \\
&T\bar{\psi}\gamma^\mu\psi T^{-1} = \bar{\psi}\left(-\gamma^1\gamma^3\right)\gamma^{\mu*}\gamma^1\gamma^3\psi(-t,\vec{x}) = \begin{cases} \bar{\psi}\gamma^0\psi(-t,\vec{x}) \\ -\bar{\psi}\gamma^i\psi(-t,\vec{x}) \end{cases} \\
&C\bar{\psi}\gamma^\mu\psi C^{-1} = \left(-i\gamma^0\gamma^2\psi\right)^{\mathrm{T}}\gamma^\mu\left(-\bar{\psi}i\gamma^0\gamma^2\right)^{\mathrm{T}} = -\left(\bar{\psi}i\gamma^0\gamma^2\gamma^{\mu\mathrm{T}}i\gamma^0\gamma^2\psi\right)^{\mathrm{T}} = -\bar{\psi}\gamma^\mu\psi
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&P\bar{\psi}\gamma^\mu\gamma^5\psi P^{-1} = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^5\gamma^0\psi = \begin{cases} -\bar{\psi}\gamma^0\psi(t,-\vec{x}) \\ \bar{\psi}\gamma^i\psi(t,-\vec{x}) \end{cases} \\
&T\bar{\psi}\gamma^\mu\gamma^5\psi T^{-1} = \left(\gamma^1\gamma^3\psi\right)^\dagger\gamma^0\gamma^{\mu*}\gamma^{5*}\gamma^1\gamma^3\psi = \begin{cases} \bar{\psi}\gamma^0\psi(-t,\vec{x}) \\ -\bar{\psi}\gamma^i\psi(-t,\vec{x}) \end{cases} \\
&C\bar{\psi}\gamma^\mu\gamma^5\psi C^{-1} = -\left(\bar{\psi}i\gamma^0\gamma^2\gamma^{5\ \mathrm{T}}\gamma^{\mu\mathrm{T}}i\gamma^0\gamma^2\psi\right)^{\mathrm{T}} = \bar{\psi}\gamma^\mu\gamma^5\psi
\end{aligned}
\tag{15}
$$

习题 3

在标量 Yukawa 理论中，拉氏量为:

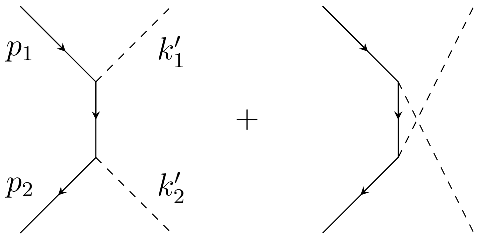
$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - g\phi\bar{\psi}\psi
\tag{16}
$$

请分别计算 $e^+e^- \to \phi\phi$ 和 $e^-\phi \to e^-\phi$ 的最低阶非平庸 S 矩阵元模方 $\bar{\sum}_s|M|^2$ ，需要对矩阵元自旋求和，及对初态自旋求平均，结果以 Mandelstam 变量表达。如果可能的话，请用交叉对称性（crossing symmetry）解释这两个结果的关系。

解：

对于 $e^+e^- \to \phi\phi$

$$i\mathcal{M}(e^+e^- \to \phi\phi) =$$



即，

$$i\mathcal{M} = \bar{v}_{s_2}(p_2)(-ig)\frac{i(\not{p}_1 - \not{k}_1' + m)}{t - m^2}(-ig)u_{s_1}(p_1) + \bar{v}_{s_2}(p_2)(-ig)\frac{i(\not{p}_1 - \not{k}_2' + m)}{u - m^2}(-ig)u_{s_1}(p_1) \tag{17}$$

简记，$u_1 \equiv u_{s_1}(p_1)$ 和 $v_2 \equiv v_{s_2}(p_2)$。使用 $\not{p}_1 u_1 = m u_1$ 简化，

$$\mathcal{M} = g^2 \bar{v}_2 \left[\frac{\not{k}_1' - 2m}{t - m^2} + \frac{\not{k}_2' - 2m}{u - m^2}\right] u_1 \tag{18}$$

$$\mathcal{M}^* = g^2 \bar{u}_1 \left[\frac{\not{k}_1' - 2m}{t - m^2} + \frac{\not{k}_2' - 2m}{u - m^2}\right] v_2 \tag{19}$$

因此，

$$\begin{aligned}|\mathcal{M}|^2 = &+ \frac{g^4}{(m^2 - t)^2}\operatorname{Tr}\left[(v_2\bar{v}_2)(\not{k}_1' - 2m)(u_1\bar{u}_1)(\not{k}_1' - 2m)\right] \\ &+ \frac{g^4}{(m^2 - u)^2}\operatorname{Tr}\left[(v_2\bar{v}_2)(\not{k}_2' - 2m)(u_1\bar{u}_1)(\not{k}_2' - 2m)\right] \\ &+ \frac{g^4}{(m^2 - t)(m^2 - u)}\operatorname{Tr}\left[(v_2\bar{v}_2)(\not{k}_1' - 2m)(u_1\bar{u}_1)(\not{k}_2' - 2m)\right] \\ &+ \frac{g^4}{(m^2 - t)(m^2 - u)}\operatorname{Tr}\left[(v_2\bar{v}_2)(\not{k}_2' - 2m)(u_1\bar{u}_1)(\not{k}_1' - 2m)\right].\end{aligned} \tag{20}$$

对初态自旋求平均，末态求和，

$$\bar{\sum_s}|\mathcal{M}|^2 = g^4\left[\frac{\langle\Phi_{tt}\rangle}{(m^2 - t)^2} + \frac{\langle\Phi_{uu}\rangle}{(m^2 - u)^2} + \frac{\langle\Phi_{tu}\rangle + \langle\Phi_{ut}\rangle}{(m^2 - t)(m^2 - u)}\right], \tag{21}$$

其中，

$$\begin{aligned}\langle\Phi_{tt}\rangle &= \frac{1}{4}\operatorname{Tr}\left[(\not{p}_2 - m)(\not{k}_1' - 2m)(\not{p}_1 + m)(\not{k}_1' - 2m)\right], \\ \langle\Phi_{uu}\rangle &= \frac{1}{4}\operatorname{Tr}\left[(\not{p}_2 - m)(\not{k}_2' - 2m)(\not{p}_1 + m)(\not{k}_2' - 2m)\right], \\ \langle\Phi_{tu}\rangle &= \frac{1}{4}\operatorname{Tr}\left[(\not{p}_2 - m)(\not{k}_1' - 2m)(\not{p}_1 + m)(\not{k}_2' - 2m)\right], \\ \langle\Phi_{ut}\rangle &= \frac{1}{4}\operatorname{Tr}\left[(\not{p}_2 - m)(\not{k}_2' - 2m)(\not{p}_1 + m)(\not{k}_1' - 2m)\right].\end{aligned} \tag{22}$$

而，

$$
\begin{aligned}
\langle\Phi_{tt}\rangle =&\frac{1}{4}\operatorname{Tr}\left[\not{p}_2\not{k}'_1\not{p}_1\not{k}'_1\right]+\frac{1}{4}m^2\operatorname{Tr}\left[4\not{p}_1\not{p}_2+2\not{p}_1\not{k}'_1+2\not{p}_1\not{k}'_1-2\not{p}_2\not{k}'_1-2\not{p}_2\not{k}'_1-\not{k}'_1\not{k}'_1\right]-m^4\operatorname{Tr}1\\
=&2\left(p_1k'_1\right)\left(p_2k'_1\right)-\left(p_1p_2\right)k'^2_1-m^2\left(4p_1p_2+4p_1k'_1-4p_2k'_1-k^2_1\right)-4m^4\\
=&\frac{1}{2}\left(t-m^2-m^2\right)\left(u-m^2-M^2\right)-\frac{1}{2}\left(s-2m^2\right)m^2\\
&-m^2\left[4\left(m^2-\frac{1}{2}s\right)+2\left(t-m^2-m^2\right)-2\left(u-m^2-M^2\right)+m^2\right)\Bigg]-4m^4\\
=&-\frac{1}{2}\left[-tu+m^2(9t+u)\right]
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\langle\Phi_{tu}\rangle =&\frac{1}{4}\operatorname{Tr}\left[\not{p}_2\not{k}'_1\not{p}_1\not{k}'_2\right]+\frac{1}{4}m^2\operatorname{Tr}\left[4\not{p}_1\not{p}_2+\not{2}_1\not{k}'_1+2\not{p}_1\not{k}'_2-2\not{p}_2\not{k}'_1-2\not{p}_2\not{k}'_2-\not{k}'_1\not{k}'_2\right]-m^4\operatorname{Tr}1\\
=&\left(p_1k'_1\right)\left(p_2k'_2\right)+\left(p_1k'_2\right)\left(p_2k'_1\right)-\left(p_1p_2\right)\left(k'_1k'_2\right)\\
&-m^2\left(4p_1p_2+2p_1k'_1+2p_1k'_2-2p_2k'_1-2p_2k'_2-k'_1k'_2\right)-4m^4\\
=&\frac{1}{4}\left(t-m^2-m^2\right)^2+\frac{1}{4}\left(u-m^2-m^2\right)^2-\frac{1}{4}\left(s-2m^2\right)\left(s-2m^2\right)\\
&-m^2\left[4\left(m^2-\frac{1}{2}s\right)-\left(m^2-\frac{1}{2}s\right)\right]-4m^4\\
=&-\frac{1}{2}\left[tu+3m^2(t+u)\right].
\end{aligned}
\tag{24}
$$

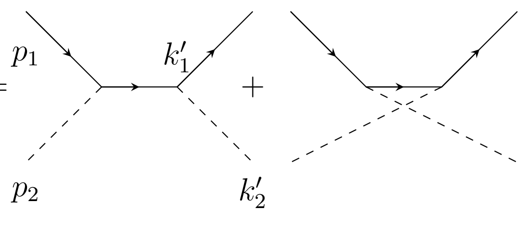
通过 $t\leftrightarrow u$,

$$\langle\Phi_{tt}\rangle=-\frac{1}{2}\left[-tu+m^2(9u+t)\right] \tag{25}$$

$$\langle\Phi_{tu}\rangle=-\frac{1}{2}\left[tu+3m^2(t+u)\right]. \tag{26}$$

对于 $e^-\phi\to e^-\phi$,

$$i\mathcal{M}'(e^-\phi\to e^-\phi)=$$



即，

$$i\mathcal{M}'=\bar{u}_{s_2}\left(k'_1\right)(-ig)\frac{i(\not{p}_1+\not{p}_2+m)}{s-m^2}(-ig)u_{s_1}\left(p_1\right)+\bar{u}_{s_2}\left(k'_1\right)(-ig)\frac{i(\not{p}_1-\not{k}'_2+m)}{u-m^2}(-ig)u_{s_1}\left(p_1\right) \tag{27}$$

同理，

$$\bar{\sum_s}|\mathcal{M}'|^2=g^4\left[\frac{\langle\Phi_{ss}\rangle}{\left(m^2-s\right)^2}+\frac{\langle\Phi_{uu}\rangle}{\left(m^2-u\right)^2}+\frac{\langle\Phi_{su}\rangle+\langle\Phi_{ut}\rangle}{\left(m^2-s\right)\left(m^2-u\right)}\right], \tag{28}$$

其中，

$$
\begin{aligned}
\langle\Phi_{ss}\rangle &= \frac{1}{2}\operatorname{Tr}\left[(\not{k}_1'+m)(\not{p}_2+2m)(\not{p}_1+m)(\not{p}_2+2m)\right] = -su+m^2(9s+u),\\
\langle\Phi_{uu}\rangle &= \frac{1}{2}\operatorname{Tr}\left[(\not{k}_1'+m)(\not{k}_2'-2m)(\not{p}_1+m)(\not{k}_2'-2m)\right] = -su+m^2(9u+s),\\
\langle\Phi_{su}\rangle &= \frac{1}{2}\operatorname{Tr}\left[(\not{k}_1'+m)(\not{p}_2+2m)(\not{p}_1+m)(\not{k}_2'-2m)\right] = su+3m^2(s+u),\\
\langle\Phi_{us}\rangle &= \frac{1}{2}\operatorname{Tr}\left[(\not{k}_1'+m)(\not{k}_2'-2m)(\not{p}_1+m)(\not{p}_2+2m)\right] = su+3m^2(s+u).
\end{aligned}
\tag{29}
$$

除了自旋平均多出的 $\frac{1}{2}$ 因子外，cross symmetry 满足的 $p_2 \leftrightarrow -k_1'$ 在（29）和（22）公式中显然体现出来。

习题 4

如下相互作用拉氏量描述了 $\pi^- \to \mu^-\bar{\nu}_\mu$ 的衰变:

$$\mathcal{L}_I = 2c_1 G_F f_\pi \partial_\mu \phi \bar{\psi}_{mu}\gamma^\mu P_L \psi_{\nu_\mu} + h.c.$$

其中 ϕ 是复标量场，ψ_{mu} 是 muon 场，ψ_{ν_μ} 是 muon 中微子场。$G_F = 1.166\times10^{-5}\mathrm{GeV}^{-2}$，$G_F = 1.166\times10^{-5}\mathrm{GeV}^{-2}$ 是 Cabibbo 角，f_π 称作 pion 衰变常数。muon 质量为 105.7MeV，pion 质量为 139.6MeV 。请计算 $\pi^- \to \mu^-\bar{\nu}_\mu$ 的总衰变宽度 Γ，并与实验测量得到的 pion 寿命 $T = \frac{\hbar}{\Gamma} \sim 2.603\times10^{-8}$ 比较抽取 f_π 的值。对 $\pi^- \to e^-\overline{\nu_e}$ 也做同样计算，所有参数不变，仅将电子质量设为 0.511MeV 。并尝试解释两个过程总衰变宽度的差异。

解:

记 $g \equiv c_1 G_\mathrm{F} f_\pi$；顶点为 $(ig)(ik_\mu)\gamma^\mu(1-\gamma_5) = -g\not{k}(1-\gamma_5)$, k 为介子动量。则

$$i\mathcal{M}(\pi^-(k)\to\mu^-(p_1)\bar{\nu}_\mu(p_2)) = -ig\bar{u}_1\not{k}(1-\gamma_5)v_2 \tag{30}$$

对初态自旋求平均，末态求和，

$$
\begin{aligned}
\bar{\sum}|\mathcal{M}|^2 &= g^2m_\mu^2\operatorname{Tr}\left[(\not{p}_1+m_\mu)(1-\gamma_5)(\not{p}_2)(1+\gamma_5)\right]\\
&= g^2m_\mu^2\operatorname{Tr}\left[(\not{p}_1+m_\mu)(\not{p}_2)(1+\gamma_5)(1+\gamma_5)\right]\\
&= 2g^2m_\mu^2\operatorname{Tr}\left[(\not{p}_1+m_\mu)(\not{p}_2)(1+\gamma_5)\right]\\
&= 2g^2m_\mu^2\operatorname{Tr}\left[\not{p}_1\not{p}_2\right]\\
&= 2g^2m_\mu^2(-4p_1p_2)\\
&= 4g^2m_\mu^2\left[-(p_1+p_2)^2+p_1^2+p_2^2\right]\\
&= 4g^2m_\mu^2\left(-k^2+p_1^2+p_2^2\right)\\
&= 4g^2m_\mu^2\left(m_\pi^2-m_\mu^2\right).
\end{aligned}
\tag{31}
$$

那么得到 $\Gamma = \bar{\sum}|\mathcal{M}|^2|\mathbf{p}_1|/8\pi m_\pi^2$, 和 $|\mathbf{p}_1| = \left(m_\pi^2-m_\mu^2\right)/2m_\pi^2$, 则

$$\Gamma = \frac{g^2m_\mu^2}{4\pi m_\pi^3}\left(m_\pi^2-m_\mu^2\right)^2 \approx 2.528\times10^{-17}\mathrm{GeV} \tag{32}$$

由此可以得到 $f_\pi = 93.14\mathrm{MeV}$。同理对衰变道 $\pi^- \to e^-\overline{\nu_e}$，可以得到 $\Gamma = 5.200 \times 10^{-31}\mathrm{GeV}$。此差异是由于弱衰变只有和左手费米子的耦合项，而手征对称性由 $m\bar{\psi}\psi$ 破坏，同时衰变宽度正比于破坏程度导致。