Την Τρίτη μάθημα 8:30 χωρίς διάλειμμα

Σήμα - σύστημα

$$\boxed{\underbrace{g}_{\text{εξαρτημένη}} = f(\underbrace{t}_{\text{ανεξάρτητη}})} \qquad g = f(\vec{r}, t) \qquad \vec{E}(\vec{r}, t)$$

Αναλογικό

Αν t συνεχής $\in \mathbb{R}$

και y συνεχής $\in \mathbb{R}$

```
draw[->] (0,-1) -- (0,2); draw[->] (-2,0) -- (2,0);

draw[blue, very thick] plot [smooth, tension=1, domain=-2:2, samples=9] (x,1+rand/2) node[below] g(x, y);
```

Διακριτού χρόνου / Διακριτό (discrete)

t διακριτό $\to \mathbb{Z},\ n \in \mathbb{Z}$

g συνεχής $\in \mathbb{R}$

```
[scale=0.7] draw[->,gray] (0,0) -- (0,4.5); draw[->,gray] (-6,0) -- (6,0);
draw(-5,0) node[below] −5; filldraw[black] (-5,0) -- (-5,(1+rand)*2) circle (1.5pt); draw(-4,0) node[below] −4; filldraw[black] (-4,0) -- (-4,(1+rand)*2) circle (1.5pt); draw(-3,0) node[below] −3; filldraw[black] (-3,0) -- (-3,(1+rand)*2) circle (1.5pt); draw(-2,0) node[below] −2; filldraw[black] (-2,0) -- (-2,(1+rand)*2) circle (1.5pt); draw(-1,0) node[below] −1; filldraw[black] (-1,0) -- (-1,(1+rand)*2) circle (1.5pt); draw(0,0) node[below] 0; filldraw[black] (0,0) -- (0,(1+rand)*2) circle (1.5pt); draw(1,0) node[below] 1; filldraw[black] (1,0) -- (1,(1+rand)*2) circle (1.5pt); draw(2,0) node[below] 2; filldraw[black] (2,0) -- (2,(1+rand)*2) circle (1.5pt); draw(3,0) node[below] 3; filldraw[black] (3,0) -- (3,(1+rand)*2) circle (1.5pt); draw(4,0) node[below] 4; filldraw[black] (4,0) -- (4,(1+rand)*2) circle (1.5pt); draw(5,0) node[below] 5; filldraw[black] (5,0) -- (5,(1+rand)*2) circle (1.5pt); ;

draw(0,-1) node[below] 7     ;
```

Κβαντισμένο

$n \in \mathbb{Z}$

g διακριτή

```
draw[->,gray] (0,0) -- (0,4.5); draw[->,gray] (-4,0) -- (4,0);
draw[gray,dashed] (-4,1) -- (4,1); draw[gray,dashed] (-4,2) -- (4,2); draw[gray,dashed] (-4,3) -- (4,3);

filldraw[black] (-3,0) node[below] −3 -- (-3,1) circle (1.5pt); filldraw[black] (-2,0) node[below] −2 -- (-2,2) circle (1.5pt); filldraw[black] (-1,0) node[below] −1 -- (-1,1) circle (1.5pt); filldraw[black] ( 0,0) node[below] 0 -- (0,0) circle (1.5pt); filldraw[black] ( 1,0) node[below] 1 -- (1,1.5) circle (1.5pt); filldraw[black] ( 2,0) node[below] 2 -- (2,3) circle (1.5pt); filldraw[black] ( 3,0) node[below] 3 -- (3,2) circle (1.5pt); draw(1,1) node[cross=10pt,red!50!black] ;
```

Στοχαστικό Περιέχει και τις τρεις κατηγορίες

Σύστημα

```
[scale=0.8] draw(-1,1) rectangle (3,-1) node[midway] system;

draw[->] (-5,0) -- (-1,0) node[midway,above] x(t) node[midway,below] input signal; draw[->] (3,0) -- (7,0) node[midway,above] y(t) node[midway,below] output signal;
```

Περιοδικά σήματα

Αν $\exists T \in \mathbb{R} : \forall t \in \mathbb{R} \quad x(t) = x(t+T)$ τότε $x(t)$ **περιοδικό σήμα** με περίοδο T.

Ή θα είναι 0, ή θα συνεχιστεί για πάντα.

$$\int_{-T/2}^{T/2} x(t)\,\mathrm{d}t = \int_{t_0-T/2}^{t_0+T/2} x(t)\,\mathrm{d}t\ \forall t$$

Η σύνθεση μιας συνάρτησης με μια περιοδική συνάρτηση είναι περιοδική;

Απόδ. Έστω g μία περιοδική συνάρτηση:

$$\begin{aligned}(f \circ g)(x) &= f\big(g(x)\big) = f\big(g(x+T)\big) = \\ &= \big(f \circ g\big)(x+T)\end{aligned}$$

Συμμετρίες

- Αν $x(t) = x(-t)\ \forall t$ τότε η $x(t)$ λέγεται **άρτια συνάρτηση** (even function).
- Αν $x(t) = -x(t)\ \forall t$ τότε η $x(t)$ λέγεται **περιττή συνάρτηση** (odd function).

$$\forall x(t) \quad \exists\, x_0(t), x_e(t) : x(t) = x_e(t) + x_0(t)$$

Απόδ.

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$
$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$\underbrace{x_{\ e}}_{\text{άρτια}} y_e = z_e$$
$$x_o y_o = z_e$$
$$x_e y_0 = z_0$$
$$\int_{-A}^{A} x_0(t)\,\mathrm{d}t = 0$$
$$\int_{-\infty}^{\infty} x_0(t)\,\mathrm{d}t = ?\ (\text{εξαρτάται})$$
$$\lim_{A\to\infty} \int_{-A}^{A} x_0(t)\,\mathrm{d}t = 0 \quad (\text{principal Cauchy value})$$

Χαρακτηριστικά σήματα

1) **Εκθετικό σήμα**

$$x(t) = ce^{at} \quad a \in \mathbb{R} \quad c > 0$$

draw[->,gray] (0,0) -- (0,4); *draw*[->,gray] (-2,0) -- (4,0) node[below right] t;
draw(1,3) node $c > 0$;

draw[xscale=3,domain=-0.7:1.3,smooth,variable=x,blue] plot (x,exp(x)); *draw*[xscale=3,domain=1.3:-1.2,dashed,smooth,variable=x,blue] plot (x,exp(-x)); *draw*(4,1) node[anchor=north east,blue] $a < 0$; *draw*(4.5,3) node[anchor=north east,blue] $a > 0$;

$$x(t) = ce^{(\sigma t + j\omega)t} = ce^{\sigma t} e^{j\omega t} = ce^{\sigma t}\left[\cos(\omega t) + j\sin(\omega t)\right]$$

2) **(Συν)ημιτονοειδή σήματα**

$$x(t) = A\cos(\omega t \pm \phi) = a\mathrm{Re}\left\{e^{j(\omega t + \phi)}\right\} = A\frac{e^{j(\omega t \pm \phi)} + e^{-j(\omega t \pm \phi)}}{2}$$

[scale=1.3] *draw*[->,gray] (0,-2) -- (0,2); *draw*[->,gray] (-2,0) -- (4,0) node[below right] t;

draw[scale=1,domain=0:4,samples=200,smooth,variable=x,blue,thick] plot (x,sin((xr)*20)*exp(-x));

3) **Δέλτα Dirac $\delta(t)$**

[scale=1.3] *draw*[->,gray] (0,-2) -- (0,2) node[black,below right] $\delta(t)$; *draw*[->,gray] (-1,0) -- (1,0) node[below right] t; *draw*(0,0) node[below left] 0;

draw[very thick,blue,->] (0,0) -- (0,1); [scale=1.3] *filldraw*[fill=green!20] (-0.5,0) rectangle (0.5,1) node[above right] $1/T$ node[midway,right] 1;

draw[->,gray] (0,-2) -- (0,2); *draw*[->,gray] (-1.5,0) -- (2,0);

draw(current bounding box.east) node[above] $\lim_{T\to 0} P_T(t) = \delta(t)$; [scale=1.3] *filldraw*[scale=1,domain=-2:2,samples=200,smooth,variable=x,fill=green!20] plot (x,exp(-x*x)) node[above right] $G_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{\sigma^2}}$ node[midway,above left] 1;

draw[->,gray] (0,-2) -- (0,2); *draw*[->,gray] (-2,0) -- (2,0);

draw[<->,gray] (0.7,exp(-pow(0.7,2))) -- (-0.7,exp(-pow(0.7,2))) node[above left] σ;

draw(current bounding box.east) node[below left] $\lim_{T\to 0} G_\sigma(t) = \delta(t)$;

Ορ.

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)\ \forall f(t)$$

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}t = f(\tau)$$

$$\boxed{\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau = f(t)}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}\tau = f(t)$$

Ιδιότητες της $\delta(t)$

1. **Κλιμάκωση**

$$a \in \mathbb{R} : \delta(at) = \frac{1}{|a|}\delta(t)$$

Απόδ.

$$\underbrace{\int_{-\infty}^{\infty} \phi(t)\boxed{\delta(at)}\,\mathrm{d}t}_{\substack{at=\xi \\ \mathrm{d}t = \frac{\mathrm{d}\xi}{a}}} = \int_{-\infty(a)}^{\infty(a)} \phi\left(\frac{\xi}{a}\right)\delta(\xi)\frac{\mathrm{d}\xi}{a} = \frac{1}{|a|}\int_{-\infty}^{\infty}\frac{\phi\left(\frac{\xi}{a}\right)}{|a|}\delta(\xi)\,\mathrm{d}\xi = \frac{\phi(0)}{|a|} = \int_{-\infty}^{\infty}\phi(t)\boxed{\frac{\delta(t)}{|a|}}\mathrm{d}t$$

2. $f(t)\delta(t) = f(0)\delta(t)$
3. $f(t)\delta(t-\xi) = f(g)\delta(t-\xi)$

```
[scale=0.8] draw(-1,1) rectangle (3,-1) node[midway] system;
draw[->] (-5,0) -- (-1,0) node[midway,above] x(t) node[midway,below] input signal; draw[->] (3,0) -- (7,0) node[midway,above] y(t) node[midway,below]
output signal;
```

$y(t) = \mathscr{L}\left\{x(t)\right\}$
$\forall x_1(t)\ x_n(t)$
$y_1(t) = \mathscr{L}\left\{x_1(t)\right\}$
$y_2(t) = \mathscr{L}\left\{x_2(t)\right\}$

Για const $a_1, a_2 \in \mathbb{R}$

$$x(t) = a_1x_1(t) + a_2x_2(t)$$
$$y(t) = \mathscr{L}\left\{x(t)\right\}$$

ανν

$$y(t) = a_1y_1(t) + a_2y_2(t)$$

τότε

$$\mathscr{L} : \text{γραμμικό σύστημα}$$

- $g(t) = \mathscr{L}\left\{x(t)\right\}$
  $x'(t) = x(t-\tau)$
  ανν $y'(t) = \mathscr{L}\left\{x'(t)\right\} = \mathscr{L}\left\{x(t-\tau)^2\right\} = y(t-\tau)$
  τότε το σύστημα $\mathscr{L}$ είναι αμετάβλητο κατά τη μετατόπιση.

```
[scale=0.8] draw(-1,-1) rectangle (3,1) node[above right] ℒ {} node[midway]      & AKM;
draw[->] (-5,0) -- (-1,0) node[midway,above] δ(t) node[midway,below] input signal; draw[->] (3,0) -- (14,0) node[midway,above] h(t) node[midway,below]
(impulse response);
```

Υποστηρίζω ότι ένα γραμμικό & ΑΚΜ σύστημα περιγράφεται πλήρως από την κρουστική απόκριση $h(t)$.

Απόδ. Από παραπάνω, γνωρίζουμε ότι $x(t) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}t$

$$
\begin{aligned}
y(t) &= \mathscr{L}\left\{y(t)\right\} = \mathscr{L}\left\{\int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}\tau\right\} \\
&\overset{\text{linearity}}{=} \int_{-\infty}^{\infty} \mathscr{L}\left\{x(\tau)\delta(t-\tau)\right\} \\
&= \int_{-\infty}^{\infty} x(\tau)\mathscr{L}\left\{\delta(t-\tau)\right\}\mathrm{d}\tau \\
&\overset{\text{AKM}}{\underset{TSI}{=}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau \\
y(t) &= \int_{-\infty}^{\infty} x(\tau)\underbrace{h(t-\tau)}_{\text{linear time-shift invariant}}\mathrm{d}\tau
\end{aligned}
$$

- $\delta(t) = \delta(-t)$ άρτια συνάρτηση
- $\delta^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\delta(t)$, για την οποία αποδεικνύεται ότι:

$$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)\,\mathrm{d}t = (-1)^n\ \phi^{(n)}(t)\Big|_{t=0}$$

Βηματική Συνάρτηση (Unit Step Function)

$$\mathrm{u}(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \mathrm{u}(t)\phi(t)\,\mathrm{d}t = \mathscr{N}_{\mathrm{u}}\left\{\phi(t)\right\} = \int_{0}^{\infty} \underbrace{\phi(t)}_{\text{number}}\mathrm{d}t$$

```
[xlabel=t ,ylabel=u(t) ,axis lines = center ,ymax=1.5 ,ytick=0,1 ,xtick=0,1 ] addplot+[const plot, no marks,ultra thick] coordinates (-4,0) (0,1)
(4,1);
```

$$
\begin{aligned}
\delta(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{u}(t) \\
\mathrm{u}(t) &= \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \int_{0}^{\infty} \delta(t-\xi)\,\mathrm{d}\xi
\end{aligned}
$$

Ράμπα

$$\mathrm{r}(t) = \int_{-\infty}^{t} \mathrm{u}(\tau)\,\mathrm{d}\tau = \begin{cases} t & t \geq 0 \\ 0 & \text{else} \end{cases} = t\mathrm{u}(t)$$

```
draw[->] (0,-0.5) -- (0,4) node[right] r(t); draw[->] (-2,0) -- (4,0) node[below] t;
draw(0,0) node[below right] 0;
draw[very thick,blue] (-2,0) -- (0,0) -- (3.5,3.5);
```

$$\mathrm{u}(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{r}(t)$$

Ορθογωνικός παλμός (Rectangular Pulse function)

$$\mathrm{p}_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$

```
draw[->] (0,-0.5) -- (0,3) node[right] r(t); draw[->] (-2,0) -- (2,0) node[below] t;

draw[very thick,blue] (-1,0) node[below,black] −a -- (-1,1.5) node[above left, black] 1 -- (1,1.5) -- (1,0) node[below,black] a;

draw[->] (0,-0.5) -- (0,2); draw[->] (-2,0) -- (3,0) node[below] t;
draw[very thick,blue] (-3,0) -- (-1,0) node[below,black] −a -- (-1,1) -- (3,1) node[above right] u(t + a);     draw[->] (0,-0.5) -- (0,2);
draw[->] (-2,0) -- (3,0) node[below] t;

draw[very thick,blue] (-3,0) -- (1,0) node[below,black] a -- (1,1) -- (3,1) node[above right] u(t − a);
```

$$\begin{aligned} \mathrm{p}_a(t) &= \mathrm{u}(t+a) - \mathrm{u}(t-a) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{p}_a(t) &= \delta(t+a) - \delta(t-a) \end{aligned}$$

```
draw[->] (0,-0.5) -- (0,3) node[right] d/dt p_a(t); draw[->] (-2,0) -- (2,0) node[below] t;
draw[very thick,blue,->] (-1,0) node[below,black] −a -- (-1,1.5) node[above] δ(t + a);

draw[very thick,blue,->] (1,0) node[above,black] a -- (1,-1.5) node[below] −δ(t − a);
```

Τριγωνικός Παλμός (Triangular Pulse function)

$$\mathrm{p}_{\mathrm{tr},a} = \begin{cases} 1 - \frac{|t|}{a} & |t| < a \\ 0 & |t| > a \end{cases}$$

```
[scale=1.2] draw[->] (0,-0.5) -- (0,2) node[right] p_tr,a(t); draw[->] (-2,0) -- (2,0) node[below] t;

draw[very thick,blue] (-2,0) -- (-1,0) node[below,black] −a -- (0,1.5) node[above left, black] 1 -- (1,0) node[below,black] a -- (2,0);
```

$$p_{\mathrm{tr},a}(t) = \frac{1}{a}\left[\mathrm{r}(t+a) + \mathrm{r}(t-a) - 2\mathrm{r}(t)\right]$$

Χαρακτηριστικά Μεγέθη

1) **Μέση τιμή (Mean Value)**

$$\overline{x(t)} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x(t)\,\mathrm{d}t$$

Αν περιοδική τότε

$$\begin{aligned} \bar{x}(t) &= \frac{1}{T} = \int_0^T x(t)\,\mathrm{d}t \\ &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t \end{aligned}$$

2) **Ενεργός τιμή (Root Mean Square Value)**

$$\overline{\overline{x(t)}} = \left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t\right]^{1/2}$$

Αν ημιτονοειδές σήμα $\bar{\bar{x}}(t) = \frac{x_{\max}}{\sqrt{2}}$

3) **Ενέργεια - Ισχύς**

- Στιγμιαία ισχύς (Instant power)

$$p(t) = x^2(t)$$

- Μέση ισχύς (Mean power)

$$\overline{p(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t = \left(\overline{\overline{x(t)}}\right)^2$$

- Ενέργεια (Energy)

$$W = \int_{t_1}^{t_2} p(t)\,\mathrm{d}t = \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t = (t_2 - t_1)\left(\overline{\overline{x(t)}}\right)^2$$

$$\textbf{Σήματα} \begin{cases} \textbf{Σήμα ενέργειας} \text{ αν } \lim\limits_{T\to\infty} W < \infty \\ \\ \textbf{Σήμα ισχύος} \text{ αν } \lim\limits_{T\to\infty} \overline{p(t)} > 0 \\ \text{Υπάρχουν και σήματα που δεν είναι ούτε ενέργειας, ούτε ισχύος.} \end{cases}$$

Συνέλιξη

$$x(t) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau$$

$$h(t) = \mathscr{L}\left\{\delta(t)\right\}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \underbrace{x(t)}_{\text{είσοδος}} \underbrace{\ast}_{\text{συνέλιξη}} \underbrace{h(t)}_{\text{κρουστική απόκριση}}$$

Συνέλιξη - Convolution

$$z(t) = x(t) \ast y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau$$

- $x(t) \ast y(t) = y(t) \ast x(t)$ **Αντιμεταθετική**

$$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\lambda)y(\lambda)[-\,\mathrm{d}\lambda] = \int_{-\infty}^{\infty} y(\lambda)x(t-\lambda)\,\mathrm{d}\lambda = y(t) \ast x(t)$$

- $x_1(t) \ast \left[x_2(t) \ast x_3(t)\right] = \left[x_1(t) \ast x_2(t)\right] \ast x_3(t)$ **Προσεταιριστική**

Παρ.

$$f_1(t) = 2(1-t)\left[\mathrm{u}(t) - u(t-1)\right]$$
$$f_2(t) = \mathrm{u}(t) - \mathrm{u}(t-\tau)$$

Γραφική μέθοδος υπολογισμού συνέλιξης

Παρατηρώ ότι:

$$\int_{-\infty}^{\infty} f(t,\tau)\mathrm{u}(t-\xi)\mathrm{u}(\phi-\tau)\,\mathrm{d}\tau = \int_{\xi}^{\phi} f(t,\tau)\,\mathrm{d}\tau\mathrm{u}(\phi-\xi)$$

$$\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} \underbrace{2(1-\tau)}_{x(\tau)} \big[\mathrm{u}(\tau)-\mathrm{u}(\tau+1)\big]\big[\mathrm{u}(t-\tau)-\mathrm{u}(t-\tau-2)\big]\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\big[\mathrm{u}(\tau)\mathrm{u}(t-\tau)-\mathrm{u}(\tau-1)\mathrm{u}(t-\tau)-\mathrm{u}(\tau)\mathrm{u}(t-\tau-2)+\mathrm{u}(\tau-1)\mathrm{u}(t-\tau-2)\big]\,\mathrm{d}\tau \\
&= \int_{0}^{t} x(\tau)\,\mathrm{d}\tau\mathrm{u}(t) - \int_{1}^{x} x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-1) - \int_{0}^{t-2} x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-2) + \int_{1}^{t-2} x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-3) \\
&= (2t-t^2)\mathrm{u}(t) - \Big[2t-t^2-1\Big]\mathrm{u}(t-1) - \Big[2(t-2)-(t-2)^2\Big]\mathrm{u}(t-2) + \Big[2(t-2)-(t-2)^2-1\Big]\mathrm{u}(t-
\end{aligned}$$

Ex

$$\begin{aligned}
f_1(t) &= e^t\mathrm{u}(-t) \\
f_2(t) &= \mathrm{u}(t+2)-u(t+1) \\
f &= f_1 * f_2
\end{aligned}$$

$$\begin{aligned}
f &= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}\big(-(t-\tau)+2\big)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(\tau-t+2)\,\mathrm{d}\tau \\
&= \int_{t-2}^{0} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t-2) \\
&= e^{\tau}\big|_{t-2}^{0}\,\mathrm{u}(2-t) \\
&= \Big[1-e^{t-2}\Big]\mathrm{u}(2-t)
\end{aligned}$$