Estimation and Kalman Filtering (Part 3)

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Correlation: estimation error w/ another random variable (#4)

Random variables Z, P, Q

$$\Sigma_{(Z-\mathcal{L}(Z|P)),Q} = \Sigma_{ZQ} - \Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ}$$

Proof: $$[Z - \mathcal{L}(Z|P)](Q-\bar{q})^T = \left[Z - \left(\Sigma_{ZP}\Sigma_P^{-1}(P-\bar{p}) + \bar{z}\right)\right](Q-\bar{q})^T$$
$$= \left[(Z-\bar{z}) - \Sigma_{ZP}\Sigma_P^{-1}(P-\bar{p})\right](Q-\bar{q})^T$$

Now take expectation...

for $Q := P$, this gives $\mathbb{E}\,[Z - \mathcal{L}(Z|P)]\,(P-\bar{p})^T = 0$

for $Q = CZ$, this gives $\mathbb{E}\,[Z - \mathcal{L}(Z|P)]\,(CZ-C\bar{z})^T = \Sigma_{Z-\mathcal{L}(Z|P)}C^T$

for $Q = CZ$, this gives $\mathbb{E}\,[Z - \mathcal{L}(Z|P)]\,(Q-\bar{q})^T = \Sigma_{Z-\mathcal{L}(Z|P)}C^T$

for $Q = CZ + W$, with $\mathbb{E}W = 0$, $\mathbb{E}[W(Z-\bar{z})^T] = 0$, $\mathbb{E}[W(P-\bar{p})^T] = 0$, this gives

$$\mathbb{E}\,[Z - \mathcal{L}(Z|P)]\,(Q-\bar{q})^T = \Sigma_{Z-\mathcal{L}(Z|P)}C^T$$

$$\begin{aligned}\mathbb{E}[Z - \mathcal{L}(Z|P)]Q^T &= \\ &= \mathbb{E}\left([(Z-\bar{z}) - \Sigma_{Z,P}\Sigma_P^{-1}(P-\bar{p})][(Z-\bar{z})^TC^T + W^T]\right) \\ &= (\Sigma_Z - \Sigma_{Z,P}\Sigma_P^{-1}\Sigma_{Z,P}^T)C^T + 0 \\ &= \Sigma_{Z-\mathcal{L}(Z|P)}C^T\end{aligned}$$

linear minimum-variance estimator: “recursive” update (fact #5)

$$\mathcal{L}(Z|P,Q) \equiv \mathcal{L}(Z|\begin{bmatrix} P \\ Q \end{bmatrix}) \overset{1}{=} [\Sigma_{ZP}\ \Sigma_{ZQ}] \begin{bmatrix} \Sigma_P & \Sigma_{PQ} \\ \Sigma_{PQ}^T & \Sigma_Q \end{bmatrix}^{-1} \begin{bmatrix} P-\bar{p} \\ Q-\bar{q} \end{bmatrix} + \bar{z}$$

Matrix inversion lemma $\begin{bmatrix} \Sigma_P & \Sigma_{PQ} \\ \Sigma_{PQ}^T & \Sigma_Q \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma_P^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_P^{-1}\Sigma_{PQ} \\ -I \end{bmatrix} (\Sigma_Q - \Sigma_{PQ}^T\Sigma_P^{-1}\Sigma_{PQ})^{-1} \begin{bmatrix} \Sigma_{PQ}^T\Sigma_P^{-1} & -I \end{bmatrix}$

$$= \Sigma_{ZP}\Sigma_P^{-1}(P-\bar{p})+\bar{z} + [\Sigma_{ZP}\ \Sigma_{ZQ}] \begin{bmatrix} \Sigma_P^{-1}\Sigma_{PQ} \\ -I \end{bmatrix} (\Sigma_Q - \Sigma_{PQ}^T\Sigma_P^{-1}\Sigma_{PQ})^{-1} \begin{bmatrix} \Sigma_{PQ}^T\Sigma_P^{-1} & -I \end{bmatrix} \begin{bmatrix} P-\bar{p} \\ Q-\bar{q} \end{bmatrix}$$

$$= \mathcal{L}(Z|P) + \left(\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ} - \Sigma_{ZQ}\right)\left(\Sigma_Q - \Sigma_{PQ}^T\Sigma_P^{-1}\Sigma_{PQ}\right)^{-1}\left(\Sigma_{PQ}^T\Sigma_P^{-1}(P-\bar{p}) + \bar{q} - Q\right)$$

$$\overset{1}{=} \mathcal{L}(Z|P) + (\Sigma_{ZQ} - \Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ})(\Sigma_Q - \Sigma_{QP}\Sigma_P^{-1}\Sigma_{QP}^T)^{-1}(Q - \mathcal{L}(Q|P))$$

$$\boxed{\begin{aligned} \mathcal{L}(Z|P,Q) &= \mathcal{L}(Z|P) + (\Sigma_{ZQ}-\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ})(\Sigma_Q-\Sigma_{QP}\Sigma_P^{-1}\Sigma_{QP}^T)^{-1}(Q-\mathcal{L}(Q|P)) \\ &\overset{1,4}{=} \mathcal{L}(Z|P) + \Sigma_{Z-\mathcal{L}(Z|P),Q}\Sigma_{Q-\mathcal{L}(Q|P)}^{-1}(Q-\mathcal{L}(Q|P)) \end{aligned}}$$

Variance: expressed with the recursive update (fact #6)

$$\Sigma_{Z-\mathcal{L}(Z|P,Q)} = \Sigma_Z - \begin{bmatrix} \Sigma_{ZP} & \Sigma_{ZQ} \end{bmatrix} \begin{bmatrix} \Sigma_P & \Sigma_{PQ} \\ \Sigma_{PQ}^T & \Sigma_Q \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{PZ} \\ \Sigma_{QZ} \end{bmatrix}$$

$$\begin{bmatrix} \Sigma_P & \Sigma_{PQ} \\ \Sigma_{PQ}^T & \Sigma_Q \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma_P^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_P^{-1}\Sigma_{PQ} \\ -I \end{bmatrix} \left(\Sigma_Q - \Sigma_{PQ}^T \Sigma_P^{-1} \Sigma_{PQ}\right)^{-1} \begin{bmatrix} \Sigma_{PQ}^T \Sigma_P^{-1} & -I \end{bmatrix}$$

Matrix inversion lemma, again

$$= \Sigma_Z - \Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PZ}$$
$$- \left(\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ} - \Sigma_{ZQ}\right)\left(\Sigma_Q - \Sigma_{PQ}^T\Sigma_P^{-1}\Sigma_{PQ}\right)^{-1}\left(\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ} - \Sigma_{ZQ}\right)^T$$

$$\boxed{\begin{aligned} \Sigma_{Z-\mathcal{L}(Z|P,Q)} &= \Sigma_{Z-\mathcal{L}(Z|P)} - \left(\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ} - \Sigma_{ZQ}\right)\Sigma_{Q-\mathcal{L}(Q|P)}^{-1}\left(\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ} - \Sigma_{ZQ}\right)^T \\ &\overset{4}{=} \Sigma_{Z-\mathcal{L}(Z|P)} - \Sigma_{Z-\mathcal{L}(Z|P),Q}\Sigma_{Q-\mathcal{L}(Q|P)}^{-1}\Sigma_{Z-\mathcal{L}(Z|P),Q}^T \end{aligned}}$$

Special Case: recursive update with Q = CZ + DW

$$\mathcal{L}(Z|P,Q) \quad \text{where} \quad Q = CZ + DW, \ \Sigma_{W,P} = 0, \Sigma_{W,Z} = 0 \qquad \mu_W = 0$$

$$\mathcal{L}(Q|P) = C\mathcal{L}(Z|P) + D\mathcal{L}(W|P) = C\mathcal{L}(Z|P)$$

$$\begin{aligned} \Sigma_Q &= \mathbb{E}(C(Z-\bar{z}) + DW)((Z-\bar{z})^T C^T + W^T D^T) \\ &= C\Sigma_Z C^T + D\Sigma_W D^T \end{aligned}$$

$$\begin{aligned} \Sigma_{Q-\mathcal{L}(Q|P)} &= \Sigma_Q - \Sigma_{Q,P}\Sigma_P^{-1}\Sigma_{Q,P}^T \\ &= C\Sigma_Z C^T + D\Sigma_W D^T - C\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{ZP}^T C^T \\ &= C\left(\Sigma_Z - \Sigma_{ZP}\Sigma_P^{-1}\Sigma_{ZP}^T\right)C^T + D\Sigma_W D^T \\ &= C\Sigma_{Z-\mathcal{L}(Z|P)}C^T + D\Sigma_W D^T \end{aligned}$$

$$\begin{aligned} \Sigma_{Q,P} &= \mathbb{E}(C(Z-\bar{z}) + DW)(P-\bar{p})^T \\ &= C\Sigma_{Z,P} \end{aligned}$$

$$\Sigma_{(Z-\mathcal{L}(Z|P)),Q} = \Sigma_{(Z-\mathcal{L}(Z|P))}C^T$$

Special Case: recursive update with Q = CZ + DW (fact #7)

$$\mathcal{L}(Z|P,Q) \quad \text{where} \quad Q = CZ + DW,\ \Sigma_{W,P} = 0, \Sigma_{W,Z} = 0 \qquad \mu_W = 0$$

$$\mathcal{L}(Z|P,Q) \overset{5}{=} \mathcal{L}(Z|P) + \Sigma_{Z-\mathcal{L}(Z|P),Q} \qquad \Sigma^{-1}_{Q-\mathcal{L}(Q|P)} \qquad (Q - \mathcal{L}(Q|P))$$

$$\boxed{\mathcal{L}(Z|P,Q) = \mathcal{L}(Z|P) + \Sigma_{Z-\mathcal{L}(Z|P)} C^T \left(C\Sigma_{Z-\mathcal{L}(Z|P)}C^T + D\Sigma_W D^T\right)^{-1} (Q - C\mathcal{L}(Z|P))}$$

$$\Sigma_{Z-\mathcal{L}(Z|P,Q)} \overset{6}{=} \Sigma_{Z-\mathcal{L}(Z|P)} \ - \ \Sigma_{Z-\mathcal{L}(Z|P),Q}\Sigma^{-1}_{Q-\mathcal{L}(Q|P)}\Sigma^T_{Z-\mathcal{L}(Z|P),Q}$$

$$\boxed{\Sigma_{Z-\mathcal{L}(Z|P,Q)} = \Sigma_{Z-\mathcal{L}(Z|P)} - \Sigma_{Z-\mathcal{L}(Z|P)} C^T \left(C\Sigma_{Z-\mathcal{L}(Z|P)}C^T + D\Sigma_W D^T\right)^{-1} C\Sigma^T_{Z-\mathcal{L}(Z|P)}}$$

Notation and variables of specific interest

There are 3 important signals evolving in time: x, w and y. For each of them, use the same notation for estimates of them, in terms of the values of $y_0, y_1, \ldots, y_j$

$$\hat{x}_{k|j} := \mathcal{L}(x_k | y_0, \ldots, y_j)$$
$$\hat{y}_{k|j} := \mathcal{L}(y_k | y_0, \ldots, y_j)$$
$$\hat{w}_{k|j} := \mathcal{L}(w_k | y_0, \ldots, y_j)$$

Similarly, use a common notation for the variance of the error between x_k, y_k, w_k and the corresponding estimate

$$\Sigma^x_{k|j} := \Sigma_{x_k - \mathcal{L}(x_k|y_0,\ldots,y_j)} = \Sigma_{x_k - \hat{x}_{k|j}}$$
$$\Sigma^y_{k|j} := \Sigma_{y_k - \mathcal{L}(y_k|y_0,\ldots,y_j)} = \Sigma_{y_k - \hat{y}_{k|j}}$$
$$\Sigma^w_{k|j} := \Sigma_{w_k - \mathcal{L}(w_k|y_0,\ldots,y_j)} = \Sigma_{w_k - \hat{w}_{k|j}}$$

Used throughout the derivation:

$$\hat{x}_{k|k}, \Sigma^x_{k|k}$$
$$\hat{x}_{k|k-1}, \Sigma^x_{k|k-1}$$
$$\hat{x}_{k+1|k}, \Sigma^x_{k+1|k}$$

$$\hat{y}_{k|k-1}, \Sigma^y_{k|k-1} \qquad e_k := y_k - \hat{y}_{k|k-1}$$

$$\hat{w}_{k|k}, \Sigma^w_{k|k}$$
$$\hat{w}_{k|k-1}, \Sigma^w_{k|k-1}$$

Fact #8

$$\Sigma_{x_k, w_k} = 0 \qquad \Sigma_{\mathcal{Y}_{k-1}, w_k} = 0$$

$$x_k = \underbrace{\mathcal{A}_{k,0}}_{M_k} x_0 + \underbrace{\begin{bmatrix} \mathcal{A}_{k,1}E_0 & \mathcal{A}_{k,2}E_1 & \mathcal{A}_{k,3}E_2 & \cdots & \mathcal{A}_{k,k-1}E_{k-2} & E_{k-1} \end{bmatrix}}_{N_k} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{k-2} \\ w_{k-1} \end{bmatrix}$$

$$x_k = M_k x_0 + N_k \mathcal{W}_{k-1}$$

$$\mathcal{Y}_{k-1} = R_k x_0 + S_k \mathcal{W}_{k-1}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{k-2} \\ y_{k-1} \end{bmatrix} = \underbrace{\begin{bmatrix} C_0 \\ C_1\mathcal{A}_{1,0} \\ C_2\mathcal{A}_{2,0} \\ \vdots \\ C_{k-2}\mathcal{A}_{k-2,0} \\ C_{k-1}\mathcal{A}_{k-1,0} \end{bmatrix}}_{R_k} x_0 + \underbrace{\begin{bmatrix} F_0 & 0 & 0 & \cdots & 0 & 0 \\ C_1E_0 & F_1 & 0 & \cdots & 0 & 0 \\ C_2\mathcal{A}_{2,1}E_0 & C_2E_1 & F_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{k-2}\mathcal{A}_{k-2,1}E_0 & C_{k-2}\mathcal{A}_{k-2,2}E_1 & C_{k-2}\mathcal{A}_{k-2,3}E_2 & \cdots & F_{k-2} & 0 \\ C_{k-1}\mathcal{A}_{k-1,1}E_0 & C_{k-1}\mathcal{A}_{k-1,2}E_1 & C_{k-1}\mathcal{A}_{k-1,3}E_2 & \cdots & C_{k-1}E_{k-2} & F_{k-1} \end{bmatrix}}_{S_k} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{k-2} \\ w_{k-1} \end{bmatrix}$$

Fact #9

$$\hat{w}_{k|k-1} = 0$$

$$Z := w_k$$
$$P := \mathcal{Y}_{k-1} (= R_k x_0 + S_k \mathcal{W}_{k-1})$$

$$\Rightarrow \quad \Sigma_{Z,P} = 0, \quad \mu_Z = 0$$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{k-2} \\ y_{k-1} \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \mathcal{A}_{1,0} \\ C_2 \mathcal{A}_{2,0} \\ \vdots \\ C_{k-2} \mathcal{A}_{k-2,0} \\ C_{k-1} \mathcal{A}_{k-1,0} \end{bmatrix} x_0 + \begin{bmatrix} F_0 & 0 & 0 & \cdots & 0 & 0 \\ C_1 E_0 & F_1 & 0 & \cdots & 0 & 0 \\ C_2 \mathcal{A}_{2,1} E_0 & C_2 E_1 & F_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{k-2} \mathcal{A}_{k-2,1} E_0 & C_{k-2} \mathcal{A}_{k-2,2} E_1 & C_{k-2} \mathcal{A}_{k-2,3} E_2 & \cdots & F_{k-2} & 0 \\ C_{k-1} \mathcal{A}_{k-1,1} E_0 & C_{k-1} \mathcal{A}_{k-1,2} E_1 & C_{k-1} \mathcal{A}_{k-1,3} E_2 & \cdots & C_{k-1} E_{k-2} & F_{k-1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{k-2} \\ w_{k-1} \end{bmatrix}$$

$$\begin{aligned} \hat{w}_{k|k-1} &= \mathcal{L}(w_k | \mathcal{Y}_{k-1}) \\ &= \mathcal{L}(Z|P) \\ &\overset{1}{=} \Sigma_{Z,P} \Sigma_P^{-1} (P - \mu_P) + \mu_Z \\ &= 0 \end{aligned}$$

Fact #10

$$\boxed{\hat{y}_{k|k-1} = C_k \hat{x}_{k|k-1}}$$

$$\boxed{e_k = y_k - C_k \hat{x}_{k|k-1}}$$

$$y_k = C_k x_k + F_k w_k$$

$$\begin{aligned}
\hat{y}_{k|k-1} &:= \mathcal{L}(y_k | \mathcal{Y}_{k-1}) \\
&= \mathcal{L}(C_k x_k + F_k w_k | \mathcal{Y}_{k-1}) \\
&= \mathcal{L}(C_k x_k | \mathcal{Y}_{k-1}) + \mathcal{L}(F_k w_k | \mathcal{Y}_{k-1}) \\
&= C_k \mathcal{L}(x_k | \mathcal{Y}_{k-1}) + F_k \mathcal{L}(w_k | \mathcal{Y}_{k-1}) \\
&= C_k \hat{x}_{k|k-1} + F_k \hat{w}_{k|k-1} \\
&\overset{9}{=} C_k \hat{x}_{k|k-1}
\end{aligned}$$

$$\begin{aligned}
e_k &= y_k - \hat{y}_{k|k-1} \\
&= y_k - C_k \hat{x}_{k|k-1}
\end{aligned}$$

Fact #11

$$\boxed{\mathbb{E}\left[(x_k - \hat{x}_{k|k-1})w_k^T\right] = 0}$$

x_k is a linear combination of x_0 and $w_0, w_1, \ldots, w_{k-1}$

$\hat{x}_{k|k-1}$ is a linear combination of $y_0, y_1, \ldots, y_{k-1}$

$\Longrightarrow$ $y_0, y_1, \ldots, y_{k-1}$ are linear combinations of x_0 and $w_0, w_1, \ldots, w_{k-1}$

$\Longrightarrow$ $(x_k - \hat{x}_{k|k-1})$ is a linear combination of x_0 and $w_0, w_1, \ldots, w_{k-1}$ and hence, uncorrelated with w_k

$$Z := x_k$$
$$P := \mathcal{Y}_{k-1}$$
$$Q := w_k \qquad (\mu_Q = 0)$$

$$\begin{aligned}
\mathbb{E}\left[(x_k - \hat{x}_{k|k-1})w_k^T\right] &= \mathbb{E}\left[(Z - \mathcal{L}(Z|P))(Q - \mu_Q)^T\right] \\
&\overset{4}{=} \Sigma_{ZQ} - \Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ} \\
&= \Sigma_{x_k, w_k} - \Sigma_{ZP}\Sigma_P^{-1}\Sigma_{\mathcal{Y}_{k-1}, w_k} \\
&\overset{8}{=} 0
\end{aligned}$$

Fact #12

$$\boxed{\Sigma^y_{k|k-1} = C_k \Sigma^x_{k|k-1} C_k^T + F_k W_k F_k^T}$$

$$\Sigma^y_{k|k-1} := \Sigma_{y_k - \hat{y}_{k|k-1}} = \mathbb{E}\left[e_k e_k^T\right]$$

$$\begin{aligned} y_k - \hat{y}_{k|k-1} &\overset{10}{=} (C_k x_k + F_k w_k) - C_k \hat{x}_{k|k-1} \\ &= C_k (x_k - \hat{x}_{k|k-1}) + F_k w_k \end{aligned}$$

$$\begin{aligned} \Sigma_{y_k - \hat{y}_{k|k-1}} &= \mathbb{E}\left[C_k (x_k - \hat{x}_{k|k-1}) + F_k w_k\right]\left[C_k (x_k - \hat{x}_{k|k-1}) + F_k w_k\right]^T \\ &= C_k \Sigma^x_{k|k-1} C_k^T + F_k W_k F_k^T + C_k \mathbb{E}\left[(x_k - \hat{x}_{k|k-1}) w_k^T\right] F_k^T + (\cdot)^T \\ &\overset{11}{=} C_k \Sigma^x_{k|k-1} C_k^T + F_k W_k F_k^T \end{aligned}$$

Fact #13

$$\boxed{\Sigma_{e_k,w_k} = \mathbb{E}(e_k w_k^T) = F_k W_k}$$

Both e_k and w_k have zero mean, so

$$\Sigma_{e_k,w_k} = \mathbb{E}(e_k w_k^T)$$

$$\begin{aligned}
\mathbb{E}(e_k w_k^T) &\overset{10}{=} \mathbb{E}\left[(C_k(x_k - \hat{x}_{k|k-1}) + F_k w_k) w_k^T\right] \\
&= C_k \mathbb{E}\left[(x_k - \hat{x}_{k|k-1}) w_k^T\right] + F_k \mathbb{E}\left[w_k w_k^T\right] \\
&\overset{11}{=} F_k W_k
\end{aligned}$$

Fact #14

$$\Sigma_{(x_k - \hat{x}_{k|k-1}), y_k} = \Sigma^x_{k|k-1} C_k^T$$

fact #4

Random variables Z, P, Q

$$\Sigma_{(Z-\mathcal{L}(Z|P), Q} = \Sigma_{ZQ} - \Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ}$$

for $Q = CZ + W$, with

$\mathbb{E}W = 0$, $\mathbb{E}[W(Z-\bar{z})^T] = 0$, $\mathbb{E}[W(P-\bar{p})^T] = 0$, this gives

$$\mathbb{E}\left[Z - \mathcal{L}(Z|P)\right](Q - \bar{q})^T = \Sigma_{Z-\mathcal{L}(Z|P)} C^T$$

$$\begin{aligned} Z &:= x_k \\ P &:= \mathcal{Y}_{k-1} \\ Q &:= y_k \\ &= C_k x_k + F_k w_k \\ &= C_k Z + F_k w_k \end{aligned}$$

$$\mathcal{L}(Z|P) = \hat{x}_{k|k-1}$$

$$\begin{aligned} \Sigma_{(x_k - \hat{x}_{k|k-1}), y_k} &= \Sigma_{Z-\mathcal{L}(Z|P), Q} \\ &= \mathbb{E}\left[(Z - \mathcal{L}(Z|P))(Q - \mu_Q)^T\right] \\ &\overset{4}{=} \Sigma^x_{k|k-1} C_k^T \end{aligned}$$

Fact #15

$$\Sigma_{(x_k - \hat{x}_{k|k-1}),e_k} = \Sigma^x_{k|k-1} C_k^T$$

$$e_k \overset{10}{=} C_k(x_k - \hat{x}_{k|k-1}) + F_k w_k$$

$$\begin{aligned}
\Sigma_{(x_k - \hat{x}_{k|k-1}),e_k} &= \mathbb{E}\left[(x_k - \hat{x}_{k|k-1})((x_k - \hat{x}_{k|k-1})^T C_k^T + w_k^T F_k^T)\right] \\
&= \mathbb{E}\left[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T\right] C_k^T \\
&\quad + \mathbb{E}\left[(x_k - \hat{x}_{k|k-1}) w_k^T\right] F_k^T \\
&\overset{11}{=} \Sigma^x_{k|k-1} C_k^T \;+\; 0 \\
&= \Sigma^x_{k|k-1} C_k^T
\end{aligned}$$

Fact #16

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} \left(y_k - \hat{y}_{k|k-1}\right)$$

$$Z := x_k \qquad \mathcal{L}(Z|P) = \hat{x}_{k|k-1} \qquad \mathcal{L}(Z|P,Q) = \hat{x}_{k|k}$$

$$P := \mathcal{Y}_{k-1}$$

$$Q := y_k \qquad \mathcal{L}(Q|P) = \hat{y}_{k|k-1}$$

$$\mathcal{L}(Z|P,Q) \overset{5}{=} \mathcal{L}(Z|P) + \Sigma_{Z-\mathcal{L}(Z|P),Q} \Sigma^{-1}_{Q-\mathcal{L}(Q|P)} \left(Q - \mathcal{L}(Q|P)\right)$$

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$$\hat{x}_{k|k} \qquad \hat{x}_{k|k-1} \qquad \Sigma^x_{k|k-1} C_k^T \qquad \left(\Sigma^y_{k|k-1}\right)^{-1} \qquad y_k - \hat{y}_{k|k-1}$$

$$(= y_k - C_k \hat{x}_{k|k-1})$$

$$(= e_k)$$

Fact 16 (continued - notation for a later slide)

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \underbrace{\Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1}\right)^{-1}}_{:= H_k} \left(y_k - \hat{y}_{k|k-1}\right)$$

$$= \hat{x}_{k|k-1} + H_k e_k$$

Fact #17

$$\boxed{\hat{w}_{k|k} = W_k F_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} e_k}$$

$$\begin{aligned} Z &:= w_k \\ P &:= \mathcal{Y}_{k-1} \\ Q &:= y_k \\ &= C_k x_k + F_k w_k \end{aligned}$$

$$\mathcal{L}(Z|P) = \hat{w}_{k|k-1} \overset{9}{=} 0$$

$$Q - \mathcal{L}(Q|P) = y_k - \hat{y}_{k|k-1}$$

$$\begin{aligned} \Sigma_{Z,Q} &= \Sigma_{w_k (C_k(x_k - m_k) + F_k w_k)^T} \\ &\overset{8}{=} W_k F_k^T \end{aligned}$$

$$\Sigma_{Z,P} \overset{8}{=} 0$$

$$\mathcal{L}(Z|P,Q) \overset{5}{=} \mathcal{L}(Z|P) + (\Sigma_{ZQ} - \Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ})(\Sigma_Q - \Sigma_{QP}\Sigma_P^{-1}\Sigma_{QP}^T)^{-1}(Q - \mathcal{L}(Q|P))$$

$$\hat{w}_{k|k} \qquad \hat{w}_{k|k-1} = 0 \qquad \underbrace{W_k F_k^T \qquad \left(\Sigma^y_{k|k-1}\right)^{-1}}_{:= G_k} \qquad y_k - \hat{y}_{k|k-1}$$

$$(= y_k - C_k \hat{x}_{k|k-1})$$

$$(= e_k)$$

Fact #18

$$\boxed{\Sigma^w_{k|k} = W_k - W_k F_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} F_k W_k}$$

$$\begin{aligned} Z &:= w_k \\ P &:= \mathcal{Y}_{k-1} \\ Q &:= y_k \end{aligned}$$

$$\begin{aligned} \mathcal{L}(Z|P) &= \hat{w}_{k|k-1} \overset{9}{=} 0 \\ Q - \mathcal{L}(Q|P) &= y_k - \hat{y}_{k|k-1} \\ Z - \mathcal{L}(Z|P) &= Z = w_k \\ \mathcal{L}(Z|P,Q) &= \hat{w}_{k|k} \end{aligned}$$

$$\begin{aligned} \Sigma_{Z,Q} &\overset{8}{=} W_k F_k^T \\ \Sigma_{Z,P} &\overset{8}{=} 0 \end{aligned}$$

$$\Sigma_{Z-\mathcal{L}(Z|P,Q)} \overset{6}{=} \Sigma_{Z-\mathcal{L}(Z|P)} - (\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ} - \Sigma_{ZQ})\Sigma^{-1}_{Q-\mathcal{L}(Q|P)}(\Sigma_{ZP}\Sigma_P^{-1}\Sigma_{PQ} - \Sigma_{ZQ})^T$$

$$\Sigma^w_{k|k} \qquad W_k \qquad W_k F_k^T \qquad \left(\Sigma^y_{k|k-1}\right)^{-1} \qquad F_k W_k$$

Fact #19

$$\boxed{\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + E_k W_k F_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} e_k}$$

$$x_{k+1} = A_k x_k + E_k w_k$$

$$\begin{aligned}
\hat{x}_{k+1|k} &:= \mathcal{L}(x_{k+1}|\mathcal{Y}_k) \\
&= \mathcal{L}(A_k x_k + E_k w_k|\mathcal{Y}_k) \\
&= \mathcal{L}(A_k x_k|\mathcal{Y}_k) + \mathcal{L}(E_k w_k|\mathcal{Y}_k) \\
&= A_k \mathcal{L}(x_k|\mathcal{Y}_k) + E_k \mathcal{L}(w_k|\mathcal{Y}_k) \\
&= A_k \hat{x}_{k|k} + E_k \hat{w}_{k|k} \\
&\overset{17}{=} A_k \hat{x}_{k|k} + \underbrace{E_k W_k F_k^T}_{S_k} \left(\Sigma^y_{k|k-1}\right)^{-1} e_k
\end{aligned}$$

For later use, define

$$S_k := E_k W_k F_k^T$$

Fact #20

$$\boxed{\mathbb{E}\left(x_k - \hat{x}_{k|k}\right)\left(w_k - \hat{w}_{k|k}\right) = -\Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} F_k W_k}$$

$$\mathbb{E}\left(x_k - \hat{x}_{k|k}\right)\left(w_k - \hat{w}_{k|k}\right)^T \stackrel{16}{=} \mathbb{E}\left(x_k - \hat{x}_{k|k-1} - H_k e_k\right)\left(w_k - W_k F_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} e_k\right)^T$$

$$= \mathbb{E}\left(x_k - \hat{x}_{k|k-1}\right) w_k^T$$

$$-\mathbb{E}\left[\left(x_k - \hat{x}_{k|k-1}\right) e_k^T (\Sigma^y_{k|k-1})^{-1} F_k W_k\right]$$

$$-\mathbb{E}\left[H_k e_k w_k^T\right]$$

$$+\mathbb{E}\left[H_k e_k e_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} F_k W_k\right]$$

recall, $H_k := \Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1}\right)^{-1}$

$$\stackrel{11,13,15}{=} -\Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} F_k W_k - H_k F_k W_k + H_k \Sigma^y_{k|k-1} \left(\Sigma^y_{k|k-1}\right)^{-1} F_k W_k$$

$$= -\Sigma^x_{k|k-1} C_k^T (\Sigma^y_{k|k-1})^{-1} F_k W_k$$

Fact #21

$$\Sigma^x_{k+1|k} = A_k \Sigma^x_{k|k} A_k^T + E_k \left(W_k - W_k F_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} F_k W_k \right) E_k^T$$
$$- A_k \Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} F_k W_k E_k^T$$
$$- E_k W_k F_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} C_k \Sigma^x_{k|k-1} A_k^T$$

$$x_{k+1} - \hat{x}_{k+1|k} = A_k (x_k - \hat{x}_{k|k}) + E_k (w_k - \hat{w}_{k|k})$$

$$\begin{aligned}
\Sigma^x_{k+1|k} = \mathbb{E}\left[A_k (x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T A_k^T\right] &\longrightarrow A_k \Sigma^x_{k|k} A_k^T \\
+\mathbb{E}\left[E_k (w_k - \hat{w}_{k|k})(w_k - \hat{w}_{k|k})^T E_k^T\right] &\xrightarrow{18} E_k \left(W_k - W_k F_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} F_k W_k \right) E_k^T \\
+\mathbb{E}\left[A_k (x_k - \hat{x}_{k|k})(w_k - \hat{w}_{k|k})^T E_k^T\right] &\xrightarrow{20} -A_k \Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} F_k W_k E_k^T \\
+\mathbb{E}\left[E_k (w_k - \hat{w}_{k|k})(x_k - \hat{x}_{k|k})^T A_k^T\right] &\xrightarrow{20} (\cdot)^T
\end{aligned}$$

Everything all together

$$\hat{x}_{0|-1} = m_0$$
$$\Sigma^x_{0|-1} = \Sigma_0$$

Just the starting point (known statistics of the initial condition).
Notate as this, and that helps us move to the first step and beyond

$$\hat{y}_{k|k-1} = C_k \hat{x}_{k|k-1} \qquad e_k = y_k - C_k \hat{x}_{k|k-1}$$

Output residual

$$\Sigma^y_{k|k-1} = C_k \Sigma^x_{k|k-1} C_k^T + F_k W_k F_k^T$$

Residual covariance

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} (y_k - \hat{y}_{k|k-1})$$

State update

$$\Sigma^x_{k|k} = \Sigma^x_{k|k-1} - \Sigma^x_{k|k-1} C_k^T (C_k \Sigma^x_{k|k-1} C_k^T + F_k W_k F_k^T)^{-1} C_k \Sigma^x_{k|k-1}$$

Updated error variance

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + E_k W_k F_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} e_k$$

State prediction

$$\begin{aligned}\Sigma^x_{k+1|k} = {} & A_k \Sigma^x_{k|k} A_k^T + E_k \left(W_k - W_k F_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} F_k W_k\right) E_k^T \\ & - A_k \Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} F_k W_k E_k^T \\ & - E_k W_k F_k^T \left(\Sigma^y_{k|k-1}\right)^{-1} C_k \Sigma^x_{k|k-1} A_k^T\end{aligned}$$

Variance of prediction error

Iterate with $k = 0, 1, \ldots,$ starting from

$$\hat{x}_{0|-1} = m_0$$
$$\Sigma^x_{0|-1} = \Sigma_0$$

Reorder/group to variances and estimates

$$\Sigma^y_{k|k-1} = C_k \Sigma^x_{k|k-1} C_k^T + F_k W_k F_k^T$$

$$\Sigma^x_{k|k} = \Sigma^x_{k|k-1} - \Sigma^x_{k|k-1} C_k^T (\Sigma^y_{k|k-1})^{-1} C_k \Sigma^x_{k|k-1}$$

$$\begin{aligned}\Sigma^x_{k+1|k} = {} & A_k \Sigma^x_{k|k} A_k^T + E_k \left(W_k - W_k F_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} F_k W_k \right) E_k^T \\ & - A_k \Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} F_k W_k E_k^T \\ & - E_k W_k F_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} C_k \Sigma^x_{k|k-1} A_k^T\end{aligned}$$

Iterate with $k = 0, 1, \ldots$, starting from

$$\Sigma^x_{0|-1} = \Sigma_0$$
$$\hat{x}_{0|-1} = m_0$$

Be careful, as I have used the notation L_k differently very early in this slide packet, and in the extended Batch derivation HW.

$$e_k = y_k - C_k \hat{x}_{k|k-1}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} e_k$$

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + E_k W_k F_k^T \left(\Sigma^y_{k|k-1} \right)^{-1} e_k$$

note observer structure, $A\hat{x} + L(y - C\hat{x})$

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k-1} + \overbrace{\left(A_k \Sigma^x_{k|k-1} C_k^T + E_k W_k F_k^T \right) \left(\Sigma^y_{k|k-1} \right)^{-1}}^{L_k :=} e_k$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \underbrace{\Sigma^x_{k|k-1} C_k^T \left(\Sigma^y_{k|k-1} \right)^{-1}}_{\Gamma_k :=} e_k$$