Seven

6.4 Solution:

$$
\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\omega t} dt \\
&= \int_{0}^{\infty} e^{-(a+i\omega)t} dt + \int_{-\infty}^{0} e^{(a-i\omega)t} dt \\
&= \frac{2a}{a^2+\omega^2} \ .
\end{aligned}
$$

逆变换为

$$
I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2} e^{i\omega x} d\omega \ ,
$$

被积函数在复平面上有单极点 $\omega = \pm ia$ 。当 $x > 0$ 时，将 ω 沿上半平面绕一个大圆弧构成封闭围道，根据约当引理，大圆弧积分为 0 。此时，

$$
\begin{aligned}
I &= \frac{1}{2\pi} \times 2\pi i \cdot \mathrm{Res}\left(\frac{2a}{a^2+\omega^2} e^{i\omega x}, ia \right) \\
&= e^{-ax} \ .
\end{aligned}
$$

当 $x < 0$ 时，将 ω 沿下半平面绕一个大圆弧构成封闭围道，根据约当引理，大圆弧积分为 0 。此时，

$$
\begin{aligned}
I &= -\frac{1}{2\pi} \times 2\pi i \cdot \mathrm{Res}\left(\frac{2a}{a^2+\omega^2} e^{i\omega x}, -ia \right) \\
&= e^{ax} \ .
\end{aligned}
$$

当 $x=0$ 时，

$$
\begin{aligned}
I &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+\omega^2}d\omega \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{\omega-ia}-\frac{1}{\omega+ia}\right)d\omega \\
&= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R}^{R}\left(\frac{1}{\omega-ia}-\frac{1}{\omega+ia}\right)d\omega \\
&= \frac{1}{2\pi i}\lim_{R\to\infty}\left(\ln(\omega-ia)|_{-R}^{R}-\ln(\omega+ia)|_{-R}^{R}\right) \\
&= \frac{1}{2\pi i}\lim_{R\to\infty}(\ln|R|-(\ln|R|-i\pi)-\ln|R|+\ln|R|+i\pi) \\
&= 1\ .
\end{aligned}
$$

综上，逆傅里叶变换为

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+\omega^2}e^{i\omega x}d\omega=\begin{cases} e^{-ax} & \text{if } x>0 \\ 1 & \text{if } x=0 \\ e^{ax} & \text{if } x<0 \end{cases}\ .
$$

6.5 (6.6) Solution:

$$
\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty}\mathrm{rect}(t-1)e^{-i\omega t}dt \\
&= e^{-i\omega}\int_{-\infty}^{\infty}\mathrm{rect}(t)e^{-i\omega t}dt \\
&= e^{-i\omega}\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{-i\omega t}dt \\
&= e^{-i\omega}\cdot\frac{2}{\omega}\sin\frac{\omega}{2}\ .
\end{aligned}
$$

6.7 Solution:

$$
\begin{aligned}
\hat{f}(\omega) &= \int_{-\frac{1}{a}}^{\frac{1}{a}}(1-a|t|)h\cdot e^{-i\omega t}dt \\
&= h\int_{0}^{\frac{1}{a}}(1-at)\cdot e^{-i\omega t}dt+h\int_{-\frac{1}{a}}^{0}(1+at)\cdot e^{-i\omega t}dt \\
&= \frac{2ah}{\omega^2}\left(1-\cos\frac{\omega}{a}\right)\ .
\end{aligned}
$$

6.8 Solution:

$$\begin{aligned}
\hat{f}(\vec{k}) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{1}{|\vec{r}|^2}e^{-i\vec{k}\cdot\vec{r}}d^3\vec{r} \\
&= \frac{1}{(2\pi)^3}\int\frac{1}{r^2}e^{-ikr\cos\theta}\cdot r^2\sin\theta d\theta d\varphi \\
&= \frac{2\pi}{(2\pi)^3}\int_0^{\infty}dr\int_{-1}^{1}dx\ e^{-ikrx} \\
&= \frac{1}{(2\pi)^2}\int_0^{\infty}dr\ \frac{2\sin(kr)}{kr} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dr\ \frac{\sin(kr)}{kr} \\
&= \frac{1}{(2\pi)^2k}\int_{-\infty}^{\infty}dr\ \frac{\sin(r)}{r} = \frac{1}{(2\pi)^2k}\cdot\mathrm{Im}\int_{-\infty}^{\infty}dr\ \frac{e^{ir}}{r} \\
&= \frac{1}{(2\pi)^2k}\cdot\mathrm{Im}(\pi i) \\
&= \frac{1}{4\pi k},
\end{aligned}$$

其中第二个等号是将笛卡尔坐标转化为球坐标，并以 $\vec{k}$ 方向为参照，设为 z 轴方向， $k=|\vec{k}|$ 。