Price Competition, Fluctuations and Welfare Guarantees

Moshe Babaioff, Microsoft Research. moshe@microsoft.com.
Renato Paes Leme, Google Research. renatoppl@google.com.
Balasubramanian Sivan, Microsoft Research. bsivan@microsoft.com.

In various markets where sellers compete in price, price oscillations are observed rather than convergence to equilibrium. Such fluctuations have been empirically observed in the retail market for gasoline, in airline pricing and in the online sale of consumer goods. Motivated by this, we study a model of price competition in which equilibria rarely exist. We seek to analyze the welfare, despite the nonexistence of equilibria, and present welfare guarantees as a function of the market power of the sellers.

We first study best response dynamics in markets with sellers that provide a homogeneous good, and show that except for a modest number of initial rounds, the welfare is guaranteed to be high. We consider two variations: in the first the sellers have full information about the buyer's valuation. Here we show that if there are n items available across all sellers and $n_{\max}$ is the maximum number of items controlled by any given seller, then the ratio of the optimal welfare to the achieved welfare will be at most $\log\left(\frac{n}{n-n_{\max}+1}\right)+1$. As the market power of the largest seller diminishes, the welfare becomes closer to optimal. In the second variation we consider an extended model in which sellers have uncertainty about the buyer's valuation. Here we similarly show that the welfare improves as the market power of the larger seller decreases, yet with a worse ratio of $\frac{n}{n-n_{\max}+1}$. Our welfare bounds in both cases are essentially tight. The exponential gap in welfare between the two variations quantifies the value of accurately learning the buyer's valuation in such settings.

Finally, we show that extending our results to heterogeneous goods in general is not possible. Even for the simple class of k-additive valuations, there exists a setting where the welfare approximates the optimal welfare within any non-zero factor only for $O(1/s)$ fraction of the time, where s is the number of sellers.

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1. INTRODUCTION

Price fluctuations have been observed in a variety of markets: from the traditional retail market for gasoline (Noel [2007]; Borenstein and Shepard [2002]) all the way to novel online marketplaces (Edelman and Ostrovsky [2007]). A recent WSJ article [Angwin and Mattioli 2012] tracks the price of a microwave across different online retailers (Amazon, Best Buy and Sears) and observes sellers constantly adjusting prices leading to fluctuations. The article remarks that frequent price adjustments which used to be confined to domains such as airline and hotel pricing are becoming increasingly common for all sorts of consumer goods. In this paper, we seek to analyze the efficiency of such markets despite the lack of convergence to equilibrium. One interesting feature is that the usual tools used in Price of Anarchy analysis (such as the smoothness framework) are not available and instead one needs to directly analyze the dynamics that leads to price fluctuations.

We consider a scenario with multiple sellers, each holding a set of goods, where each seller's strategy is to set a price for each of his goods. After prices are set, a buyer with a given valuation function over sets of goods (possibly representing the aggregate demand of many buyers), chooses an optimal bundle of items. Sellers move sequentially and in each time period one of them responds to the demand and the other sellers' prices by posting prices that are myopically optimal. The supply for each seller refreshes every time period to the seller's original set. In this paper we first study the special case of homogeneous goods, in which all items are identical and then explore the case of heterogeneous goods.

As an illustration consider web-publishers. Each receives a fixed number of impressions per day and sells those via posted prices[1]. After prices are posted, a DSP (Demand Side Platform, typically a network representing many advertisers), acquires impressions according to the aggregate demand expressed by the advertisers. Each publisher (seller) can observe the supply sold in each day and adjust his price. We are interested in the dynamic that arises from each seller repeatedly updating his price to best respond to other sellers. Below we discuss other markets with similar characteristics:

— *retail market for gasoline*: in each day a gas station can sell a fixed amount of gas that it is capable of storing in its tank. Periodically, stations update their price in response to the observed consumption of the previous period as well as the prices posted by other gas stations.
— *market for electronic components*: companies manufacturing phones and other electronic devices typically buy component parts (such as chips or flash memory) from other companies specialized in those. Typically those suppliers update prices to respond to the demand of the buyer and to other sellers.
— *airline tickets*: prices of airline tickets are the prototypical example of price fluctuation arising from fierce competition. According to [Economist 2010], there are roughly 1.86 million price updates per year for flights between New York City and San Francisco (across all airlines, days and fare classes).

Clearly any real world example is much more complex than our simplified model. Electronic components might not be completely identical, thus not being perfect substitutes. Airlines sell tickets for the same flight during a long period of time, and demand distribution shifts over time. Moreover, they experience seasonality and many other issues. Our model abstracts away all these issues in order to gain tractability and isolates the price competition aspect of such settings.

In modeling price updates we assume that each seller maintains a price for each of his units and at any given point in time, an arbitrary seller is allowed to change his prices. In this paper we assume that sellers use *myopic best-response*, i.e., they update their prices to optimize their revenue in the next time period based on their belief about the valuation function of the buyer and on the current prices of other sellers. In the first part of the paper, we consider the *Full Information case*, in which each seller is certain about the valuation of the buyer. In the second, we consider the *Uncertain Demand case* in which the valuation is still fixed over time, yet the sellers do not know it exactly. Rather, each seller has a belief expressed as a distribution over valuation functions of the buyer (and this belief does not rule out the actual valuation). In the Uncertain Demand case, we still assume that each seller myopically responds to the prices of the other sellers given his current belief about the demand, which is formed from his initial belief (prior) using all observations from previous time-steps.

Myopic best response dynamics has been extensively studied for many repeated game settings. It is attractive as a model of situations in which agents act rationally in the short run, but lack deep understanding of the game and the implications of their behavior in the long run. There is no coordination issue and the dynamics is

[1]in practice those are sold via an auction where the publisher can set the reserve price. Using the reserve as a posted price, however, is a good first approximation to this scenario.

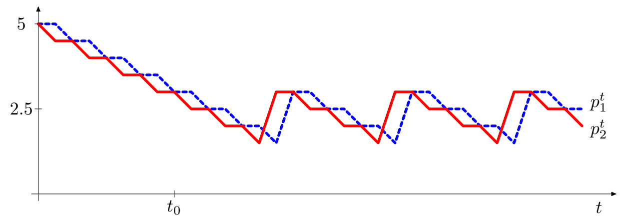

Fig. 1. The time sequence of prices posted by sellers 1 and 2, each with 2 items, when the buyer have a valuation with marginals $5, 5, 3, 1$. Notice that a best-response sequence cycles forever rather than converge to an equilibrium.

distributed in nature. In the Uncertain Demand case we still assume that sellers are completely myopic and disregard the long term implications of learning. Nevertheless, once observing new information about the demand, they do update their belief. The myopic model allows us to focus on the features arising purely from competition.

Our results and techniques for homogeneous goods. Our paper relates to the model of price competition studied in Babaioff et al. [2014]. In that paper the authors note that if each seller holds only one item, an efficient pure Nash equilibrium (NE) always exists for any combinatorial valuation of the buyer (even when items are not identical).[2] They also show that if the valuation of the buyer has decreasing marginals and one seller holds all items (monopolist case), then an equilibrium also exists but it can have welfare $\Omega(\log n)$ factor away from the welfare of the optimal allocation, where n is the number of items.

If each seller holds multiple items but no seller is a monopolist, however, a pure Nash equilibrium might not exist (actually, we show that it fails to exist unless there is an efficient Nash equilibrium, which occurs only in very restricted settings). We illustrate in Figure 1 a best response sequence that cycles. Pure Nash equilibrium might fail to exist even if the items are homogeneous, i.e., the valuation of a buyer just depends on the number of items he acquires. This fact is in line with observations that for various markets that can be approximately captured by this model, prices fluctuate rather than converge to an equilibrium.

Despite the lack of equilibrium, we still want to provide *welfare guarantees* for this setting. The guarantees we provide are as follows: we say that a game has *Eventual Welfare Guarantee* (EWG) α if for any initial prices and every sequence of best-responses, after finitely many *rounds* of price updates (where a round is a sequence of price updates in which every seller updates his price at least once (Nisan et al. [2011])), the welfare is at least an $1/\alpha$ fraction of the optimal welfare *in every time-step*. In particular, we show that if there is a total supply of n items and $n_{\max}$ is the maximum number of items controlled by any given seller, then the EWG is smaller than $1 + \log(n/(n - n_{\max} + 1))$.[3] This high welfare is reached quite fast, after number of rounds that is linear in the number of sellers and the inverse of a discretization parameter. An immediate corollary is that if no seller controls more than a c fraction of the goods, the EWG is smaller than $1 - \log(1 - c)$. These results present a bound on the welfare loss as a function of the market power of the seller with the largest supply. As that seller controls a smaller share, efficiency increases (as expected), not only at equilibrium but also when prices fluctuate (after enough price updates). For the Uncertain Demand case, we show the exponentially worse EWG bound of $n/(n-n_{\max}+1)$. If every seller controls at most a c fraction of the market, this implies an EWG of $1-c$.

[2]When the valuation is monotone there is always an equilibrium in which all items are being sold.
[3]All logarithms in this paper are natural logarithms (base e).

We also show that both bounds on the Eventual Welfare Guarantee are tight up to constant factors.

Our results for heterogeneous goods. The dynamics changes quite dramatically in the setting of heterogeneous goods. When the goods are heterogeneous, there is a whole spectrum of buyer valuations, some very complicated. One would naturally be interested in understanding simple valuation first and then move to more complicated ones. Two extreme valuations that are simple yet incomparable are unit-demand and additive valuations. These two extremes are well behaved: they both have an efficient pure Nash equilibrium. Given this, we move slightly away from these extremes to explore the simplest valuations that generalize both additive and unit-demand valuations, which are k-additive valuations: given a bundle, the buyer's valuation is additive over the k most valuable items in the bundle.

First, we show that for k-additive valuations an efficient pure Nash equilibrium need not always exist. Interestingly, in a heterogeneous goods setting, we observe that increasing competition by splitting the goods held by a monopolist among different sellers could hurt welfare! Whenever a pure Nash equilibrium exists, the welfare obtained is very close to optimal welfare, and we obtain a tight characterization of welfare in any pure Nash equilibrium. But when it doesn't exist, the welfare obtained is very poor in a strong sense: we show that the Eventual Welfare Guarantee is unbounded. Further, there exist best response sequences with s sellers where a non-zero approximation to optimal welfare is obtained only in $O(\frac{1}{s})$ fraction of the time. This implies that the positive results for the homogeneous case cannot be extended to heterogeneous items, even when restricting to the relatively simple k-additive valuations.

Relation to other equilibrium concepts. Before we proceed, we would like to comment on our choice of Eventual Welfare Guarantee rather than more traditional alternatives. A common choice in settings where pure Nash equilibria don't exist is to study mixed Nash equilibria, in which each agent plays according to a distribution. A mixed Nash would be characterized by a setting in which each agent has a distribution over actions (in our case, each seller has a distribution over prices) and in each round he plays an independent sample from this distribution. Although a game-theoretically sound concept, it doesn't seem to reflect the empirical behavior of sellers in various markets, since the prices used in a certain time period are very related to prices used in the previous one. This is unlike the case of sellers sampling independently from their mixed strategies, which results in frequent large changes of prices, unrelated to recent prices. The same argument can be made about other static equilibrium notions such as correlated and coarse correlated equilibrium.

A popular dynamic alternative is to consider outcomes of no-regret dynamics – see Nadav and Piliouras [2010] and Immorlica et al. [2010] for a discussion of no-regret outcomes in price competition and Roughgarden [2009]; Syrgkanis and Tardos [2013] for examples of this analysis in various other settings. While those model strategy updates that are dependent on the outcome of previous steps, the analysis generates welfare guarantees *on average*, i.e., if agents play long enough, the average welfare is at least a certain factor of the optimal welfare. Our analysis provides guarantees that hold for *each* time period after a certain point. The concept of *sink equilibria* proposed by Goemans et al. [2005] is closely related yet weaker, it only provides *average* guarantees over the Markov chain defined by the best response graph.

A main difference between our approach and the no-regret learning/sink-equilibria approach is that in the latter, the welfare bounds are studied by analyzing a static *limit object*. For the no-regret learning, the limit object is typically a coarse correlated equilibrium and for sink equilibria, it is the stationary state of a certain Markov chain. One drawback of such approaches is that although they provide a good description of the limiting behavior of the process, they don't offer intuitions on the transient behavior of the dynamic.

Related work. The work in out-of-equilibrium versions of price competition started by the work of Edgeworth [Edgeworth 1897] who criticized the prediction in the oligopoly models of Cournot and Bertrand that prices converge to an equilibrium. Edgeworth pointed out that various small changes in model caused the equilibrium no longer to exist. This idea was later developed by Shubik [Shubik 1959], Shapley [Shapley 1957] and others. We refer to [Vives 1993] for a survey on the modern theory of oligopolies. Various early concepts on oligopolies and price fluctuations received a formal treatment in a sequence of papers by Maskin and Tirole [Maskin and Tirole 1988a,b, 1987]. Our paper studies similar models but seeks to quantify the welfare loss from the resulting dynamic on a per-step basis instead of characterizing equilibria.

Another stream of related work is the study of the outcome of best response dynamics in algorithmic game theory: Mirrokni and Vetta [2004] study the relation between Nash equilibria and outcomes of best response dynamics. They show that even when all pure and mixed Nash equilibria have high welfare, there can exist sequences of best response dynamics which cycle through states of very low welfare. Roughgarden [2009] shows that for the special case of potential games, the smoothness framework can be used to provide efficiency guarantees for best-response dynamics. Recently, Fanelli et al. [2012] study the rate in which best-response dynamics converge for potential games.

Also related is the work of Nisan et al. [2011] who characterize a class of games for which it is rational for agents to best respond. The authors also look at the same question from the mechanism design perspective and seek to design games that converge to desirable outcomes. More recently, Engelberg et al. [2013] give conditions for games converge under any sequence of best responses.

Another popular alternative, that relaxes best-response dynamics and models bounded rationality and bounded computational abilities, is logit dynamics [McFadden 1974]. Here, at each time-step, one among the n agents in the game is chosen uniformly at random, and the chosen agent i plays a strategy s with probability proportional to $e^{\beta u_i(s,s_{-i})}$, where β is a parameter that models the level of rationality, with higher β being more rational, and $u_i(\cdot)$ is the utility of agent i. This dynamics defines a Markov chain over the strategy profiles, and this Markov chain has a unique stationary distribution, which is called as the logit equilibrium [Auletta et al. 2013b]. Given this, the primary questions that are studied here are the rate at which the dynamics converges to this logit equilibrium [Auletta et al. 2011], the existence of metastable distributions in cases where the dynamics takes a long time to converge [Auletta et al. 2012b], what happens if players can sumultaneously update their strategies [Auletta et al. 2012a, 2013a]. Although the literature in this area is taking steps to explore the transient dynamics in cases where the dynamics takes a long time to converge [Ferraioli and Ventre 2012; Auletta et al. 2012b], the main difference from our approach is that our work explicitly studies the transient dynamics and gives guarantee on a per-round basis, rather than the stationary or metastable distribution that is accomplished in time.

While in this paper we study the welfare guarantees in presence of price competition, one could also study seller revenue in presence of price competition. See [Greenwald et al. 1999; Greenwald and Kephart 1999; Kephart and Greenwald 1999] for a one-shot version Nash equilibrium anaylsis of a price competition game between sellers, using various price update strategies.

2. PRELIMINARIES

2.1. Homogeneous goods

Pricing game with full information. A full information pricing game is defined by $s \geq 2$ sellers, with each seller $i \in [s]$ holding n_i units of a homogeneous good for which he has no value, and a buyer with a publicly known valuation $v : \{0, 1, \ldots, n\} \to \mathbb{R}_+$, where $n := \sum_{i=1}^{s} n_i$ denotes the total supply of the good. We define $n_{\max}$ to be the maximal supply of any single seller, $n_{\max} = \max_{i=1}^{s} n_i$. We assume that the valuation function is (i) monotone, i.e., $v(k) \leq v(k+1), \forall k$; (ii) is normalized at zero, i.e., $v(0) = 0$ and (iii) obeys the decreasing marginal returns property, i.e., $m_1 \geq m_2 \geq \ldots m_n$ where we denote by $m_k = v(k) - v(k-1)$ the k-th marginal value. We also use $v(\ell|k)$ to denote the marginal value of ℓ items with respect to a set of k items, i.e., $v(\ell|k) = v(k+\ell) - v(k)$.

Let $N_i = \{1 + \sum_{r=1}^{i-1} n_r, \ldots, n_i + \sum_{r=1}^{i-1} n_r\}$ denote the set of n_i units that seller i holds. Let p_j denote the price of item j, and let $\mathbf{p}_i$ denote the vector of prices posted by seller i. The utility of seller i is his total revenue, given by $u_i^{X^v}(\mathbf{p}) = \sum_{j \in {X_i}^v(\mathbf{p})} p_j$ where ${X_i}^v(\mathbf{p})$ is the bundle of goods purchased by the buyer with valuation v from seller i, when facing price vector $\mathbf{p}$.

We assume that the buyer responds to prices by picking a bundle maximizing his total value minus his payment. Since the items are homogeneous and the marginals decreasing, the optimal choice of the buyer is to follow a *Greedy Algorithm*[4]: start with an empty set of items and recursively choose one of the cheapest items available. If the price of this item is smaller than the marginal value, include the item in the selected set and repeat. If the price is larger, then stop. If the price of this item equal to the marginal value, either include the item in the selected set and repeat, or stop. Any specific Greedy Algorithm needs to determine how to break ties between items of the same price, as well as deciding when to stop picking items for which the price is equal to the marginal value. In fact, all optimal sets can be described as outcomes of a greedy algorithm for some tie breaking rule and some rule of picking items with zero contribution to the utility of the buyer. For clarity of presentation we fix the tie breaking rule used by the buyer as follows[5]: between two items of the same price he breaks ties lexicographically (consistent with the lexicographic order over the sellers). Also, he will always pick items with zero marginal utility, i.e., items for which the price is equal to the marginal value. We define $X^v(\mathbf{p})$ as the set purchased by the buyer with value v when facing a price vector $\mathbf{p}$ according to the specific greedy algorithm discussed above. When this is clear from the context, we omit v and refer to $X^v(\mathbf{p})$ as $X(\mathbf{p})$.

We define the welfare associated with a strategy profile as the sum of utilities of all agents involved (the buyer and all the sellers): $W(\mathbf{p}) = v(|X^v(\mathbf{p})|)$.

Discretization. We choose in this paper to model all valuations and prices as integral multiples of a (smallest) monetary unit ϵ (e.g., 1 cent in the case of US dollars). To simplify, we also assume that $1/\epsilon$ is an integer. We are primarily interested in studying price update dynamics in this paper, where sellers frequently undercut each other. A non-trivial change in price is necessary to meaningfully describe the rate of progress of such update dynamics. Moreover, the constraint of pricing in multiples of a fixed quantity is present in essentially every (electronic or not) commercial platforms.

[4]The fact that a greedy algorithm computes the utility maximizing bundle for the buyer holds more generally when the items are heterogeneous and the buyer has a gross substitutes valuation. See Paes Leme [2013].
[5]See Appendix A for a short discussion of random tie breaking.

Notation. We refer to the discretized domain as $\epsilon\mathbb{Z}_+ := \{0, \epsilon, 2\epsilon, 3\epsilon, \ldots\}$. We assume from this point on that for all $k \in \{0, 1, \ldots, n\}$, $v(k) \in \epsilon\mathbb{Z}_+$ and that the strategy of a seller consists of choosing a vector $\mathbf{p}_i \in \epsilon\mathbb{Z}_+^{n_i}$.

We will refer to the set of valuation functions as $\mathbf{V}$, i.e., $\mathbf{V}$ is the set of valuation $v : \{0, 1, \ldots, n\} \to \epsilon\mathbb{Z}_+$ that satisfy (i) monotonicity, (ii) normalization at zero and (iii) decreasing marginal values.

Eventual Welfare Guarantee. The traditional solution concept for full information games is that of the pure Nash equilibrium: a price vector is an equilibrium if no seller can change the prices of the goods he controls and improve his utility. Formally: $\mathbf{Nash}^{X^v} = \{\mathbf{p} \in \epsilon\mathbb{Z}_+^n : u_i^{X^v}(\mathbf{p}_i, \mathbf{p}_{-i}) \geq u_i^{X^v}(\tilde{\mathbf{p}}_i, \mathbf{p}_{-i}), \forall i \in [s], \tilde{\mathbf{p}}_i \in \epsilon\mathbb{Z}_+^{n_i}\}$.
As we show in Section 3, a pure Nash equilibrium does not always exist in the pricing game defined above. Given the absence of a pure strategy Nash equilibrium, and the motivation to model price fluctuations observed in online markets, we propose a notion on how to measure welfare which we call *Eventual Welfare Guarantee*.

First, we define a *best response sequence*: We say that a sequence of vectors of prices $\mathbf{p}^0, \mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^t, \ldots$ is a *best-response sequence* if for each time-step t there is a seller $i(t)$ such that $\mathbf{p}_{i(t)}^t \in \text{BR}_{i(t)}^{X^v}(\mathbf{p}_{-i(t)}^{t-1})$ and $\mathbf{p}_{i'}^t = \mathbf{p}_{i'}^{t-1}$ for all $i' \neq i(t)$, where $\text{BR}_i^{X^v}(\mathbf{p}_{-i})$ is the set of best-responses of seller i:

$$\text{BR}_i^{X^v}(\mathbf{p}_{-i}) := \text{argmax}_{\mathbf{p}_i \in \epsilon\mathbb{Z}_+^{n_i}} u_i^{X^v}(\mathbf{p}_i, \mathbf{p}_{-i})$$

When the seller has more than one best-response, we will assume ties are broken favoring uniform price vectors over discriminatory prices, favoring the sale of more items and any remaining tie is broken according to an arbitrary tie breaking rule. At the end of each time-step $t \geq 0$, the buyer buys his favorite bundle $X^v(\mathbf{p}^t)$, and the sellers' supplies are replenished so that seller i has n_i units.

We say that a best-response sequence is *fair*, if every seller is allowed to best-respond infinitely often, i.e, for all $i' \in [s]$, $\{t; i(t) = i'\}$ is an infinite set. Given a fair best response sequence, we divide the time in rounds, where a *round* is a minimal interval in which each seller has a chance to play at least once. Formally, we define the beginning time r_ℓ of the ℓ-th round recursively: $r_0 = 0$, and $r_\ell = \min\{r; \forall i' \in [s] \exists t \in [r_{\ell-1}, r) \text{ s.t. } i(t) = i'\}$.

We say that a game has *Eventual Welfare Guarantee* α if there is a finite $\tilde{\ell}$ such that from round $\tilde{\ell}$ onwards, for every fair best-response sequence, the welfare is at least a $\frac{1}{\alpha}$ fraction of the optimal welfare $W^* = v(n)$. Formally: for every $t \geq r_{\tilde{\ell}}$, $W(\mathbf{p}^t) \geq \frac{1}{\alpha} W^*$.

Pricing game with uncertain demand. In the basic model we assumed that the valuation of the buyer is fixed and known by all the sellers. In the uncertain demand model we still assume that the buyer picks an optimal set with respect to some fixed true underlying valuation v^* but this valuation is not known to the sellers. Instead, each seller has a (possibly different) belief expressed as a probability distribution over possible valuations of the buyer. We assume the beliefs are consistent in the sense that they assign non-zero probability to the true valuation v^*.

An important special case of the model is the case where there is a common prior and the valuation of the buyer is drawn from this common prior at time-step zero and used for all subsequent time-steps. In this special case, the prior beliefs are consistent by definition.

We model the belief of seller i at time t as a probability distribution $\mathbf{B}_i^t$ over $\mathbf{V}$, the set of all valuation functions. Given $v \in \mathbf{V}$, we denote by $\mathbf{B}_i^t(v)$ the probability assigned by seller i to valuation v at time t. We call $\mathbf{B}_i^0$, the prior of seller i and say that the prior is consistent if $\mathbf{B}_i^0(v^*) > 0$. We observe that since $\mathbf{V}$ is a countable discrete set, a seller believing that some valuation is feasible is equivalent to assigning positive probability to it, unlike in continuous settings.

Given a sequence of prices $\mathbf{p}^0, \mathbf{p}^1, \mathbf{p}^2, \ldots$, in each time-step t, the buyer faces prices $\mathbf{p}^t$ and he picks his optimal set $X^{v^*}(\mathbf{p}^t)$ according to his true valuation function v^*. Upon observing the set S^t picked by the buyer in time-step t, each seller i updates his belief in a Bayesian way to take into account the purchasing information at time t

$$\mathbf{B}_i^t(v) = \frac{\mathbf{1}\{S^t = X^v(\mathbf{p}^t)\} \cdot \mathbf{B}_i^{t-1}(v)}{\sum_{v' \in \mathbf{V}} \mathbf{1}\{S^t = X^{v'}(\mathbf{p}^t)\} \cdot \mathbf{B}_i^{t-1}(v')}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. The assumption that for every seller has a consistent prior guarantees that the update is well defined.

We say that the sequence of prices is a *best response sequence* if for each time-step t, there is an seller $i(t)$ such that $\mathbf{p}_{i(t)}^t \in \mathrm{BR}_{i(t)}^{X, \mathbf{B}_{i(t)}^{t-1}}(\mathbf{p}_{-i(t)}^{t-1})$ and $\mathbf{p}_{i'}^t = \mathbf{p}_{i'}^{t-1}$ for $i' \neq i(t)$, where:

$$\mathrm{BR}_i^{X, \mathbf{B}_i^{t-1}}(\mathbf{p}_{-i}) := \operatorname{argmax}_{\mathbf{p}_i} \mathbb{E}_{v \sim \mathbf{B}_i^{t-1}} \left[u_i^{X^v}(\mathbf{p}_i, \mathbf{p}_{-i}) \right]$$

In other words, the best responding seller picks a price vector that optimizes his expected utility according to his belief and given the prices of all other sellers. We stress that sellers are behaving myopically and are not explicitly trying to learn the buyer valuation (say by exploring various prices that are not currently optimal).

Given the definition of a best response sequence in the uncertain demand setting, we define a fair best response sequence and the concept of Eventual Welfare Guarantee in the same way as for a full information game.

2.2. Heterogeneous goods

Pricing game with full information. We define a full information pricing game with heterogeneous goods analogously to that for homogeneous goods. Our terminology doesn't distinguish between different goods and multiple copies of the same good. There are s sellers with each holding an arbitrary subset of the universe of goods. We assume the buyer's valuation function to be k-additive: his value for a bundle of goods is the sum of the values of his k most valuable goods in that bundle. With a such a valuation function, the optimal choice of a buyer while responding to a vector of prices is to still follow a *Greedy Algorithm*: start with an empty set, and recursively pick the item that gives the largest non-negative utility (item's value minus its price), and continue till the set is of size k or all items offering non-negative utility are exhausted. As before, while there are several ways of breaking ties in a Greedy Algorithm, for clarity we stick to the following tie-breaking rules for the buyer. If at any step in the greedy algorithm multiple items offer the same non-negative utility, the buyer prefers the item from the lexicographically most preferred seller. Among multiple items from the same seller that offer the same utility, the buyer prefers the most expensive item. The buyer buys items of zero utility (as long as that doesn't make his purchased set larger than size k). Any remaining tie is broken arbitrarily.

3. FULL INFORMATION GAME FOR HOMOGENEOUS GOODS

We start the section by showing that pure Nash equilibrium might fail to exist in the full information game, and then present our upper bound on the Eventual Welfare Guarantee of this game. We conclude by showing that the bound is tight.

3.1. Nonexistence of Pure Nash Equilibria

Before proceeding to discuss Eventual Welfare Guarantee of the full information pricing game, we show that pure Nash equilibria (NE) need not always exist for such games. When every seller has only one unit ($s = n$), there is always a pure Nash equilibrium in which every seller prices at the n-th marginal m_n and sells his unit. The next example shows this is no longer the case when sellers have at least two units each, and this is true for any tie breaking rule. We prove this in Appendix B.

Example 3.1. Consider a setting with two sellers, each holding 2 units of the same good. The buyer has diminishing marginal valuations, of $5, 5, 3, 1$, i.e., $v(1) = 5$, $v(2) = 10$, $v(3) = 13$, $v(4) = 14$. This setting does not have any pure Nash equilibrium.

In Figure 1 we depict a price cycle resulting from a best-response sequence between the two sellers in the previous example. In time-step t, seller 1 posts a price of p_1^t for both of his items and in the next time-step $t + 1$, seller two posts a price of p_2^{t+1} for his items. We assume that seller 2 best-responds in odd time-steps and seller 1 in even ones. We set $\epsilon = 0.5$ and break ties toward seller 1. Note that within 8 time-steps (marked as t_0 in the figure) 3 units are sold, and that the sold quantity will stay at 3 forever in this example (although prices will cycle forever). The welfare when 3 items are sold is $13/14$ fraction of the optimal welfare (93%). For this example, our main theorem presents a guarantee of a slightly smaller fraction: $(1 + \log(4/3))^{-1}$ = 77%.

The fact that there is no pure Nash equilibrium for the above example is no coincidence. We next show that the full information game will rarely have a pure equilibrium. Unless there is an efficient equilibrium in which all units are sold at the marginal price of the last unit m_n, there is no pure Nash equilibria at all. This is in line with Edgeworth's original observation [1897] for a different oligopoly model.

PROPOSITION 3.2. *Assume that the buyer has a decreasing marginal valuation with $m_n = v(n) - v(n-1) > 0$ and there are at least 2 sellers. Fix an arbitrary deterministic tie breaking rule for the buyer and assume that $\epsilon > 0$ is small enough. Either the only pure Nash equilibrium is for all sellers to price at m_n and sell all units (an efficient equilibrium), or there is no pure Nash equilibrium.*

PROOF. First observe that in any efficient Nash equilibrium the price of every seller must be m_n as otherwise either the buyer will drop some units (if the price is larger), or a seller can increase his revenue (if the price is lower).

Assume that $\epsilon < m_n/(4n)$. Consider any pure Nash equilibrium which is not efficient, that is, there is at least one seller i that is not selling his entire supply. Assume that the buyer is buying q units in total.

We first observe that all items must be sold for about the same price (prices of sold units can differ by at most ϵ). Assume that in a pure equilibrium the most expensive unit is sold at price h (note that $h \geq m_n$). Then any sold unit is priced either at h or at $h - \epsilon$. This is so as if some unit is priced at $p < h - \epsilon$ then the seller selling this unit can increase his revenue by increasing the price of this unit to $h - \epsilon$ and still sell (as the buyer must be running a greedy algorithm as presented above).

Next, observe that at any pure Nash equilibrium every seller is selling at least one item and making positive revenue, as if a seller prices all his units at $\epsilon < m_n$ the buyer will always buy from him. We conclude that other than seller i that is not selling all his units, there is at least one other seller that sells at least one unit for price of at least $h - \epsilon$. Consider a deviation by seller i changing all the prices of his units to $h - 2\epsilon$. Since the buyer must be using a greedy algorithm, this seller must be selling the minimum between n_i and q, which is strictly larger than the quantity he was selling before. His utility increased by at least $h - 2\epsilon - 2(n_i - 1)\epsilon$ as he gains at least $h - 2\epsilon$ for extra units sold, and loses at most $2(n_i - 1)\epsilon$ for units sold before and for which he has reduces his price. As $\epsilon < m_n/(4n)$ it holds that $h - 2\epsilon - 2(n_i - 1)\epsilon \geq m_n - 2\epsilon n_i > m_n - 2n(m_n/(4n)) > 0$, thus this is a beneficial deviation. This contradicts the fact that the this was a pure Nash equilibrium. $\square$

3.2. Bounding the Eventual Welfare Guarantee

In this section we prove the main theorem in the paper, which is a bound on the Eventual Welfare Guarantee of the full information game in the homogeneous goods setting.

THEOREM 3.3. *Assume that the buyer has a decreasing marginal valuation. The Eventual Welfare Guarantee of the full information game is at most* $1 + \log\left(\frac{n}{n - n_{\max} + 1}\right)$. *Moreover, such welfare is achieved after at most* $s \cdot v(1)/\epsilon$ *rounds.*

We note that this theorem generalizes Theorem 23 in [Babaioff et al. 2014], which gives an $O(\log n)$ bound for the Price of Anarchy for the monopolist case.

In what follows, we will prove a sequence of lemmas leading to our main result. We remind the reader that the buyer upon being faced with a price vector $\mathbf{p}$, chooses a set $X(\mathbf{p})$ using the greedy algorithm with tie breaking rules as discussed in Section 2.

LEMMA 3.4. *Let* $p^*(v, \mathbf{p})$ *denote the price of the most expensive item that a buyer with value* v *purchased at prices* $\mathbf{p}$. *Then, at any other price vector* $\tilde{\mathbf{p}}$ *that has at least* $|X(\mathbf{p})|$ *prices no larger than* $p^*(v, \mathbf{p})$, *the number of items sold is at least* $|X(\mathbf{p})|$.

PROOF. At prices $\mathbf{p}$, the buyer obtained a non-negative utility from purchasing his $|X(\mathbf{p})|$-th unit at $p^*(v, \mathbf{p})$. Therefore, because of decreasing marginal utility, the utility obtained from buying kth unit for any $k \leq |X(\mathbf{p})|$ at a price at most $p^*(v, \mathbf{p})$ is also non-negative. Thus, the greedy procedure will not terminate before purchasing at least $|X(\mathbf{p})|$ units. $\square$

The fact that the buyer purchases items following the greedy algorithm gives the seller no incentive to price different units differently.

LEMMA 3.5. *For every price vector* $\mathbf{p}_{-i}$ *of other sellers, every best-response price vector of seller* i *prices all his sold units at the same price.*

PROOF. Consider any candidate price vector $\mathbf{p}_i$ for seller i where i makes a non-zero revenue. Let h be the price of the most expensive unit that seller i sells at $(\mathbf{p}_i, \mathbf{p}_{-i})$. Consider raising the price of every unit of seller i to h. By Lemma 3.4, the total number of sold units will not fall down. The number of units $|X_i(\mathbf{p})|$ sold from i also will not fall down: for it to fall down, the buyer should have compensated by buying some previously unsold unit from some seller other than i. But the greedy procedure implies that every unsold unit from sellers other than i at the original price vector $\mathbf{p}$, is either more expensive than h or is priced at h but held by a lesser preferred seller. So $|X_i(\mathbf{p})|$ will not go down either. Thus pricing all sold units at h strictly increases seller i's revenue (if at least two of i's units were sold at $(\mathbf{p}_i, \mathbf{p}_{-i})$), and this proves the lemma. $\square$

Recall our assumption that sellers prefer uniform price vectors over non-uniform price vectors and prefer selling larger number of items. Lemma 3.5 states that the assumption that a seller prices uniformly all his sold units, is actually without loss of generality. So our assumption that each seller prices all his units (not just sold units) uniformly boils down to assuming that all unsold units are priced the same as the sold units. Pricing uniformly is a natural assumption in many settings, and often also a constraint of the marketplace. The second assumption that each seller prefers selling the larger number of items implies that among all utility equivalent prices, the seller picks the lowest price.

Last Considered Seller. For a given price vector $\mathbf{p}$ where the prices are uniform within a seller, the greedy procedure visits sellers in the order of increasing prices breaking ties lexicographically. We call the s-th seller in this order as the *last considered seller*. Note that the greedy algorithm could sometimes stop with an earlier seller (as the marginal utility becomes negative), but the last considered seller is still defined to be the s-th seller in the order.

Let μ_i denote the number of items that seller i would sell to a buyer who has already in possession of $n - n_i$ items, i.e., the number of items sold by a monopolist seller to a buyer with valuation $v(\cdot|n - n_i)$ (breaking ties towards larger quantity by our assumption). Formally, μ_i is the largest element in the set $\text{argmax}_k[v(k|n - n_i)]$.

The main step towards our main theorem (Theorem 3.3) is to show that after $s \cdot v(1)/\epsilon$ rounds, at least $\min_i(n - n_i + \mu_i)$ are sold, and the quantity sold will never drop below that amount in later time-steps. We actually prove a stronger version of Theorem 3.3 given by the following proposition:

PROPOSITION 3.6. *For every starting price* $\mathbf{p}^0$*, and every possible order of best-responses from sellers, the number of units sold reaches* $\min_i(n - n_i + \mu_i)$ *after at most* $s \cdot v(1)/\epsilon$ *rounds, and never falls below that amount. The social welfare after at most* $s \cdot v(1)/\epsilon$ *rounds is at least*
$\left[1 + \log\left(\frac{n}{\min_i(n-n_i+\mu_i)}\right)\right]^{-1} \cdot v(n)$ *and never falls below.*

The proof of the proposition follows directly from the series of lemmas below.

LEMMA 3.7. *When seller* i*'s best-response to prices* $\mathbf{p}_{-i}$ *makes him the last considered seller in the buyer's greedy procedure, he sells exactly* μ_i *items (regardless of* $\mathbf{p}_{-i}$*).*

PROOF. We prove this by showing that whenever seller i's best response to $\mathbf{p}_{-i}$ makes him the last considered seller, his revenue is identical to what a monopolist seller would get when dealing with a buyer with valuation $v(\cdot|n - n_i)$. This proves that the number of items sold by the best-responding seller and the monopolistic seller is the same (the latter is μ_i by definition) because we have assumed that sellers break tie towards selling more items. To prove the claim, note that the monopolist seller could not have made strictly smaller profit than the best responding seller i because he could have just used the price of the latter and obtained as much profit. Similarly, the best-responding seller i could not have made strictly smaller profit than the monopolist seller i: the former could have always used the latter's price. If this price resulted in the best-responding seller i being the last considered seller, he would have by definition of μ_i sold exactly μ_i items and made the monopolist's profit. If this price resulted in seller i being considered before the last seller, he would have sold at least μ_i items, and thus making at least as much profit. $\square$

LEMMA 3.8. *For every starting price* $\mathbf{p}^0$ *and every possible order of best-responses from sellers, the number of units sold reaches* $\min_i(n - n_i + \mu_i)$ *within the first* $s \cdot v(1)/\epsilon$ *rounds, and never falls below after that.*

PROOF. We prove this in two claims. First, we show that the number of units sold reaches $\min_i(n - n_i + \mu_i)$ within the first $s \cdot v(1)/\epsilon$ rounds, and second, we show that it doesn't fall below $\min_i(n - n_i + \mu_i)$. In fact, we need only $s(v(1)/\epsilon - 1)$ rounds to reach $\min_i(n - n_i + \mu_i)$.

We begin with the first claim. Consider two cases.

Case 1. In the first $s(v(1)/\epsilon - 1)$ rounds, some seller's best response made him the last considered seller in the buyer's greedy procedure. The seller, say i, who made this choice did so because the buyer at this price will exhaust the supply of every other seller, and buy a few items from seller i too. The exact number sold by seller i will be μ_i as shown in Lemma 3.7, and the total number sold is $n - n_i + \mu_i$.

Case 2. In no time step in the first $s(v(1)/\epsilon - 1)$ rounds did any seller best respond to become (or remain) the last considered seller in the buyer's greedy procedure. Let h^r be the largest price in the price vector at the end of rs rounds (so h^0 would the largest price in $\mathbf{p}^0$). We show by induction on r that either $h^r \leq \max(h^0 - r\epsilon, \epsilon)$ or the number of items sold already reached $\min_i(n - n_i + \mu_i)$ at some time step in the first r rounds. Thus by the end $s(v(1)/\epsilon - 1)$ rounds, we are guaranteed to have sold at least $\min_i(n - n_i + \mu_i)$ units at some time step, proving the claim. The base case of $r = 0$ is trivial. Assume that for $r \leq k$ the number of items sold never reached $\min_i(n - n_i + \mu_i)$ in the first r rounds, and that $h^r \leq \max(h^0 - r\epsilon, \epsilon)$ for $r \leq k$. Consider $r = k + 1$. By the definition of case 2, at no time step in any of the rounds between $ks + 1$ to $ks + s$ (both inclusive) did the largest price go strictly above h^k as that would put this in case 1. Furthermore, in each of these s rounds, some seller who is currently the last considered seller in the buyer's greedy procedure has to best-respond, and by definition of case 2, this seller has to strictly reduce his price below h^k. Also, this seller will never again raise his price to h^k in these s rounds as it will again violate case 2. Thus in each of the s rounds between $ks + 1$ and $ks + s$, as long as there is at least one seller priced at h^k, at least one seller reduces his price by at least ϵ. Since there are s sellers, the number of sellers priced at h^k is at most s, and thus after s rounds, the highest price would have fallen down by at least ϵ, thus proving the inductive step.

We now prove the second claim, namely, the number of units sold doesn't fall below $\min_i(n - n_i + \mu_i)$ once it reaches this quantity. At every time step t after this quantity has been sold, the best-responding seller i's action could either make (or retain) him the last considered seller in which case the number sold is $n - n_i + \mu_i$ by Lemma 3.7. If not, the best-responding seller i's action will satisfy the condition of Lemma 3.4, in which case by Lemma 3.4 the number sold will not fall below what is currently getting sold. □

LEMMA 3.9. *When the number of items sold is at least* $\min_i(n - n_i + \mu_i)$*, the welfare is at least* $\left[1 + \log\left(\frac{n}{\min_i(n - n_i + \mu_i)}\right)\right]^{-1} \cdot v(n)$.

PROOF. Let t^* be the smallest time step at which the number of units sold reaches $\min_i(n - n_i + \mu_i)$. Let W^t denote the social welfare at the end of time-step t, and let $\underline{W} = \min_{t \geq t^*} W^t$. Let $i^* \in \operatorname{argmin}_i (n - n_i + \mu_i)$. Note that when $n - n_{i^*} + \mu_{i^*}$ units are sold, by the decreasing marginals property, the social welfare is at least $(n - n_{i^*} + \mu_{i^*}) \cdot m_{n - n_{i^*} + \mu_{i^*}}$ where m_k denotes the marginal of the k-th unit for the buyer. If more items are sold, the welfare doesn't drop, and therefore we have $\underline{W} \geq (n - n_{i^*} + \mu_{i^*}) \cdot m_{n - n_{i^*} + \mu_{i^*}}$.

By the definition of a monopolist who sells to a buyer with valuation $v(\cdot|n - n_i)$, we have that $\mu_i \cdot m_{n - n_i + \mu_i} \geq k \cdot m_{n - n_i + k}$ for all $k \in \{1, 2, \ldots, n_i\}$. Therefore, by the decreasing marginals property, we have that for all k such that $\mu_i \leq k \leq n_i$:

$$(n - n_i + \mu_i) m_{n - n_i + \mu_i} \geq (n - n_i + k) m_{n - n_i + k}$$

Since at least $n - n_{i^*} + \mu_{i^*}$ units are sold, $\underline{W}$ will be short of $v(n)$ by at most the welfare of the remaining marginals. I.e., we have $v(n) - \underline{W} \leq \sum_{k=\mu_{i^*}+1}^{n_{i^*}} m_{n-n_{i^*}+k}$. Thus,

$$\begin{aligned} v(n) - \underline{W} &\leq \sum_{k=\mu_{i^*}+1}^{n_{i^*}} m_{n-n_{i^*}+k} \\ &\leq \sum_{k=\mu_{i^*}+1}^{n_{i^*}} \frac{(n - n_{i^*} + \mu_{i^*}) m_{n-n_{i^*}+\mu_{i^*}}}{n - n_{i^*} + k} \\ &\leq \log\left(\frac{n}{n - n_{i^*} + \mu_{i^*}}\right)(n - n_{i^*} + \mu_{i^*}) m_{n-n_{i^*}+\mu_{i^*}} \\ &\leq \log\left(\frac{n}{n - n_{i^*} + \mu_{i^*}}\right)\underline{W} \end{aligned}$$

where the second inequality follows from the previous expression and the third from the inequality $\sum_{j=r+1}^{t} 1/j \leq \log(t/r)$ for any integers t and n. Rearranging the terms, we obtain the desired result:

$$\underline{W} \geq \frac{v(n)}{1 + \log\left(\frac{n}{n-n_{i^*}+\mu_{i^*}}\right)} = \frac{v(n)}{1 + \log\left(\frac{n}{\min_i(n-n_i+\mu_i)}\right)}$$

□

Lemmas 3.8 and 3.9 together prove Proposition 3.6 and Theorem 3.3. The following corollary is an immediate consequence of Theorem 3.6 for the case where all sellers have the same quantity to sell. The corollary illustrates that if each seller holds the same number of items, then the Eventual Welfare Guarantee depends only on the number of sellers. Moreover, the equilibrium is efficient in the limit when the number of sellers grows to infinity.

COROLLARY 3.10. *If each seller holds the same number of items, then the Eventual Welfare Guarantee is bounded by* $1 + \log\left(1 + \frac{1}{s-1}\right)$. *In particular, it tends to* 1 *as* $s \to \infty$.

COROLLARY 3.11. *If there is a constant* $c \in (0,1)$*, such that no seller controls more than a* c *fraction of items in the market, i.e.,* $n_i \leq c \cdot n$*, then the Eventual Welfare Guarantee is bounded by a constant:* $1 - \log(1-c)$. *In particular, it tends to* 1 *as* $c \to 0$.

3.3. Lower bound on the Eventual Welfare Guarantee

We next present a lower bound showing that the efficiency bound of Theorem 3.3 is asymptotically tight when $\frac{n}{n-n_{\max}}$ grows large, i.e., the nearly-monopolistic setting.

THEOREM 3.12. *Fix any number of sellers* s *and any positive supplies* $\{n_i\}_{i\in[s]}$ *such that* $\frac{n}{n-n_{\max}} > 3$. *There exists a decreasing marginal valuation for the buyer such that for any sequence of best-responses from sellers,*[6] *the social welfare after any number of rounds is at most*
$\left[\frac{1}{5}\log\left(\frac{n}{n-n_{\max}}\right) - 1\right]^{-1} \cdot v(n)$.

PROOF. Recall that $n_{\max} = \max_i \; n_i$ and $n = \sum_i n_i$. To prove the claim we need to show that the ratio of $v(n)$ to the welfare in every round is at least $\frac{1}{5}\left(\log\left(\frac{n}{n-n_{\max}}\right) - 1\right)$.

[6] and any lexicographic order that determines the tie breaking rule

Recall that we have assumed that $\frac{n}{n-n_{\max}} > 3$ and thus $n_{\max} > 2n/3$. Let $r = n - n_{\max} < n/3$ be the total supply of all sellers but the one with $n_{\max}$ units. Assume that $\epsilon < 1$ is small enough. Consider the following valuation of the buyer. He values each of the first $2r$ units for $1+\epsilon$ each, he values the next r items for $1/3$ each, the next r for $1/4$ each etc., till we reach n units (note that the last bunch might have less than r units). We claim that at most $2r$ items will ever be sold. Indeed, if $\ell > 2r$ items are sold, then the seller with $n_{\max}$ units must be selling at least $\ell - r > r$ units (as all other sellers combined supply only r units). To sell such a quantity he must be pricing his items at price at most $\lceil(\ell-r)/r\rceil^{-1}$, since this is the $(\ell-r)$-th marginal of the buyer, therefore, getting revenue at most $(\ell-r)/\lceil(\ell-r)/r\rceil \le r$. However, by posting price $1+\epsilon$, this seller is guaranteed revenue at least $(1+\epsilon)r$.

As at most $2r$ items are ever sold, the welfare at every round is at most $2r(1+\epsilon)$, while the welfare $v(n)$ is at least $r(2+\sum_{j=3}^{\lfloor n/r\rfloor}\frac{1}{j}) \ge r(\frac{1}{2}+\sum_{j=1}^{\lfloor n/r\rfloor}\frac{1}{j}) \ge r(\frac{1}{2}+\log(\lfloor n/r\rfloor)) \ge r\left(\log(n/r)-\frac{1}{2}\right)$ since $\log(\lfloor n/r\rfloor) \ge \log(n/(2r)) = \log(n/r) - 1$ because $n > 2r$. We conclude that the ratio of $v(n)$ to the welfare in every round is at least
$\frac{\log(n/r)-\frac{1}{2}}{2(1+\epsilon)} \ge \frac{1}{4(1+\epsilon)}\left(\log\left(\frac{n}{n-n_{max}}\right)-1\right)$
$\ge \frac{1}{5}\left(\log\left(\frac{n}{n-n_{max}}\right)-1\right)$. □

4. PRICING GAME WITH UNCERTAIN DEMAND FOR HOMOGENEOUS GOODS

In this section we drop the assumption that the valuation of the buyer is common knowledge among the sellers. Instead, while we still assume that the buyer has an intrinsic valuation $v^* \in \mathbf{V}$, we no longer assume that it is known to the sellers. Rather, we assume that each seller has a prior belief $\mathbf{B}_i^0$, expressed as a probability distribution over $\mathbf{V}$. Note that the initial beliefs of different sellers might be different. At each step, sellers observe the quantity sold and update their beliefs in a Bayesian way.

We remind the reader that $\mathbf{V}$ is a set of functions mapping quantities to values in $\epsilon\mathbb{Z}_+$, and thus $\mathbf{V}$ is a countable infinite set. In our main theorem for the pricing game with uncertain demand present below, we assume that the prior belief of each seller has finite support, i.e., it places positive probability on finitely many valuation functions. A natural example of a finite set of beliefs is the one where all sellers have an upper bound M on the total value of the buyer $v(n)$, that is, $\mathbf{B}_i^0(v) = 0$ for all v such that $v(n) > M$. In this case the beliefs form a finite set.

THEOREM 4.1. *If the prior belief of each seller has finite support and all priors are consistent*[7]*, then the Eventual Welfare Guarantee of the uncertain demand game is at most* $\alpha = \frac{n}{n-n_{\max}+1}$. *Moreover, in any best response sequence, the welfare is lower than an* $1/\alpha$ *fraction of the optimal welfare in at most* $\frac{s\cdot K}{\epsilon}\cdot\sum_i|\mathit{supp}(\mathbf{B}_i^0)|$ *rounds, where* $\mathit{supp}(\mathbf{B}_i^0) = \{v \in \mathbf{V}; \mathbf{B}_i^0(v) > 0\}$ *and* $K = \max_{v\in\cup_i \mathit{supp}(\mathbf{B}_i^0)} v(1)$. *If all sellers start with the same initial belief* B^0, *the number of rounds will decrease to* $\frac{s\cdot K}{\epsilon}\cdot|\mathit{supp}(\mathbf{B}^0)|$.

We prove Theorem 4.1 in Appendix C.

We note two main differences between Theorem 4.1 and its counterpart in the full information game (Theorem 3.3). The first is the welfare ratio: in the nearly-monopolsitic setting in which one seller controls a large fraction of the market, i.e. $\frac{n}{n-n_{\max}+1} \to \infty$, the bound is exponentially worse than the full information bound. We will show in Theorem 4.4 that welfare can indeed be very low if beliefs do not converge to the true valuation. This difference highlights that the efficiency of the market can exponentially

[7]i.e., they place positive probability on the true valuation of the buyer

improve if sellers can accurately learn the valuation of the buyer. For the other extreme case where no seller controls a significant fraction of the market, i.e. $n_{\max} = \epsilon n$, the two bounds become approximately $1 + \epsilon$ as $\epsilon \to 0$.

The second difference is that although both theorems guarantee that best response dynamics will lead to good welfare states in finite time, the first bound guarantees that except for possibly the first $t' = \frac{s \cdot v(1)}{\epsilon}$ rounds, the welfare will be high. Theorem 4.1 gives a much weaker guarantee: it shows that welfare will be good for all but at most $t'' = \frac{s \cdot K}{\epsilon} \cdot \sum_i |\mathrm{supp}(\mathbf{B}_i^0)|$ rounds. Yet it doesn't guarantee those will be the first rounds. We leave the question of getting a stronger convergence guarantee for this setting as an open problem.

We next present corollaries for the uncertain demand setting that are parallel to Corollaries 3.10 and 3.11.

COROLLARY 4.2. *If every seller controls the same number of items, then the Eventual Welfare Guarantee is at most $1 + \frac{1}{s-1}$. In particular, it tends to 1 as $s \to \infty$.*

COROLLARY 4.3. *If every seller controls at most a c fraction of the market, then the Eventual Welfare Guarantee is at most $\frac{1}{1-c}$. In particular, it tends to 1 as $c \to 0$.*

And finally, we show that the bound we have presented for The Eventual Welfare Guarantee of uncertain demand games is essentially tight. We show that it can be as large as $\Theta(n)$ even when the beliefs of all sellers are identical. The proof of this Theorem (Theorem 4.4) follows directly from the example below.

THEOREM 4.4. *The Eventual Welfare Guarantee of an uncertain demand game can be as large as $\Theta(n)$.*

Example 4.5. Consider a buyer with a valuation v^* over n items with marginals $1+\epsilon, 1+\epsilon, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots, \frac{1}{4}$. Now, let v be the valuation function with marginals $1+\epsilon, 1+\epsilon, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}$. Consider also two sellers holding $n_1 = 1$ and $n_2 = n - 1$ items and with beliefs $\mathbf{B}_1^0 = \mathbf{B}_2^0$ assigning probability δ to v^* and $1 - \delta$ to v. If δ is small enough, both sellers will best respond assuming that the buyer chooses how many items to buy according to v. Since for this best-response sequence more than 3 items will never be sold (since the situation here is the same as in the example used in the proof of Theorem 3.12), the sellers will never be able to refine their beliefs. This leads to an best-response dynamics in which the welfare is less than 3 for each time-step, while the optimal welfare is $\frac{n}{4}$.

5. HETEROGENEOUS GOODS

As explained in the introduction, the pricing game with heterogeneous goods has very different properties from that for homogeneous goods. A good place to start studying heterogeneous goods is the case of two extremes: a buyer with additive valuations, and a buyer with unit-demand valuations. In both these cases, it turns out that an efficient pure Nash equilibrium is guaranteed to exist.

FACT 5.1. *The full information pricing game with sellers holding heterogeneous goods, and the buyer having additive valuation, is guaranteed to have an efficient pure Nash equilibrium.*

FACT 5.2. *The full information pricing game with sellers holding heterogeneous goods, and the buyer having unit-demand valuation, is guaranteed to have an efficient pure Nash equilibrium.*

We prove these facts in Appendix D. Given that the two extremes are well behaved, we move to explore k-additive valuations, the simplest generalization of the two extremes. We already see that there need not exist efficient equilibria, even when an equilibrium exists.

FACT 5.3. *The full information pricing game with just two sellers, and a* 3*-additive buyer need not have an efficient pure Nash equilibrium. There is such a setting in which equilibrium exists but is not efficient.*[8]

PROOF. This fact is again easy to see. Consider two sellers with seller 1 holding 1 item of value ϵ to the buyer and seller 2 holding 3 items of value 2ϵ each to the buyer. The buyer breaks ties towards seller 1. If seller 1 prices his item at ϵ, then for seller 2, the best response is to price all the items at 2ϵ. This pricing pair is an equilibrium, and it is easy to verify that in any equilibrium the two sellers price the sold items the same and have the same utility as in this equilibrium. This results in a buyer welfare of 5ϵ, which is strictly smaller than the optimal welfare of 6ϵ. □

An interesting fact about the example in the proof of Fact 5.3 is that if instead of 2 sellers, all the 4 items were held by a single monopolist seller, the social welfare would have been at its optimal level (6ϵ), strictly larger than with two competing sellers. While one might intuitively think that splitting the items between two sellers could only increase welfare as it increases competition, this example shows that it could also *decrease welfare in equilibrium* (note that equilibrium welfare is well defined when all the sellers' and buyer's utilities are same across all equilibria, like in the example used in the proof of Fact 5.3).

Remark 5.4. When the items held by a monopolist are split between two sellers, equilibrium welfare could decrease.

The proof of Fact 5.3 presents an example with equilibrium welfare loss of ϵ. Can the loss be much larger? In Theorem 5.6 we show that with a k-additive buyer, whenever there exists an equilibrium, the loss in welfare cannot get larger than $2k\epsilon$. Thus, the loss in welfare vanishes with ϵ, and the welfare *in equilibrium* is almost optimal. Yet, unlike in the unit-demand and additive cases, for k-additive valuations equilibrium might fail to exist and thus the welfare guarantee is much weaker as it relies on assuming that an equilibrium is reached, but for some valuations that might not be the case. What happens when equilibrium is not reached? Is welfare still guaranteed to be high? We next show that in some settings, welfare can be very low.

5.1. Out of equilibrium cycling

We show that even for a k-additive buyer, when equilibria don't exist, welfare can be very poor in a strong sense. First, the Eventual Welfare Guarantee can be arbitrarily large. Further, only in $\frac{1}{s}$ fraction of the time does the welfare reach even a non-zero fraction of the optimal welfare (and thus, the difference between the welfare most of the time, and the optimal welfare, is very large).

THEOREM 5.5. *The full information pricing game with heterogeneous goods, and a* k*-additive buyer does not always have a pure Nash equilibrium. When a pure Nash equilibrium does not exist, the Eventual Welfare Guarantee is* ∞*. Further, there exist best response sequences where the welfare obtained approximates the optimal welfare within a non-zero factor only for* $O(\frac{1}{s})$ *fraction of the time.*

PROOF. Consider s sellers with seller 1 holding one item item of a large value V, and sellers 2 to $s-1$ holding one item each of a small value, say 10. Seller s holds

[8]The equilibrium is unique up to price of the unsold item, which can be increased with no change to the utilities of both sellers.

$s = \frac{1}{\epsilon}$ items of value 10 each. The buyer is s-additive, and prefers sellers in the order $1, \ldots, s$. Sellers best respond in the order $s, \ldots, 1$. Let the price sequence begin from all prices being ∞. First, seller s sets a price of 10 for each of his items and gets the maximal possible revenue of $10s$ for his items. Then seller $s-1$ updates his price from ∞ to 10, following by seller $s-2$ and so on till seller 2 updates his price to 10. Then, seller 1 will update his price from ∞ to V. In the second best response cycle, seller s slashes his price down to $10 - \epsilon$ each, and so will sellers $s-1$ to 2 in sequence, and then seller 1 will slash his price down to $V - \epsilon$. This price competition will continue until seller s slashes his prices down to 10ϵ each, and so do sellers $s-1$ to 2, and seller 1 sets his price at $V - 10 + 10\epsilon$. At this point, seller s will no more drop his price, but raise the prices of all his items to 10 each, upon which sellers $s-1$ to 2 will update their price to 10, and seller 1 will update his price to V, bringing the sequence back to where it started. Thus in this infinite looping, only at the stage immediately after seller 1 responded does the item of value V ever get bought. At other times, only items of value 10 get bought leading to social welfare of $10s$. Since V can be made arbitrarily big compared to s, a non-zero approximation to welfare is attained only when seller 1 responds, which happens $O(\frac{1}{s})$ fraction of the time.

To see that no pure equilibrium exists in this setting, let p be the price of the most expensive item sold by seller s in some equilibrium. Note that $p > 0$ because seller s is guaranteed to get a revenue of at least 10 by just selling one item priced at 10. Given this, sellers $2, \ldots, s-1$ best respond by setting prices of p each, and 1 sets a price of $V - 10 + p$. This would mean that in any equilibrium the buyer buys exactly one item from seller s, and thus s has to price it at $p = 10$, and sellers $2, \ldots, s-1$ best respond with a price of 10, and 1 sets a price of V. But given this, seller s is not best responding with a price of 10: he can sell all his items at a price of $10 - \epsilon$ each. $\square$

5.2. In equilibrium bounds for a k-additive buyer

We next show that with a k-additive buyer, whenever there exists an equilibrium, the loss in welfare cannot get larger than $2k\epsilon$. This is almost tight: we show an example where there can be an additive loss of $k\epsilon$. As a corollary, we get that whenever all items are of value at least $r\epsilon$ with $r > 2$, the Price of Anarchy in a k-additive setting is at most $\frac{r}{r-2} = 1 + \frac{2}{r-2}$, which is bounded from above by 3 and for any fixed valuation decreases monotonically to 1 as ϵ goes to 0 (which implies that r grows to infinity).

THEOREM 5.6. *Every pure Nash equilibrium in a full information pricing game with heterogeneous goods and a k-additive buyer obtains a welfare of at least optimal welfare less $2k\epsilon$.*

We prove the theorem in Appendix D. The lost welfare of $2k\epsilon$ is tight up to constant factors. Here is an example where losing welfare of magnitude $k\epsilon$ is inevitable.

Example 5.7. Consider two sellers. The first holds k items of value 0 each, and the second holds k items of value ϵ each. Suppose the first seller is more preferred. Then, the first seller selling all his items at 0, and the second seller posting a price of ϵ for all his items is an equilibrium with a welfare of 0, causing an additive loss in welfare of $k\epsilon$.

The example shows that the Price of Anarchy can be unbounded. Yet, this happens when the value of items is very low (only ϵ). We next show that this cannot happen when the value is large relative to ϵ, in all such settings the PoA is small.

COROLLARY 5.8. *For $r > 2$, if all items are of value at least $r\epsilon$, the pure Nash price of anarchy of the full information pricing game with heterogeneous goods and a k-additive buyer is at most $\frac{r}{r-2}$. As r increases, the PoA monotonically decreases, approaching* 1.

6. CONCLUSION AND OPEN PROBLEMS

In this paper we consider a model of price competition that exhibits price fluctuations and we provide welfare guarantees, despite the lack of convergence to equilibrium. We did so in a very simple model of seller behavior: myopic best response. It would be interesting to try to do the same for more sophisticated models of behavior, such as Sequential Equilibrium [Kreps and Wilson 1982] or Markov Perfect Equilibrium [Maskin and Tirole 2001]. In such models a seller would reason not only about his utility in the next round but also about the influence of his action on the long-run utility – measured, say, by time discounted payoffs. For the uncertain demand setting, sellers would also face the trade-off between learning more about the buyer valuation and maximizing gains given their current information.

Our results also highlight the value of accurately knowing the valuation function of the buyer. In the setting with uncertainty, the Eventual Welfare Guarantee can be exponentially worse than in the full information setting. Are there natural learning strategies that sellers could employ that could overcome this gap and at the same time preserve some sort of equilibrium behavior ?

Another direction to extend in the setting of heterogeneous items is to understand what buyer valuations yield good Eventual Welfare Guarantee. An important ingredient in our analysis is that we have a precise description of how a buyer reacts to price updates. For heterogeneous items, understanding the effect of price updates is considerably more challenging.

One other feature absent from our model is the intrinsic costs of sellers. For example, in the retail market for gasoline the cost is typically the wholesale price. In the case of publishers and impressions, the cost of publishers to serve ads corresponds to the dis-utility of the website-users in seeing the ads. It would be interesting to see how non-zero costs would alter the model.

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A. RANDOM TIE BREAKING

We next show that with random tie breaking, as long as ϵ is small enough, no best response will result in a tie. One can use this to derive similar bounds as we do in the paper.

A random tie breaking is such that items of the same price are considered in a uniformly random order by the greedy algorithm.

PROPOSITION A.1. *Consider the full information game. Assume that the buyer has a decreasing marginal valuation with $m_n = v(n) - v(n-1) > 0$ and there are at least 2 sellers. In the full information game with the random tie breaking rule and small enough ϵ, any best response of a seller will never result with an item of this seller that is priced at exactly the same price as an item offered by another seller, unless this item is sold with probability 0 or 1.*

PROOF. Consider such a best response in which the item is priced at p and sold with probability $z > 0$, $z < 1$. We argue that this seller can strictly increase his utility by decreasing the price of that item by ϵ, as long as ϵ is small enough.

Observe that before the change in price, any item with price smaller than p is sold, and any item with price larger than p is not sold. This is still true after the price decrease, and in particular, the item for which the price has decreased is sold with probability 1. Assume that there is a total of $k > 1$ items priced at p before the price decrease, out of which $m < k$ are from the considered seller. Assume that $r < k$ items are sold at price p before the price decrease. The probability of each of the items priced at p to be sold before the price change is $z = r/k$, and after the price change the probability each of the items priced at p to be sold is $(r-1)/(k-1)$, and now the seller is pricing only $m-1$ items at this price.

The increase in utility for this seller is:

$$(p-\epsilon) + p\left(\frac{(m-1)(r-1)}{k-1} - \frac{mr}{k}\right) = p\left(\frac{(k-m)(k-r)}{k(k-1)}\right) - \epsilon$$

which is positive for small enough ϵ as long as p is no too small, but $p \geq m_n$ since otherwise $z = 1$. □

B. NONEXISTENCE OF PURE NASH EQUILIBRIA IN FULL INFORMATION GAME FOR HOMOGENEOUS GOODS

We now show that for the example 3.1 presented in Section 3 there does not exist a pure Nash equilibrium.

EXAMPLE 3.1. *Consider a setting with two sellers, each holding 2 units of the same good. The buyer has diminishing marginal valuations, of $5, 5, 3, 1$, i.e., $v(1) = 5$, $v(2) = 10$, $v(3) = 13$, $v(4) = 14$. This setting does not have any pure Nash equilibrium.*

Clearly, the number of units sold at any pure NE cannot be 1 because every seller can always make a non-zero revenue. It cannot be two either. There are two ways in which two goods can be sold. Both goods sold from the same seller, which is not possible at any equilibrium. The second way is to have just one good sold from each seller. This can happen only if the unsold item from each seller is priced above 3. Either seller can price the unsold item at 3 (and the sold item also at 3) to get a revenue of 6 which is larger than 5 which is the maximum they would have gotten previously. Next, the number sold at pure NE cannot be three too. Without loss of generality, we assume that this was reached with seller 1 selling two items and seller 2 selling one item. Let p_1 and p_2 denote the prices used by seller 1. Given this, seller 2 can obtain a revenue of at least $p_1+p_2-2\epsilon$ by pricing at $p_1-\epsilon$ and $p_2-\epsilon$. If $p_1+p_2 > 3+2\epsilon$, then seller 2 now gets a revenue strictly larger than 3 which is the maximum he could have gotten by selling one item when three items were totally sold. On the other hand, if $p_1+p_2 \leq 3+2\epsilon$, seller 1 could have priced his items at 3 and 1 and gotten a total revenue of $4 > 3+2\epsilon = p_1+p_2$. Finally, the number of goods sold at any pure NE cannot be 4 because that would imply that all the four units were priced at 1, and one of the sellers can instead price both his items at 3, sell one item and get a larger revenue of 3 instead of 2.

C. PRICING GAME WITH UNCERTAIN DEMAND AND HOMOGENEOUS GOODS

Uniform pricing. In the uncertain demand setting it is no longer true that a seller always has a best response that sets uniform price for all his units. Consider for example a monopolist seller whose belief about the valuation of the buyer is as follows: he believes that with probability $.99$ the seller has marginals $1, 1, 0$ and with probability $.01$ he has marginals $10, 10, 10$. Then, his unique best response is to set prices $1, 1, 10$ for his items. However, we can argue that if a seller has no uniform price best-response, after posting these prices he will necessarily learn something about the buyer's valuation.

At any given time-step t, given the set of beliefs $\mathbf{B}_1^{t-1}, \ldots, \mathbf{B}_s^{t-1}$ formed by the sellers in the previous time-step, we say that a price vector $\mathbf{p}$ is *informative* if for at least one buyer, there are $v, v' \in \text{supp}(\mathbf{B}_i^{t-1})$ such that $|X^v(\mathbf{p})| \neq |X^{v'}(\mathbf{p})|$. Otherwise we say that the price vector is *uninformative*. We say that seller i has an *uninformative best response* to $\mathbf{p}_{-i}$ if i has a best response $\mathbf{p}_i$ such that the price vector $(\mathbf{p}_i, \mathbf{p}_{-i})$ is uninformative.

We observe now that if $\mathbf{p}^1, \mathbf{p}^2, \ldots$ is a best-response sequence, there can be at most $\sum_{i=1}^s |\text{supp}(\mathbf{B}_i^0) - 1|$ informative prices, since if $\mathbf{p}^t$ is informative, then $\sum_{i=1}^s |\text{supp}(\mathbf{B}_i^t)| < \sum_{i=1}^s |\text{supp}(\mathbf{B}_i^{t-1})|$ and once $|\text{supp}(\mathbf{B}_i^t)| = 1$ for all i, then all prices are uninformative. We assume that the seller favors uniform price uninformative best responses over non-uniform price uninformative ones. In other words, when an uninformative uniform best response is available, the seller will never play a non-uniform best response unless it is informative. The following lemma shows that it is indeed the case that when a seller has an uninformative best response, he has an uniform one as well.

LEMMA C.1. *If a seller has an uninformative best-response, then he also has a uniform price best-response. Therefore, if sellers break ties in favor of uniform prices, there will be at most $\sum_{i=1}^s |supp(\mathbf{B}_i^t)| - s$ time-steps in which it does not hold that every seller will post uniform prices across his items.*

PROOF. If a seller has an uninformative best-response $\mathbf{p}_i$, then for every $v \in \text{supp}(\mathbf{B}_i^{t-1})$, $X^v(\mathbf{p}_i, \mathbf{p}_{-i}^{t-1})$ is the same. In particular, the most expensive item bought has the same price for every v. So we can use the exact same argument as in Lemma 3.5 to guarantee that there is an uniform best-response. $\square$

LEMMA C.2. *Consider a (possibly infinite) interval in a best-response sequence where the price posted in each time-step is uninformative. Then in every time-step in* *the interval that is after the first $\frac{K \cdot s}{\epsilon}$ rounds, it holds that at least $n - n_{\max} + 1$ items will be sold.*

PROOF. We note that Lemma 3.4 is a statement about the buyer behavior, which is the same in this setting – so the lemma continues to hold. Lemma C.1 guarantees that the conclusion of Lemma 3.5 holds in the uncertain demand setting if the prices being posted are uninformative. Finally, the exact same argument used in Lemma 3.8 can be replayed to show that in less than $\frac{s \cdot K}{\epsilon}$ rounds, there will be a time-step in which all sellers except one will sell their entire supply, making the total amount sold at least $n - n_i + \tilde{\mu}_i$ and from that point on this amount will be sold for all subsequent time-steps. Where $\tilde{\mu}_i \geq 1$ is the amount that a monopolist seller would sell at the optimal price when facing a buyer with valuation $v(\cdot|n - n_i)$ drawn from the distribution representing the limit belief of seller i. Note that since all prices from the interval are uninformative, the belief of the seller is constant inside that interval. $\square$

We observe that the limit belief of each seller might not be accurate, in which case $\tilde{\mu}_i$ might be far from the value of μ_i taken with respect to the true valuation v^* (as we demonstrate in Example 4.5). The fact that the last considered seller doesn't best respond properly is the main reason why the bound is linear rather than logarithmic.

PROOF OF THEOREM 4.1. The previous lemma guarantees that in all but $\frac{s \cdot K}{\epsilon} \cdot \sum_i |\text{supp}(\mathbf{B}_i^0)|$ rounds, $n - n_{\max} + 1$ items are sold. Since v has the decreasing marginals property, then for every time-step t that is not one of these rounds $W^t \geq v(n - n_{\max} + 1) \geq \frac{n - n_{\max} + 1}{n} \cdot v(n)$. This completes the proof of the theorem.

Theorem 4.1 is stated for the case that sellers are completely myopic and do not post non-optimal prices only for the sake of learning the valuation of the buyer (exploring). We remark that the same proof (and thus the same result) also holds if sellers *do* try to learn the valuation of the buyer by sometimes posting informative prices that are not best responses for them. This is true as long as it holds that in every time-step in which a seller is not exploring and in which prices are uninformative, the seller posts a uniform-price best response (which we know that is available to him).

D. HETEROGENEOUS GOODS

FACT 5.1. *The full information pricing game with sellers holding heterogeneous goods, and the buyer having additive valuation, is guaranteed to have an efficient pure Nash equilibrium.*

FACT 5.2. *The full information pricing game with sellers holding heterogeneous goods, and the buyer having unit-demand valuation, is guaranteed to have an efficient pure Nash equilibrium.*

PROOF OF FACTS 5.1 AND 5.2. These facts are easy to see. In the additive valuation case, it is an equilibrium for each seller to price every good he owns exactly at the buyer's value for that good. An additive buyer breaking ties towards buying items of zero marginal utility will buy the entire set of items, getting the optimal social welfare. In the unit-demand case, if there is only one type of good, clearly all sellers pricing it at 0 is an efficient pure Nash equilibrium. Suppose there were two or more types of good, let v_{max} the buyer's value for his most preferred item type and let i^* denote the most preferred seller (lexicographically least) holding such a good. Let v_2 be the most valuable good owned by some seller other than i^*. Then i^* pricing all his items at $v_{max} - v_2$, and the other sellers pricing all their items at 0 is an efficient equilibrium.

THEOREM 5.6. *Every pure Nash equilibrium in a full information pricing game with heterogeneous goods and a k-additive buyer obtains a welfare of at least optimal welfare less $2k\epsilon$.*

PROOF. Let W^* be a welfare maximizing set of items defined by mimicking the buyer's greedy algorithm and his tie breaking procedure. Start with an empty set, and recursively pick the item of maximum value till the set is of size k or all items get exhausted. When multiple items are of same value, pick the item from the most preferred seller, and when multiple items from same seller are of same value, break ties arbitrarily. This arbitrary tie-breaking rule makes W^* non-unique. While this non-uniqueness is straight-forward to handle[9], for clarity of proof, we fix some arbitrary tie breaking rule and assume that W^* is uniquely defined.

Let I_i be the set of items owned by seller i, and let $I = \cup_i I_i$. Fix an arbitrary pure strategy equilibrium and let S_i be the set of items sold by seller i in this equilibrium. Let $S = \cup_i S_i$. Note that $|S| = \min(k, |I|)$, i.e., when k or more items are available, no equilibrium will involve the buyer buying less than k items. Let $\underline{v}$ be the value of the least valued item that gets sold.

We prove this theorem by proving several claims in sequence. All of them will refer to the order in which the buyer approached sellers according to his greedy procedure on utilities. Thus it is useful to keep in mind the global order of utilities offered by items in equilibrium (where the ordering is defined with ties broken as defined before). Let $1, 2, \ldots, s$ denote the lexicographical ordering of sellers. Let ℓ be the seller who was approached last by the greedy procedure, and, sold at least one item. Note that ℓ is not necessarily equal to s.

CLAIM 1. *In any pure equilibrium, all items sold by a seller offer equal utility for the buyer.*

PROOF. Suppose not: consider an arbitrary seller whose sold items offered different utilities for the buyer. The seller could consider reducing the utilities of all items to match that of the sold item offering the lowest utility. The fact that the buyer, among items of same utility, picks from the lexicographically most favorable seller, and the fact that the lowest utility item was previously sold, together imply that the seller continues to sell the same set of items as before, and gets a higher revenue. This contradicts that the seller is best-responding. □

COROLLARY D.1. *In any pure equilibrium, all items sold by a seller are sold in one contiguous stretch.*

The corollary implies that each seller is approached at most once by the algorithm, so we can talk about the order that sellers are approached (also note that every seller that is approached, sells at least one item).

CLAIM 2. *In any pure equilibrium, all sellers $i \neq \ell$ sell all items from $W^* \cap I_i$ that have values at least $\underline{v} + 2\epsilon$.*

PROOF. Suppose there is a seller i with unsold items of value at least $\underline{v} + 2\epsilon$. In this case, seller i could consider dropping the price of such an unsold item in steps of ϵ. This process will increase the utility of that item, until it matches that of a sold item offering least utility. Due to the "2ϵ" assumption in the lemma and that the item under consideration is from W^*, it follows that this utility matching will be achieved at a price of 2ϵ or larger. At this point, the item is either already sold, or by dropping the price an ϵ more, it definitely gets sold, improving the revenue of seller i. This proves the claim. □

[9]For instance, note that all W^*'s that satisfy this definition, not only get the same welfare, but also contain the same number of items from every seller, and contain the same value profile of items from each seller.

There is a close connection between the lexicographical preference ordering of sellers, and the order in which the sellers are approached by the buyer. We explore this in the next claim.

CLAIM 3. *For any two sellers i, j that are approached by the buyer, if $i < j < \ell$, or if $\ell < i < j$, then i is approached before j. Additionally, every item sold by i offers the same utility as every item sold by j.*

PROOF. Consider sellers i, j that were approached by the buyer, and say $i < j < \ell$. Suppose j was approached before i. This is possible only if j offered a strictly larger utility than i. In such a case, j could reduce its utility to that offered by i. This makes j get more revenue. This follows by noting that j still manages to sell all the items he previously sold because the seller ℓ who offers an equal or lesser utility (because ℓ is approached later) is less preferred than j yet is still selling at least one item. Similar argument holds for $i, j > \ell$ too except that to argue that j manages to sell all items he previously sold, we must note that i must have offered strictly larger utility to have been considered before ℓ. Thus even if j matches i's utility, he still manages to be considered before ℓ because of higher utility offered to the buyer.

From the argument above, it follows that utility offered by i's sold items is at least that offered by j's sold items. It is easy to see that seller i's sold items will not offer a strictly higher utility than that of j because even if i matches j's utility, i's items will be considered before j's. □

COROLLARY D.2. *All the sold items from sellers in the set $\{1, \ldots, \ell - 1\}$ offer the same utility, and the sold items from sellers in $\{\ell + 1, \ldots, s\}$ offer the same utility.*

CLAIM 4. *Let u be the utility offered by seller ℓ. Then every sold item offers utility u or $u + \epsilon$.*

PROOF. Suppose no seller was approached before ℓ, then the claim is trivial. Let ℓ^- denote the largest numbered seller in $\{1, \ldots, \ell - 1\}$ who sells at least one item, and let ℓ^+ be the largest numbered seller in $\{\ell + 1, \ldots, s\}$ who sells at least one item. By Claim 3, the seller who was approached just before ℓ will be either ℓ^+ or ℓ^-. Suppose it was ℓ^+, then seller ℓ^+ should offer a strictly larger utility than u, but need not offer more than $u + \epsilon$. Thus, by Corollary D.2, all sold items from sellers $\ell + 1, \ldots, s$ offer utility $u + \epsilon$. Given this, seller ℓ^- need not offer a utility higher than $u + \epsilon$, and he will not offer a smaller utility either, for otherwise he would have been considered after ℓ^+. This completes the first case. Consider the other case where the seller approached just before ℓ was ℓ^-. Then ℓ^- will offer a utility of exactly u, and by Corollary D.2, so will all the sold items from sellers $\{1, 2, \ldots, \ell - 1\}$. The seller ℓ^+ should offer utility $u + \epsilon$ to be considered before ℓ^-, and by Corollary D.2, so will all the sold items from sellers $\ell + 1, \ldots, s$. □

Now consider seller ℓ. He currently sells the most valuable set of $|S_\ell|$ items I_ℓ (if not he can raise his revenue by switching to sell more valuable items at an increased price while giving the buyer the same utility). Suppose all the items from $W^* \cap I_\ell$ are sold, then the Theorem stands proved by Claim 2. If ℓ had unsold items in $W^* \cap I_\ell$ of value at most $\underline{v} + \epsilon$, each such item causes at most an additive ϵ welfare loss, which our Theorem accommodates. Thus, assume that all unsold items in $W^* \cap I_\ell$ are of value at least $\underline{v} + 2\epsilon$. Without loss of generality, ℓ can adjust the prices of all unsold items in $W^* \cap I_\ell$ so that they all offer the same utility u offered by the sold items in S_ℓ. This is because ℓ can drop the prices of unsold items in $W^* \cap I_\ell$, and along the lines of the argument in Claim 2, at a price no smaller than 2ϵ the utility of unsold items will match the utility u offered by items in S_ℓ. Now, by claim 4, all sold items offer utility of u or $u + \epsilon$. Seller ℓ has the option of dropping the prices of all items in $W^* \cap I_\ell$ by 2ϵ, to make their utility $u + 2\epsilon$, there by selling all items in $W^* \cap I_\ell$. Seller ℓ chose not to do this implying that ℓ does not see any increase in revenue from doing this change. This means:

$$\sum_{j \in W^* \cap (I_\ell \setminus S_\ell)} (v_j - u) - 2\epsilon |W^* \cap I_\ell| \leq 0 \tag{1}$$

where v_j is the buyer's value for item j.

Recall that $\underline{v}$ denote the value of the least valued item that gets sold in ALG. Since u is the smallest utility that is offered in ALG, it follows that $u \leq \underline{v}$. This, combined with (1) gives

$$\sum_{j \in W^* \cap (I_\ell \setminus S_\ell)} v_j \leq \sum_{j \in W^* \cap (I_\ell \setminus S_\ell)} \underline{v} + 2\epsilon |W^* \cap I_\ell| \tag{2}$$

From Claim 2, it follows that for all sellers other than ℓ, we have

$$\sum_{i \neq \ell} \sum_{j \in W^* \cap (I_i \setminus S_i)} v_j \leq \sum_{i \neq \ell} \sum_{j \in W^* \cap (I_i \setminus S_i)} (\underline{v} + \epsilon) \tag{3}$$

The optimal welfare can be written as

$$\begin{aligned} OPT &= \sum_i \sum_{j \in W^*} v_j \\ &= \sum_i \left(\sum_{j \in W^* \cap S_i} v_j \right) + \sum_i \left(\sum_{j \in W^* \cap (I_i \setminus S_i)} v_j \right) \\ &\leq \sum_i \left(\sum_{j \in W^* \cap S_i} v_j \right) + \sum_i \sum_{j \in W^* \cap (I_i \setminus S_i)} \underline{v} + 2k\epsilon \\ &\leq ALG + 2k\epsilon \end{aligned}$$

The first inequality in the chain above follows just by combining (2) and (3), and the fact that $|W^*| \leq k$. The second follows from the definition of $\underline{v}$, namely, all sold items in ALG are of value at least $\underline{v}$. $\square$