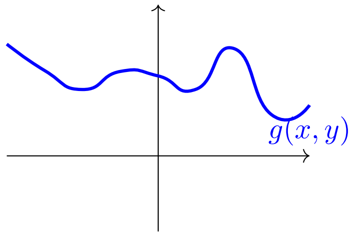
Την Τρίτη μάθημα 8:30 χωρίς διάλειμμα

Σήμα - σύστημα

$$\boxed{\underbrace{g}_{\text{εξαρτημένη}} = f(\underbrace{t}_{\text{ανεξάρτητη}})} \qquad g = f(\vec{r}, t) \qquad \vec{E}(\vec{r}, t)$$

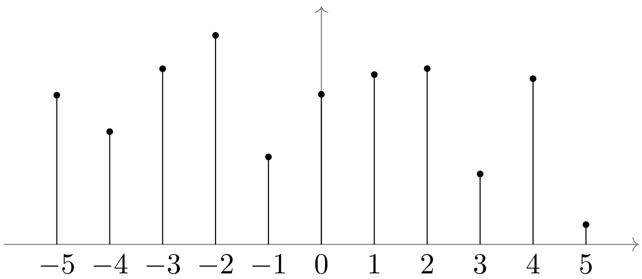
Αναλογικό

Αν t συνεχής $\in \mathbb{R}$
και y συνεχής $\in \mathbb{R}$



Διακριτού χρόνου / Διακριτό (discrete)

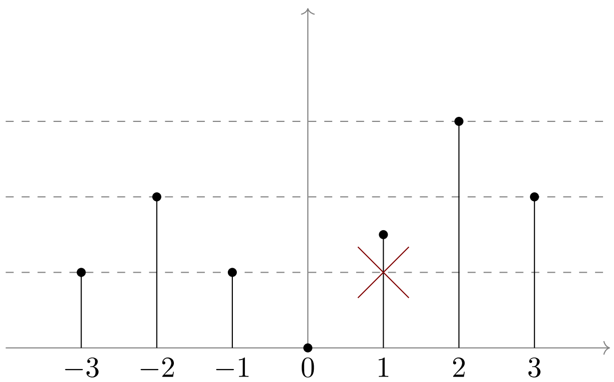
t διακριτό $\to \mathbb{Z},\ n \in \mathbb{Z}$
g συνεχής $\in \mathbb{R}$



7ο εξάμηνο

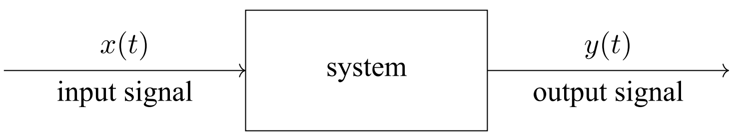
Κβαντισμένο

$n \in \mathbb{Z}$
g διακριτή



Στοχαστικό Περιέχει και τις τρεις κατηγορίες

0.1 Σύστημα



0.2 Περιοδικά σήματα

Αν $\exists T \in \mathbb{R} : \forall t \in \mathbb{R} \quad x(t) = x(t+T)$ τότε $x(t)$ **περιοδικό σήμα** με περίοδο T.
Ή θα είναι 0, ή θα συνεχιστεί για πάντα.

$$\int_{-T/2}^{T/2} x(t)\,\mathrm{d}t = \int_{t_0-T/2}^{t_0+T/2} x(t)\,\mathrm{d}t\ \forall t$$

Η σύνθεση μιας συνάρτησης με μια περιοδική συνάρτηση είναι περιοδική;

Απόδ. Έστω g μία περιοδική συνάρτηση:

$$\begin{aligned}\big(f \circ g\big)(x) &= f\,\big(g(x)\big) = f\,\big(g(x+T)\big) = \\ &= \big(f \circ g\big)(x+T)\end{aligned}$$

0.3 Συμμετρίες

- Αν $x(t) = x(-t)\ \forall t$ τότε η $x(t)$ λέγεται **άρτια συνάρτηση** (even function).
- Αν $x(t) = -x(t)\ \forall t$ τότε η $x(t)$ λέγεται **περιττή συνάρτηση** (odd function).

$\forall x(t) \quad \exists\, x_0(t), x_e(t) : x(t) = x_e(t) + x_0(t)$

Απόδ.

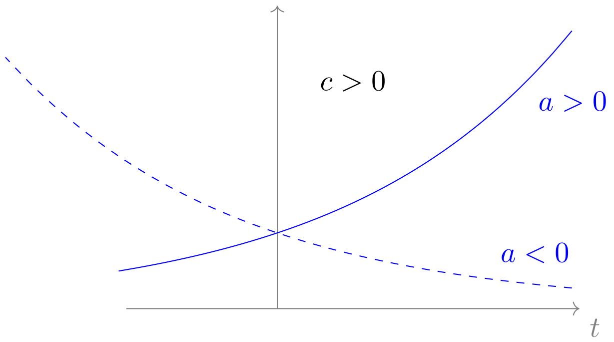
$$\begin{aligned}x_e(t) &= \frac{x(t)+x(-t)}{2} \\ x_o(t) &= \frac{x(t)-x(-t)}{2}\end{aligned}$$

$$\begin{aligned}x\underbrace{\ {}_e\ }_{\text{άρτια}} y_e &= z_e \\ x_o y_o &= z_e \\ x_e y_0 &= z_0 \\ \int_{-A}^{A} x_0(t)\,\mathrm{d}t &= 0 \\ \int_{-\infty}^{\infty} x_0(t)\,\mathrm{d}t &=?\ \text{(εξαρτάται)} \\ \lim_{A\to\infty}\int_{-A}^{A} x_0(t)\,\mathrm{d}t &= 0 \quad \text{(principal Cauchy value)}\end{aligned}$$

Χαρακτηριστικά σήματα

1) **Εκθετικό σήμα**

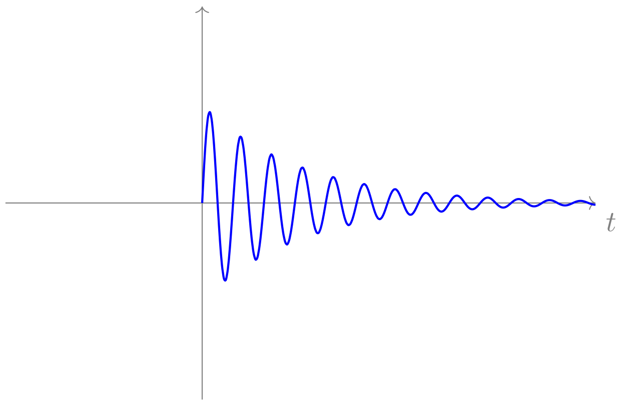
$$x(t) = ce^{at} \quad a \in \mathbb{R} \quad c > 0$$



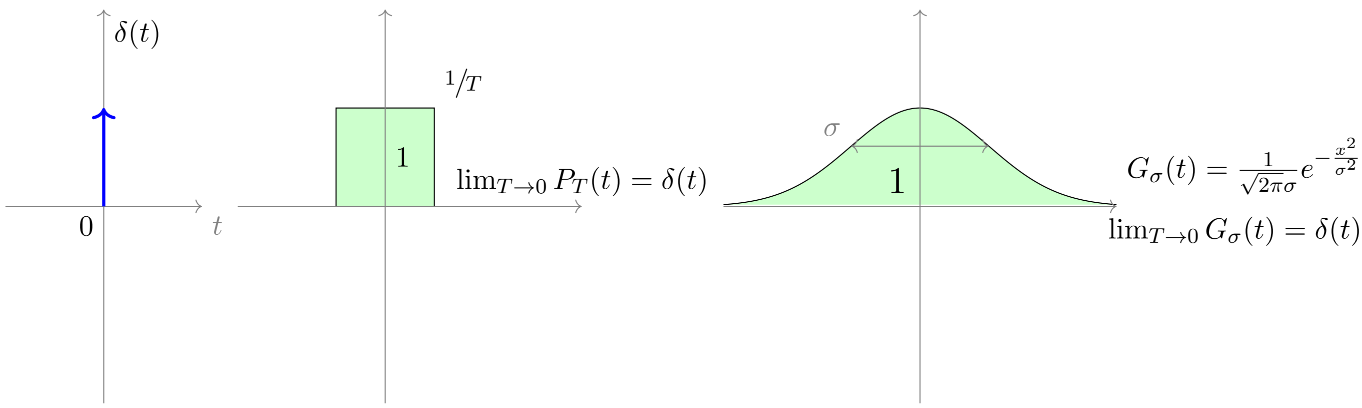
$$x(t) = ce^{(\sigma t + j\omega)t} = ce^{\sigma t}e^{j\omega t} = ce^{\sigma t}\left[\cos(\omega t) + j\sin(\omega t)\right]$$

2) **(Συν)ημιτονοειδή σήματα**

$$x(t) = A\cos(\omega t \pm \phi) = a\mathrm{Re}\left\{e^{j(\omega t + \phi)}\right\} = A\frac{e^{j(\omega t \pm \phi)} + e^{-j(\omega t \pm \phi)}}{2}$$



3) **Δέλτα Dirac $\delta(t)$**



Ορ.

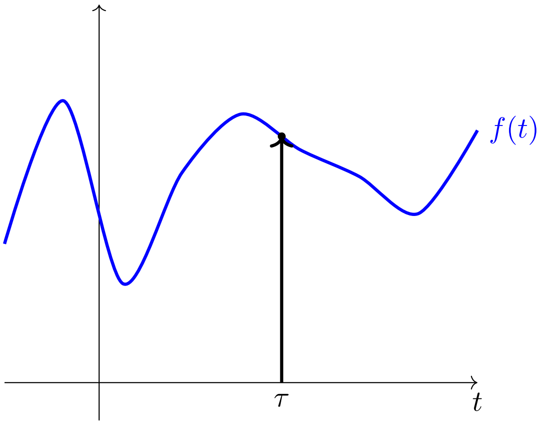
$$\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)\ \forall f(t)$$

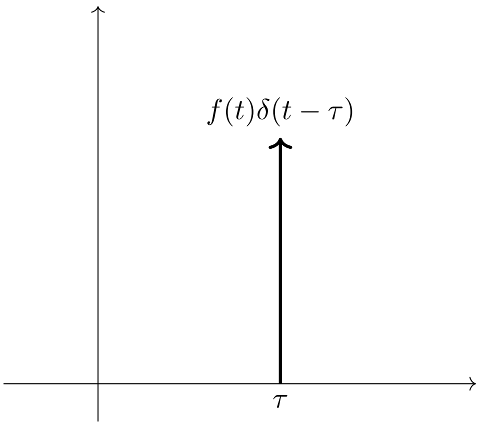
$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}t = f(\tau)$$

$$\boxed{\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau = f(t)}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}\tau = f(t)$$





Ιδιότητες της $\delta(t)$

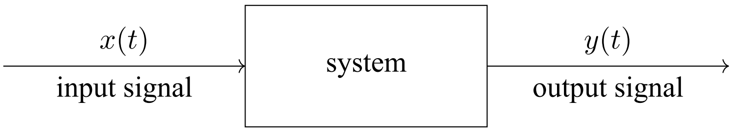
1. **Κλιμάκωση**

$$a \in \mathbb{R} : \delta(at) = \frac{1}{|a|}\delta(t)$$

Απόδ.

$$\underbrace{\int_{-\infty}^{\infty} \phi(t)\boxed{\delta(at)}\,\mathrm{d}t}_{\substack{at=\xi \\ \mathrm{d}t = \frac{\mathrm{d}\xi}{a}}} = \int_{-\infty_{(a)}}^{\infty_{(a)}} \phi\left(\frac{\xi}{a}\right)\delta(\xi)\frac{\mathrm{d}\xi}{a} = \frac{1}{|a|}\int_{-\infty}^{\infty} \frac{\phi\left(\frac{\xi}{a}\right)}{|a|}\delta(\xi)\,\mathrm{d}\xi = \frac{\phi(0)}{|a|} = \int_{-\infty}^{\infty} \phi(t)\boxed{\frac{\delta(t)}{|a|}}\mathrm{d}t$$

2. $f(t)\delta(t) = f(0)\delta(t)$
3. $f(t)\delta(t-\xi) = f(g)\delta(t-\xi)$



$y(t) = \mathscr{L}\left\{x(t)\right\}$
$\forall x_1(t)\ x_n(t)$
$y_1(t) = \mathscr{L}\left\{x_1(t)\right\}$
$y_2(t) = \mathscr{L}\left\{x_2(t)\right\}$

Για const $a_1, a_2 \in \mathbb{R}$

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$
$$y(t) = \mathscr{L}\left\{x(t)\right\}$$

ανν

$$y(t) = a_1 y_1(t) + a_2 y_2(t)$$

τότε

$$\mathscr{L}: \text{ γραμμικό σύστημα}$$

- $g(t) = \mathscr{L}\left\{x(t)\right\}$
  $x'(t) = x(t-\tau)$
  ανν $y'(t) = \mathscr{L}\left\{x'(t)\right\} = \mathscr{L}\left\{x(t-\tau)^2\right\} = y(t-\tau)$
  τότε το σύστημα $\mathscr{L}$ είναι αμετάβλητο κατά τη μετατόπιση.

Υποστηρίζω ότι ένα γραμμικό & ΑΚΜ σύστημα περιγράφεται πλήρως από την κρουστική απόκριση $h(t)$.

Απόδ. Από παραπάνω, γνωρίζουμε ότι $x(t) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}t$

$$\begin{aligned}
y(t) &= \mathscr{L}\left\{y(t)\right\} = \mathscr{L}\left\{\int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}\tau\right\} \\
&\overset{\text{linearity}}{=} \int_{-\infty}^{\infty} \mathscr{L}\left\{x(\tau)\delta(t-\tau)\right\} \\
&= \int_{-\infty}^{\infty} x(\tau)\mathscr{L}\left\{\delta(t-\tau)\right\}\mathrm{d}\tau \\
&\underset{TSI}{\overset{\text{AKM}}{=}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau \\
y(t) &= \int_{-\infty}^{\infty} x(\tau)\underbrace{h(t-\tau)}_{\text{linear time-shift invariant}}\mathrm{d}\tau
\end{aligned}$$

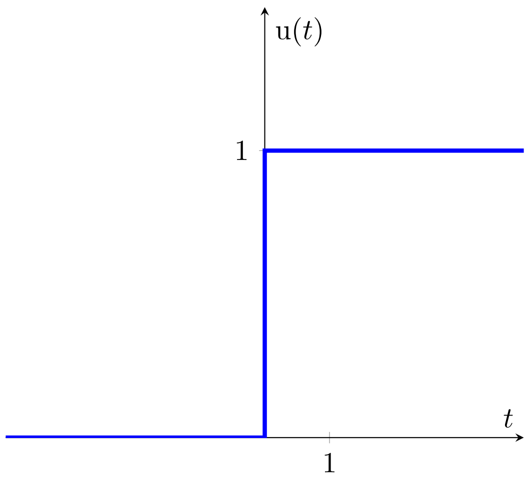
- $\delta(t) = \delta(-t)$ άρτια συνάρτηση
- $\delta^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\delta(t)$, για την οποία αποδεικνύεται ότι:

$$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)\,\mathrm{d}t = (-1)^n\ \phi^{(n)}(t)\Big|_{t=0}$$

0.3.1 Βηματική Συνάρτηση (Unit Step Function)

$$\mathrm{u}(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \mathrm{u}(t)\phi(t)\,\mathrm{d}t = \mathscr{N}_{\mathrm{u}}\left\{\phi(t)\right\} = \int_{0}^{\infty} \underbrace{\phi(t)}_{\text{number}}\,\mathrm{d}t$$

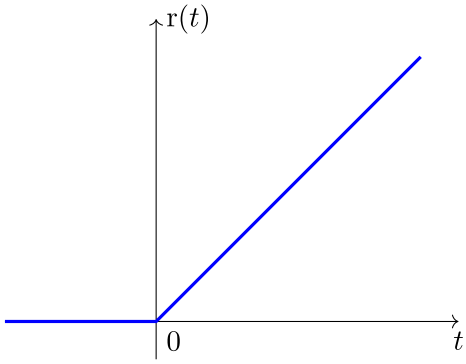


$$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{u}(t)$$

$$\mathrm{u}(t) = \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \int_{0}^{\infty} \delta(t-\xi)\,\mathrm{d}\xi$$

0.3.2 Ράμπα

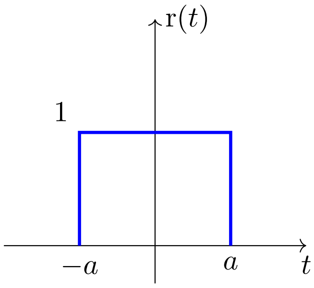
$$\mathrm{r}(t) = \int_{-\infty}^{t} \mathrm{u}(\tau)\,\mathrm{d}\tau = \begin{cases} t & t \geq 0 \\ 0 & \text{else} \end{cases} = t\mathrm{u}(t)$$

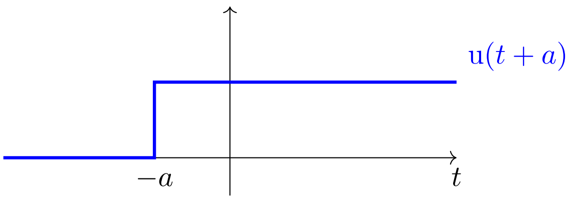


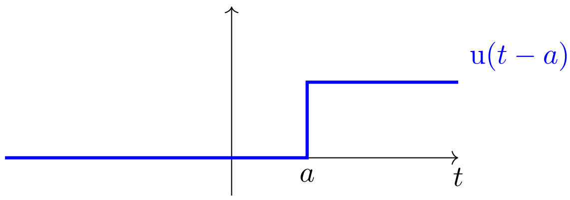
$$\mathrm{u}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{r}(t)$$

0.3.3 Ορθογωνικός παλμός (Rectangular Pulse function)

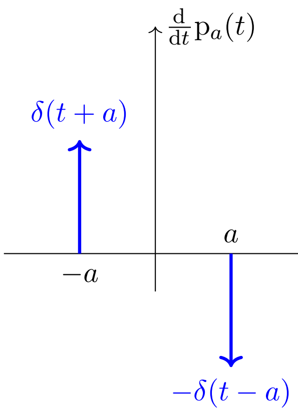
$$\mathrm{p}_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$





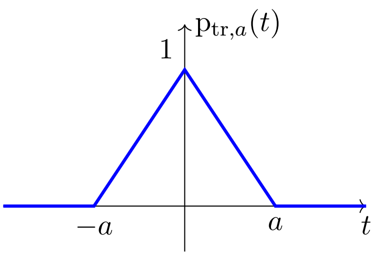


$$\begin{aligned} \mathrm{p}_a(t) &= \mathrm{u}(t+a) - \mathrm{u}(t-a) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{p}_a(t) &= \delta(t+a) - \delta(t-a) \end{aligned}$$

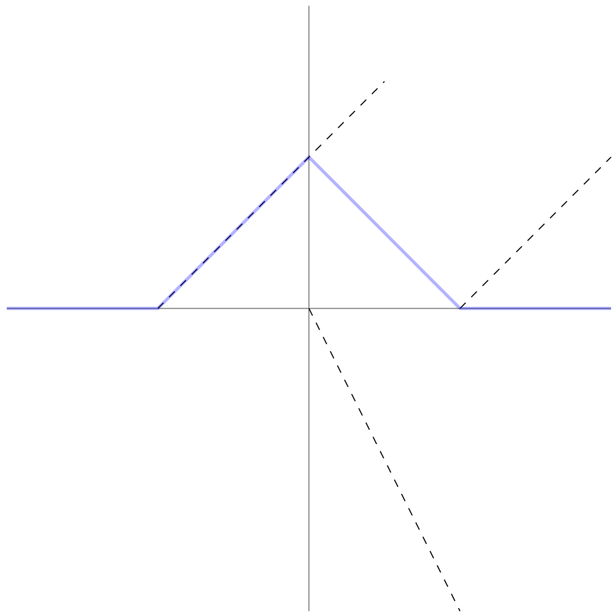


0.3.4 Τριγωνικός Παλμός (Triangular Pulse function)

$$\mathrm{p}_{\mathrm{tr},a} = \begin{cases} 1 - \frac{|t|}{a} & |t| < a \\ 0 & |t| > a \end{cases}$$



$$p_{\mathrm{tr},a}(t) = \frac{1}{a}\left[\mathrm{r}(t+a) + \mathrm{r}(t-a) - 2\mathrm{r}(t)\right]$$



0.4 Χαρακτηριστικά Μεγέθη

1) **Μέση τιμή (Mean Value)**

$$\overline{x(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t)\,\mathrm{d}t$$

Αν περιοδική τότε

$$\begin{aligned} \bar{x}(t) &= \frac{1}{T} = \int_{0}^{T} x(t)\,\mathrm{d}t \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t \end{aligned}$$

2) **Ενεργός τιμή (Root Mean Square Value)**

$$\overline{\overline{x(t)}} = \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t \right]^{1/2}$$

Αν ημιτονοειδές σήμα $\bar{\bar{x}}(t) = \frac{x_{\max}}{\sqrt{2}}$

3) **Ενέργεια - Ισχύς**

- Στιγμιαία ισχύς (Instant power)

$$p(t) = x^2(t)$$

- Μέση ισχύς (Mean power)

$$\overline{p(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t = \left(\overline{\overline{x(t)}}\right)^2$$

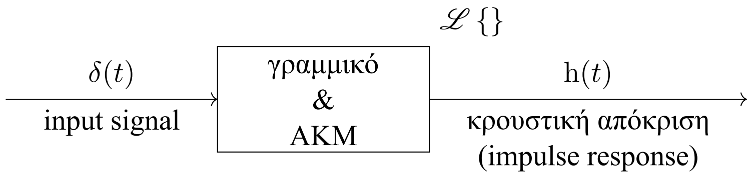
- Ενέργεια (Energy)

$$W = \int_{t_1}^{t_2} p(t)\,\mathrm{d}t = \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t = (t_2 - t_1)\left(\overline{\overline{x(t)}}\right)^2$$

$$\textbf{Σήματα} \begin{cases} \textbf{Σήμα ενέργειας} \text{ αν } \lim\limits_{T\to\infty} W < \infty \\ \\ \textbf{Σήμα ισχύος} \text{ αν } \lim\limits_{T\to\infty} \overline{p(t)} > 0 \\ \text{Υπάρχουν και σήματα που δεν είναι ούτε ενέργειας, ούτε ισχύος.} \end{cases}$$

0.5 Συνέλιξη

$$x(t) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau$$



$$h(t) = \mathscr{L}\left\{\delta(t)\right\}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \underbrace{x(t)}_{\text{είσοδος}}\ \underbrace{*}_{\text{συνέλιξη}}\ \underbrace{h(t)}_{\text{κρουστική απόκριση}}$$

Συνέλιξη - Convolution

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau$$

- $x(t) * y(t) = y(t) * x(t)$ **Αντιμεταθετική**

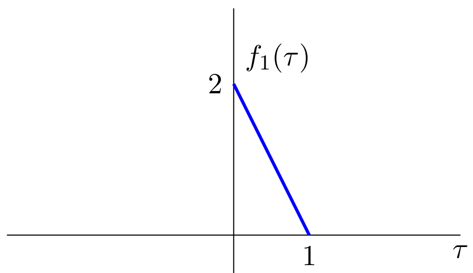
$$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\lambda)y(\lambda)[-\,\mathrm{d}\lambda] = \int_{-\infty}^{\infty} y(\lambda)x(t-\lambda)\,\mathrm{d}\lambda = y(t) * x(t)$$

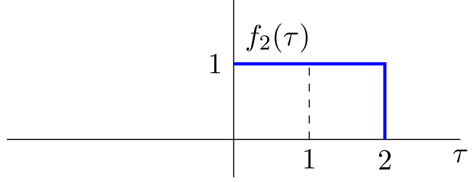
- $x_1(t) * \left[x_2(t) * x_3(t)\right] = \left[x_1(t) * x_2(t)\right] * x_3(t)$ **Προσεταιριστική**

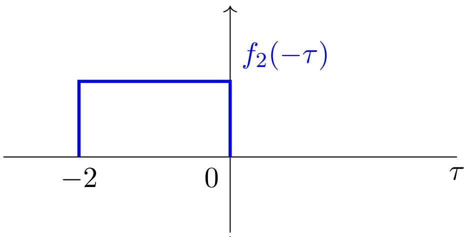
Παρ.

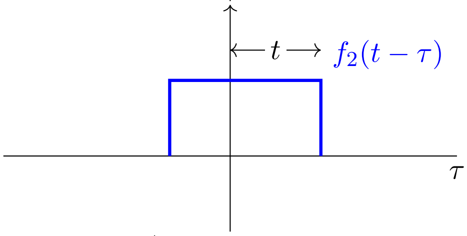
$$f_1(t) = 2(1-t)\left[\mathrm{u}(t) - u(t-1)\right]$$
$$f_2(t) = \mathrm{u}(t) - \mathrm{u}(t-\tau)$$

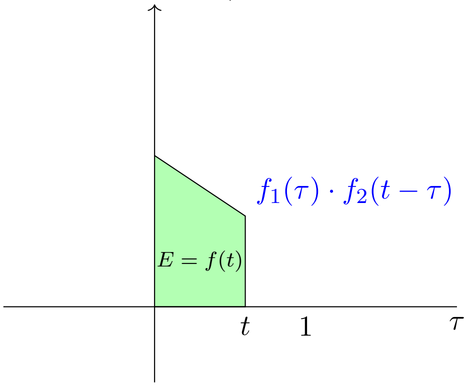
Γραφική μέθοδος υπολογισμού συνέλιξης

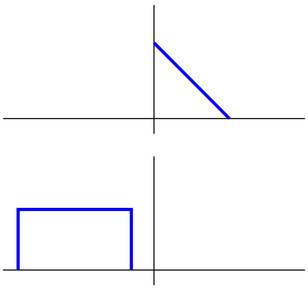
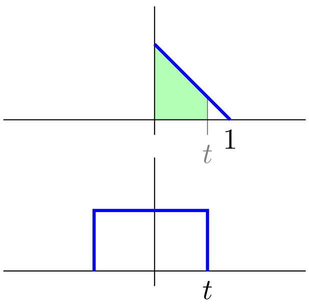


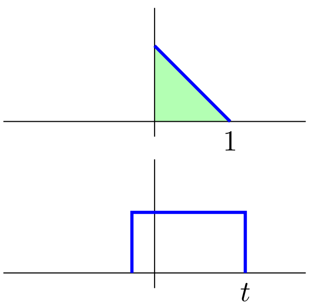






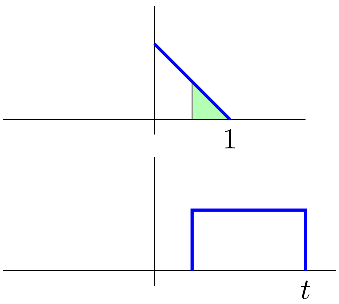


- $t < 0$  $f(t) = 0$
- $0 < t < 1$ 

$f(t) = \frac{1}{2}(2 + 2 - 2t)t = (2 - t)t$
- $1 < t < 2$ 

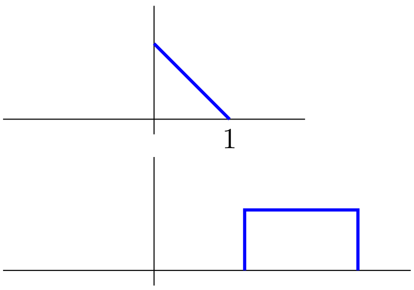
$f(t) = 1$

- $2 < t < 3$

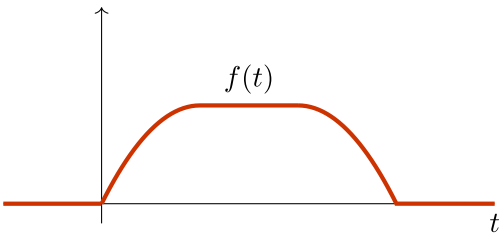


$$f(t) = \frac{(t-1)\cdot 2\cdot\big(1-(t-2)\big)}{2} = (t-1)(3-t)$$

- $t > 3$



$$f(t) = 0$$



Αναλυτική μέθοδος Παρατηρώ ότι:

$$\int_{-\infty}^{\infty} f(t,\tau)\mathrm{u}(t-\xi)\mathrm{u}(\phi-\tau)\,\mathrm{d}\tau = \int_{\xi}^{\phi} f(t,\tau)\,\mathrm{d}\tau \mathrm{u}(\phi-\xi)$$

$$\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} \underbrace{2(1-\tau)}_{x(\tau)} \big[\mathrm{u}(\tau)-\mathrm{u}(\tau+1)\big]\,\big[\mathrm{u}(t-\tau)-\mathrm{u}(t-\tau-2)\big]\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\,\big[\mathrm{u}(\tau)\mathrm{u}(t-\tau)-\mathrm{u}(\tau-1)\mathrm{u}(t-\tau)-\mathrm{u}(\tau)\mathrm{u}(t-\tau-2)+\mathrm{u}(\tau-1)\mathrm{u}(t-\tau-2)\big]\,\mathrm{d}\tau \\
&= \int_{0}^{t} x(\tau)\,\mathrm{d}\tau \mathrm{u}(t) - \int_{1}^{x} x(\tau)\,\mathrm{d}\tau \mathrm{u}(t-1) - \int_{0}^{t-2} x(\tau)\,\mathrm{d}\tau \mathrm{u}(t-2) + \int_{1}^{t-2} x(\tau)\,\mathrm{d}\tau \mathrm{u}(t-3) \\
&= (2t-t^2)\mathrm{u}(t)-\big[2t-t^2-1\big]\mathrm{u}(t-1)-\big[2(t-2)-(t-2)^2\big]\mathrm{u}(t-2)+\big[2(t-2)-(t-2)^2-1\big]\mathrm{u}(t-3)
\end{aligned}$$

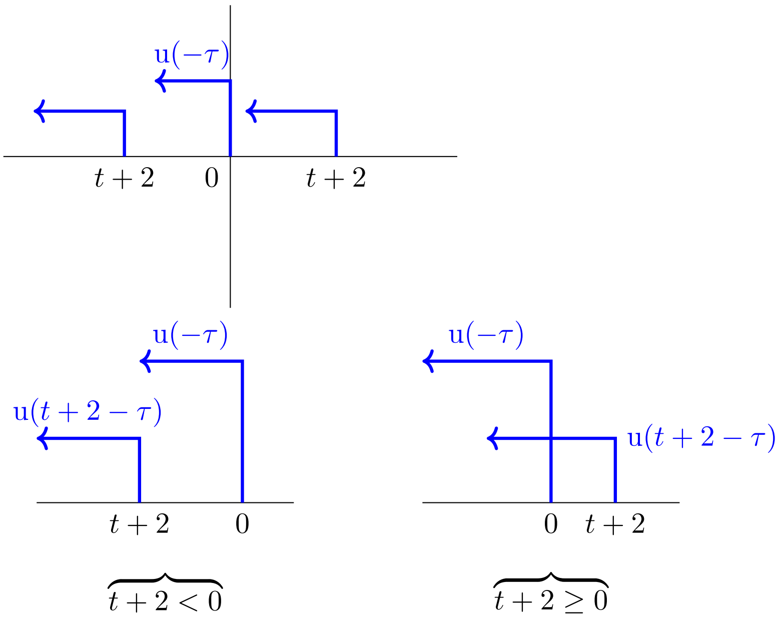
Ex

$$\begin{aligned}
f_1(t) &= e^t\mathrm{u}(-t) \\
f_2(t) &= \mathrm{u}(t+2) - u(t+1) \\
f &= f_1 * f_2
\end{aligned}$$

$$
\begin{aligned}
f &= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}\left(-(t-\tau)+2\right)\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(\tau-t+2)\,\mathrm{d}\tau \\
&= \int_{t-2}^{0} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t-2) \\
&= e^{\tau}\big|_{t-2}^{0}\,\mathrm{u}(2-t) \\
&= \left[1-e^{t-2}\right]\mathrm{u}(2-t)
\end{aligned}
$$

$$
\int_{-\infty}^{\infty} f(t,\tau)\mathrm{u}(\tau-\xi)\mathrm{u}(\phi-\tau)\,\mathrm{d}\tau = \int_{\xi}^{\phi} f(t,\tau)\,\mathrm{d}\tau\mathrm{u}(\phi-\xi)
$$

Ex.



$$
\begin{aligned}
x(t) &= e^{t}\mathrm{u}(-t) \\
y(t) &= \mathrm{u}(t+2) \\
z(t) &= x(t)*y(t) = \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}\left[(t-\tau)+2\right]\mathrm{d}\tau = \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(t+2-\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{\tau}\left[1-\mathrm{u}(t)\right]\mathrm{u}(t+2-\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(t+2-\tau)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(\tau)u(t+2-\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{t+2} e^{\tau}\,\mathrm{d}\tau\,\mathrm{u}\left(\cancelto{1}{t+2-(-\infty)}\right) - \int_{0}^{t+2} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t+2) \\
&= e^{t+2} - \left[e^{t+2}-1\right]\mathrm{u}(t+2)
\end{aligned}
$$

Ex.

$$
\begin{aligned}
x(t) &= e^t \mathrm{u}(-t)\\
y(t) &= \mathrm{u}(t+2) - \mathrm{u}(t+1)\\
z(t) &= x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\left[\mathrm{u}(t-\tau+2) - \mathrm{u}(t-\tau+1)\right]\mathrm{d}\tau\\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(t-\tau+2)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(-\tau)\mathrm{u}(t-\tau+1)\,\mathrm{d}\tau\\
&= \int_{-\infty}^{\infty} e^{\tau}\left[1-\mathrm{u}(\tau)\right]\mathrm{u}(t-\tau+2)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\left[1-\mathrm{u}(\tau)\right]\mathrm{u}(t-\tau+1)\,\mathrm{d}\tau\\
&= \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(t-\tau+2)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(\tau)\mathrm{u}(t-\tau+2)\,\mathrm{d}\tau - \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(t-\tau+1)\,\mathrm{d}\tau + \int_{-\infty}^{\infty} e^{\tau}\mathrm{u}(\tau)\mathrm{u}(t-\tau+1\\
&= \int_{-\infty}^{t+1} e^{\tau}\,\mathrm{d}\tau - \int_{-\infty}^{t+1} e^{\tau}\,\mathrm{d}\tau - \int_{0}^{t+2} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t+2) + \int_{0}^{t+1} e^{\tau}\,\mathrm{d}\tau\mathrm{u}(t+1)\\
&= \int_{t+1}^{t+2} e^{\tau}\,\mathrm{d}\tau - \left[e^{t+2}-1\right]\mathrm{u}(t+2) + \left[e^{t+1}-1\right]\mathrm{u}(t+1)
\end{aligned}
$$

$\exists h(t)$ ανν LTI

$$
\begin{aligned}
y(t) &= x(t) * h(t)\\
&= \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\,\mathrm{d}\tau\\
&= \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\,\mathrm{d}\tau
\end{aligned}
$$

Έστω ότι η $x(t) = e^{j\omega t}$

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(t) e^{j\omega(t-\tau)}\,\mathrm{d}\tau = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\,\mathrm{d}\tau\\
&= x(t)\underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\,\mathrm{d}\tau}_{h(t)\xrightarrow{FT}H(\omega)}
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t}\\
y(t) &= A_1 e^{j\omega_1 t} H(\omega_1) + A_2 e^{j\omega_2 t} H(\omega_2)
\end{aligned}
$$

Κεφάλαιο 1 Συναρτηστιακοί χώροι

Διανυσματικός χώρος S

$\bar{x}, \quad \bar{y} \quad S$

Εσωτερικό γινόμενο

$\langle \bar{x}, \bar{y} \rangle \in \mathbb{C}$

1) $\langle \bar{x}, \bar{y} \rangle = \langle \bar{y}, \bar{x} \rangle^*$

2) $c \langle \bar{x}, \bar{y} \rangle = \langle c\bar{x}, \bar{y} \rangle$

3) $\langle \bar{x} + \bar{y}, \bar{z} \rangle = \langle \bar{x}, \bar{z} \rangle + \langle \bar{y}, \bar{z} \rangle$

4) $\langle \bar{x}, \bar{x} \rangle \geq 0$ με $\langle \bar{x}, \bar{x} \rangle = 0$ ανν $\bar{x} = \bar{0}$

Νόρμα

$\bar{x} \in S$

$||\bar{x}|| \geq 0$

1) $||\bar{x}|| = 0$ ανν $\bar{x} = \bar{0}$

2) $||a\bar{x}|| = |a| ||\bar{x}|| \quad x \in \mathbb{C}$

3) $||\bar{x} + \bar{y}|| \leq ||\bar{x}|| + ||\bar{y}||$

Μέτρο: Απόσταση μεταξύ $\bar{x}, \bar{y} \in S$

1) $d(\bar{x}, \bar{y}) \geq 0 \qquad d(\bar{x}, \bar{y}) = 0$ ανν $\bar{x} = \bar{y}$

2) $d(\bar{x}, \bar{y}) = d(\bar{y}, \bar{x})$

3) $d(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{z}) + d(\bar{y}, \bar{z}) \quad \bar{z} \in S$

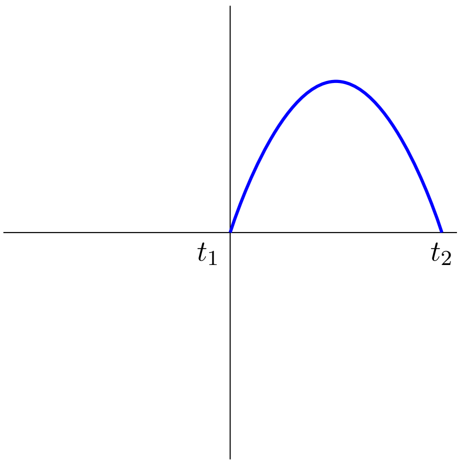
Συναρτησιακός χώρος

$$x(t), y(t) \in S = \big\{ x(t)/x(t) : [t_1, t_2] \to \mathbb{R} \big\}$$

$$\big\langle x(t), y(t) \big\rangle = \int_{t_1}^{t_2} x(t) y(t) \, \mathrm{d}t$$

$$\big|\big|x(t)\big|\big| = \left[\int_{t_1}^{t_2} x^2(t) \, \mathrm{d}t \right]^{1/2}$$

$$d\big(x(t), y(t)\big) = \left[\int_{t_1}^{t_2} \big[x(t) - y(t)\big]^2 \, \mathrm{d}t \right]^{1/2}$$



$$\begin{aligned}
&\text{Αν } \big\langle \phi_1(t), \phi_2(t) \big\rangle = 0 \quad \phi_1(t) \perp \phi_2(t) \\
&\phantom{\text{Αν }} \big\langle \phi_1(t), \phi_1(t) \big\rangle = 1 \quad \phi_1(t) \text{ κανονική}
\end{aligned}$$

$\tilde{a}$ και $d(\vec{a}, \tilde{a})$

$$\begin{aligned}
&\tilde{a} = k\hat{x} \\
&\vec{a} = a_x\hat{x} + a + y\hat{y} \\
&\vec{a} - \tilde{a} = (a_x - k)\hat{x} - a_y\hat{y} \\
&d(\vec{a}, \tilde{a}) = \sqrt{(a_x - k)^2 + a_y^2} \\
&\frac{\mathrm{d}}{\mathrm{d}k}\big(d(\vec{a}, \tilde{a})\big) = \frac{a_x - k}{\cdots} = 0 \implies k = a_k = \tilde{a} \cdot \hat{x} \\
&\vec{a} \cdot \hat{x} = a_x
\end{aligned}$$

Μη κάθετα διανύσματα

$$\begin{aligned}
&\vec{a} = a_x\hat{x} + a_y\hat{y} \\
&\vec{a} \cdot \vec{a} = (a_x - k)\hat{x} + a_y\hat{y} \\
&\Big(\big[(a_x - k)\hat{x} + a_y\hat{y}\big] \cdot \big[(a_x - k)\hat{x} + a_y\hat{y}\big]\Big)^{1/2} \\
&\Big[(a_x - k)^2 + a_y^2 + 2(a_x - k)a_y\hat{x} \cdot \hat{y}\Big]^{1/2} \\
&\tilde{a} = (\vec{a} \cdot \hat{x})\hat{x} \neq a_x
\end{aligned}$$

$$f(t) = \sum_{n=0}^{\infty} a_n\phi_n(t) \quad t \in \Delta$$

$\hat{f}(t) = \sum_{n=0}^{M} \hat{a}_n\phi_n(t)$ βέλτιστη, ώστε η απόσταση με την f να είναι ελάχιστη

$$\vec{?} = \int_\Delta \left[f(t)\right]$$

??????

$$\vec{i}^2 = \int_\Delta f^2(t)\,\mathrm{d}t + \int_\Delta \sum_{i=1}^{?} \left[\hat{a}_n \phi_n(t)\right]^2 + \int_\Delta \left[\sum_{?}^{?} \sum_{?}^{?} \hat{a}_? \hat{a}_? \phi_n(t) \phi_?(t)\right] \mathrm{d}t - 2 \int_\Delta \sum_{?=0}^{M} \hat{a}_n f(t) \phi_n(t)\,\mathrm{d}t$$

$$= \int_\Delta f^2(t)\,\mathrm{d}t + \sum_{n=0}^{\infty} \tilde{a} \int \phi_n^2(t)\,\mathrm{d}t + 2 \sum n = n^N \sum_{???}^{M} \hat{a}_i \hat{a}_m \int_\Delta \phi_2(t) \phi_m(t)\,\mathrm{d}t - 2 \sum_{h=0}^{M} \hat{a}_n \int_\Delta f(t) \phi_n(t)\,\mathrm{d}t$$

$$\frac{\mathrm{d}\vec{i}}{\mathrm{d}a_i} = 2\hat{a} \int_\Delta \phi(t)\,\mathrm{d}t + 2 \sum \phi_i(t) \phi_n(t)\,\mathrm{d}t - 2 \int_\Delta f(t) \phi_n(t)\,\mathrm{d}t$$

Έστω ϕ_i μοναδιαία $\iff \int_\Delta \phi^2(t)\,\mathrm{d}t = 1$ και ϕ_i ορθογώνια $\iff \int_\Delta \phi_1 \phi_2(t) = 0$

Αν η $\{\phi(t)\}$ ορθοκανονική

$$2\vec{a}_i - 2 \int_\Delta f(t) \phi(t)\,\mathrm{d}t = 0 \implies \hat{a}_i = \int_\Delta f(t) \phi(t)\,\mathrm{d}t$$

$$2a \langle \phi, \phi \rangle + \sum \sum \hat{a}_n \langle \phi_i, \phi_n \rangle - 2 \langle f, \phi \rangle = 0$$

$$f(t) = \sum_{a=0} a_n \phi_n(t)$$

$$\int_\Delta f(t) \phi(t)\,\mathrm{d}t = a_i$$

$$\langle f, \phi_i \rangle = \left\langle \sum_{n=0}^{\infty} a_i \phi_n, \phi_i \right\rangle = \sum_{??}^{\infty} a_n \langle \phi_n, \phi_i \rangle$$

Ηθικό δίδαγμα: Αν η βάση του χώρου είναι ορθοκανονική και μας ζητηθεί να υπολογίσουμε μία προσέγγιση της συνάρτησης σε έναν υποχώρο, μπορούμε άμεσα να υπολογίσουμε την προβολή της συνάρτησης πάνω στη βάση.

Ex. $f(t) = e^{-3t}\mathrm{u}(t) \qquad \phi_1(t) = e^{t}\mathrm{u}(t) \ \&\ \phi_2(t) = e^{-2t}\mathrm{u}(t)$

$$\hat{f}(t) = a_1 e^{-t}\mathrm{u}(t) + a_2 e^{-2t}\mathrm{u}(t)$$

$$\int \left[a_1 \phi_1 + a_2 \phi_2 - f\right] \phi_1\,\mathrm{d}t = 0$$

$$\int_0^\infty \left[a_1 e^{-t} + a_2 e^{-2t} - e^{-3t}\right] e^{-t}\,\mathrm{d}t = 0 \implies$$

$$a_1 \int_0^\infty e^{-2t}\,\mathrm{d}t + a_2 \int_0^\infty e^{-3t}\,\mathrm{d}t - \int_0^\infty e^{-4t}\,\mathrm{d}t = 0 \implies \boxed{\frac{a_1}{2} + \frac{a_2}{3} - \frac{1}{4} = 0}$$

$$\int \left[a_1 e^{-t} + a_2 e^{-2t} - e^{-3t}\right] e^{-2t}\,\mathrm{d}t = 0 \implies \boxed{\frac{a_1}{2} + \frac{a_2}{4} - \frac{1}{5} = 0}$$

$$a_1 = {}^3\!/\!_{10},\ a_2 = {}^6\!/\!_5$$