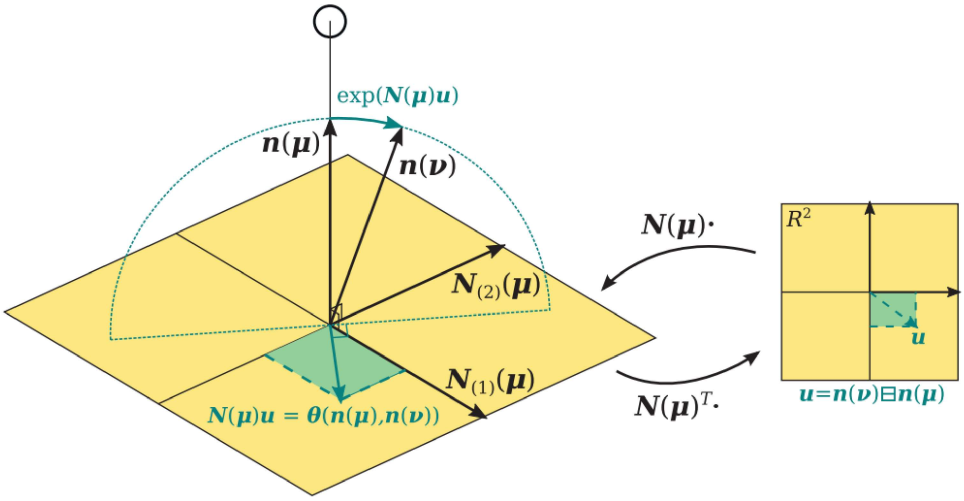

图 1: $\boldsymbol{\mu}$ 表示一种旋转变换。将 $\boldsymbol{N}(\boldsymbol{\mu})$ 作为旋转变换的切空间，$\boldsymbol{u}$ 为旋转变换由李群到切空间李代数的表达。因此，若切空间内的 $\boldsymbol{u}$ 产生变化，就会对应旋转变换在李群的变化。

1 ROVIO

1.1 路标点运动学方程推导

路标点运动学方程是文章 [1] 中的创新点和核心组成部分，理解和掌握作者开发出的算法是有利于应用 ROVIO 的。

1.1.1 背景知识

这里，我们需要理解三个非常重要的量，$\boldsymbol{\mu}$, $\boldsymbol{n}(\boldsymbol{\mu})$ 和 $\boldsymbol{N}(\boldsymbol{\mu})$，其含义包括：

- $\boldsymbol{\mu} \in SO(3)$, 是一个在三维空间内的旋转变换
- $\boldsymbol{n}(\boldsymbol{\mu}) := \boldsymbol{\mu}(\boldsymbol{e}_z) \in S^2 \subset \mathbb{R}^3$, $\boldsymbol{N}(\boldsymbol{\mu}) := [\boldsymbol{\mu}(\boldsymbol{e}_x), \boldsymbol{\mu}(\boldsymbol{e}_y)] \in \mathbb{R}^{3\times 2}$，其中 $\boldsymbol{e}_{x/y/z} \in \mathbb{R}^3$ 是任意直角坐标系的单位基坐标；S^2 指的是 2-sphere, 意思自由度为 2 且存在于 3D 空间内的群（本人认为可将 $\boldsymbol{n}(\boldsymbol{\mu})$ 理解为一个存在于李群中的三维单位向量，即它是一个三维单位向量，但不能像普通向量那样在向量空间内随意变化，因为它受限于李群）。$\boldsymbol{n}(\boldsymbol{\mu})$ 和 $\boldsymbol{N}(\boldsymbol{\mu})$ 可以理解为两个算子，前者是说一个三维直解坐票系经过 $\boldsymbol{\mu}$ 的旋转变换后会得到一个新的坐标表达（一个新的 3×3 的矩阵），取其第三列；后者则是取其前两列。 具体可以这么理解：对于任一定义好在三维空间内的惯性坐标 $\{\boldsymbol{e}_{x0}, \boldsymbol{e}_{y0}, \boldsymbol{e}_{z0}\}$（其中 $\boldsymbol{e}_{x0}, \boldsymbol{e}_{y0}, \boldsymbol{e}_{z0}$ 为单位化的基坐标, 因此它们都是 3×1 的向量），将其进行 $\boldsymbol{\mu}$ 的旋转变换，得到新的基坐标表达式 $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$。然后，令 $\boldsymbol{n}(\boldsymbol{\mu}) = \boldsymbol{e}_z$, 且 $\boldsymbol{N}(\boldsymbol{\mu}) = [\boldsymbol{e}_x, \boldsymbol{e}_y]$, 这两个等式在后面的推导过程中起着至关重要的作用

根据上述内容，我们可以知道 $[\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z]^{3\times 3}$ 仍然是一个直角坐标系，进一步展开有

$$\boldsymbol{e}_{x/y/z} = \begin{bmatrix} e_{x1/y1/z1} & e_{x2/y2/z2} & e_{x3/y3/z3} \end{bmatrix}^T \tag{1}$$

，所以

$$\boldsymbol{n}(\boldsymbol{\mu})=\begin{bmatrix}e_{z1}\\e_{z2}\\e_{z3}\end{bmatrix},\quad \boldsymbol{N}(\boldsymbol{\mu})=\begin{bmatrix}e_{x1}&e_{y1}\\e_{x2}&e_{y2}\\e_{x3}&e_{y3}\end{bmatrix}\tag{2}$$

也就是说

$$\begin{bmatrix}\boldsymbol{e}_x&\boldsymbol{e}_y&\boldsymbol{e}_z\end{bmatrix}=\begin{bmatrix}\boldsymbol{N}(\boldsymbol{\mu})&\boldsymbol{n}(\boldsymbol{\mu})\end{bmatrix}\tag{3}$$

我们可得到

$$\boldsymbol{N}(\boldsymbol{\mu})^T\boldsymbol{N}(\boldsymbol{\mu})=\boldsymbol{I}\in\mathbb{R}^{2\times2}\tag{4}$$

证明如下

$$\begin{bmatrix}\boldsymbol{e}_x&\boldsymbol{e}_y&\boldsymbol{e}_z\end{bmatrix}^T\begin{bmatrix}\boldsymbol{e}_x&\boldsymbol{e}_y&\boldsymbol{e}_z\end{bmatrix}=\begin{bmatrix}\boldsymbol{N}(\boldsymbol{\mu})^T\\\boldsymbol{n}(\boldsymbol{\mu})^T\end{bmatrix}\begin{bmatrix}\boldsymbol{N}(\boldsymbol{\mu})&\boldsymbol{n}(\boldsymbol{\mu})\end{bmatrix}=\begin{bmatrix}\boldsymbol{N}(\boldsymbol{\mu})^T\boldsymbol{N}(\boldsymbol{\mu})&\boldsymbol{N}(\boldsymbol{\mu})^T\boldsymbol{n}(\boldsymbol{\mu})\\\boldsymbol{n}(\boldsymbol{\mu})^T\boldsymbol{N}(\boldsymbol{\mu})&\boldsymbol{n}(\boldsymbol{\mu})^T\boldsymbol{n}(\boldsymbol{\mu})\end{bmatrix}=\boldsymbol{I}\tag{5}$$

即等式 (6) 得证。此外，

$$\boldsymbol{N}(\boldsymbol{\mu})^T\boldsymbol{n}(\boldsymbol{\mu})^\times\boldsymbol{n}(\boldsymbol{\mu})^\times+\boldsymbol{N}(\boldsymbol{\mu})^T=\begin{bmatrix}E_{11}&E_{12}&E_{13}\\E_{21}&E_{22}&E_{23}\end{bmatrix}=\boldsymbol{0}\tag{6}$$

$$\begin{aligned}
E_{11}&=e_{x1}-e_{z2}(e_{x1}e_{z2}-e_{x2}e_{z1})-e_{z3}(e_{x1}e_{z3}-e_{x3}e_{z1})\\
E_{12}&=e_{x2}+e_{z1}(e_{x1}e_{z2}-e_{x2}e_{z1})-e_{z3}(e_{x2}e_{z3}-e_{x3}e_{z2})\\
E_{13}&=e_{x3}+e_{z1}(e_{x1}e_{z3}-e_{x3}e_{z1})+e_{z2}(e_{x2}e_{z3}-e_{x3}e_{z2})\\
E_{21}&=e_{y1}-e_{z2}(e_{y1}e_{z2}-e_{y2}e_{z1})-e_{z3}(e_{y1}e_{z3}-e_{y3}e_{z1})\\
E_{22}&=e_{y2}+e_{z1}(e_{y1}e_{z2}-e_{y2}e_{z1})-e_{z3}(e_{y2}e_{z3}-e_{y3}e_{z2})\\
E_{24}&=e_{y3}+e_{z1}(e_{y1}e_{z3}-e_{y3}e_{z1})+e_{z2}(e_{y2}e_{z3}-e_{y3}e_{z2})
\end{aligned}$$

因为 $[\boldsymbol{e}_x,\boldsymbol{e}_y,\boldsymbol{e}_z]$ 是直角坐标系，所以 $\boldsymbol{e}_{x/y/z}\times\boldsymbol{e}_{y/z/x}=\boldsymbol{e}_{z/x/y}$, 即

$$\boldsymbol{e}_x=\begin{bmatrix}e_{y2}e_{z3}-e_{y3}e_{z2}\\e_{y3}e_{z1}-e_{y1}e_{z3}\\e_{y1}e_{z2}-e_{y2}e_{z1}\end{bmatrix},\ \boldsymbol{e}_y=\begin{bmatrix}e_{x3}e_{z2}-e_{x2}e_{z3}\\e_{x1}e_{z3}-e_{x3}e_{z1}\\e_{x2}e_{z1}-e_{x1}e_{z2}\end{bmatrix},\ \boldsymbol{e}_z=\begin{bmatrix}e_{x2}e_{y3}-e_{x3}e_{y2}\\e_{x3}e_{y1}-e_{x1}e_{y3}\\e_{x1}e_{y2}-e_{x2}e_{y1}\end{bmatrix}\tag{7}$$

将公式 (7) 代入 (4) 中可证得上述结论。同理，利用 (7) 我们还可以得到如下结论（同上理，易证，此处省略）:

$$\boldsymbol{N}(\boldsymbol{\mu})^T\boldsymbol{n}(\boldsymbol{\mu})^\times\boldsymbol{n}(\boldsymbol{\mu})=\boldsymbol{0}\in\mathbb{R}^{2\times1}\tag{8}$$

$$\boldsymbol{n}(\boldsymbol{\mu})^T\boldsymbol{n}(\boldsymbol{\mu})^\times=\boldsymbol{0}\in\mathbb{R}^{1\times3}\tag{9}$$

等式 (4), (6), (8) 以及 (9) 在后面的模型推导中的数学变换都将发挥作用。

1.1.2 具体推导过程

路标点的运动学方程的推导中，还会大量运用到一些三维旋转的微积分运算，主要可参考 [2]。路标点 $\boldsymbol{\mu}$ 在 2D 切空间的运动学模型，在论文内是（此处省去了噪声项）

$$\dot{\boldsymbol{\mu}}=\boldsymbol{N}(\boldsymbol{\mu})^T\left(\hat{\boldsymbol{\omega}}_{\mathcal{C}}+\boldsymbol{n}(\boldsymbol{\mu})^\times\frac{\hat{\boldsymbol{v}}_{\mathcal{C}}}{d(\rho)}\right)\tag{10}$$

推导过程如下:

- 路标点的模型包含在
$$\frac{\mathrm{d}}{\mathrm{d}t}\{_{\mathcal{I}}\boldsymbol{r}_{\mathcal{IF}} = _{\mathcal{I}}\boldsymbol{r}_{\mathcal{IC}} + \boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\} \tag{11}$$
- 假设路标点在惯性坐标系下静止不动，则 $_{\mathcal{I}}\dot{\boldsymbol{r}}_{\mathcal{IF}} = 0$。另外，$_{\mathcal{I}}\dot{\boldsymbol{r}}_{\mathcal{IC}} = _{\mathcal{I}}\boldsymbol{v}_{\mathcal{IC}}$
-
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\right) = \frac{\partial\left(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\right)}{\partial \boldsymbol{q}_{\mathcal{CI}}^{-1}}\frac{\partial \boldsymbol{q}_{\mathcal{CI}}^{-1}}{\partial \boldsymbol{q}_{\mathcal{CI}}}\frac{\mathrm{d}\boldsymbol{q}_{\mathcal{CI}}}{\mathrm{d}t} + \frac{\partial\left(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\right)}{\partial \boldsymbol{\mu}}\frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}t} + \frac{\partial\left(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\right)}{\partial \rho}\dot{\rho} \tag{12}$$
$$\begin{aligned}\frac{\partial\left(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\right)}{\partial \boldsymbol{q}_{\mathcal{CI}}^{-1}} &= \frac{\partial\left(\boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}}^{-1})\right)(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))}{\partial \boldsymbol{q}_{\mathcal{CI}}^{-1}} = -(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho)))^{\times} \\ &= -(\boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}}^{-1})(\boldsymbol{n}(\boldsymbol{\mu})d(\rho)))^{\times} = -\boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}}^{-1})\boldsymbol{n}(\boldsymbol{\mu})^{\times}\boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}})d(\rho) \qquad (13)\\ \frac{\partial \boldsymbol{q}_{\mathcal{CI}}^{-1}}{\partial \boldsymbol{q}_{\mathcal{CI}}} &= -\boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}})^{T}, \quad \frac{\mathrm{d}\boldsymbol{q}_{\mathcal{CI}}}{\mathrm{d}t} = {_{\mathcal{C}}}\boldsymbol{\omega}_{\mathcal{CI}} = -{_{\mathcal{C}}}\boldsymbol{\omega}_{\mathcal{IC}} \qquad (14)\\ \frac{\partial\left(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\right)}{\partial \boldsymbol{\mu}} &= \boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}}^{-1})d(\rho)\frac{\partial \boldsymbol{n}(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}} \qquad (15)\end{aligned}$$

这里，在公式 (15) 中，要求解 $\boldsymbol{n}(\boldsymbol{\mu})$ 对 $\boldsymbol{\mu}$ 的偏导数，换言之，$\boldsymbol{\mu}$ 的微小变化会引起 $\boldsymbol{n}(\boldsymbol{\mu})$ 怎样的变化。我们知道,$\boldsymbol{\mu}$ 被定义为三维空间内的旋转变换，属于李群中的量，它的微小变化是很难在状态估算或者优化问题中起到什么作用的，因为它是非线性的。所以这里，作者提供的方案就是，将偏导的因子变为切空间内的向量 $\boldsymbol{u}$，它属于李代数，就可以做加减运算了（妙哉!）。显而易见，我们能得到
$$\frac{\partial\left(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\right)}{\partial \boldsymbol{\mu}}\frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}t} = \frac{\partial\left(\boldsymbol{q}_{\mathcal{CI}}^{-1}(\boldsymbol{n}(\boldsymbol{\mu})d(\rho))\right)}{\partial \boldsymbol{u}}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \tag{16}$$
也就是说，切空间内向量 $\boldsymbol{u}$ 的变化引起的 $\boldsymbol{n}(\boldsymbol{\mu})$ 的变化，和李群的 manifold 中 $\boldsymbol{\mu}$ 的变化引起的 $\boldsymbol{n}(\boldsymbol{\mu})$ 的变化，是可以等效的。可以将 $\boldsymbol{n}(\cdot)$ 看成一个算子，其变量属于李群，取这个李群代表的旋转阵的最后一列。所以
$$\begin{aligned}\frac{\partial \boldsymbol{n}(\boldsymbol{\mu})}{\partial \boldsymbol{u}} &= \lim_{\boldsymbol{u}\to \boldsymbol{0}}\frac{\boldsymbol{n}(\boldsymbol{\mu}\boxplus \boldsymbol{N}(\boldsymbol{\mu})\boldsymbol{u}) - \boldsymbol{n}(\boldsymbol{\mu})}{\boldsymbol{u}} \qquad (17)\\ &= \lim_{\boldsymbol{u}\to \boldsymbol{0}}\frac{\boldsymbol{n}(\exp(\boldsymbol{N}(\boldsymbol{\mu})\boldsymbol{u})\boldsymbol{\mu}) - \boldsymbol{n}(\boldsymbol{\mu})}{\boldsymbol{u}} \qquad (18)\\ &= \lim_{\boldsymbol{u}\to \boldsymbol{0}}\frac{\boldsymbol{n}((\boldsymbol{I} + (\boldsymbol{N}(\boldsymbol{\mu})\boldsymbol{u})^{\times})\boldsymbol{\mu}) - \boldsymbol{n}(\boldsymbol{\mu})}{\boldsymbol{u}} \qquad (19)\\ &= \underbrace{\lim_{\boldsymbol{u}\to \boldsymbol{0}}\frac{\boldsymbol{n}((\boldsymbol{N}(\boldsymbol{\mu})\boldsymbol{u})^{\times}\boldsymbol{\mu})}{\boldsymbol{u}} = \lim_{\boldsymbol{u}\to \boldsymbol{0}}\frac{(\boldsymbol{N}(\boldsymbol{\mu})\boldsymbol{u})^{\times}\boldsymbol{n}(\boldsymbol{\mu})}{\boldsymbol{u}}}_{\text{等式1}} \qquad (20)\\ &= -\lim_{\boldsymbol{u}\to \boldsymbol{0}}\frac{\boldsymbol{n}(\boldsymbol{\mu})^{\times}\boldsymbol{N}(\boldsymbol{\mu})\boldsymbol{u}}{\boldsymbol{u}} \qquad (21)\\ &= -\boldsymbol{n}(\boldsymbol{\mu})^{\times}\boldsymbol{N}(\boldsymbol{\mu}) \qquad (22)\end{aligned}$$

其中 “等式 1”，同样可由符号运算验证。

- 综上述步骤后，我们得到

$$\mathbf{0} = {}_{\mathcal{I}}\boldsymbol{v}_{\mathcal{IC}} - \boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}}^{-1})\boldsymbol{n}(\boldsymbol{\mu})^{\times}{}_{\mathcal{C}}\boldsymbol{\omega}_{\mathcal{IC}}d(\rho) - \boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}}^{-1})\boldsymbol{n}(\boldsymbol{\mu})^{\times}\boldsymbol{N}(\boldsymbol{\mu})d(\rho)\dot{\boldsymbol{u}} + \boldsymbol{C}(\boldsymbol{q}_{\mathcal{CI}}^{-1})\boldsymbol{n}(\boldsymbol{\mu})d'(\rho)\dot{\rho} \tag{23}$$

$$\mathbf{0} = {}_{\mathcal{C}}\boldsymbol{v}_{\mathcal{IC}} - \boldsymbol{n}(\boldsymbol{\mu})^{\times}{}_{\mathcal{C}}\boldsymbol{\omega}_{\mathcal{IC}}d(\rho) - \boldsymbol{n}(\boldsymbol{\mu})^{\times}\boldsymbol{N}(\boldsymbol{\mu})d(\rho)\dot{\boldsymbol{u}} + \boldsymbol{n}(\boldsymbol{\mu})d'(\rho)\dot{\rho} \tag{24}$$

- 在公式 (24) 左右两边分别同时左乘 $1/d(\rho)\boldsymbol{N}(\boldsymbol{\mu})^T\boldsymbol{n}(\boldsymbol{\mu})^{\times}$ 和 $1/d'(\rho)\boldsymbol{n}(\boldsymbol{\mu})^T$，并运用前面证得的等式 (4), (6)，(8) 以及 (9)，可得

$$\mathbf{0} = 1/d(\rho)\boldsymbol{N}(\boldsymbol{\mu})^T\boldsymbol{n}(\boldsymbol{\mu})^{\times}{}_{\mathcal{C}}\boldsymbol{v}_{\mathcal{IC}} + \boldsymbol{N}(\boldsymbol{\mu})^T{}_{\mathcal{C}}\boldsymbol{\omega}_{\mathcal{IC}} + \dot{\boldsymbol{u}}$$

$$\dot{\boldsymbol{u}} = -\boldsymbol{N}(\boldsymbol{\mu})^T({}_{\mathcal{C}}\boldsymbol{\omega}_{\mathcal{IC}} + 1/d(\rho)\boldsymbol{n}(\boldsymbol{\mu})^{\times}{}_{\mathcal{C}}\boldsymbol{v}_{\mathcal{IC}}) \tag{25}$$

$$\mathbf{0} = 1/d'(\rho)\boldsymbol{n}(\boldsymbol{\mu})^T{}_{\mathcal{C}}\boldsymbol{v}_{\mathcal{IC}} + \dot{\rho}$$

$$\dot{\rho} = -1/d'(\rho)\boldsymbol{n}(\boldsymbol{\mu})^T{}_{\mathcal{C}}\boldsymbol{v}_{\mathcal{IC}} \tag{26}$$

1.1.3 部分微积分运算的推导

此节内容完全依据于文章 [2]。首先，验证等式 (13), 将问题一般化，$\boldsymbol{q}$ 是两个坐标系之间的变换关系，$\boldsymbol{q}(\boldsymbol{r}) \triangleq \boldsymbol{C}(\boldsymbol{q})\boldsymbol{r}, \boldsymbol{r} \in \mathbb{R}^3$, $\boldsymbol{C}(\boldsymbol{q})$ 是 $\boldsymbol{q}$ 对应的旋转矩阵，现在要求解 $\partial\boldsymbol{q}(\boldsymbol{r})/\partial\boldsymbol{q}$, 即求解 $\partial(\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r})/\partial\boldsymbol{q}$。先求解 $\boldsymbol{C}(q)$ 第一列受到微小扰动对应的导数：

$$\begin{aligned}
\left[\frac{\partial\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r}}{\partial\boldsymbol{q}}\right]_1 &= \lim_{\epsilon\to 0}\frac{(\boldsymbol{q}\boxplus\boldsymbol{e}_1\epsilon)(\boldsymbol{r}) - \boldsymbol{q}(\boldsymbol{r})}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\boldsymbol{C}(\boldsymbol{e}_1\epsilon)\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r} - \boldsymbol{C}(\boldsymbol{q})\boldsymbol{r}}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{(\boldsymbol{I} + \boldsymbol{e}_1^{\times}\epsilon)\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r} - \boldsymbol{C}(\boldsymbol{q})\boldsymbol{r}}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\boldsymbol{e}_1^{\times}\epsilon\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r}}{\epsilon} \\
&= \boldsymbol{e}_1^{\times}\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r} \\
&= -(\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r})^{\times}\boldsymbol{e}_1
\end{aligned} \tag{27}$$

所以

$$\frac{\partial\boldsymbol{q}(\boldsymbol{r})}{\partial\boldsymbol{q}} = \frac{\partial\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r}}{\partial\boldsymbol{q}} = -(\boldsymbol{C}(\boldsymbol{q})\boldsymbol{r})^{\times} = -\boldsymbol{q}(\boldsymbol{r})^{\times} \tag{28}$$

$$\frac{\partial\boldsymbol{q}^{-1}}{\partial\boldsymbol{q}} = \frac{\partial\boldsymbol{C}(\boldsymbol{q}^{-1})}{\partial\boldsymbol{q}} \tag{29}$$

$$
\begin{aligned}
\left[\frac{\partial \boldsymbol{C}(\boldsymbol{q}^{-1})}{\partial \boldsymbol{q}}\right]_1 &= \lim_{\epsilon\to 0}\frac{(\boldsymbol{q}\boxplus \boldsymbol{e}_1\epsilon)^{-1}\boxminus \boldsymbol{q}^{-1}}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{(\exp(\boldsymbol{e}_1^{\times}\epsilon)\boldsymbol{q})^{-1}\boxminus \boldsymbol{q}^{-1}}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{(\boldsymbol{q}^{-1}\exp(-\boldsymbol{e}_1^{\times}\epsilon))\boxminus \boldsymbol{q}^{-1}}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\log(\boldsymbol{q}^{-1}\exp(-\boldsymbol{e}_1^{\times}\epsilon)\boldsymbol{q})}{\epsilon} && (30) \\
&= \lim_{\epsilon\to 0}\frac{\log(\exp(-\boldsymbol{q}^{-1}(\boldsymbol{e}_1\epsilon)))}{\epsilon} && (31) \\
&= \lim_{\epsilon\to 0}\frac{-\boldsymbol{q}^{-1}(\boldsymbol{e}_1\epsilon)}{\epsilon} = \lim_{\epsilon\to 0}\frac{-\boldsymbol{C}(\boldsymbol{q}^{-1})(\boldsymbol{e}_1\epsilon)}{\epsilon} && (32) \\
&= -\boldsymbol{C}(\boldsymbol{q})^T\boldsymbol{e}_1 && (33)
\end{aligned}
$$

所以

$$\frac{\partial \boldsymbol{q}^{-1}}{\partial \boldsymbol{q}} = -\boldsymbol{C}(\boldsymbol{q})^T \tag{34}$$

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