

A Primer on the Differential Calculus of 3D Orientations

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Abstract—The proper handling of 3D orientations is a central element in many optimization problems in engineering. Unfortunately many researchers and engineers struggle with the formulation of such problems and often fall back to suboptimal solutions. The existence of many different conventions further complicates this issue, especially when interfacing multiple differing implementations. This document discusses an alternative approach which makes use of a more abstract notion of 3D orientations. The relative orientation between two coordinate systems is primarily identified by the coordinate mapping it induces. This is combined with the standard exponential map in order to introduce representation-independent and minimal differentials, which are very convenient in optimization based methods.

I. INTRODUCTION

The primary goal of this document is to derive and summarize the most important identities for handling 3D orientations. It can readily be used as a look-up document (general identities are green (section IV), implementation dependent identities are red (section V)). In a compact theoretical part all equations get derived together with some insights in the mathematical background (section III). We believe however, that the best way to understand the concepts is to apply the presented findings on an actual system. To this end, we discuss the modeling of an Inertial Measurement Unit (IMU) driven kinematic model (section VI). Furthermore, we provide the most important proofs and derivations in order to provide some additional insights and examples. Similar elaborations on the topic exist [1], [2], [3].

An understanding of kinematics (including the concept of coordinate systems) is a prerequisite. The corresponding conventions and notations are summarized in section II. For the theoretical part some higher mathematical concepts are necessary.

II. VECTORS AND COORDINATE SYSTEMS NOTATION

In the following coordinate tuples are represented by bold lower case letters, e.g. $\mathcal{A}\mathbf{r}_{BC}$, and coordinate system are denoted by calligraphic capital letters, e.g. \mathcal{A} . The left-hand subscript of a coordinate tuple indicates the coordinate system the vector is represented in, while the right-hand subscripts indicate the 3D points related to start and end points. For instance, a term $\mathcal{A}\mathbf{r}_{BC}$ denotes the coordinates of a vector \vec{r}_{BC} (denoted with an arrow) in the Euclidean space \mathbb{R}^3 from the point \mathcal{B} to the point \mathcal{C} , represented in the coordinate system \mathcal{A} . By abuse of notation, we denote the origin associated with a specific coordinate system by the same symbol. Furthermore, the term $\Phi_{BA} \in SO(3)$ is

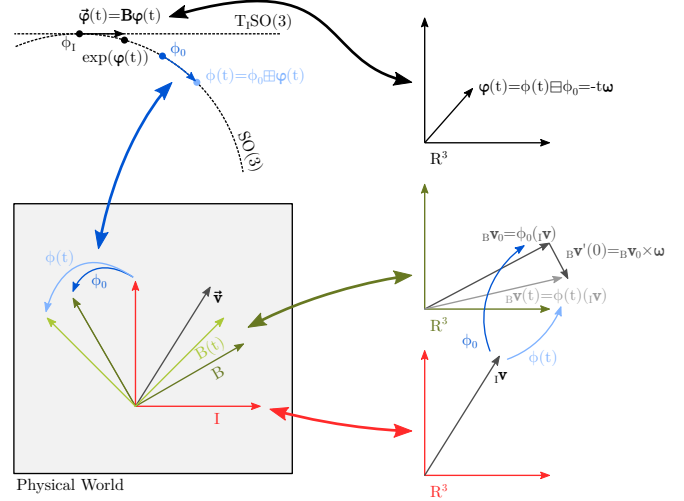


Fig. 1. This figure depicts various quantities in a setup where a coordinate system \mathcal{B} is rotated by a constant rotational velocity ω w.r.t. to an inertial coordinate system \mathcal{I} . Using coordinate systems, physical vectors can be represented through the corresponding coordinate tuples. Orientations between coordinate systems can be defined by the mapping they induce on the coordinate tuples. They are elements of $SO(3)$. Differences and derivatives of orientations can be represented in the tangential space $T_{\Phi_I}SO(3)$, which can be associated with \mathbb{R}^3 by means of a basis \mathbf{B} .

employed for representing the relative orientation of a coordinate system \mathcal{B} w.r.t. a coordinate system \mathcal{A} . Its definition is coupled to the (distance preserving) mapping of coordinate tuples and we employ the notation ${}_{\mathcal{B}}\mathbf{r}_{BC} = \Phi_{BA}({}_{\mathcal{A}}\mathbf{r}_{BC})$. We define the mapping $\mathbf{C} : SO(3) \rightarrow \mathbb{R}^{3 \times 3}$ such that $\Phi(\mathbf{r}) \triangleq \mathbf{C}(\Phi)\mathbf{r}$ (corresponding to the rotation matrix). A more complete overview of coordinate systems and rotations is given in [3].

Furthermore, the vectors $\vec{v}_{\mathcal{B}}$ and $\vec{a}_{\mathcal{B}}$ denote the absolute (w.r.t. an inertial coordinate system) velocity and acceleration of the point \mathcal{B} . The vector $\vec{\omega}_{\mathcal{AB}}$ denotes the relative angular velocity of the coordinate system \mathcal{B} w.r.t. the coordinate system \mathcal{A} . The skew symmetric matrix of a coordinate tuple $\mathbf{v} \in \mathbb{R}^3$ is denoted as $\mathbf{v}^\times \in \mathbb{R}^{3 \times 3}$ and has the property $\mathbf{v}^\times \mathbf{r} = \mathbf{v} \times \mathbf{r} \forall \mathbf{r} \in \mathbb{R}^3$, where \times denotes the Euclidean cross-product. It fulfills the following identities ($\mathbf{I} \in \mathbb{R}^{3 \times 3}$ is the identity matrix):

$$(\mathbf{v}^\times)^T = -\mathbf{v}^\times, \quad (1)$$

$$(\mathbf{v}^\times)^2 = \mathbf{v}\mathbf{v}^T - \mathbf{v}^T\mathbf{v}\mathbf{I}, \quad (2)$$

$$(\mathbf{C}(\Phi)\mathbf{v})^\times = \mathbf{C}(\Phi)\mathbf{v}^\times\mathbf{C}(\Phi)^T. \quad (3)$$

III. THEORY

The following contemplations are independent of the choice of parametrization for 3D orientations. As will follow in the next definition, 3D orientations are first only thought of as mapping.

Given a 3D rigid body with attached body-fixed coordinate system \mathcal{B} , its orientation $\Phi_{\mathcal{B}\mathcal{A}}$ w.r.t. a reference coordinate system \mathcal{A} can be defined as the mapping which maps coordinates of any fixed vector \vec{r} from \mathcal{A} to \mathcal{B} , that is,

$${}_{\mathcal{B}}\mathbf{r} = \Phi_{\mathcal{B}\mathcal{A}}({}_{\mathcal{A}}\mathbf{r}). \quad (4)$$

Together with the concatenation operation, orientations form a Lie group known as the special orthogonal group $SO(3)$. The concatenation $\circ : SO(3) \times SO(3) \rightarrow SO(3)$ comes with the following (defining) identity:

$$(\Phi_{\mathcal{C}\mathcal{B}} \circ \Phi_{\mathcal{B}\mathcal{A}})({}_{\mathcal{A}}\mathbf{r}) \triangleq \Phi_{\mathcal{C}\mathcal{B}}(\Phi_{\mathcal{B}\mathcal{A}}({}_{\mathcal{A}}\mathbf{r})). \quad (5)$$

There exists an identity element Φ_I and an inverse Φ^{-1} such that

$$\Phi_I \circ \Phi_{\mathcal{B}\mathcal{A}} = \Phi_{\mathcal{B}\mathcal{A}} \circ \Phi_I = \Phi_{\mathcal{B}\mathcal{A}}, \quad (6)$$

$$\Phi_{\mathcal{B}\mathcal{A}}^{-1} \circ \Phi_{\mathcal{B}\mathcal{A}} = \Phi_{\mathcal{B}\mathcal{A}} \circ \Phi_{\mathcal{B}\mathcal{A}}^{-1} = \Phi_I. \quad (7)$$

The Lie group $SO(3)$ is no vector space, has no addition operation, and consequently no subtraction either. This poses an issue if using orientations in filtering or optimization frameworks, which strongly rely on small differences and gradients (e.g. for linearization). Fortunately, since $SO(3)$ is a Lie group, there exists an exponential map $Exp : T_{\Phi_I}SO(3) \rightarrow SO(3)$ from a corresponding Lie algebra, which is the tangent space at the identity element Φ_I (which is isomorphic to \mathbb{R}^3). It is a smooth map which fulfills the following (uniquely) defining identities $\forall t, s \in \mathbb{R}, \forall \vec{\varphi} \in T_{\Phi_I}SO(3)$:

$$Exp((t+s)\vec{\varphi}) = Exp(t\vec{\varphi}) \circ Exp(s\vec{\varphi}), \quad (8)$$

$$d/dt(Exp(t\vec{\varphi}))|_{t=0} = \vec{\varphi}. \quad (9)$$

Elements on $T_{\Phi_I}SO(3)$ are abstract vectors and are not very suitable for actual computations. By choosing a basis $\mathbf{B} = [\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3]$ the map can be extended to \mathbb{R}^3 . We define the exponential $\exp : \mathbb{R}^3 \rightarrow SO(3)$ of a coordinate tuple $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{R}^3$ by

$$\exp(\varphi) := Exp(\vec{\varphi}_1\varphi_1 + \vec{\varphi}_2\varphi_2 + \vec{\varphi}_3\varphi_3). \quad (10)$$

There is a certain degree of freedom in the selection of the basis $[\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3]$. We choose the basis vectors $\vec{\varphi}_i$ such that $\forall i \in \{1, 2, 3\}, \forall \mathbf{v} \in \mathbb{R}^3$:

$$d/dt(Exp(t\vec{\varphi}_i)(\mathbf{v}))|_{t=0} = \mathbf{e}_i \times \mathbf{v} \quad (11)$$

where $\mathbf{e}_i \in \mathbb{R}^3$ are the standard basis vectors in \mathbb{R}^3 . This makes $\exp(\cdot)$ the unique smooth mapping that fulfills $\forall t, s \in \mathbb{R}, \forall \varphi, \mathbf{v} \in \mathbb{R}^3$:

$$\exp((t+s)\varphi) = \exp(t\varphi) \circ \exp(s\varphi) \quad (12)$$

$$d/dt(\exp(t\varphi)(\mathbf{v}))|_{t=0} = \varphi \times \mathbf{v} \quad (13)$$

We will see later, that by using this definition of the exponential \exp , its argument φ can be interpreted as the rotation vector associated with the relative orientation of two coordinate systems. There exists an open region around 0, the open ball with radius π $B_\pi(0) \subset \mathbb{R}^3$, such that the exponential is bijective and its image corresponds to all non-180°-orientations, $SO(3)^*$. Thus an inverse exists which is called the logarithm, $\log : SO(3)^* \rightarrow B_\pi(0)$.

With this we can construct boxplus and boxminus operations which adopt the function of addition and subtraction [4]:

$$\boxplus : SO(3) \times \mathbb{R}^3 \rightarrow SO(3), \quad (14)$$

$$\Phi, \varphi \mapsto \exp(\varphi) \circ \Phi,$$

$$\boxminus : SO(3) \times SO(3) \rightarrow \mathbb{R}^3, \quad (15)$$

$$\Phi_1, \Phi_2 \mapsto \log(\Phi_1 \circ \Phi_2^{-1}).$$

Similarly to regular addition and subtraction, both operators fulfill the following identities (axioms proposed by [4]):

$$\Phi \boxplus \mathbf{0} = \Phi, \quad (16)$$

$$(\Phi \boxplus \varphi) \boxminus \Phi = \varphi, \quad (17)$$

$$\Phi_1 \boxplus (\Phi_2 \boxminus \Phi_1) = \Phi_2. \quad (18)$$

This approach distinguishes between actual orientations which are on $SO(3)$ (Lie group) and differences of orientations which lie on \mathbb{R}^3 (Lie algebra, see fig. 1). The above operators take care of appropriately transforming the elements into their respective spaces and allow a smooth embedding of rotational quantities in filtering and optimization frameworks.

The definition of differentials involving orientations can be adapted by replacing the regular plus and minus operators by the above boxplus and boxminus operators. For instance the differential of a mapping $f_1 : \mathbb{R} \rightarrow SO(3)$ can be defined as:

$$\frac{\partial}{\partial x} f_1(x) := \lim_{\epsilon \rightarrow 0} \frac{f_1(x + \epsilon) \boxminus f_1(x)}{\epsilon}. \quad (19)$$

The same can be done for the other case where we have a mapping $f_2 : SO(3) \rightarrow \mathbb{R}$:

$$\frac{\partial}{\partial \Phi} f_2(\Phi) := \lim_{\epsilon \rightarrow 0} \left[\frac{f_2(\Phi \boxplus (\mathbf{e}_1 \epsilon)) - f_2(\Phi)}{\epsilon}, \frac{f_2(\Phi \boxplus (\mathbf{e}_2 \epsilon)) - f_2(\Phi)}{\epsilon}, \frac{f_2(\Phi \boxplus (\mathbf{e}_3 \epsilon)) - f_2(\Phi)}{\epsilon} \right]^T. \quad (20)$$

IV. IMPLEMENTATION-INDEPENDENT IDENTITIES

Some identities directly follow from the above considerations and are *independent* of the actual choice of the underlying orientation representation. By concatenating the exponential and the coordinate mapping we retrieve the well known Rodriguez' formula (see Appendix II-A):

$$\mathbf{C}(\varphi) := \mathbf{C}(\exp(\varphi)) \quad (21)$$

$$= \mathbf{I} + \frac{\sin(\|\varphi\|)\varphi^\times}{\|\varphi\|} + \frac{(1 - \cos(\|\varphi\|))\varphi^{\times^2}}{\|\varphi\|^2},$$

$$\mathbf{C}(\varphi) \approx \mathbf{I} + \varphi^\times, \quad (\|\varphi\| \approx 0). \quad (22)$$

This shows that the argument of the exponential, φ , can be

interpreted as the coordinate tuple of the (*passive*) rotation vector associated with the relative orientation of two coordinate systems. Thus, if the corresponding coordinate systems are known we can write:

$$\Phi_{\mathcal{B}\mathcal{A}} = \exp(\mathcal{B}\varphi_{\mathcal{B}\mathcal{A}}) = \exp(\mathcal{A}\varphi_{\mathcal{B}\mathcal{A}}). \quad (23)$$

We can also derive the following (adjoint related) identity (see Appendix II-B):

$$\exp(\Phi(\varphi)) = \Phi \circ \exp(\varphi) \circ \Phi^{-1}. \quad (24)$$

Useful identities can be derived for derivatives involving orientations (see Appendix I):

$$\partial/\partial t (\Phi_{\mathcal{B}\mathcal{A}}(t)) = -\mathcal{B}\omega_{\mathcal{A}\mathcal{B}}(t), \quad (25)$$

$$\partial/\partial \mathbf{r} (\Phi(\mathbf{r})) = \mathbf{C}(\Phi), \quad (26)$$

$$\partial/\partial \Phi (\Phi(\mathbf{r})) = -(\Phi(\mathbf{r}))^\times, \quad (27)$$

$$\partial/\partial \Phi (\Phi^{-1}) = -\mathbf{C}(\Phi)^T, \quad (28)$$

$$\partial/\partial \Phi_1 (\Phi_1 \circ \Phi_2) = \mathbf{I}, \quad (29)$$

$$\partial/\partial \Phi_2 (\Phi_1 \circ \Phi_2) = \mathbf{C}(\Phi_1), \quad (30)$$

$$\partial/\partial \varphi (\exp(\varphi)) = \mathbf{\Gamma}(\varphi), \quad (31)$$

$$\partial/\partial \Phi (\log(\Phi)) = \mathbf{\Gamma}^{-1}(\log(\Phi)). \quad (32)$$

The derivative of the exponential map is given by the Jacobian $\mathbf{\Gamma}(\varphi) \in \mathbb{R}^{3 \times 3}$ which has the following analytical expression:

$$\mathbf{\Gamma}(\varphi) = \mathbf{I} + \frac{(1 - \cos(\|\varphi\|))\varphi^\times}{\|\varphi\|^2} + \frac{(\|\varphi\| - \sin(\|\varphi\|))\varphi^\times{}^2}{\|\varphi\|^3}, \quad (33)$$

$$\mathbf{\Gamma}(\varphi) \approx \mathbf{I} + 1/2\varphi^\times, \quad (\|\varphi\| \approx 0). \quad (34)$$

V. QUATERNION IMPLEMENTATION

The above discussion is completely decoupled from any actual orientation parameterization. It is valid whether Euler-angles, rotation matrices, quaternions, or other representations are employed. In the following we provide one possible *implementation* of 3D orientations along with the means to check its correctness. Here we propose the use of unit quaternions following the *Hamilton* convention [5] and we will discuss the implementation of the different operations that are required. For more details on the differences between existing quaternion conventions we refer the reader to [6]. A unit quaternion is composed of a real part, $q_0 \in \mathbb{R}$, and an imaginary part, $\check{\mathbf{q}} \in \mathbb{R}^3$, which meet $q_0^2 + \|\check{\mathbf{q}}\|^2 = 1$. We write $\Phi = (q_0, \check{\mathbf{q}})$.

A. Coordinates Mapping and Rotation Matrix

For arbitrary coordinate systems, \mathcal{A} and \mathcal{B} , with relative orientation $\Phi_{\mathcal{B}\mathcal{A}} = (q_0, \check{\mathbf{q}})$ the coordinates of a vector $\vec{\mathbf{r}}$ can be mapped as:

$$\Phi_{\mathcal{B}\mathcal{A}}(\mathcal{A}\mathbf{r}) = (2q_0^2 - 1)\mathcal{A}\mathbf{r} + 2q_0\check{\mathbf{q}}^\times \mathcal{A}\mathbf{r} + 2\check{\mathbf{q}}(\check{\mathbf{q}}^T \mathcal{A}\mathbf{r}). \quad (35)$$

From this, we can directly derive the expression for the associated rotation matrix:

$$\mathbf{C}(\Phi_{\mathcal{B}\mathcal{A}}) = (2q_0^2 - 1)\mathbf{I} + 2q_0\check{\mathbf{q}}^\times + 2\check{\mathbf{q}}\check{\mathbf{q}}^T. \quad (36)$$

B. Concatenation

The concatenation of two unit quaternions $\Phi_1 = (q_0, \check{\mathbf{q}})$ and $\Phi_2 = (p_0, \check{\mathbf{p}})$ is given by:

$$\Phi_1 \circ \Phi_2 = (q_0 p_0 - \check{\mathbf{q}}^T \check{\mathbf{p}}, q_0 \check{\mathbf{p}} + p_0 \check{\mathbf{q}} + \check{\mathbf{q}} \times \check{\mathbf{p}}). \quad (37)$$

C. Exponential and Logarithm

Given a $\varphi \in \mathbb{R}^3$, the exponential map to a unit quaternion is given by:

$$\exp(\varphi) = (q_0, \check{\mathbf{q}}) = \left(\cos(\|\varphi\|/2), \sin(\|\varphi\|/2) \frac{\varphi}{\|\varphi\|} \right) \quad (38)$$

$$\exp(\varphi) \approx (1, \varphi/2), \quad (\|\varphi\| \approx 0). \quad (39)$$

The above small angle approximation is required to avoid numerical instabilities (typically for angles below 10^{-4} rad). The corresponding logarithm is given by:

$$\log(\Phi) = 2 \operatorname{atan2}(\|\check{\mathbf{q}}\|, q_0) \frac{\check{\mathbf{q}}}{\|\check{\mathbf{q}}\|}, \quad (40)$$

$$\log(\Phi) \approx \check{\mathbf{q}}, \quad (\|\check{\mathbf{q}}\| \approx 0). \quad (41)$$

D. Consistency Tests

The consistency of the implementation can be tested through the following unit tests:

	$\Phi()$	\mathbf{C}	\circ	\exp	\log
$\mathbf{C}(\Phi)\mathbf{r} = \Phi(\mathbf{r})$					
$(\Phi_1 \circ \Phi_2)(\mathbf{r}) = \Phi_1(\Phi_2(\mathbf{r}))$					
$\mathbf{C}(\exp(\varphi)) = \mathbf{C}(\varphi)$					
$\Phi = \exp(\log(\Phi))$					

Theoretically, the tests should be carried out for all possible values of $\Phi, \Phi_1, \Phi_2 \in SO(3)$, $\mathbf{r}, \varphi \in \mathbb{R}^3$. In practice, testing various samples, including very small angles, should be sufficient. The third test compares against Rodriguez' formula (eq. (21)). In case one of the tests failed the shaded cells provide a hint on the possible error sources.

VI. SIMPLE MODELING EXAMPLE

In this section we will present how to apply the above notation and convention to an actual system modeling task. We want to estimate the position, velocity (expressed in \mathcal{B} to simplify the Jacobians), and orientation of a robot using an IMU and a generic position and orientation sensor (pose sensor). To avoid complicated modeling or specific knowledge about your motion model the IMU can be used to do a prediction of the state. This is very common in visual-inertial state estimation e.g. [7]. In the following, we first show how to use the IMU for predicting the state and then show the necessary parts to perform an update with the pose sensor.

A. Continuous Time Description

Lets assume we have an IMU driven dynamic system with inertial coordinate system \mathcal{I} and IMU-fixed coordinate system \mathcal{B} where we want to estimate the motion. Considering additive biases, ${}_B\mathbf{b}_f$ and ${}_B\mathbf{b}_\omega$, and using continuous-time

white noise processes, $\mathcal{B}\mathbf{n}_f$, $\mathcal{B}\mathbf{n}_\omega$, $\mathcal{B}\mathbf{n}_{bf}$, $\mathcal{B}\mathbf{n}_{b\omega}$, we can model the IMU measurements, $\mathcal{B}\tilde{\mathbf{f}}_B$ and $\mathcal{B}\tilde{\boldsymbol{\omega}}_B$, as:

$$\mathcal{B}\tilde{\mathbf{f}}_B = \Phi_{IB}^{-1}(\mathcal{I}\mathbf{a}_B - \mathcal{I}\mathbf{g}) + \mathcal{B}\mathbf{b}_f + \mathcal{B}\mathbf{n}_f, \quad (42)$$

$$\mathcal{B}\tilde{\boldsymbol{\omega}}_B = \mathcal{B}\boldsymbol{\omega}_{IB} + \mathcal{B}\mathbf{b}_\omega + \mathcal{B}\mathbf{n}_\omega, \quad (43)$$

$$\mathcal{B}\dot{\mathbf{b}}_f = \mathcal{B}\mathbf{n}_{bf}, \quad (44)$$

$$\mathcal{B}\dot{\mathbf{b}}_\omega = \mathcal{B}\mathbf{n}_{b\omega}, \quad (45)$$

where $\mathcal{I}\mathbf{g}$ is the gravity vector expressed in the inertial frame. We add the IMU biases to the state \mathbf{x} and end up with the full state given by

$$\mathbf{x} = (\mathcal{I}\mathbf{r}_{IB}, \mathcal{B}\mathbf{v}_B, \Phi_{IB}, \mathcal{B}\mathbf{b}_f, \mathcal{B}\mathbf{b}_\omega). \quad (46)$$

The resulting continuous-time equations of motion can be written as:

$$\mathcal{I}\dot{\mathbf{r}}_{IB} = \Phi_{IB}(\mathcal{B}\mathbf{v}_B + \mathcal{B}\mathbf{n}_v), \quad (47)$$

$$\begin{aligned} \mathcal{B}\dot{\mathbf{v}}_B &= d/dt (\Phi_{IB}^{-1}(\mathcal{I}\mathbf{v}_B)) \\ &= \Phi_{IB}^{-1}(\mathcal{I}\dot{\mathbf{v}}_B) - (\Phi_{IB}^{-1}(\mathcal{I}\mathbf{v}_B))^\times \mathbf{C}(\Phi_{IB})^T \mathcal{I}\boldsymbol{\omega}_{BI} \\ &= \Phi_{IB}^{-1}(\mathcal{I}\mathbf{a}_B) - \mathcal{B}\mathbf{v}_B^\times \mathcal{B}\boldsymbol{\omega}_{BI} \\ &= \Phi_{IB}^{-1}(\mathcal{I}\mathbf{g}) + \mathcal{B}\mathbf{f}_{IB} - \mathcal{B}\boldsymbol{\omega}_{IB}^\times \mathcal{B}\mathbf{v}_B, \end{aligned} \quad (48)$$

$$\dot{\Phi}_{IB} = -\mathcal{I}\boldsymbol{\omega}_{BI} = \Phi_{IB}(\mathcal{B}\boldsymbol{\omega}_{IB}), \quad (49)$$

$$\mathcal{B}\dot{\mathbf{b}}_f = \mathcal{B}\mathbf{n}_{bf}, \quad (50)$$

$$\mathcal{B}\dot{\mathbf{b}}_\omega = \mathcal{B}\mathbf{n}_{b\omega}, \quad (51)$$

with the bias and noise corrected proper acceleration and angular velocity measurements

$$\mathcal{B}\mathbf{f}_{IB} = \mathcal{B}\tilde{\mathbf{f}}_B - \mathcal{B}\mathbf{b}_f - \mathcal{B}\mathbf{n}_f, \quad (52)$$

$$\mathcal{B}\boldsymbol{\omega}_{IB} = \mathcal{B}\tilde{\boldsymbol{\omega}}_B - \mathcal{B}\mathbf{b}_\omega - \mathcal{B}\mathbf{n}_\omega. \quad (53)$$

To derive (48) we used the product rule, followed by the chain rule and the identities (25),(27),(28).

B. Euler-Forward Discretization

One of the simplest and most commonly used discretization methods is Euler-Forward discretization. Of course also other discretization schemes could be employed. After a time increment Δt , this yields (the next state is denoted by a bar, discretized noise by a hat):

$$\mathcal{I}\bar{\mathbf{r}}_{IB} = \mathcal{I}\mathbf{r}_{IB} + \Delta t \Phi_{IB}(\mathcal{B}\mathbf{v}_B + \mathcal{B}\hat{\mathbf{n}}_v), \quad (54)$$

$$\mathcal{B}\bar{\mathbf{v}}_B = \mathcal{B}\mathbf{v}_B + \Delta t (\Phi_{IB}^{-1}(\mathcal{I}\mathbf{g}) + \mathbf{f} - \boldsymbol{\omega}^\times \mathcal{B}\mathbf{v}_B) \quad (55)$$

$$\begin{aligned} \bar{\Phi}_{IB} &= \Phi_{IB} \boxplus (\Delta t \Phi_{IB}(\boldsymbol{\omega})) \\ &= \exp(\Phi_{IB}(\Delta t \boldsymbol{\omega})) \circ \Phi_{IB} \\ &= \Phi_{IB} \circ \exp(\Delta t \boldsymbol{\omega}) \circ \Phi_{IB}^{-1} \circ \Phi_{IB} \\ &= \Phi_{IB} \circ \exp(\Delta t \boldsymbol{\omega}), \end{aligned} \quad (56)$$

$$\mathcal{B}\bar{\mathbf{b}}_f = \mathcal{B}\mathbf{b}_f + \Delta t \mathcal{B}\hat{\mathbf{n}}_{bf}, \quad (57)$$

$$\mathcal{B}\bar{\mathbf{b}}_\omega = \mathcal{B}\mathbf{b}_\omega + \Delta t \mathcal{B}\hat{\mathbf{n}}_{b\omega}, \quad (58)$$

with the discretized IMU measurements (bias and noise corrected)

$$\mathbf{f} = \mathcal{B}\tilde{\mathbf{f}}_B - \mathcal{B}\mathbf{b}_f - \mathcal{B}\hat{\mathbf{n}}_f, \quad (59)$$

$$\boldsymbol{\omega} = \mathcal{B}\tilde{\boldsymbol{\omega}}_B - \mathcal{B}\mathbf{b}_\omega - \mathcal{B}\hat{\mathbf{n}}_\omega. \quad (60)$$

The noise was discretized such that if \mathbf{R}_i is the noise density of the white noise process \mathbf{n}_i then the discrete Gaussian noise $\hat{\mathbf{n}}_i$ is distributed with $\mathcal{N}(0, \mathbf{R}_i/\Delta t)$.

C. Differentiation

Using the identities (25)-(32) and applying the chain rule the following Jacobians of the discrete process model can be derived (\mathbf{F} is w.r.t. the state, \mathbf{G} is w.r.t. the process noise):

$$\mathbf{F} = \begin{bmatrix} \mathbf{I} & \Delta t \mathbf{C}(\Phi_{IB}) & -\Delta t \Phi_{IB}(\mathcal{B}\mathbf{v}_B)^\times & 0 & 0 \\ 0 & \mathbf{I} - \Delta t \boldsymbol{\omega}^\times & \Delta t \mathbf{C}(\Phi_{IB})^T (\mathcal{I}\mathbf{g})^\times & -\Delta t \mathbf{I} & -\Delta t \mathcal{B}\mathbf{v}_B^\times \\ 0 & 0 & \mathbf{I} & 0 & -\Delta t \mathbf{C}(\Phi_{IB})^T \Gamma(\Delta t \boldsymbol{\omega}) \\ 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \Delta t \mathbf{C}(\Phi_{IB}) & 0 & 0 & 0 & 0 \\ 0 & -\Delta t \mathbf{I} & -\Delta t (\mathcal{B}\mathbf{v}_B)^\times & 0 & 0 \\ 0 & 0 & -\Delta t \mathbf{C}(\Phi_{IB})^T \Gamma(\Delta t \boldsymbol{\omega}) & 0 & 0 \\ 0 & 0 & 0 & \Delta t \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & \Delta t \mathbf{I} \end{bmatrix},$$

D. Measurement

For simplicity we assume a GPS position measurement $\mathcal{I}\tilde{\mathbf{r}}_{IB}$ and because this is a tutorial on orientations we also assume to have access to an orientation measurement $\tilde{\Phi}_{IB}$. The measurement equations are given by

$$\mathcal{I}\tilde{\mathbf{r}}_{IB} = \mathcal{I}\mathbf{r}_{IB} + \mathcal{I}\hat{\mathbf{n}}_p, \quad (61)$$

$$\tilde{\Phi}_{IB} = \Phi_{IB} \boxplus \mathcal{I}\hat{\mathbf{n}}_\Phi \quad (62)$$

$$= \exp(\mathcal{I}\hat{\mathbf{n}}_\Phi) \circ \Phi_{IB}, \quad (63)$$

with the discrete Gaussian measurement noise vectors $\mathcal{I}\hat{\mathbf{n}}_p$ and $\mathcal{I}\hat{\mathbf{n}}_\Phi$.

Using the identities (29),(30),(31) and because the expectation of the orientation measurement noise is zero the following Jacobians can be derived (\mathbf{H} is w.r.t. the state, \mathbf{J} is w.r.t. the update noise):

$$\mathbf{H} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 & 0 \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix},$$

E. Hints for the EKF Implementation

Now that we have derived all the required parts the well known EKF equations can be used to estimate the state. The only difference to the standard EKF is that we need to use the \boxplus operator for the innovation residual and the \boxminus operator for updating the state estimate instead of normal addition and subtraction.

VII. CONCLUSION

This document derived and summarized the main identities related to 3D orientations in robotics and other engineering fields. In particular it discussed a more abstract but convention-less notion of 3D orientations, the boxplus and boxminus operators, as well as the concept of differentials. Various differentials involving 3D orientations are derived, which can be used to compute the Jacobians of more complex models by applying the chain rule. A simple modeling example shows how to apply the introduced concepts.

APPENDIX I

DERIVATIVES INVOLVING ORIENTATIONS

1) *Time Derivative of Orientation*: Here we need the kinematic concept of angular velocities. We assume the existence of an inertial observer \mathcal{I} which observes the motion, over a duration ϵ , of a moving coordinate system $\mathcal{B}(t)$. We use the following definition of angular velocities (the negative sign is required so that the angular velocity corresponds to the *active* rotation which is measured by typical IMU devices):

$$\mathcal{B}(t)\boldsymbol{\omega}_{\mathcal{I}\mathcal{B}(t)} := -\lim_{\epsilon \rightarrow 0} \frac{\mathcal{B}(t)\boldsymbol{\varphi}_{\mathcal{B}(t+\epsilon)\mathcal{B}(t)}}{\epsilon} \quad (64)$$

Additionally we will require the limit (based on the limits (39),(41)):

$$\lim_{\epsilon \rightarrow 0} \frac{\log(\exp(\epsilon\boldsymbol{\varphi}_1) \circ \exp(\epsilon\boldsymbol{\varphi}_2))}{\epsilon} = \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2. \quad (65)$$

With this we can derive the derivative of an orientation $\Phi_{\mathcal{B}(t),\mathcal{A}(t)}$ w.r.t. time t (used identities: (19),(15),(23),(24),(64),(65)):

$$\begin{aligned} \frac{\partial}{\partial t} \Phi_{\mathcal{B}(t),\mathcal{A}(t)} &= \lim_{\epsilon \rightarrow 0} \frac{\Phi_{\mathcal{B}(t+\epsilon),\mathcal{A}(t+\epsilon)} \boxminus \Phi_{\mathcal{B}(t),\mathcal{A}(t)}}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left((\Phi_{\mathcal{B}(t+\epsilon)\mathcal{B}(t)} \circ \Phi_{\mathcal{B}(t),\mathcal{A}(t)} \circ \Phi_{\mathcal{A}(t),\mathcal{A}(t+\epsilon)}) \right. \\ &\quad \left. \boxminus \Phi_{\mathcal{B}(t),\mathcal{A}(t)} \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\log(\Phi_{\mathcal{B}(t+\epsilon)\mathcal{B}(t)} \circ \Phi_{\mathcal{B}(t),\mathcal{A}(t)} \right. \\ &\quad \left. \circ \Phi_{\mathcal{A}(t),\mathcal{A}(t+\epsilon)} \circ \Phi_{\mathcal{B}(t),\mathcal{A}(t)}^{-1}) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\log(\exp(\mathcal{B}(t)\boldsymbol{\varphi}_{\mathcal{B}(t+\epsilon)\mathcal{B}(t)}) \circ \Phi_{\mathcal{B}(t),\mathcal{A}(t)} \right. \\ &\quad \left. \circ \exp(\mathcal{A}(t)\boldsymbol{\varphi}_{\mathcal{A}(t),\mathcal{A}(t+\epsilon)}) \circ \Phi_{\mathcal{B}(t),\mathcal{A}(t)}^{-1}) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\log(\exp(\mathcal{B}(t)\boldsymbol{\varphi}_{\mathcal{B}(t+\epsilon)\mathcal{B}(t)}) \right. \\ &\quad \left. \circ \exp(\mathcal{B}(t)\boldsymbol{\varphi}_{\mathcal{A}(t),\mathcal{A}(t+\epsilon)})) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\log(\exp(-\epsilon \mathcal{B}(t)\boldsymbol{\omega}_{\mathcal{I}\mathcal{B}(t)}) \right. \\ &\quad \left. \circ \exp(\epsilon \mathcal{B}(t)\boldsymbol{\omega}_{\mathcal{I}\mathcal{A}(t)})) \right) \\ &= -\mathcal{B}(t)\boldsymbol{\omega}_{\mathcal{I}\mathcal{B}(t)} + \mathcal{B}(t)\boldsymbol{\omega}_{\mathcal{I}\mathcal{A}(t)} \\ &= -\mathcal{B}(t)\boldsymbol{\omega}_{\mathcal{A}(t)\mathcal{B}(t)} \end{aligned} \quad (66)$$

2) *Derivative of Inverse*: Here we derive the derivative of the inverse of an orientation (used identities: (19),(20),(15),(14),(24)):

$$\begin{aligned} \left[\frac{\partial}{\partial \Phi} \Phi^{-1} \right]_i &= \lim_{\epsilon \rightarrow 0} \frac{(\Phi \boxplus e_i \epsilon)^{-1} \boxminus \Phi^{-1}}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\log(\Phi^{-1} \circ \exp(-e_i \epsilon) \circ \Phi)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\log(\exp(-\Phi^{-1}(e_i)\epsilon))}{\epsilon} \\ &= -\Phi^{-1}(e_i) = -C(\Phi)^T e_i. \\ \frac{\partial}{\partial \Phi} \Phi^{-1} &= -C(\Phi)^T. \end{aligned} \quad (67)$$

3) *Derivative of Coordinate Map*: The map of an orientation applied to a coordinate tuple can be differentiated w.r.t. the orientation itself. This yields (used identities: (20),(14),(5),(22)):

$$\begin{aligned} \left[\frac{\partial}{\partial \Phi} \Phi(\mathbf{r}) \right]_i &= \lim_{\epsilon \rightarrow 0} \frac{(\Phi \boxplus e_i \epsilon)(\mathbf{r}) - \Phi(\mathbf{r})}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{C(e_i \epsilon)C(\Phi)\mathbf{r} - C(\Phi)\mathbf{r}}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{(\mathbf{I} + e_i^\times \epsilon)C(\Phi)\mathbf{r} - C(\Phi)\mathbf{r}}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{e_i^\times \epsilon C(\Phi)\mathbf{r}}{\epsilon} \\ &= -(C(\Phi)\mathbf{r})^\times e_i. \\ \frac{\partial}{\partial \Phi} \Phi(\mathbf{r}) &= -(C(\Phi)\mathbf{r})^\times. \end{aligned} \quad (68)$$

4) *Concatenation - Left*: The concatenation of two orientations can be differentiated w.r.t. the involved orientations. We first derive the derivative w.r.t. the left orientation (used identities: (19),(20),(15),(14)):

$$\begin{aligned} \left[\frac{\partial}{\partial \Phi_1} \Phi_1 \circ \Phi_2 \right]_i &= \lim_{\epsilon \rightarrow 0} \frac{((\Phi_1 \boxplus e_i \epsilon) \circ \Phi_2) \boxminus (\Phi_1 \circ \Phi_2)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\log(\exp(e_i \epsilon) \circ \Phi_1 \circ \Phi_2 \circ \Phi_2^{-1} \circ \Phi_1^{-1})}{\epsilon} \\ &= e_i. \\ \frac{\partial}{\partial \Phi_1} \Phi_1 \circ \Phi_2 &= \mathbf{I}. \end{aligned} \quad (69)$$

5) *Concatenation - Right*: The derivative of the concatenation w.r.t. the right orientation yields (used identities: (19),(20),(15),(14),(24)):

$$\begin{aligned} \left[\frac{\partial}{\partial \Phi_2} \Phi_1 \circ \Phi_2 \right]_i &= \lim_{\epsilon \rightarrow 0} \frac{(\Phi_1 \circ (\Phi_2 \boxplus e_i \epsilon)) \boxminus (\Phi_1 \circ \Phi_2)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\log(\Phi_1 \circ \exp(e_i \epsilon) \circ \Phi_2 \circ \Phi_2^{-1} \circ \Phi_1^{-1})}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\log(\exp(\Phi_1(e_i)\epsilon))}{\epsilon} \\ &= \Phi_1(e_i) = C(\Phi_1)e_i. \\ \frac{\partial}{\partial \Phi_2} \Phi_1 \circ \Phi_2 &= C(\Phi_1). \end{aligned} \quad (70)$$

6) *Exponential Derivative*: Define:

$$\Gamma(\boldsymbol{\varphi}) := \partial / \partial \boldsymbol{\varphi} (\exp(\boldsymbol{\varphi})). \quad (71)$$

Differentiate the adjoint related identity using the chain rule and product rule (identities (71),(27) for left side, identities (29),(30),(28) for right side):

$$\partial / \partial \Phi [\exp(\Phi(\boldsymbol{\varphi}))] = \Phi \circ \exp(\boldsymbol{\varphi}) \circ \Phi^{-1}, \quad (72)$$

$$-\Gamma(\Phi(\boldsymbol{\varphi}))\Phi(\boldsymbol{\varphi})^\times = \mathbf{I} - C(\Phi)C(\boldsymbol{\varphi})C(\Phi)^T. \quad (73)$$

Set Φ to identity:

$$\Gamma(\boldsymbol{\varphi})\boldsymbol{\varphi}^\times = C(\boldsymbol{\varphi}) - \mathbf{I}. \quad (74)$$

Now consider the map $f(x) = \exp(x\varphi)$ for some arbitrary $\varphi \in \mathbb{R}^3$. The chain rule yields $f'(x) = \Gamma(x\varphi)\varphi$. Alternatively, it can be differentiated using the limit (19) (used identities: (8),(15)):

$$\begin{aligned} f'(x) &= \lim_{\epsilon \rightarrow 0} \frac{\exp((x + \epsilon)\varphi) \boxminus \exp(x\varphi)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\log(\exp(\epsilon\varphi) \circ \exp(x\varphi) \circ \exp(x\varphi)^{-1})}{\epsilon} \\ &= \varphi. \end{aligned} \quad (75)$$

Compare both derivatives at $x = 1$:

$$\Gamma(\varphi)\varphi = \varphi. \quad (76)$$

This can be combined with eq. (74) in order to obtain the following matrix equation:

$$\Gamma(\varphi) \begin{bmatrix} \varphi^\times & \varphi \end{bmatrix} = \begin{bmatrix} C(\varphi) - I & \varphi \end{bmatrix}. \quad (77)$$

Right multiply with $\begin{bmatrix} \varphi^\times & \varphi \end{bmatrix}^T$ and simplify:

$$\Gamma(\varphi)(-\varphi^{\times 2} + \varphi\varphi^T) = (I - C(\varphi))\varphi^\times + \varphi\varphi^T, \quad (78)$$

$$\Gamma(\varphi)\|\varphi\|^2 = (I - C(\varphi))\varphi^\times + \varphi\varphi^T, \quad (79)$$

$$\Gamma(\varphi) = \frac{(I - C(\varphi))\varphi^\times + \varphi\varphi^T}{\|\varphi\|^2}. \quad (80)$$

If substituting $C(\varphi)$ we obtain eq. (33).

APPENDIX II OTHER PROOFS

A. Rodriguez' Formula

From eqs. (5), (12) and (13) we obtain the following properties for $C(\varphi) = C(\exp(\varphi))$, $\forall t \in \mathbb{R}$, $\varphi, v \in \mathbb{R}^3$:

$$C((t + s)\varphi) = C(t\varphi)C(s\varphi) \quad (81)$$

$$\frac{d}{dt}(C(t\varphi)(v))|_{t=0} = \varphi \times v \quad (82)$$

For a given φ we define the curve $C_\varphi(t) := C(t\varphi)$. Using a change of coordinate $t = s + r$, we can extend the range of the differential identity $\forall t \in \mathbb{R}$, $v \in \mathbb{R}^3$:

$$\frac{d}{dt}(C(t\varphi)v) = \frac{d}{ds}(C(s\varphi)C(r\varphi)v)|_{s=0, r=t} \quad (83)$$

$$= \varphi^\times C(t\varphi)v \quad (84)$$

Thus, we obtain the following matrix differential equation:

$$\frac{d}{dt}(C_\varphi(t)) = \varphi^\times C_\varphi(t), \quad (85)$$

which has the matrix exponential solution

$$C_\varphi(t) = e^{t\varphi^\times}. \quad (86)$$

Since this is valid for arbitrary φ , we obtain:

$$C(\varphi) = e^{\varphi^\times}, \quad (87)$$

which can be shown to be the same as eq. (21) using series expansions.

B. Concatenation and Exponential – Adjoint Related

We want to prove the following identity:

$$\exp(\Phi(\varphi)) = \Phi \circ \exp(\varphi) \circ \Phi^{-1}. \quad (88)$$

Since we know that \exp is unique it is sufficient to show that the right hand side is indeed the exponential of $\Phi(\varphi)$ and thus check the defining properties. First we verify eq. (12):

$$\exp((t + s)\Phi(\varphi)) \quad (89)$$

$$= \Phi \circ \exp((t + s)\varphi) \circ \Phi^{-1} \quad (90)$$

$$= \Phi \circ \exp(t\varphi) \circ \exp(s\varphi) \circ \Phi^{-1} \quad (91)$$

$$= \Phi \circ \exp(t\varphi) \circ \Phi^{-1} \circ \Phi \circ \exp(s\varphi) \circ \Phi^{-1} \quad (92)$$

$$= \exp(t\Phi(\varphi)) \circ \exp(s\Phi(\varphi)). \quad (93)$$

Equation (13) poses a requirement on the derivative which can also be verified:

$$\frac{d}{dt}(\exp(t\Phi(\varphi))(v))|_{t=0} \quad (94)$$

$$= \frac{d}{dt}(C(\Phi)(\exp(t\varphi)(C(\Phi)^T v)))|_{t=0} \quad (95)$$

$$= C(\Phi)\varphi^\times C(\Phi)^T v \quad (96)$$

$$= (C(\Phi)\varphi)^\times v = \Phi(\varphi) \times v. \quad (97)$$

Since $\Phi \circ \exp(\varphi) \circ \Phi^{-1}$ fulfills both uniquely defining properties of the exponential it is indeed equivalent to $\exp(\Phi(\varphi))$.

REFERENCES

- [1] T. Drummond, "Lie groups, Lie algebras, projective geometry and optimization for 3D Geometry, Engineering and Computer Vision," <http://tw20g.blogspot.ch/p/notes-on-lie-groups.html>, 2014, [Online].
- [2] E. Eade, "Lie groups for Computer Vision," <http://ethaneade.com/>, 2014, [Online].
- [3] T. D. Barfoot, *State Estimation for Robotics: A Matrix Lie Group Approach*. Draft in preparation for publication by Cambridge University Press, 2016.
- [4] C. Hertzberg, R. Wagner, U. Frese, and L. Schröder, "Integrating generic sensor fusion algorithms with sound state representations through encapsulation of manifolds," *Information Fusion*, vol. 14, no. 1, pp. 57–77, 2011. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/S1566253511000571>
- [5] W. R. Hamilton, "On quaternions; or on a new system of imaginaries in algebra," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 25, no. 163, pp. 10–13, 1844.
- [6] J. Sola, "Quaternion kinematics for the error-state kf," *Laboratoire d'Analyse et d'Architecture des Systemes-Centre national de la recherche scientifique (LAAS-CNRS), Toulouse, France, Tech. Rep.*, 2012.
- [7] M. Bloesch, S. Omari, M. Hutter, and R. Siegwart, "Robust visual inertial odometry using a direct ekf-based approach," in *Intelligent Robots and Systems (IROS), 2015 IEEE/RSJ International Conference on*. IEEE, 2015, pp. 298–304.