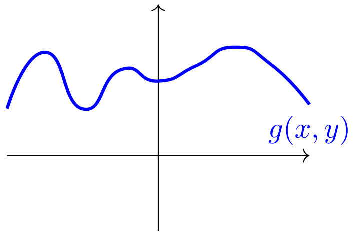
Την Τρίτη μάθημα 8:30 χωρίς διάλειμμα

Σήμα - σύστημα

$$\boxed{\underbrace{g}_{\text{εξαρτημένη}} = f(\underbrace{t}_{\text{ανεξάρτητη}})} \qquad g = f(\vec{r}, t) \qquad \vec{E}(\vec{r}, t)$$

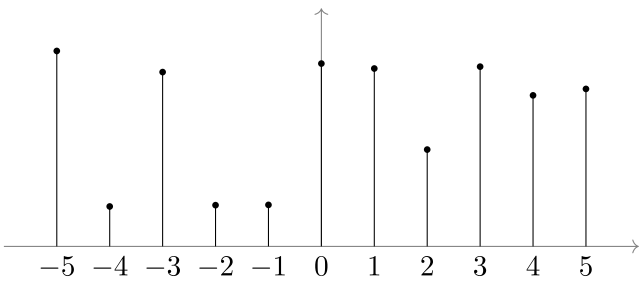
Αναλογικό

Αν t συνεχής $\in \mathbb{R}$
και y συνεχής $\in \mathbb{R}$



Διακριτού χρόνου / Διακριτό (discrete)

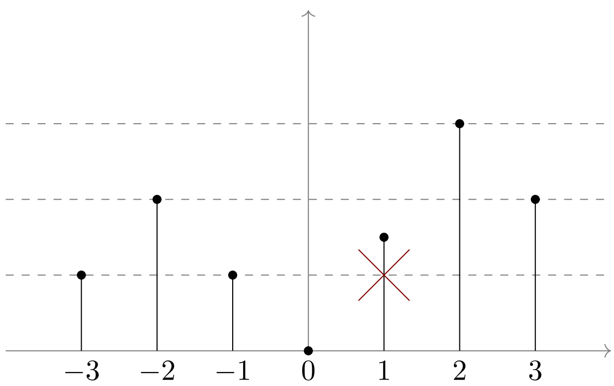
t διακριτό $\to \mathbb{Z},\ n \in \mathbb{Z}$
g συνεχής $\in \mathbb{R}$



7$^{\text{o}}$ εξάμηνο

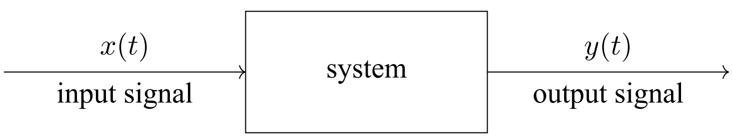
Κβαντισμένο

$n \in \mathbb{Z}$
g διακριτή



Στοχαστικό Περιέχει και τις τρεις κατηγορίες

Σύστημα



Περιοδικά σήματα

Αν $\exists T \in \mathbb{R} : \forall t \in \mathbb{R} \quad x(t) = x(t+T)$ τότε $x(t)$ **περιοδικό σήμα** με περίοδο T.
Ή θα είναι 0, ή θα συνεχιστεί για πάντα.

$$\int_{-T/2}^{T/2} x(t)\,\mathrm{d}t = \int_{t_0-T/2}^{t_0+T/2} x(t)\,\mathrm{d}t\ \forall t$$

Η σύνθεση μιας συνάρτησης με μια περιοδική συνάρτηση είναι περιοδική;

Απόδ. Έστω g μία περιοδική συνάρτηση:

$$\begin{aligned}\big(f \circ g\big)(x) &= f\left(g(x)\right) = f\left(g(x+T)\right) = \\ &= \big(f \circ g\big)(x+T)\end{aligned}$$

Συμμετρίες

- Αν $x(t) = x(-t)\ \forall t$ τότε η $x(t)$ λέγεται **άρτια συνάρτηση** (even function).
- Αν $x(t) = -x(t)\ \forall t$ τότε η $x(t)$ λέγεται **περιττή συνάρτηση** (odd function).

$\forall x(t) \quad \exists\, x_0(t), x_e(t) : x(t) = x_e(t) + x_0(t)$

Απόδ.

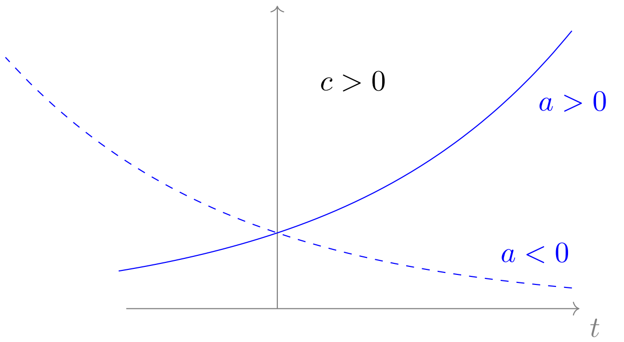
$$\begin{aligned}x_e(t) &= \frac{x(t)+x(-t)}{2} \\ x_o(t) &= \frac{x(t)-x(-t)}{2}\end{aligned}$$

$$\begin{aligned}x\underbrace{\ _e\ }_{\text{άρτια}}y_e &= z_e \\ x_o y_o &= z_e \\ x_e y_0 &= z_0 \\ \int_{-A}^{A} x_0(t)\,\mathrm{d}t &= 0 \\ \int_{-\infty}^{\infty} x_0(t)\,\mathrm{d}t &=?\ \text{(εξαρτάται)} \\ \lim_{A\to\infty}\int_{-A}^{A} x_0(t)\,\mathrm{d}t &= 0 \quad \text{(principal Cauchy value)}\end{aligned}$$

Χαρακτηριστικά σήματα

1) **Εκθετικό σήμα**

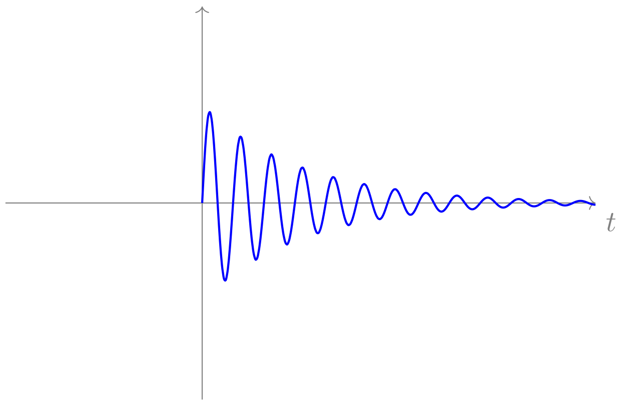
$$x(t) = ce^{at} \quad a \in \mathbb{R} \quad c > 0$$



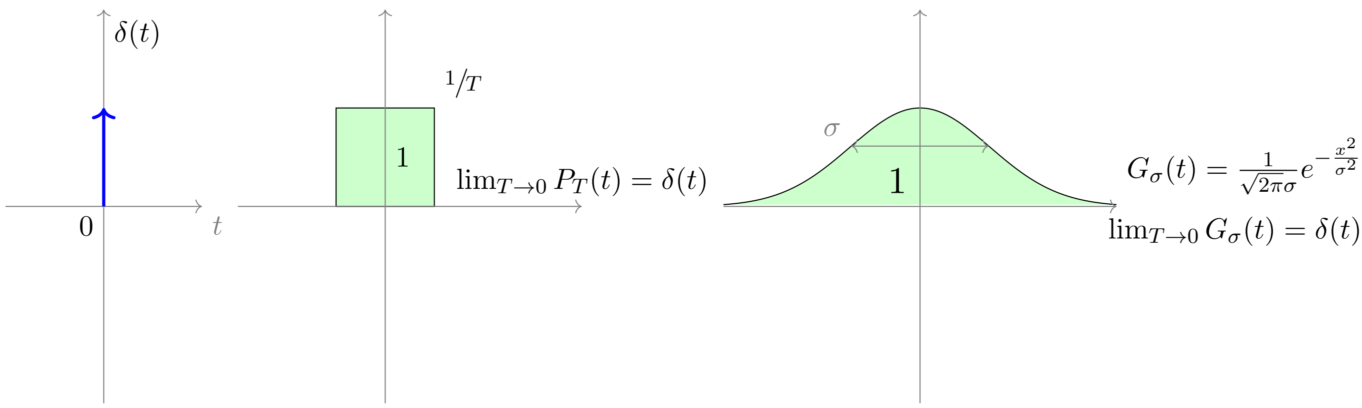
$$x(t) = ce^{(\sigma t + j\omega)t} = ce^{\sigma t}e^{j\omega t} = ce^{\sigma t}\left[\cos(\omega t) + j\sin(\omega t)\right]$$

2) **(Συν)ημιτονοειδή σήματα**

$$x(t) = A\cos(\omega t \pm \phi) = a\mathrm{Re}\left\{e^{j(\omega t+\phi)}\right\} = A\frac{e^{j(\omega t\pm\phi)} + e^{-j(\omega t\pm\phi)}}{2}$$



3) **Δέλτα Dirac $\boldsymbol{\delta(t)}$**



Ορ.

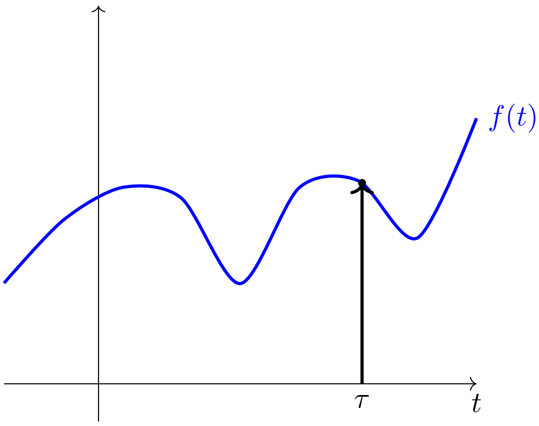
$$\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)\ \forall f(t)$$

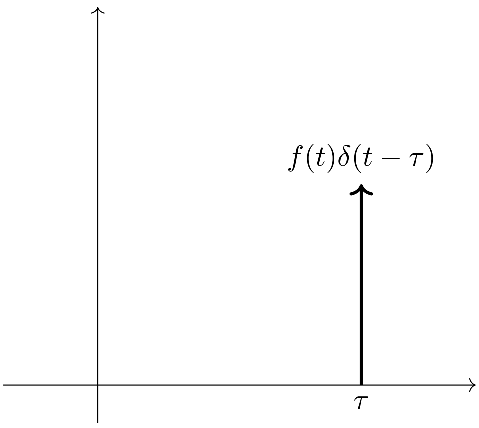
$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}t = f(\tau)$$

$$\boxed{\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau = f(t)}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}\tau = f(t)$$





Ιδιότητες της $\delta(t)$

1. **Κλιμάκωση**

$$a \in \mathbb{R} : \delta(at) = \frac{1}{|a|}\delta(t)$$

Απόδ.

$$\underbrace{\int_{-\infty}^{\infty} \phi(t)\boxed{\delta(at)}\,\mathrm{d}t}_{\substack{at=\xi \\ \mathrm{d}t=\frac{\mathrm{d}\xi}{a}}} = \int_{-\infty_{(a)}}^{\infty_{(a)}} \phi\left(\frac{\xi}{a}\right)\delta(\xi)\frac{\mathrm{d}\xi}{a} = \frac{1}{|a|}\int_{-\infty}^{\infty}\frac{\phi\left(\frac{\xi}{a}\right)}{|a|}\delta(\xi)\,\mathrm{d}\xi = \frac{\phi(0)}{|a|} = \int_{-\infty}^{\infty}\phi(t)\boxed{\frac{\delta(t)}{|a|}}\mathrm{d}t$$

2. $f(t)\delta(t) = f(0)\delta(t)$
3. $f(t)\delta(t-\xi) = f(g)\delta(t-\xi)$

$$\xrightarrow[\text{input signal}]{x(t)} \boxed{\text{system}} \xrightarrow[\text{output signal}]{y(t)}$$

$y(t) = \mathscr{L}\left\{x(t)\right\}$
$\forall x_1(t)\ x_n(t)$
$y_1(t) = \mathscr{L}\left\{x_1(t)\right\}$
$y_2(t) = \mathscr{L}\left\{x_2(t)\right\}$

Για const $a_1, a_2 \in \mathbb{R}$

$$\begin{aligned} x(t) &= a_1 x_1(t) + a_2 x_2(t) \\ y(t) &= \mathscr{L}\left\{x(t)\right\} \end{aligned}$$

ανν

$$y(t) = a_1 y_1(t) + a_2 y_2(t)$$

τότε

$$\mathscr{L}: \text{ γραμμικό σύστημα}$$

- $g(t) = \mathscr{L}\left\{x(t)\right\}$
  $x'(t) = x(t-\tau)$
  ανν $y'(t) = \mathscr{L}\left\{x'(t)\right\} = \mathscr{L}\left\{x(t-\tau)^2\right\} = y(t-\tau)$
  τότε το σύστημα $\mathscr{L}$ είναι αμετάβλητο κατά τη μετατόπιση.

$$\xrightarrow[\text{input signal}]{\delta(t)} \boxed{\text{γραμμικό \& ΑΚΜ}} \overset{\mathscr{L}\{\}}{\xrightarrow[\text{κρουστική απόκριση (impulse response)}]{\mathrm{h}(t)}}$$

Υποστηρίζω ότι ένα γραμμικό & ΑΚΜ σύστημα περιγράφεται πλήρως από την κρουστική απόκριση $h(t)$.

Απόδ. Από παραπάνω, γνωρίζουμε ότι $x(t) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}t$

$$\begin{aligned} y(t) = \mathscr{L}\left\{y(t)\right\} &= \mathscr{L}\left\{\int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}\tau\right\} \\ &\overset{\text{linearity}}{=} \int_{-\infty}^{\infty} \mathscr{L}\left\{x(\tau)\delta(t-\tau)\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathscr{L}\left\{\delta(t-\tau)\right\}\mathrm{d}\tau \\ &\underset{TSI}{\overset{\text{AKM}}{=}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau \end{aligned}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\underbrace{h(t-\tau)}_{\text{linear time-shift invariant}}\mathrm{d}\tau$$

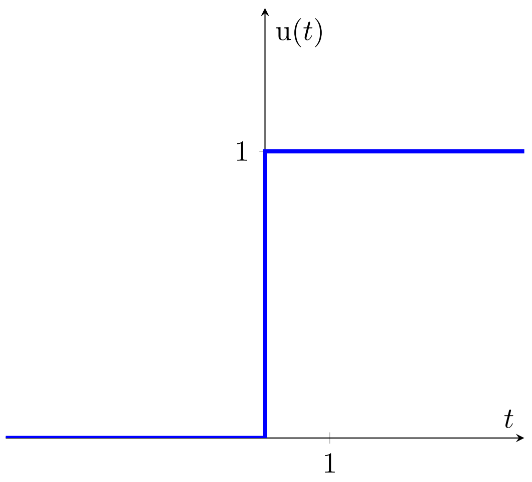
- $\delta(t) = \delta(-t)$ άρτια συνάρτηση
- $\delta^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\delta(t)$, για την οποία αποδεικνύεται ότι:

$$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)\,\mathrm{d}t = (-1)^n\ \phi^{(n)}(t)\Big|_{t=0}$$

Βηματική Συνάρτηση (Unit Step Function)

$$\mathrm{u}(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \mathrm{u}(t)\phi(t)\,\mathrm{d}t = \mathscr{N}_{\mathrm{u}}\left\{\phi(t)\right\} = \int_0^{\infty} \underbrace{\phi(t)}_{\text{number}}\,\mathrm{d}t$$

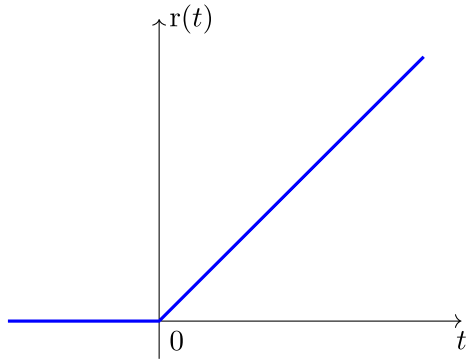


$$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{u}(t)$$

$$\mathrm{u}(t) = \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \int_0^{\infty} \delta(t-\xi)\,\mathrm{d}\xi$$

Ράμπα

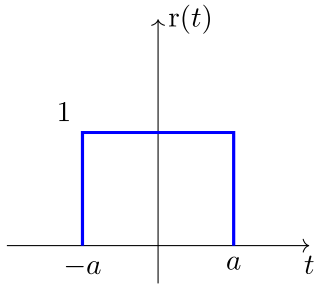
$$\mathrm{r}(t) = \int_{-\infty}^{t} \mathrm{u}(\tau)\,\mathrm{d}\tau = \begin{cases} t & t \geq 0 \\ 0 & \text{else} \end{cases} = t\mathrm{u}(t)$$

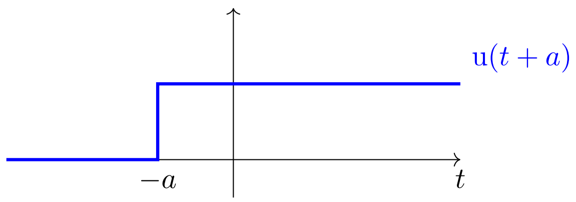


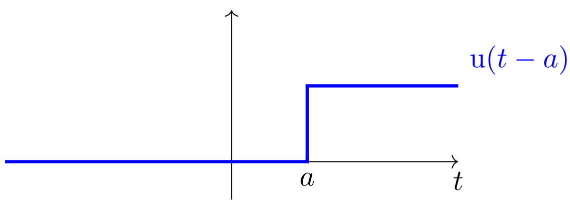
$$\mathrm{u}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{r}(t)$$

Ορθογωνικός παλμός (Rectangular Pulse function)

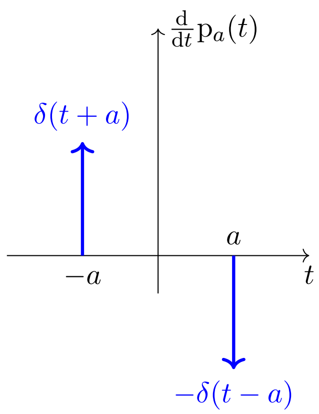
$$\mathrm{p}_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$





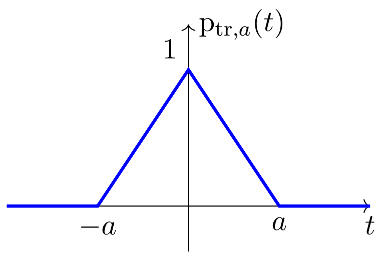


$$\begin{aligned} \mathrm{p}_a(t) =& \mathrm{u}(t+a) - \mathrm{u}(t-a) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{p}_a(t) =& \delta(t+a) - \delta(t-a) \end{aligned}$$

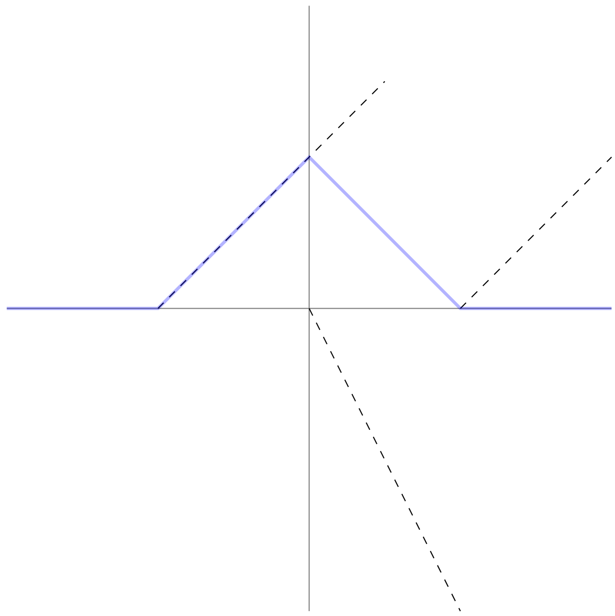


Τριγωνικός Παλμός (Triangular Pulse function)

$$\mathrm{p}_{\mathrm{tr},a} = \begin{cases} 1 - \frac{|t|}{a} & |t| < a \\ 0 & |t| > a \end{cases}$$



$$p_{\mathrm{tr},a}(t) = \frac{1}{a}\left[\mathrm{r}(t+a) + \mathrm{r}(t-a) - 2\mathrm{r}(t)\right]$$



Χαρακτηριστικά Μεγέθη

1) **Μέση τιμή (Mean Value)**

$$\overline{x(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t)\,\mathrm{d}t$$

Αν περιοδική τότε

$$\begin{aligned}\bar{x}(t) &= \frac{1}{T} = \int_{0}^{T} x(t)\,\mathrm{d}t \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t\end{aligned}$$

2) **Ενεργός τιμή (Root Mean Square Value)**

$$\overline{\overline{x(t)}} = \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t\right]^{1/2}$$

Αν ημιτονοειδές σήμα $\bar{\bar{x}}(t) = \frac{x_{\max}}{\sqrt{2}}$

3) **Ενέργεια - Ισχύς**

- Στιγμιαία ισχύς (Instant power)

$$p(t) = x^2(t)$$

- Μέση ισχύς (Mean power)

$$\overline{p(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t = \left(\overline{\overline{x(t)}}\right)^2$$

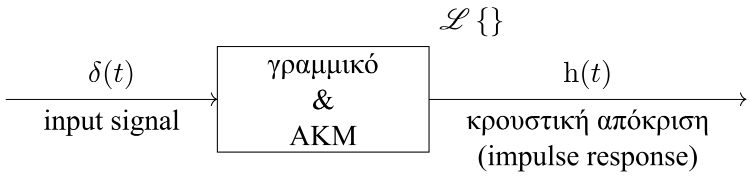
- Ενέργεια (Energy)

$$W = \int_{t_1}^{t_2} p(t)\,\mathrm{d}t = \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t = (t_2 - t_1)\left(\overline{\overline{x(t)}}\right)^2$$

Σήματα $\begin{cases} \textbf{Σήμα ενέργειας} \text{ αν } \lim\limits_{T\to\infty} W < \infty \\ \\ \textbf{Σήμα ισχύος} \text{ αν } \lim\limits_{T\to\infty} \overline{p(t)} > 0 \\ \text{Υπάρχουν και σήματα που δεν είναι ούτε ενέργειας, ούτε ισχύος.} \end{cases}$

Συνέλιξη

$$x(t) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau$$



$$h(t) = \mathscr{L}\left\{\delta(t)\right\}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \underbrace{x(t)}_{\text{είσοδος}}\ \underbrace{*}_{\text{συνέλιξη}}\ \underbrace{h(t)}_{\text{κρουστική απόκριση}}$$

Συνέλιξη - Convolution

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau$$

- $x(t) * y(t) = y(t) * x(t)$ **Αντιμεταθετική**

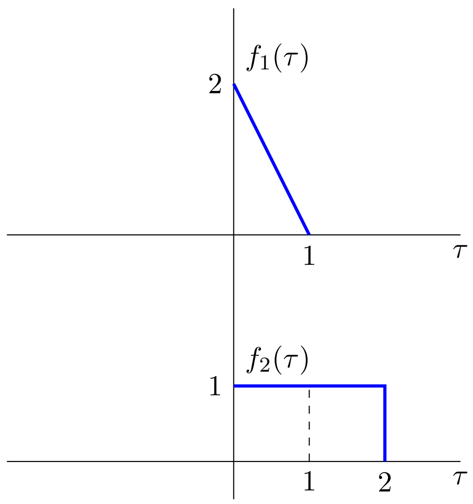
$$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\lambda)y(\lambda)[-\,\mathrm{d}\lambda] = \int_{-\infty}^{\infty} y(\lambda)x(t-\lambda)\,\mathrm{d}\lambda = y(t) * x(t)$$

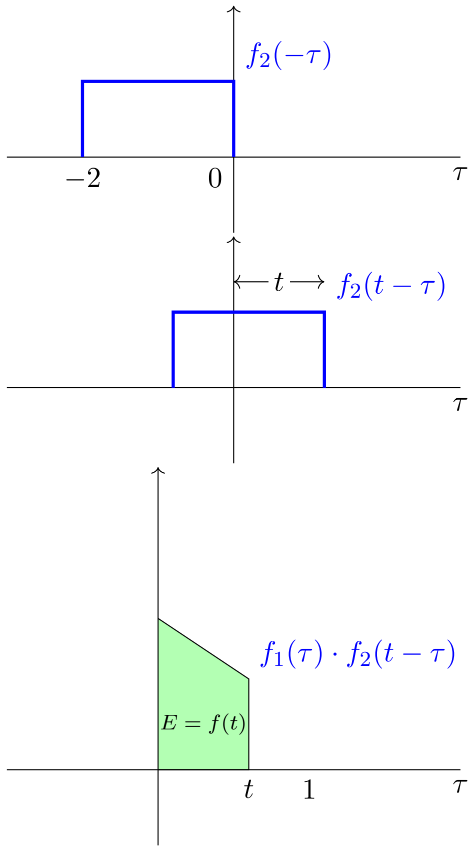
- $x_1(t) * \big[x_2(t) * x_3(t)\big] = \big[x_1(t) * x_2(t)\big] * x_3(t)$ **Προσεταιριστική**

Παρ.

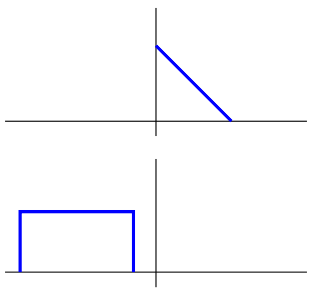
$$f_1(t) = 2(1-t)\left[\mathrm{u}(t) - u(t-1)\right]$$
$$f_2(t) = \mathrm{u}(t) - \mathrm{u}(t-\tau)$$

Γραφική μέθοδος υπολογισμού συνέλιξης



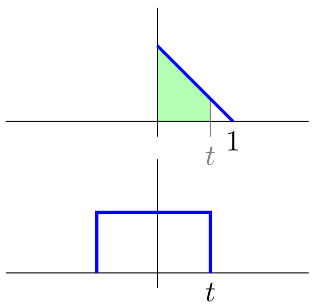


- $t<0$



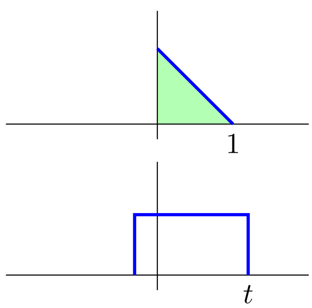
$f(t)=0$

- $0<t<1$



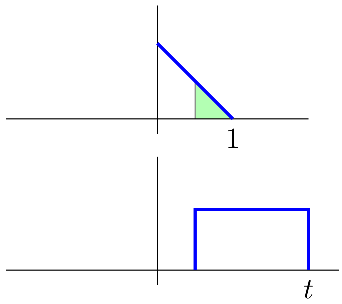
$f(t)=\frac{1}{2}(2+2-2t)t=(2-t)t$

- $1<t<2$



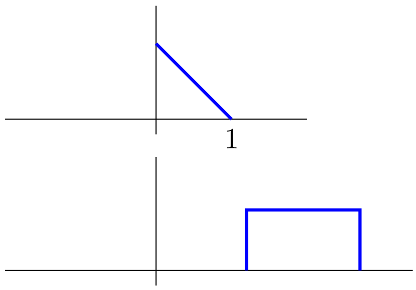
$f(t)=1$

- $2 < t < 3$

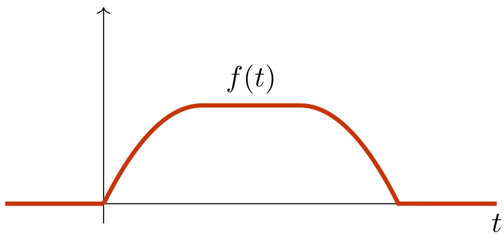


$$f(t) = \frac{(t-1)\cdot 2\cdot\big(1-(t-2)\big)}{2} = (t-1)(3-t)$$

- $t > 3$



$$f(t) = 0$$



Αναλυτική μέθοδος Παρατηρώ ότι:

$$\int_{-\infty}^{\infty} f(t,\tau)\mathrm{u}(t-\xi)\mathrm{u}(\phi-\tau)\,\mathrm{d}\tau = \int_{\xi}^{\phi} f(t,\tau)\,\mathrm{d}\tau\mathrm{u}(\phi-\xi)$$

$$\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} \underbrace{2(1-\tau)}_{x(\tau)} \big[\mathrm{u}(\tau)-\mathrm{u}(\tau+1)\big]\,\big[\mathrm{u}(t-\tau)-\mathrm{u}(t-\tau-2)\big]\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\,\big[\mathrm{u}(\tau)\mathrm{u}(t-\tau)-\mathrm{u}(\tau-1)\mathrm{u}(t-\tau)-\mathrm{u}(\tau)\mathrm{u}(t-\tau-2)+\mathrm{u}(\tau-1)\mathrm{u}(t-\tau-2)\big]\,\mathrm{d}\tau \\
&= \int_0^t x(\tau)\,\mathrm{d}\tau\mathrm{u}(t) - \int_1^x x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-1) - \int_0^{t-2} x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-2) + \int_1^{t-2} x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-3) \\
&= (2t-t^2)\mathrm{u}(t)-\big[2t-t^2-1\big]\mathrm{u}(t-1)-\big[2(t-2)-(t-2)^2\big]\mathrm{u}(t-2)+\big[2(t-2)-(t-2)^2-1\big]\mathrm{u}(t-3)
\end{aligned}$$

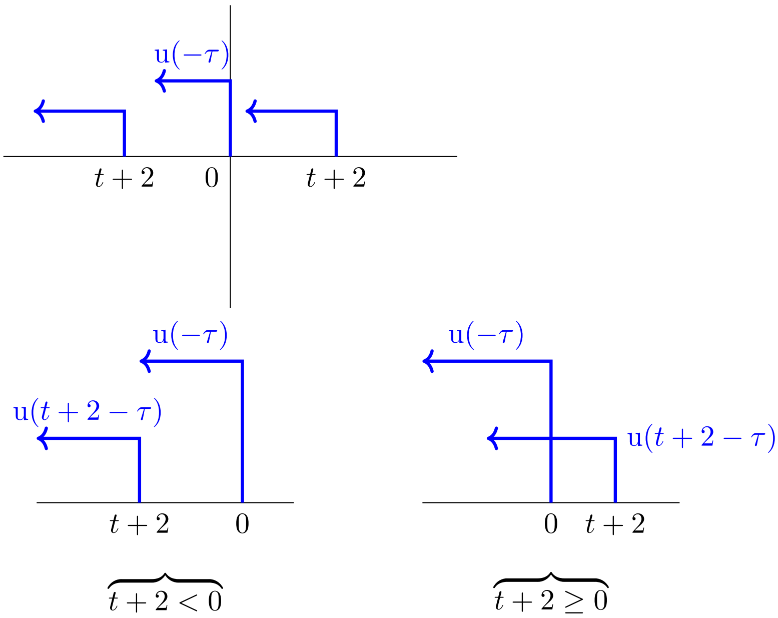
Ex

$$\begin{aligned}
f_1(t) &= e^t\mathrm{u}(-t) \\
f_2(t) &= \mathrm{u}(t+2)-u(t+1) \\
f &= f_1 * f_2
\end{aligned}$$

$$
\begin{aligned}
f &= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}\left(-(t-\tau)+2\right)\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(\tau-t+2)\,\mathrm{d}\tau \\
&= \int_{t-2}^{0} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t-2) \\
&= e^{\tau}\big|_{t-2}^{0}\,\mathrm{u}(2-t) \\
&= \left[1-e^{t-2}\right]\mathrm{u}(2-t)
\end{aligned}
$$

$$
\int_{-\infty}^{\infty} f(t,\tau)\mathrm{u}(\tau-\xi)\mathrm{u}(\phi-\tau)\,\mathrm{d}\tau = \int_{\xi}^{\phi} f(t,\tau)\,\mathrm{d}\tau\mathrm{u}(\phi-\xi)
$$

Ex.



$$
\begin{aligned}
x(t) &= e^{t}\mathrm{u}(-t) \\
y(t) &= \mathrm{u}(t+2) \\
z(t) &= x(t)*y(t) = \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}\left[(t-\tau)+2\right]\mathrm{d}\tau = \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(t+2-\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{\tau}\left[1-\mathrm{u}(t)\right]\mathrm{u}(t+2-\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(t+2-\tau)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(\tau)u(t+2-\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{t+2} e^{\tau}\,\mathrm{d}\tau\,\mathrm{u}\left(\cancelto{1}{t+2-(-\infty)}\right) - \int_{0}^{t+2} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t+2) \\
&= e^{t+2} - \left[e^{t+2}-1\right]\mathrm{u}(t+2)
\end{aligned}
$$

Ex.

$$
\begin{aligned}
x(t) &= e^{t}\mathrm{u}(-t)\\
y(t) &= \mathrm{u}(t+2)-\mathrm{u}(t+1)\\
z(t) &= x(t)*y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\left[\mathrm{u}(t-\tau+2)-\mathrm{u}(t-\tau+1)\right]\mathrm{d}\tau\\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(t-\tau+2)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(t-\tau+1)\,\mathrm{d}\tau\\
&= \int_{-\infty}^{\infty} e^{\tau}\left[1-\mathrm{u}(\tau)\right]\mathrm{u}(t-\tau+2)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\left[1-\mathrm{u}(\tau)\right]\mathrm{u}(t-\tau+1)\,\mathrm{d}\tau\\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(t-\tau+2)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(\tau)\mathrm{u}(t-\tau+2)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(t-\tau+1)\,\mathrm{d}\tau + \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(\tau)\mathrm{u}(t-\tau+1\\
&= \int_{-\infty}^{t+1} e^{\tau}\,\mathrm{d}\tau - \int_{-\infty}^{t+1} e^{\tau}\,\mathrm{d}\tau - \int_{0}^{t+2} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t+2) + \int_{0}^{t+1} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t+1)\\
&= \int_{t+1}^{t+2} e^{\tau}\,\mathrm{d}\tau - \left[e^{t+2}-1\right]\mathrm{u}(t+2) + \left[e^{t+1}-1\right]\mathrm{u}(t+1)
\end{aligned}
$$

$\exists h(t)$ ανν LTI

$$
\begin{aligned}
y(t) &= x(t)*h(t)\\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau\\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau
\end{aligned}
$$

Έστω ότι η $x(t) = e^{j\omega t}$

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(t)e^{j\omega(t-\tau)}\,\mathrm{d}\tau = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau\\
&= x(t)\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau}_{h(t)\xrightarrow{FT}H(\omega)}
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= A_1e^{j\omega_1 t} + A_2e^{j\omega_2 t}\\
y(t) &= A_1e^{j\omega_1 t}H(\omega_1) + A_2e^{j\omega_2 t}H(\omega_2)
\end{aligned}
$$

Συναρτηστιακοί χώροι

Διανυσματικός χώρος S

$\bar{x}, \quad \bar{y} \quad S$

Εσωτερικό γινόμενο

$\langle \bar{x}, \bar{y} \rangle \in \mathbb{C}$

1) $\langle \bar{x}, \bar{y} \rangle = \langle \bar{y}, \bar{x} \rangle^*$

2) $c \langle \bar{x}, \bar{y} \rangle = \langle c\bar{x}, \bar{y} \rangle$

3) $\langle \bar{x} + \bar{y}, \bar{z} \rangle = \langle \bar{x}, \bar{z} \rangle + \langle \bar{y}, \bar{z} \rangle$

4) $\langle \bar{x}, \bar{x} \rangle \geq 0$ με $\langle \bar{x}, \bar{x} \rangle = 0$ ανν $\bar{x} = \bar{0}$

Νόρμα

$\bar{x} \in S$

$||\bar{x}|| \geq 0$

1) $||\bar{x}|| = 0$ ανν $\bar{x} = \bar{0}$

2) $||a\bar{x}|| = |a| ||\bar{x}|| \quad x \in \mathbb{C}$

3) $||\bar{x} + \bar{y}|| \leq ||\bar{x}|| + ||\bar{y}||$

Μέτρο: Απόσταση μεταξύ $\bar{x}, \bar{y} \in S$

1) $d(\bar{x}, \bar{y}) \geq 0 \qquad d(\bar{x}, \bar{y}) = 0$ ανν $\bar{x} = \bar{y}$

2) $d(\bar{x}, \bar{y}) = d(\bar{y}, \bar{x})$

3) $d(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{z}) + d(\bar{y}, \bar{z}) \quad \bar{z} \in S$

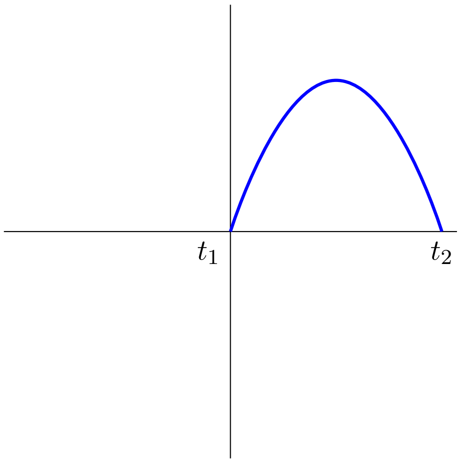
Συναρτησιακός χώρος

$$x(t), y(t) \in S = \big\{ x(t) / x(t) : [t_1, t_2] \to \mathbb{R} \big\}$$

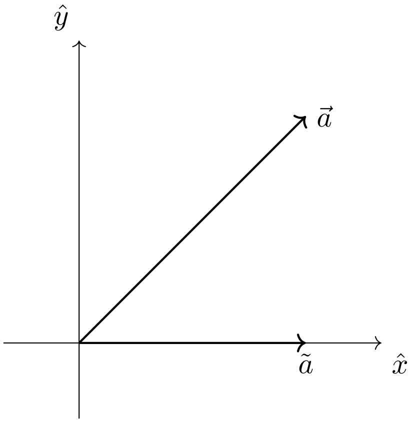
$$\big\langle x(t), y(t) \big\rangle = \int_{t_1}^{t_2} x(t) y(t) \, \mathrm{d}t$$

$$\big|\big| x(t) \big|\big| = \left[\int_{t_1}^{t_2} x^2(t) \, \mathrm{d}t \right]^{1/2}$$

$$d\big(x(t), y(t)\big) = \left[\int_{t_1}^{t_2} \big[x(t) - y(t) \big]^2 \, \mathrm{d}t \right]^{1/2}$$



$$
\begin{aligned}
\text{Αν } & \big\langle \phi_1(t), \phi_2(t) \big\rangle = 0 \quad \phi_1(t) \perp \phi_2(t) \\
& \big\langle \phi_1(t), \phi_1(t) \big\rangle = 1 \quad \phi_1(t) \text{ κανονική}
\end{aligned}
$$



Τερατοχώρος

$\hat{x}, \hat{y}$ όχι εξαρτημένα (συνευθειακά)
Ποια είναι η καλύτερη προσέγγιση για το $\vec{a}$ εφ' όσον δεν υπάρχει το $\vec{y}$;
$\tilde{a}$ best γιατί $\mathrm{d}(\vec{a}, \tilde{a})$ min.
Άρα:

$$
\begin{aligned}
& \tilde{a} = k\hat{x} \\
& \vec{a} = a_x\hat{x} + a_y\hat{y} \\
& \vec{a} - \tilde{a} = (a_x - k)\hat{x} - a_y\hat{y} \\
& d(\vec{a}, \tilde{a}) = \sqrt{(a_x - k)^2 + a_y^2} \\
& \tfrac{\mathrm{d}}{\mathrm{d}k}\big(d(\vec{a}, \tilde{a})\big) = \frac{a_x - k}{\dots} = 0 \implies k = a_x = \tilde{a} \cdot \hat{x} \\
& \vec{a} \cdot \hat{x} = a_x
\end{aligned}
$$

Η βέλτιστη έκφραση του $\vec{a}$ στο δισδιάστατο χώρο είναι το ίδιο το $\vec{a}$.

Μη κάθετα διανύσματα

$$\vec{a} = a_x\hat{x} + a_y\hat{y}$$
$$\vec{a} - \tilde{a} = (a_x - k)\hat{x} + a_y\hat{y}$$
$$\mathrm{d}(\vec{a},\tilde{a}) = ||\vec{a}-\tilde{a}|| = \sqrt{(\vec{a}-\tilde{a})(\vec{a}-\tilde{a})} = \Big(\big[(a_x-k)\hat{x}+a_y\hat{y}\big]\cdot\big[(a_x-k)\hat{x}+a_y\hat{y}\big]\Big)^{1/2}$$
$$\Big[(a_x-k)^2 + a_y^2 + 2(a_x-k)a_y\hat{x}\cdot\hat{y}\Big]^{1/2}$$
$$\vec{a}_{\text{best}} = (\vec{a}\cdot\hat{x})\hat{x} \neq a_x$$
$$\boxed{\vec{a}\cdot\hat{x} = a_x + a_y\cos\phi \neq a_x}$$

Συναρτηστιακός κόσμος $\phi_n(t)$ παράγουν χώρο με το μηχανισμό:

$$f(t) = \sum_{n=0}^{\infty} a_n\phi_n(t) \quad t\in\Delta$$

$\phi_n(t)$ ανεξάρτητες μεταξύ τους (βάση απειροδιάστατου χώρου)

$$\hat{f}(t) = \sum_{n=0}^{M} \underbrace{\hat{a}_n}_{\neq a_n,\ \text{επειδή η βάση δεν είναι ορθοκανονική}} \phi_n(t)$$ βέλτιστη, ώστε η απόσταση με την f να είναι ελάχιστη

$$\begin{aligned}
\overbrace{I^2}^{\text{σφάλμα}} &= \int_\Delta \Big[f(t)-\hat{f}(t)\Big]^2 \,\mathrm{d}t \\
&= \int_\Delta \left[\sum_{n=0}^{+\infty} a_n\phi_n(t) - \sum_{n=0}^{M}\hat{a}_n\phi(t)\right]^2 \mathrm{d}t \\
&= \int_\Delta f^2(t)\,\mathrm{d}t + \int_\Delta \left(\sum_{n=0}^{M}\hat{a}_n\phi_n(t)\right)^2 \mathrm{d}t - 2\int_\Delta \left[f(t)\sum_{n=0}^{M}\hat{a}_n\phi_n(t)\right]\mathrm{d}t
\end{aligned}$$

Άρα:

$$\begin{aligned}
I^2 &= \int_\Delta f^2(t)\,\mathrm{d}t + \int_\Delta \sum_{n=0}^{M}\big[\hat{a}_n\phi_n(t)\big]^2\,\mathrm{d}t + 2\int_\Delta\left[\sum_{n=0}^{M}\sum_{m=n+1}^{M}\hat{a}_n\cdot\hat{a}_m\phi_n(t)\phi_m(t)\right]\mathrm{d}t - 2\int_\Delta\sum_{n=0}^{M}\hat{a}_n f(t)\phi_n(t)\,\mathrm{d}t \\
&= \int_\Delta f^2(t)\,\mathrm{d}t + \sum_{n=0}^{M}\hat{a}_n^2\int_\Delta\phi_n^2(t)\,\mathrm{d}t + 2\sum_{n=0}^{M}\sum_{n=m+1}^{M}\hat{a}_n\hat{a}_m\int_\Delta\phi_n(t)\phi_m(t)\,\mathrm{d}t - 2\sum_{n=0}^{M}\hat{a}_n\int_\Delta f(t)\phi_n(t)\,\mathrm{d}t
\end{aligned}$$

$$\frac{\mathrm{d}(I^2)}{\mathrm{d}\underbrace{\hat{a}_i}_{\text{από 0 έως } n}} = 2\hat{a}_i\int_\Delta\phi_i^2(t)\,\mathrm{d}t + 2\sum_{m\neq i}\hat{a}_m\int_\Delta\phi_i(t)\phi_m(t)\,\mathrm{d}t - 2\int_\Delta f(t)\phi(t)\,\mathrm{d}t = 0$$

Έστω ϕ_i μοναδιαία $\iff \int_\Delta \phi^2(t)\,\mathrm{d}t = 1$ και ϕ_i ορθογώνια $\iff \int_\Delta \phi_1\phi_2(t) = 0$
Αν η $\big\{\phi(t)\big\}$ ορθοκανονική

$$2\vec{a}_i - 2\int_\Delta f(t)\phi(t)\,\mathrm{d}t = 0 \implies \hat{a}_i = \int_\Delta f(t)\phi(t)\,\mathrm{d}t$$

$$2a\left\langle\phi,\phi\right\rangle + \sum\sum\hat{a}_n\left\langle\phi_i,\phi_n\right\rangle - 2\left\langle f,\phi\right\rangle = 0$$

$$f(t) = \sum_{a=0} a_n \phi_n(t)$$

$$\int_\Delta f(t)\phi(t)\,\mathrm{d}t = a_i$$

$$\langle f, \phi_i \rangle = \left\langle \sum_{n=0}^{\infty} a_i \phi_n, \phi_i \right\rangle = \sum_{??}^{\infty} a_n \langle \phi_n, \phi_i \rangle$$

Ηθικό δίδαγμα: Αν η βάση του χώρου είναι ορθοκανονική και μας ζητηθεί να υπολογίσουμε μία προσέγγιση της συνάρτησης σε έναν υποχώρο, μπορούμε άμεσα να υπολογίσουμε την προβολή της συνάρτησης πάνω στη βάση.

Ex. $f(t) = e^{-3t}\mathrm{u}(t) \qquad \phi_1(t) = e^{t}\mathrm{u}(t) \;\&\; \phi_2(t) = e^{-2t}\mathrm{u}(t)$

$$\hat{f}(t) = a_1 e^{-t}\mathrm{u}(t) + a_2 e^{-2t}\mathrm{u}(t)$$

$$\int \left[a_1\phi_1 + a_2\phi_2 - f\right]\phi_1\,\mathrm{d}t = 0$$

$$\int_0^\infty \left[a_1 e^{-t} + a_2 e^{-2t} - e^{-3t}\right] e^{-t}\,\mathrm{d}t = 0 \implies$$

$$a_1 \int_0^\infty e^{-2t}\,\mathrm{d}t + a_2 \int_0^\infty e^{-3t}\,\mathrm{d}t - \int_0^\infty e^{-4t}\,\mathrm{d}t = 0 \implies \boxed{\frac{a_1}{2} + \frac{a_2}{3} - \frac{1}{4} = 0}$$

$$\int \left[a_1 e^{-t} + a_2 e^{-2t} - e^{-3t}\right] e^{-2t}\,\mathrm{d}t = 0 \implies \boxed{\frac{a_1}{2} + \frac{a_2}{4} - \frac{1}{5} = 0}$$

$$a_1 = {}^{3}\!/_{10},\; a_2 = {}^{6}\!/_{5}$$

Ex. $f(t) = e^{-3t}\mathrm{u}(t) \qquad \phi_1(t) = e^{t}\mathrm{u}(t) \;\&\; \phi_2(t) = e^{-2t}\mathrm{u}(t) \;\&\; \phi_3(t) = e^{-3t}\mathrm{u}(t)$

$f(t) = \sum_{n=0}^{N} a_n \phi_n(t) \quad \{\phi_n(t)\}$ ορθοκανονική

$$E \stackrel{\triangle}{=} \int_\Delta f^2(t)\,\mathrm{d}t = \int \left(\sum a_n\phi_n(t) \sum a_n\phi_n(t)\right)\mathrm{d}t +$$

$$= \sum_{n=0}^{\infty} a_n \int_\Delta \cancel{\phi_n^2(t)\,\mathrm{d}t}^{\,1} + \sum_{m\neq n} a_i a_n \int_? \cancel{\Delta\phi_i(t)\phi_n(t)}^{\,0}$$

$$E = \sum_{a=m}^{\Sigma} a_n^2 \quad \text{Parseval} \int \text{theorem}$$