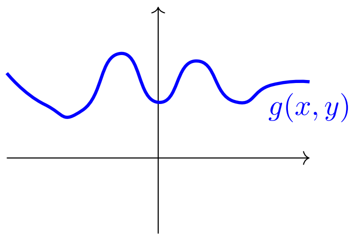
Την Τρίτη μάθημα 8:30 χωρίς διάλειμμα
Σήμα - σύστημα

$$\boxed{\underbrace{g}_{\text{εξαρτημένη}} = f(\underbrace{t}_{\text{ανεξάρτητη}})} \qquad g = f(\vec{r}, t) \qquad \vec{E}(\vec{r}, t)$$

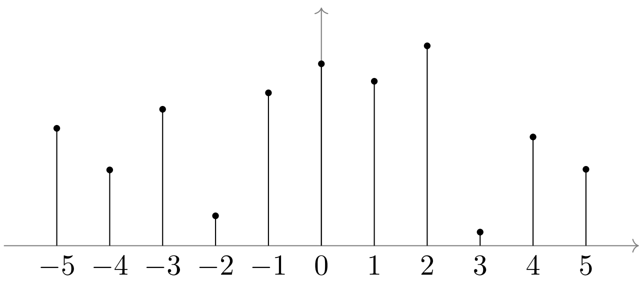
**Αναλογικό**

Αν $t$ συνεχής $\in \mathbb{R}$
και $y$ συνεχής $\in \mathbb{R}$




**Διακριτού χρόνου / Διακριτό (discrete)**

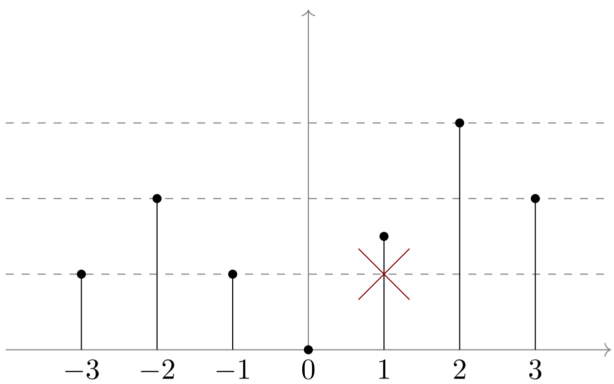
$t$ διακριτό $\to \mathbb{Z},\ n \in \mathbb{Z}$
$g$ συνεχής $\in \mathbb{R}$




$7^{\text{ο}}$ εξάμηνο

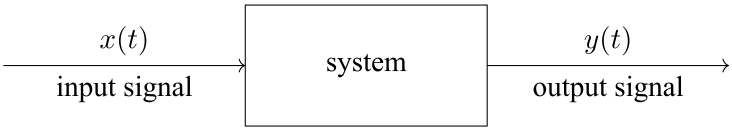
**Κβαντισμένο**

$n \in \mathbb{Z}$
$g$ διακριτή




**Στοχαστικό** Περιέχει και τις τρεις κατηγορίες

## Σύστημα




## Περιοδικά σήματα

Αν $\exists T \in \mathbb{R} : \forall t \in \mathbb{R} \quad x(t) = x(t+T)$ τότε $x(t)$ **περιοδικό σήμα** με περίοδο $T$.
Ή θα είναι 0, ή θα συνεχιστεί για πάντα.

$$\int_{-T/2}^{T/2} x(t)\,\mathrm{d}t = \int_{t_0-T/2}^{t_0+T/2} x(t)\,\mathrm{d}t\ \forall t$$

Η σύνθεση μιας συνάρτησης με μια περιοδική συνάρτηση είναι περιοδική;

**Απόδ.** Έστω $g$ μία περιοδική συνάρτηση:

$$\begin{aligned}\big(f \circ g\big)(x) &= f\left(g(x)\right) = f\left(g(x+T)\right) = \\ &= \big(f \circ g\big)(x+T)\end{aligned}$$

## Συμμετρίες

- Αν $x(t) = x(-t)\ \forall t$ τότε η $x(t)$ λέγεται **άρτια συνάρτηση** (even function).
- Αν $x(t) = -x(t)\ \forall t$ τότε η $x(t)$ λέγεται **περιττή συνάρτηση** (odd function).

$$\forall x(t) \quad \exists\, x_0(t), x_e(t) : x(t) = x_e(t) + x_0(t)$$

**Απόδ.**

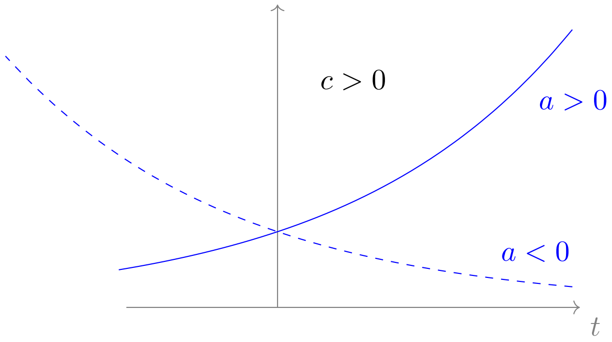
$$x_e(t) = \frac{x(t) + x(-t)}{2}$$
$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$\underbrace{x_e}_{\text{άρτια}} y_e = z_e$$
$$x_o y_o = z_e$$
$$x_e y_0 = z_0$$
$$\int_{-A}^{A} x_0(t)\,\mathrm{d}t = 0$$
$$\int_{-\infty}^{\infty} x_0(t)\,\mathrm{d}t = ?\ \text{(εξαρτάται)}$$
$$\lim_{A\to\infty} \int_{-A}^{A} x_0(t)\,\mathrm{d}t = 0 \quad \text{(principal Cauchy value)}$$

**Χαρακτηριστικά σήματα**

1) **Εκθετικό σήμα**

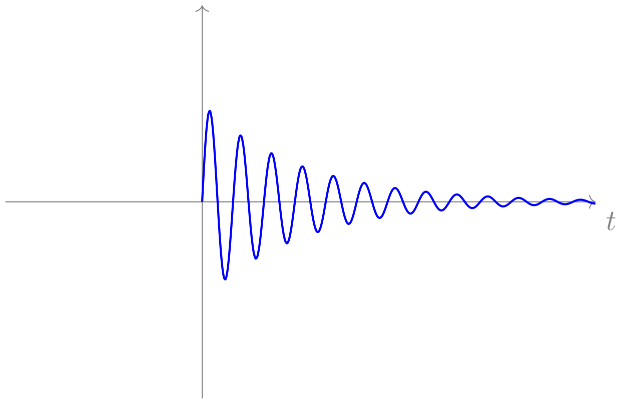
$$x(t) = ce^{at} \quad a \in \mathbb{R} \quad c > 0$$



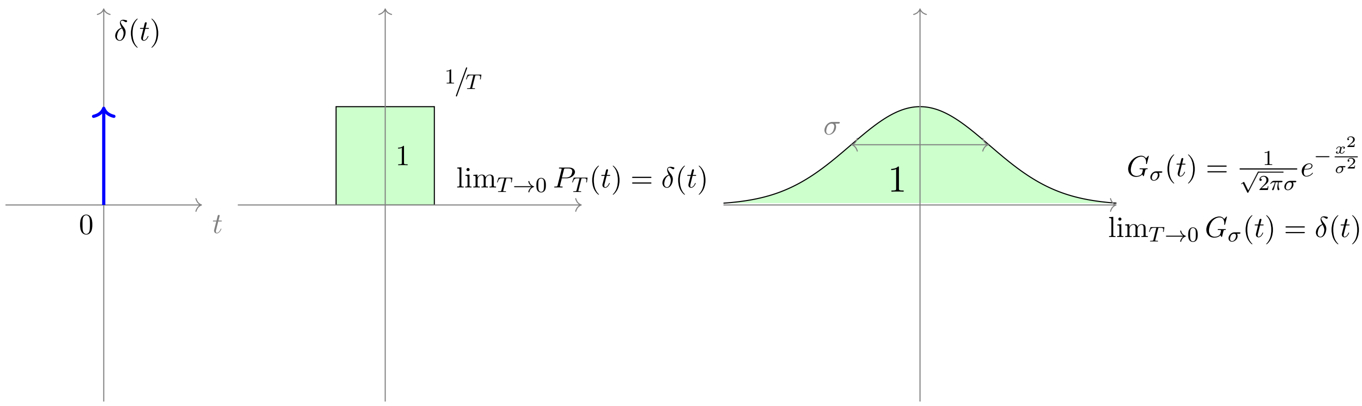

$$x(t) = ce^{(\sigma t + j\omega)t} = ce^{\sigma t}e^{j\omega t} = ce^{\sigma t}\left[\cos(\omega t) + j\sin(\omega t)\right]$$

2) **(Συν)ημιτονοειδή σήματα**

$$x(t) = A\cos(\omega t \pm \phi) = a\mathrm{Re}\left\{e^{j(\omega t + \phi)}\right\} = A\frac{e^{j(\omega t \pm \phi)} + e^{-j(\omega t \pm \phi)}}{2}$$




3) **Δέλτα Dirac $\boldsymbol{\delta(t)}$**




**Ορ.**

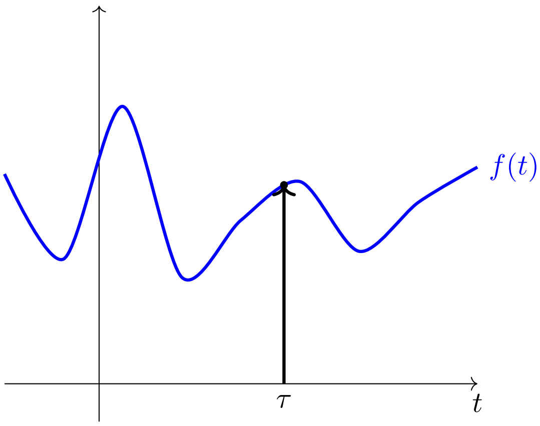
$$\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)\ \forall f(t)$$

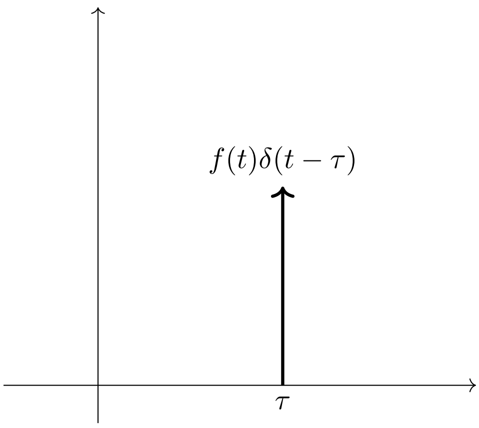
$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}t = f(\tau)$$

$$\boxed{\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau = f(t)}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}\tau = f(t)$$




### Ιδιότητες της $\delta(t)$

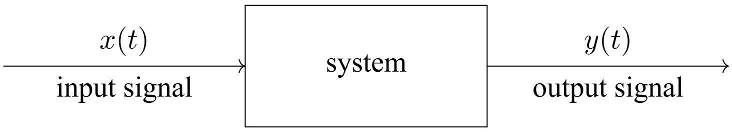
1. **Κλιμάκωση**

$$a \in \mathbb{R} : \delta(at) = \frac{1}{|a|}\delta(t)$$

**Απόδ.**

$$\underbrace{\int_{-\infty}^{\infty} \phi(t)\boxed{\delta(at)}\,\mathrm{d}t}_{\substack{at=\xi \\ \mathrm{d}t=\frac{\mathrm{d}\xi}{a}}} = \int_{-\infty_{(a)}}^{\infty_{(a)}} \phi\left(\frac{\xi}{a}\right)\delta(\xi)\frac{\mathrm{d}\xi}{a} = \frac{1}{|a|}\int_{-\infty}^{\infty}\frac{\phi\left(\frac{\xi}{a}\right)}{|a|}\delta(\xi)\,\mathrm{d}\xi = \frac{\phi(0)}{|a|} = \int_{-\infty}^{\infty}\phi(t)\boxed{\frac{\delta(t)}{|a|}}\mathrm{d}t$$

2. $f(t)\delta(t) = f(0)\delta(t)$
3. $f(t)\delta(t-\xi) = f(g)\delta(t-\xi)$




$y(t) = \mathscr{L}\left\{x(t)\right\}$
$\forall x_1(t)\ x_n(t)$
$y_1(t) = \mathscr{L}\left\{x_1(t)\right\}$
$y_2(t) = \mathscr{L}\left\{x_2(t)\right\}$

Για const $a_1, a_2 \in \mathbb{R}$

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$
$$y(t) = \mathscr{L}\left\{x(t)\right\}$$

ανν

$$y(t) = a_1 y_1(t) + a_2 y_2(t)$$

τότε

$$\mathscr{L}: \text{ γραμμικό σύστημα}$$

- $g(t) = \mathscr{L}\left\{x(t)\right\}$
  $x'(t) = x(t-\tau)$
  ανν $y'(t) = \mathscr{L}\left\{x'(t)\right\} = \mathscr{L}\left\{x(t-\tau)^2\right\} = y(t-\tau)$
  τότε το σύστημα $\mathscr{L}$ είναι αμετάβλητο κατά τη μετατόπιση.


Υποστηρίζω ότι ένα γραμμικό & ΑΚΜ σύστημα περιγράφεται πλήρως από την κρουστική απόκριση $h(t)$.

**Απόδ.** Από παραπάνω, γνωρίζουμε ότι $x(t) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}t$

$$\begin{aligned}
y(t) &= \mathscr{L}\left\{y(t)\right\} = \mathscr{L}\left\{\int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}\tau\right\} \\
&\overset{\text{linearity}}{=} \int_{-\infty}^{\infty} \mathscr{L}\left\{x(\tau)\delta(t-\tau)\right\} \\
&= \int_{-\infty}^{\infty} x(\tau)\mathscr{L}\left\{\delta(t-\tau)\right\}\mathrm{d}\tau \\
&\underset{TSI}{\overset{\text{AKM}}{=}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau \\
y(t) &= \int_{-\infty}^{\infty} x(\tau)\underbrace{h(t-\tau)}_{\text{linear time-shift invariant}}\mathrm{d}\tau
\end{aligned}$$

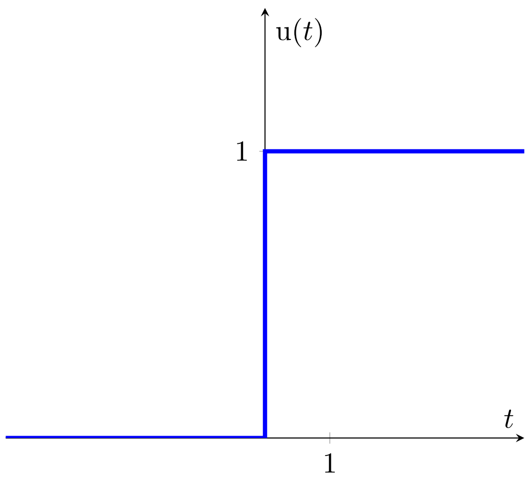
- $\delta(t) = \delta(-t)$ άρτια συνάρτηση
- $\delta^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\delta(t)$, για την οποία αποδεικνύεται ότι:

$$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)\,\mathrm{d}t = (-1)^n\ \phi^{(n)}(t)\Big|_{t=0}$$

**Βηματική Συνάρτηση (Unit Step Function)**

$$\mathrm{u}(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \mathrm{u}(t)\phi(t)\,\mathrm{d}t = \mathscr{N}_{\mathrm{u}}\left\{\phi(t)\right\} = \int_{0}^{\infty} \underbrace{\phi(t)}_{\text{number}}\,\mathrm{d}t$$

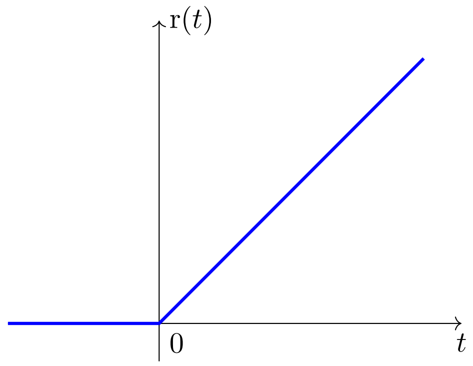



$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{u}(t)$

$$\mathrm{u}(t) = \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \int_{0}^{\infty} \delta(t-\xi)\,\mathrm{d}\xi$$

**Ράμπα**

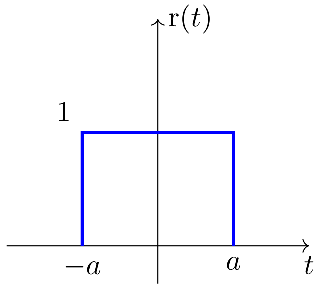
$$\mathrm{r}(t) = \int_{-\infty}^{t} \mathrm{u}(\tau)\,\mathrm{d}\tau = \begin{cases} t & t \geq 0 \\ 0 & \text{else} \end{cases} = t\mathrm{u}(t)$$

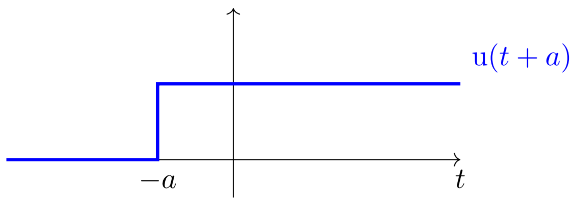


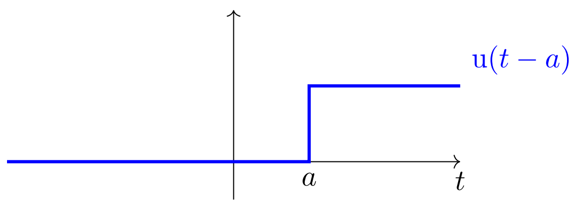

$\mathrm{u}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{r}(t)$

**Ορθογωνικός παλμός (Rectangular Pulse function)**

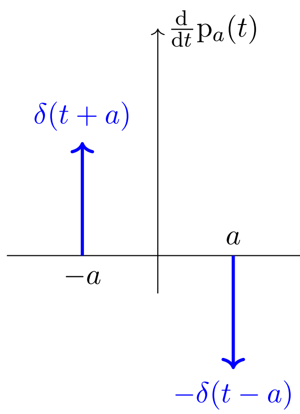
$$\mathrm{p}_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$



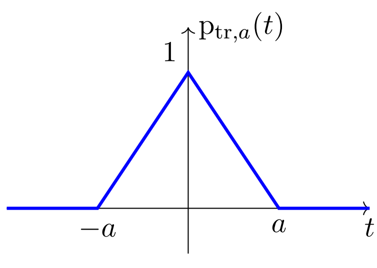




$$\begin{aligned} \mathrm{p}_a(t) =& \mathrm{u}(t+a) - \mathrm{u}(t-a) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{p}_a(t) =& \delta(t+a) - \delta(t-a) \end{aligned}$$

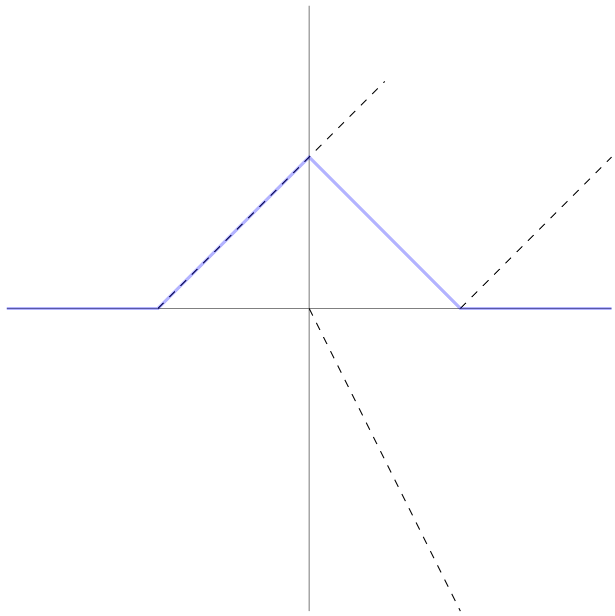



**Τριγωνικός Παλμός (Triangular Pulse function)**

$$\mathrm{p}_{\mathrm{tr},a} = \begin{cases} 1 - \frac{|t|}{a} & |t| < a \\ 0 & |t| > a \end{cases}$$




$$p_{\mathrm{tr},a}(t) = \frac{1}{a}\left[\mathrm{r}(t+a) + \mathrm{r}(t-a) - 2\mathrm{r}(t)\right]$$



## Χαρακτηριστικά Μεγέθη

1) **Μέση τιμή (Mean Value)**

$$\overline{x(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t)\,\mathrm{d}t$$

Αν περιοδική τότε

$$\begin{aligned}\bar{x}(t) &= \frac{1}{T} = \int_{0}^{T} x(t)\,\mathrm{d}t \\ &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t\end{aligned}$$

2) **Ενεργός τιμή (Root Mean Square Value)**

$$\overline{\overline{x(t)}} = \left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t\right]^{1/2}$$

Αν ημιτονοειδές σήμα $\bar{\bar{x}}(t) = \frac{x_{\max}}{\sqrt{2}}$

3) **Ενέργεια - Ισχύς**

- Στιγμιαία ισχύς (Instant power)

$$p(t) = x^2(t)$$

- Μέση ισχύς (Mean power)

$$\overline{p(t)} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t = \left(\overline{\overline{x(t)}}\right)^2$$

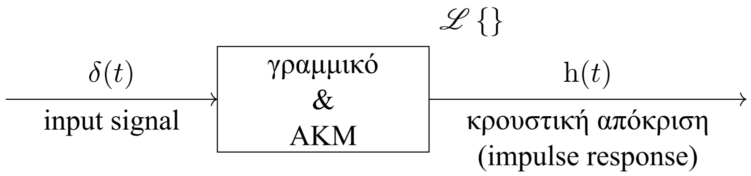
- Ενέργεια (Energy)

$$W = \int_{t_1}^{t_2} p(t)\,\mathrm{d}t = \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t = (t_2 - t_1)\left(\overline{\overline{x(t)}}\right)^2$$

**Σήματα** $\begin{cases} \textbf{Σήμα ενέργειας} \text{ αν } \lim_{T\to\infty} W < \infty \\ \\ \textbf{Σήμα ισχύος} \text{ αν } \lim_{T\to\infty} \overline{p(t)} > 0 \\ \text{Υπάρχουν και σήματα που δεν είναι ούτε ενέργειας, ούτε ισχύος.} \end{cases}$

**Συνέλιξη**

$$x(t) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau$$




$$h(t) = \mathscr{L}\left\{\delta(t)\right\}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \underbrace{x(t)}_{\text{είσοδος}}\ \underbrace{*}_{\text{συνέλιξη}}\ \underbrace{h(t)}_{\text{κρουστική απόκριση}}$$

**Συνέλιξη - Convolution**

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau$$

- $x(t) * y(t) = y(t) * x(t)$ **Αντιμεταθετική**

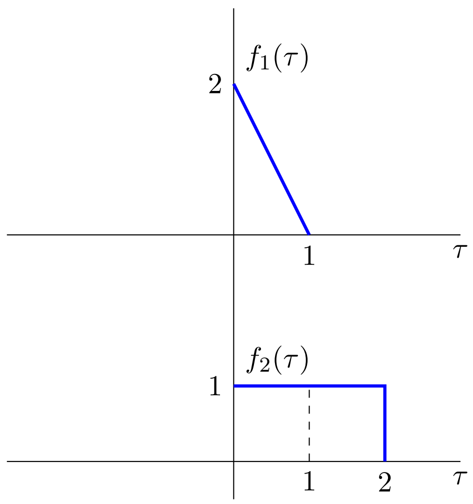
$$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\lambda)y(\lambda)[-\,\mathrm{d}\lambda] = \int_{-\infty}^{\infty} y(\lambda)x(t-\lambda)\,\mathrm{d}\lambda = y(t) * x(t)$$

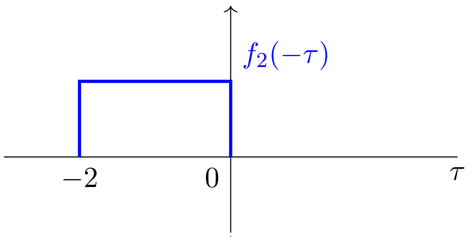
- $x_1(t) * \left[x_2(t) * x_3(t)\right] = \left[x_1(t) * x_2(t)\right] * x_3(t)$ **Προσεταιριστική**

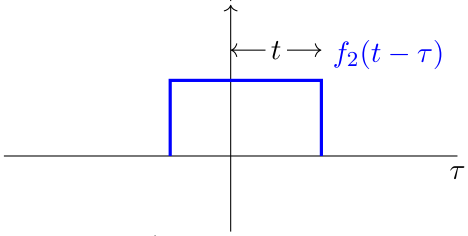
**Παρ.**

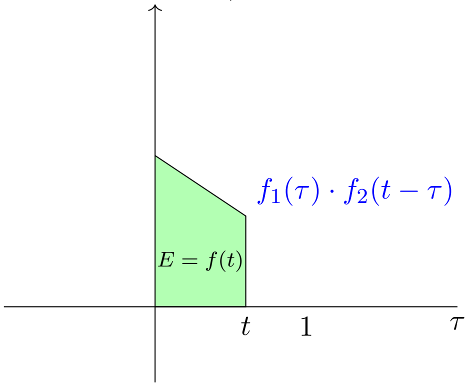
$$f_1(t) = 2(1-t)\left[\mathrm{u}(t) - u(t-1)\right]$$
$$f_2(t) = \mathrm{u}(t) - \mathrm{u}(t-\tau)$$

**Γραφική μέθοδος υπολογισμού συνέλιξης**

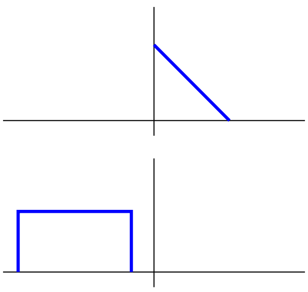




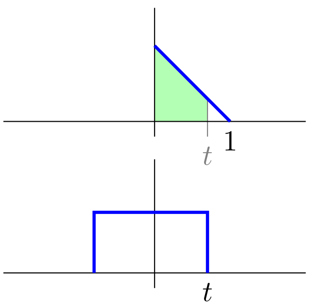




- $t < 0$



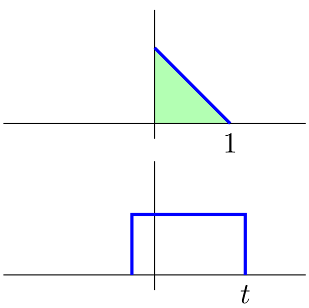
$f(t) = 0$

- $0 < t < 1$



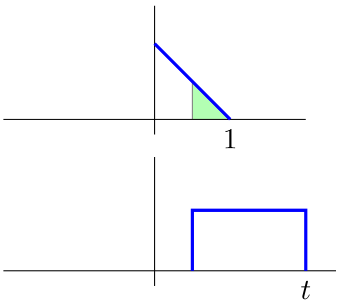

$f(t) = \frac{1}{2}(2 + 2 - 2t)t = (2 - t)t$

- $1 < t < 2$



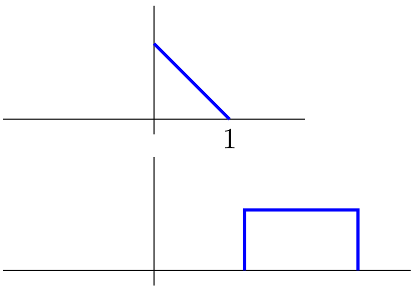

$f(t) = 1$

- $2 < t < 3$

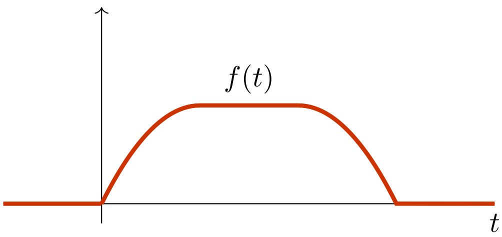



$f(t) = \frac{(t-1)\cdot 2\cdot\left(1-(t-2)\right)}{2} = (t-1)(3-t)$

- $t > 3$




$f(t) = 0$




**Αναλυτική μέθοδος** Παρατηρώ ότι:

$$\int_{-\infty}^{\infty} f(t,\tau)\mathrm{u}(t-\xi)\mathrm{u}(\phi-\tau)\,\mathrm{d}\tau = \int_{\xi}^{\phi} f(t,\tau)\,\mathrm{d}\tau\mathrm{u}(\phi-\xi)$$

$$\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} \underbrace{2(1-\tau)}_{x(\tau)} \left[\mathrm{u}(\tau)-\mathrm{u}(\tau+1)\right] \left[\mathrm{u}(t-\tau)-\mathrm{u}(t-\tau-2)\right] \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\mathrm{u}(\tau)\mathrm{u}(t-\tau) - \mathrm{u}(\tau-1)\mathrm{u}(t-\tau) - \mathrm{u}(\tau)\mathrm{u}(t-\tau-2) + \mathrm{u}(\tau-1)\mathrm{u}(t-\tau-2)\right] \mathrm{d}\tau \\
&= \int_0^t x(\tau)\,\mathrm{d}\tau\mathrm{u}(t) - \int_1^x x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-1) - \int_0^{t-2} x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-2) + \int_1^{t-2} x(\tau)\,\mathrm{d}\tau\mathrm{u}(t-3) \\
&= (2t-t^2)\mathrm{u}(t) - \left[2t-t^2-1\right]\mathrm{u}(t-1) - \left[2(t-2)-(t-2)^2\right]\mathrm{u}(t-2) + \left[2(t-2)-(t-2)^2-1\right]\mathrm{u}(t-
\end{aligned}$$

**Ex**

$$\begin{aligned}
f_1(t) &= e^t\mathrm{u}(-t) \\
f_2(t) &= \mathrm{u}(t+2) - u(t+1) \\
f &= f_1 * f_2
\end{aligned}$$

$$
\begin{aligned}
f &= \int_{-\infty}^{\infty} e^{\tau} \mathrm{u}(-\tau) \mathrm{u}\left(-(t-\tau)+2\right) \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{\tau} \mathrm{u}(-\tau) \mathrm{u}(\tau - t + 2) \, \mathrm{d}\tau \\
&= \int_{t-2}^{0} e^{\tau} \, \mathrm{d}\tau \mathrm{u}(t-2) \\
&= e^{\tau}\big|_{t-2}^{0} \mathrm{u}(2-t) \\
&= \left[1 - e^{t-2}\right] \mathrm{u}(2-t)
\end{aligned}
$$