期中测试

1. Solution:

$$
\begin{aligned}
11^{1/11} &= 10^{1/11}\left(1+\frac{1}{10}\right)^{1/11} \\
&= 1^{1/11}\times 10^{1/11}\left(1+\frac{1}{11}\times\frac{1}{10}+\frac{1}{2}\times\frac{1}{11}\left(\frac{1}{11}-1\right)\left(\frac{1}{10}\right)^2+o(10^{-5})\right) \\
&\approx 1^{1/11}\times 10^{1/11}\left(1+0.0091-0.0004\right) \\
&\approx e^{i2k\pi/11}\times 1.23\times 1.01 \\
&\approx 1.24\times e^{i2k\pi/11}\ ,
\end{aligned}
$$

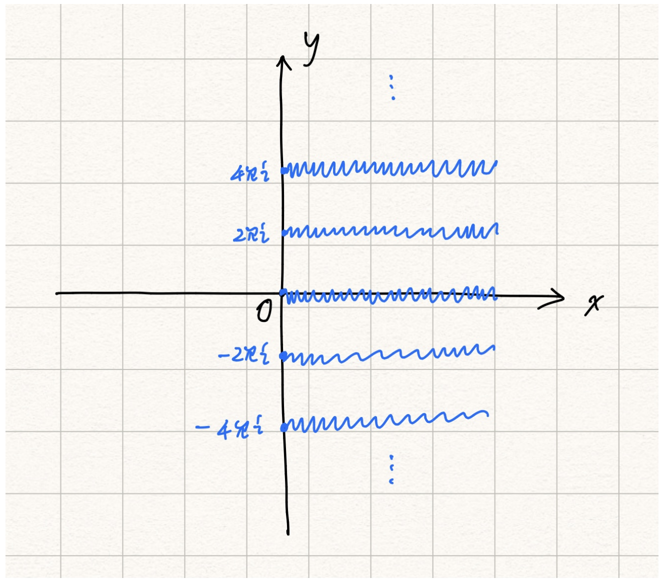
其中 k 取 $k=0,1,2,\cdots,10$.

2.(i) Solution:

$$
\begin{aligned}
u_x &= \cos x\cosh y\ , \qquad & u_y &= \sin x\sinh y\ , \\
v_y &= \cos x\cosh y\ , \qquad & v_x &= -\sin x\sinh y\ .
\end{aligned}
$$

所以 u_x,u_y,v_x,v_y 在 $\mathbb{C}$ 上存在且连续，并且满足 $u_x=v_y,u_y=-v_x$ 。所以 $f(x,y)$ 是解析函数。

(ii) Solution: 枝点为 $z=2k\pi i\ (k\in\mathbb{Z}),\infty$ 。割线可取



3. Solution:

$$v_y = u_x = 2e^{x^2-y^2}[x\cos(2xy) - y\sin(2xy)] \ ,$$
$$v_x = -u_y = 2e^{x^2-y^2}[y\cos(2xy) + x\sin(2xy)] \ ,$$

所以

$$\begin{aligned} dv(x,y) &= v_x \mathrm{d}x + v_y \mathrm{d}y \\ &= \mathrm{d}\left(e^{x^2-y^2}\sin(2xy)\right) \ . \end{aligned}$$

所以虚部

$$v(x,y) = e^{x^2-y^2}\sin(2xy) + C \ ,$$

其中 C 是个实常数。

4. Solution: 因为

$$(z-a)(\bar{z}-\bar{a}) = R^2 \ ,$$

所以

$$\bar{z} = \frac{R^2}{z-a} + \bar{a} \ ,$$
$$d\bar{z} = -\frac{R^2}{(z-a)^2} dz \ .$$

所以要求的积分

$$\begin{aligned} I &= -R^2 \int_C \frac{P(z)}{(z-a)^2} dz \\ &= -(2\pi i) \cdot R^2 P'(a) \ , \end{aligned}$$

其中第二个等号用了“ $P(z)$ 是多项式函数”条件和柯西公式。

5. Solution:

$$\begin{aligned} f(z) &= \sin\left(\frac{z-1+1}{1-z}\right) = -\sin\left(1+\frac{1}{z-1}\right) \\ &= -\sin 1 \cos\left(\frac{1}{z-1}\right) - \cos 1 \sin\left(\frac{1}{z-1}\right) \\ &= -\sin 1 \times \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{(z-1)^{2k}} - \cos 1 \times \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{(z-1)^{2k+1}} \ . \end{aligned}$$

6. Solution:

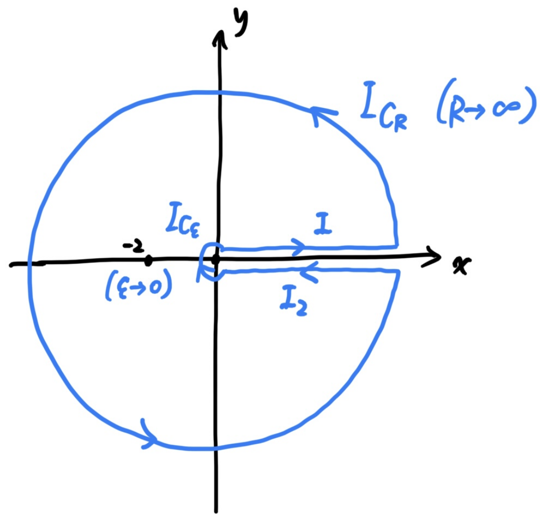
$$\begin{aligned}
I &= \int_0^\infty \frac{\sin^3 x}{x^3} dx \\
&= \int_0^\infty \frac{-\frac{1}{4}\sin(3x) + \frac{3}{4}\sin x}{x^3} dx \\
&= -\frac{1}{4}\int_0^\infty \frac{\sin(3x)}{x^3} dx + \frac{3}{4}\int_0^\infty \frac{\sin x}{x^3} dx \\
&= -\frac{9}{4}\int_0^\infty \frac{\sin x}{x^3} dx + \frac{3}{4}\int_0^\infty \frac{\sin x}{x^3} dx \\
&= -\frac{3}{2}\int_0^\infty \frac{\sin x}{x^3} dx \\
&= -\frac{3}{4}\int_{-\infty}^\infty \frac{\sin x}{x^3} dx \\
&= -\frac{3}{4} \cdot \mathrm{Im} \int_{-\infty}^\infty \frac{e^{ix}}{x^3} dx \\
&= -\frac{3}{4} \cdot \mathrm{Im} \left(-\frac{1}{2}\int_{-\infty}^\infty \frac{e^{ix}}{x} dx \right) \\
&= -\frac{3}{4} \cdot \mathrm{Im} \left(-\frac{1}{2} \times \pi i \right) \\
&= \frac{3}{8}\pi \ ,
\end{aligned}$$

其中倒数第三个等号用了两次分部积分。

7. Solution: 设

$$I = \int_0^\infty \frac{\ln^3 x}{x^2 + a^2} dx \ , \qquad I_1 = \int_0^\infty \frac{\ln^2 x}{x^2 + a^2} dx \ .$$

对被积函数 $f(z) = \ln^3 z/(z^2 + a^2)$ （割线取 $[0, +\infty)$ ）做如下围道积分：



则有

$$\begin{aligned}
I + I_{C_R} + I_2 + I_{C_\varepsilon} &= 2\pi i \cdot \left[\mathrm{Res}(\frac{\ln^3 z}{z^2+a^2}, ae^{i\pi/2}) + \mathrm{Res}(\frac{\ln^3 z}{z^2+a^2}, ae^{i3\pi/2}) \right] \\
&= \frac{\pi}{a}\left[\left(\ln a + i\frac{\pi}{2}\right)^3 - \left(\ln a + i\frac{3\pi}{2}\right)^3 \right] \\
&= \frac{\pi}{a}\left[i\left(-3\pi(\ln a)^2 + \frac{13\pi^3}{4}\right) + 6\pi^2 \ln a \right] \ , \qquad (1)
\end{aligned}$$

其中

$$
\begin{aligned}
I_2 &= -\int_0^\infty \frac{(\ln\rho + i2\pi)^3}{\rho^2+a^2}d\rho \\
&= -\int_0^\infty \frac{(\ln\rho)^3}{\rho^2+a^2}d\rho - i6\pi\int_0^\infty \frac{(\ln\rho)^2}{\rho^2+a^2}d\rho + 12\pi^2\int_0^\infty \frac{\ln\rho}{\rho^2+a^2}d\rho + i8\pi^3\int_0^\infty \frac{1}{\rho^2+a^2}d\rho \\
&= -I - i6\pi\cdot I_1 + i8\pi^3\times\frac{\pi}{2a} + 12\pi^2\int_0^\infty \frac{\ln\rho}{\rho^2+a^2}d\rho\ ,
\end{aligned}
$$

$$
\begin{aligned}
|I_{C_R}| &= \left|\int_0^{2\pi} \frac{\ln^3(Re^{i\varphi})}{R^2e^{2i\varphi}+a^2} iRe^{i\varphi}d\varphi\right| \\
&\le \int_0^{2\pi} \frac{|\ln R + i\varphi|^3 R}{R^2-a^2}d\varphi \\
&\sim \int_0^{2\pi} \frac{\ln^3 R}{R}d\varphi \\
&= 2\pi\cdot\frac{\ln^3 R}{R} \\
&= 0\ ,
\end{aligned}
$$

（以上用了 $R\to\infty$ 条件），

$$
\begin{aligned}
|I_{C_\varepsilon}| &= \left|\int_0^{2\pi} \frac{\ln^3(\varepsilon e^{i\varphi})}{\varepsilon^2e^{2i\varphi}+a^2} i\varepsilon e^{i\varphi}d\varphi\right| \\
&\le \int_0^{2\pi} \frac{|\ln\varepsilon + i\varphi|^3 \varepsilon}{a^2-\varepsilon^2}d\varphi \\
&\sim \int_0^{2\pi} \frac{|\ln\varepsilon|^3\varepsilon}{a^2}d\varphi \\
&= 0\ .
\end{aligned}
$$

（以上用了 $\varepsilon\to 0$ 条件），所以

$$
I_{C_R} = 0 = I_{C_\varepsilon}\ .
$$

综上，

$$
\begin{aligned}
I + I_{C_R} + I_2 + I_{C_\varepsilon} &= -i6\pi\cdot I_1 + i8\pi^3\times\frac{\pi}{2a} + 12\pi^2\int_0^\infty \frac{\ln\rho}{\rho^2+a^2}d\rho \\
&= -i6\pi\cdot I_1 + i\frac{4\pi^4}{a} + [\text{real part}]\ ,
\end{aligned}
$$

结合 eq. (1) ，对比虚部，可求出

$$
I_1 = \frac{\pi}{2}\frac{(\ln a)^2}{a} + \frac{\pi^3}{8a}\ .
$$

8. Solution:

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\omega_0 t)H(t)e^{-(\gamma+i\omega t)t}dt \\ &= \frac{1}{2\pi}\int_{0}^{\infty} \cos(\omega_0 t)e^{-(\gamma+i\omega t)t}dt \\ &= \frac{1}{2\pi}\times\frac{\gamma+i\omega}{(\gamma+i\omega)^2+\omega_0^2}\ ,\end{aligned}$$

倒数第二个积分是用分部积分原理算的。因为

$$\begin{aligned}|\hat{f}(\omega)|^2 &\sim \frac{\gamma+i\omega}{(\gamma+i\omega)^2+\omega_0^2}\cdot\frac{\gamma-i\omega}{(\gamma-i\omega)^2+\omega_0^2} \\ &= \frac{\gamma^2+\omega^2}{(\gamma^2-\omega^2+\omega_0^2)^2+4\gamma^2\omega^2} \\ &= \frac{\gamma^2+\omega^2}{(\gamma^2+\omega^2)^2+2\omega_0^2(\gamma^2-\omega^2)+\omega_0^4} \\ &= \frac{1}{(\gamma^2+\omega^2)+\frac{\omega_0^4+4\omega_0^2\gamma^2}{\gamma^2+\omega^2}-2\omega_0^2}\ ,\end{aligned}$$

所以当

$$\gamma^2+\omega^2=\omega_0\sqrt{\omega_0^2+4\gamma^2}$$

时， $|\hat{f}(\omega)|^2$ 取得最大值，也即

$$\omega^2=\omega_0\sqrt{\omega_0^2+4\gamma^2}-\gamma^2\ .$$

（当 $0<\gamma^2\le(2+\sqrt{5})\omega_0^2$ 时， ω 才能取到该值。当 $\gamma^2>(2+\sqrt{5})\omega_0^2$ 时， $\omega=0$ 处 $|\hat{f}(\omega)|^2$ 最大。）