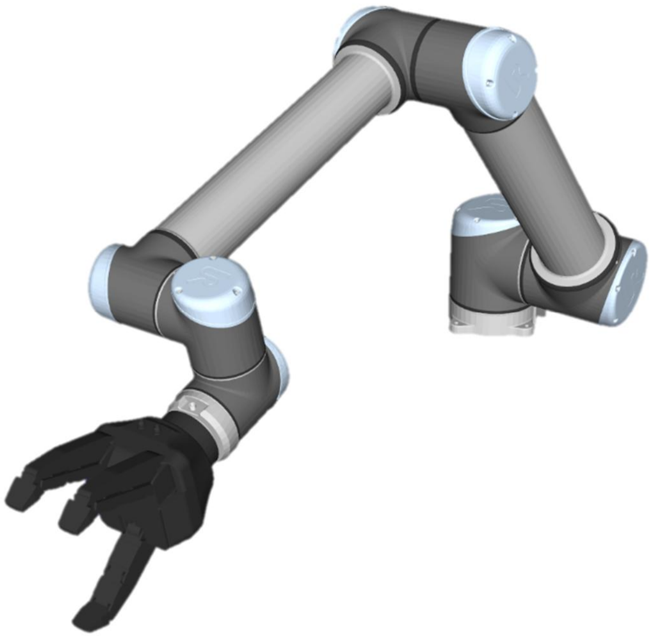
Applied Robot Design



Lecture 5 : Trajectory Planning

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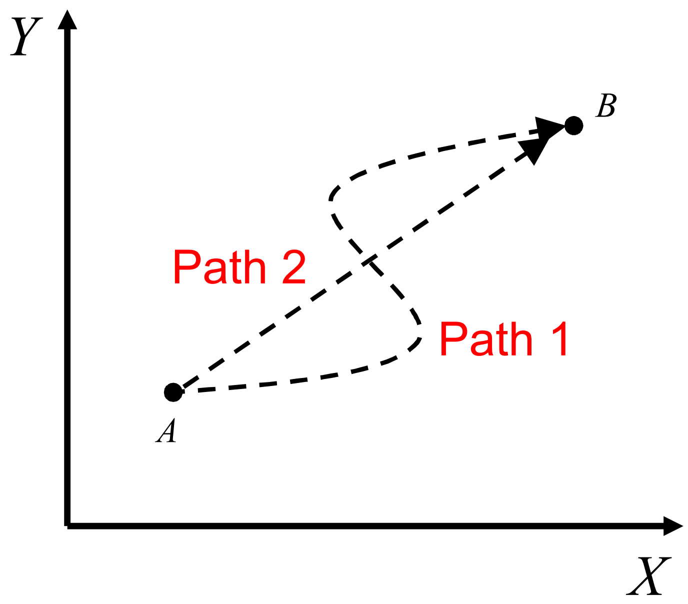
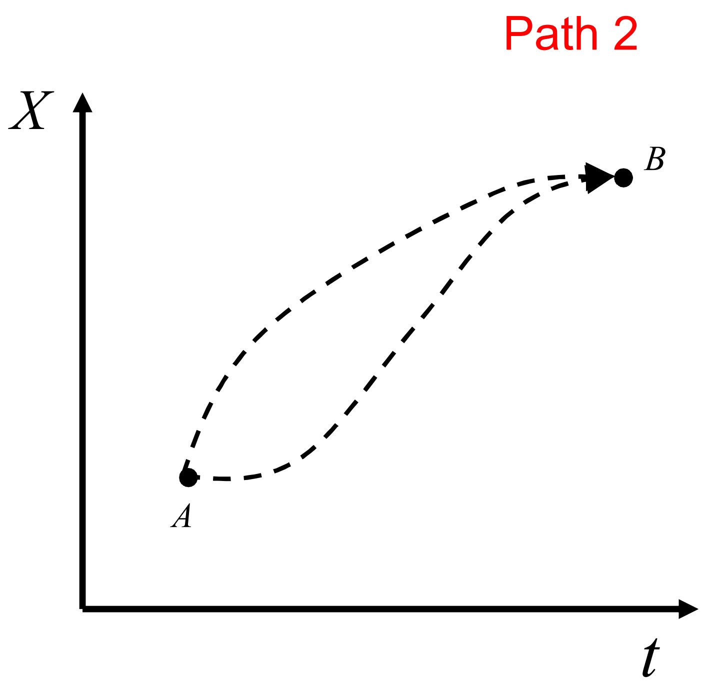
Introduction

- After planning end-effector path, we have to determine the end-effector velocities and accelerations while traversing the path.
- Trajectory planner computes a function $q^d(t)$ that completely specifies the motion of the robot as it traverses the path.





Introduction

- Example : Humanoid Upper Body Motion Generation

1) Path planning using human motion capture
2) Trajectory planning using Fourier series fitting
3) Generation of speed and amplitude adjustable motion

5.1 The Trajectory Planning Problem

A trajectory is a function of time q(t) such that $q(t_0) = q_s$ and $q(t_f) = q_f$.

In this case, $t_f - t_0$ represents the amount of time taken to execute the trajectory.

Since the trajectory is parametrized by time, we can compute velocities and accelerations along the trajectories by differentiation.

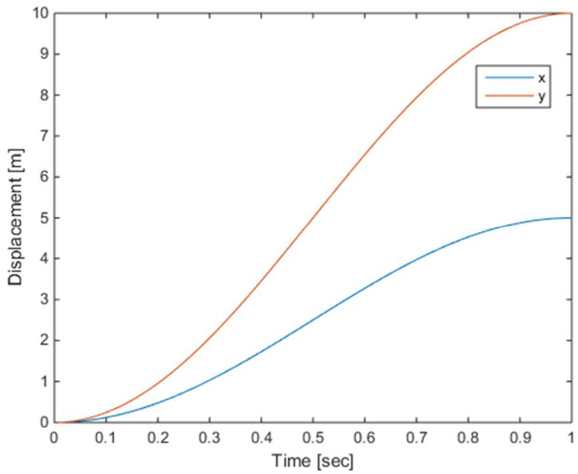
If the trajectory represents a Cartesian position of the end-effector, the [blank box] must be used to derive a sequence of joint configurations.

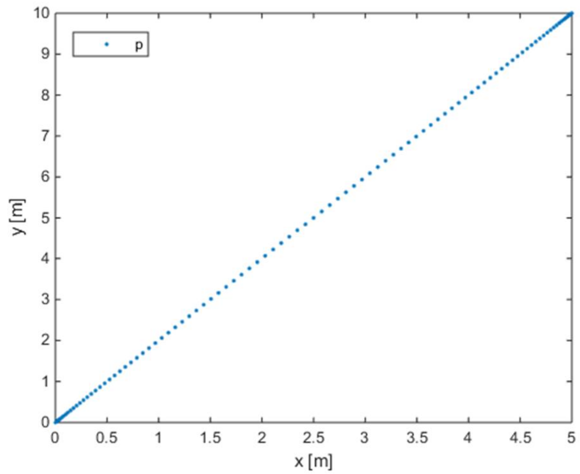
Ex.)

$$x = 5 \cdot 0.5 \cdot \left(1 - \cos\left(\frac{\pi}{1.0} t\right)\right)$$

$$y = 10 \cdot 0.5 \cdot \left(1 - \cos\left(\frac{\pi}{1.0} t\right)\right)$$

$$(0 \le t \le 1.0)$$





5.1 The Trajectory Planning Problem

It is often the case that a manipulator motion can be decomposed into segments consisting of free and guarded motions.

During the free motion the manipulator can move very fast, since no obstacles are near by, but at the start and end of motion, care must be taken to avoid obstacles.

Below, we first consider point to point motion. Once we have seen how to construct trajectories between two configurations, it is straightforward to generalize the method to the case of trajectories specified by multiple via points.

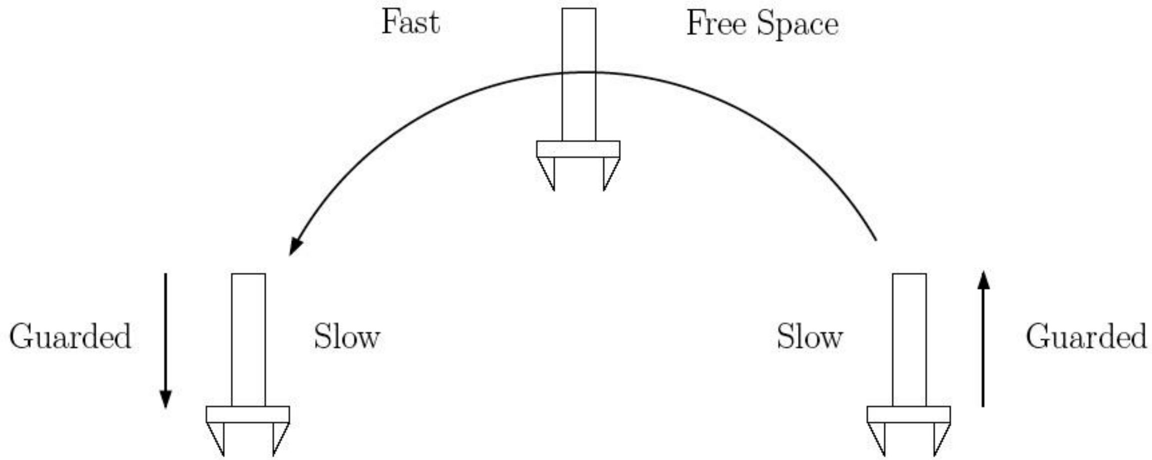
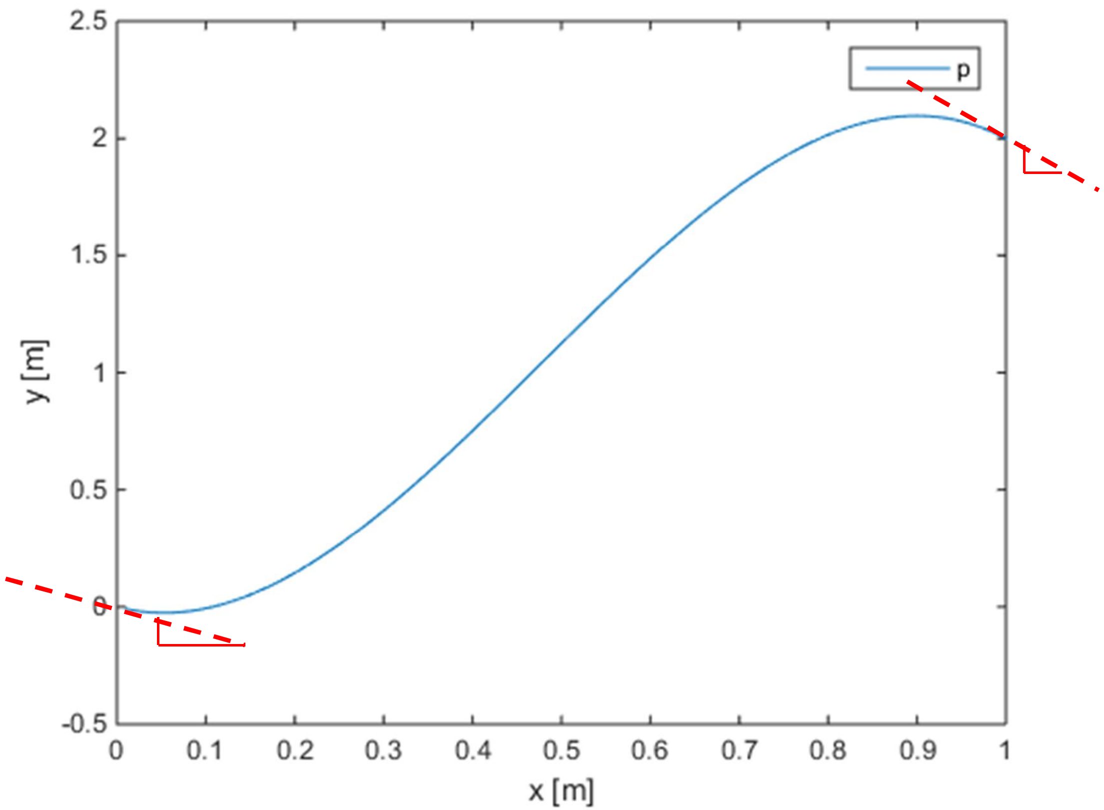


Figure 5.11: Often the end effector trajectory can be decomposed into initial and final guarded motions that are executed at low speeds, and a free motion that is executed at high speed.

5.1 The Trajectory Planning Problem

- Path Planning :



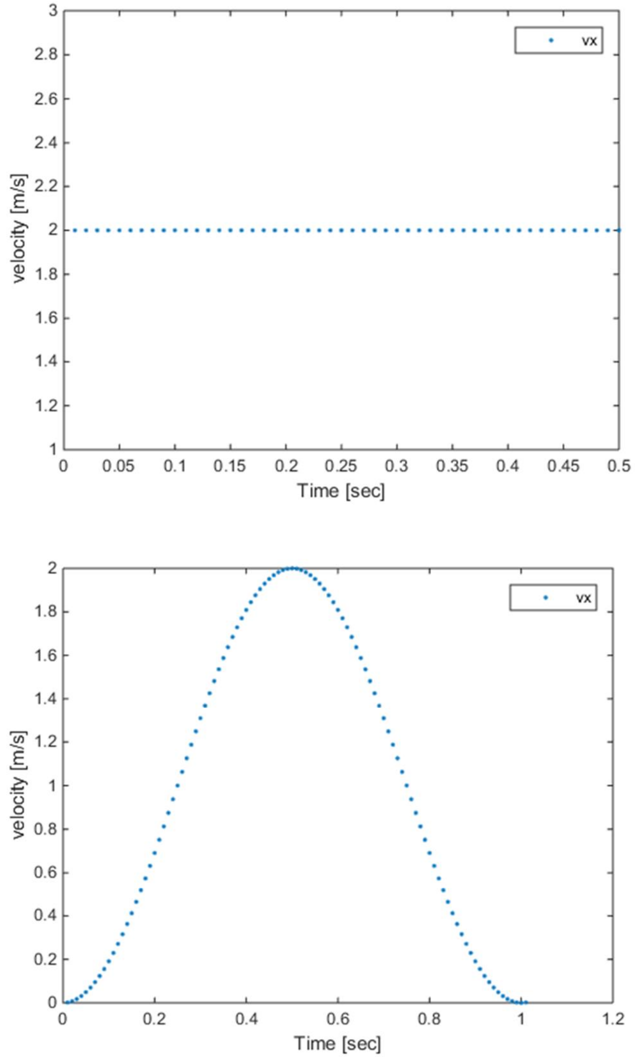

```
x0 = 0;
y0 = 0;

xf = 1;
yf = 2;

dy_dx0 = -1;
dy_dxf = -2;

A = [1 x0 x0^2 x0^3;
     0 1 2*x0 3*x0^2;
     1 xf xf^2 xf^3; 0 1 2*xf 3*xf^2];

X = [y0 dy_dx0 yf dy_dxf]';

C = inv(A)*X;

for i=1:1:100

    x(i) = 0.01*i;
    y(i) = C'*[1 x(i) x(i)^2 x(i)^3]';

end

plot(x,y);
legend('p');
xlabel('x [m]');
ylabel('y [m]');
```

Coefficients of cubic polynomial path

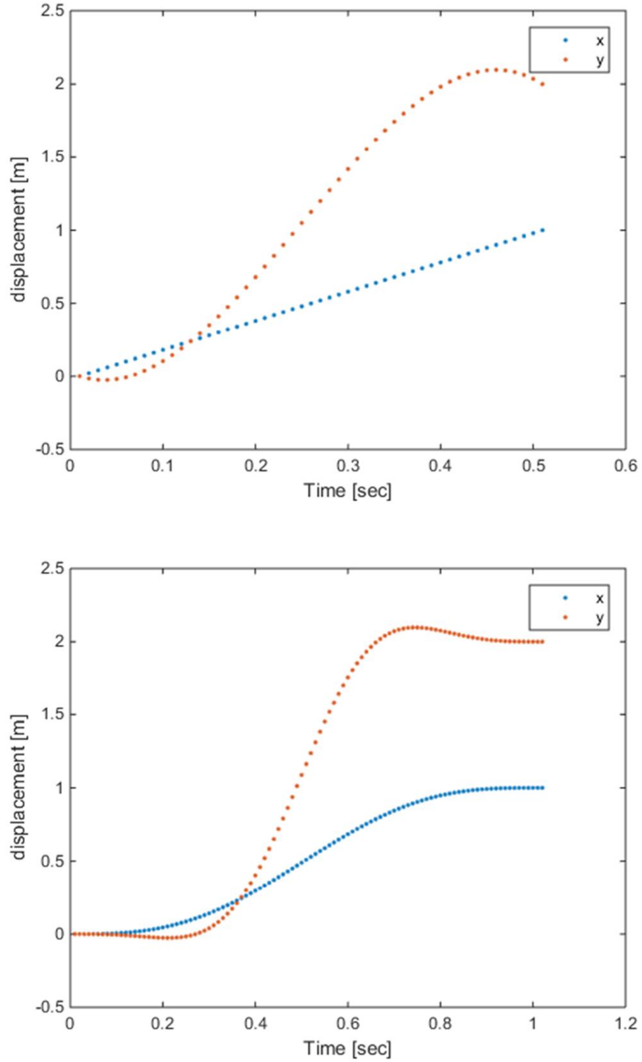
5.1 The Trajectory Planning Problem

- Trajectory Planning :

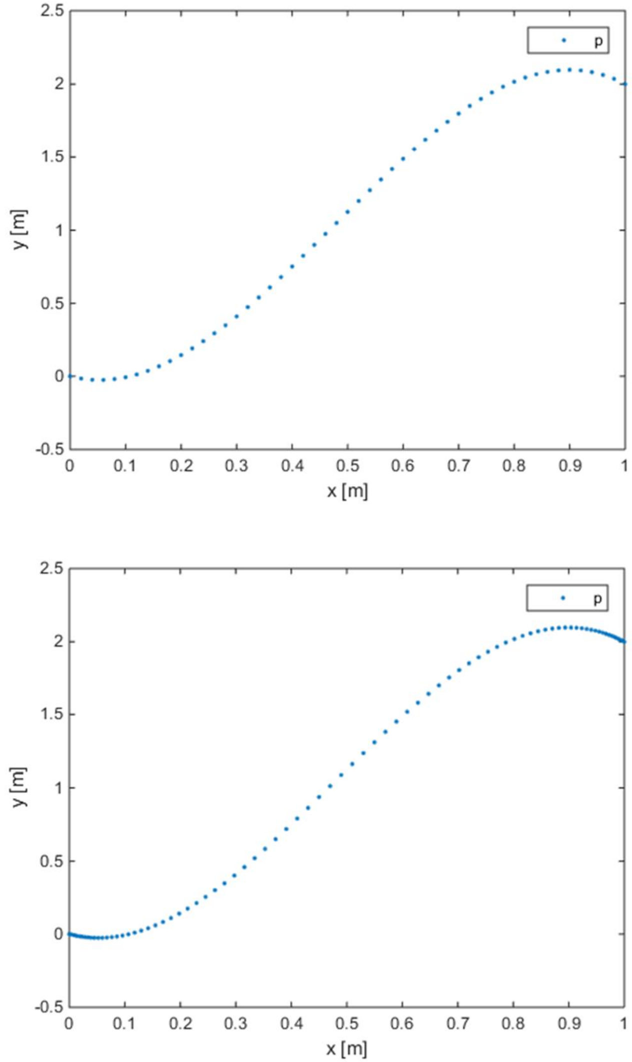
(a) X-velocity profile



(b) Trajectory



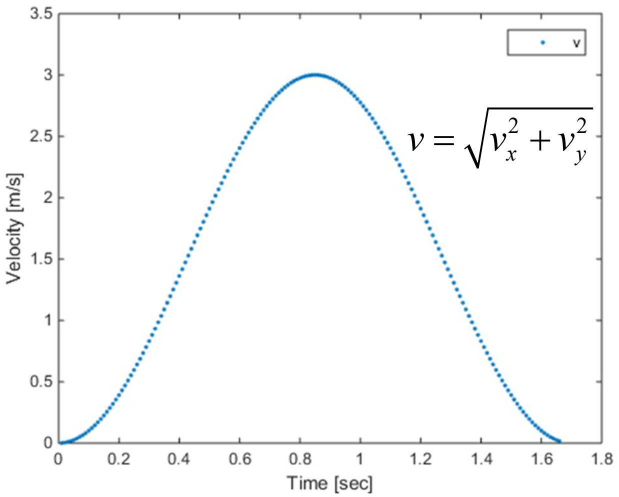
(c) Path



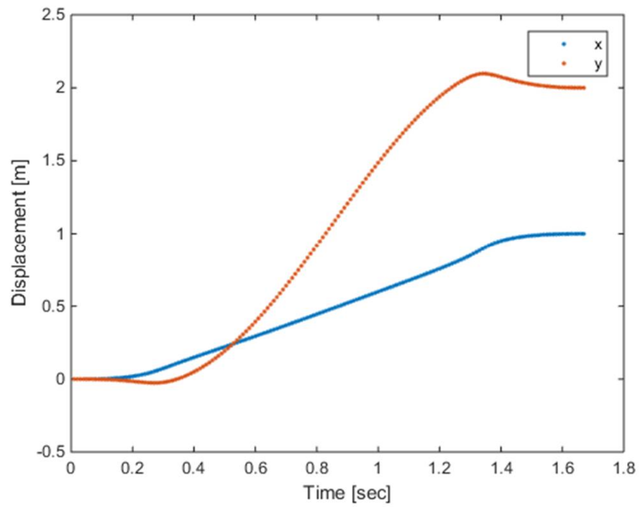
5.1 The Trajectory Planning Problem

- Trajectory Planning :

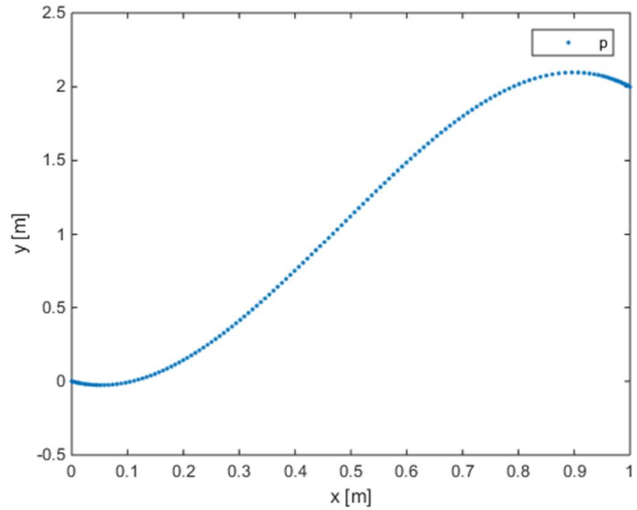
(a) Velocity profile



(b) Trajectory



(c) Path



5.2 Trajectories for Point to Point Motion

We supposed that at time t_0 the E.E. position/orientation variable satisfies

And we wish to attain the values at t_f

The right figure shows a suitable trajectory for this motion. In addition, we may wish to specify the constraints on initial and final accelerations.

$$\ddot{q}(t_0) = \alpha_0$$

$$\ddot{q}(t_f) = \alpha_f$$

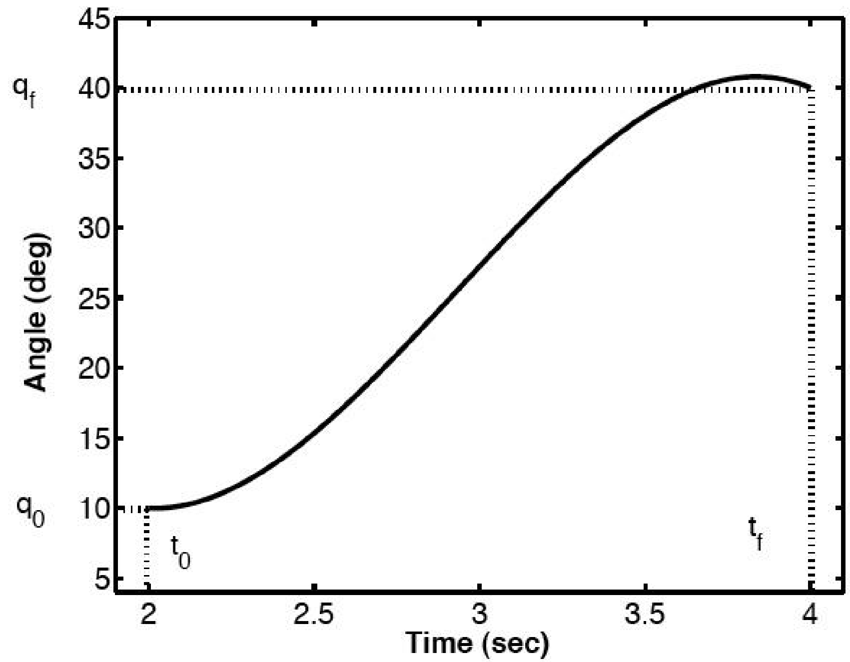


Figure 5.12: A typical joint space trajectory.

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1. Cubic Polynomial Trajectories

We consider a cubic trajectory of the form

Then the desired velocity is given as

$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

Four constraints yields four equations in four unknowns,

$$q_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3$$

$$v_0 = a_1 + 2a_2 t_0 + 3a_3 t_0^2$$

$$q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3$$

$$v_f = a_1 + 2a_2 t_f + 3a_3 t_f^2$$

These four equations can be combined into a single matrix equation

❖ The determinant of the coefficient matrix in the above equation is equal to and hence the above equation has a unique solution provided a nonzero time interval is allowed for the execution of the trajectory.

Ex. 8.1. Cubic Polynomial Trajectory

As an illustrative example, we may consider the special case that the initial and final velocities are zero. Suppose we take $t_0 = 0$ and $t_f = 1$ sec, with

$$v_0 = 0 \qquad v_f = 0$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ 0 \\ q_f \\ 0 \end{bmatrix}$$

Then,

$$a_0 = q_0$$
$$a_1 = 0$$
$$a_2 + a_3 = q_f - q_0$$
$$2a_2 + 3a_3 = 0$$

The required cubic polynomial function is therefore

The corresponding velocity and acceleration curves are given as

$$\dot{q}(t) = 6(q_f - q_0)t - 6(q_f - q_0)t^2$$
$$\ddot{q}(t) = 6(q_f - q_0) - 12(q_f - q_0)t$$

5.2 Trajectories for Point to Point Motion

When $q_0 = 10^0, \quad q_f = -20^0$,

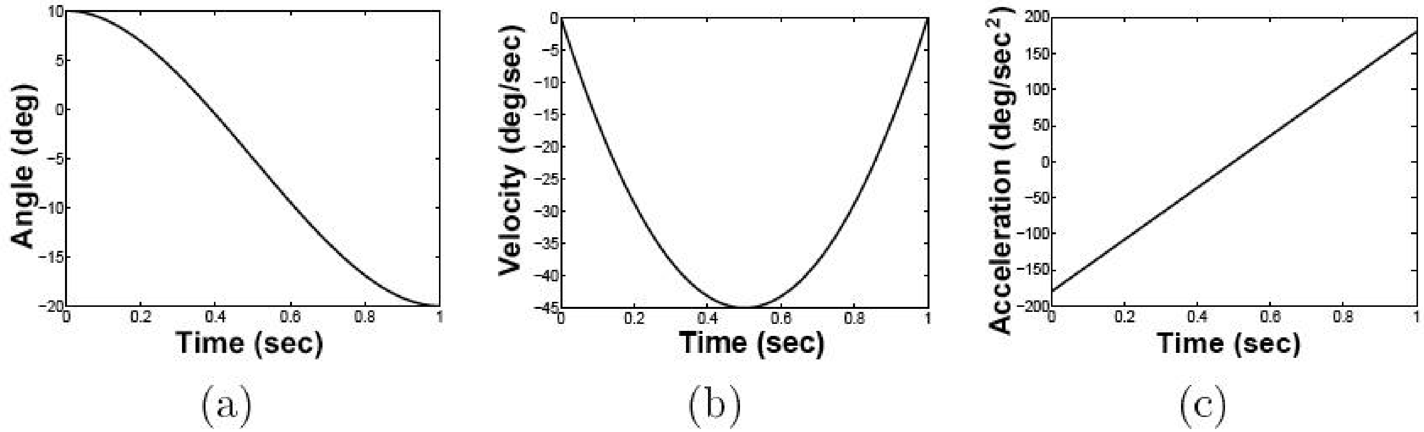
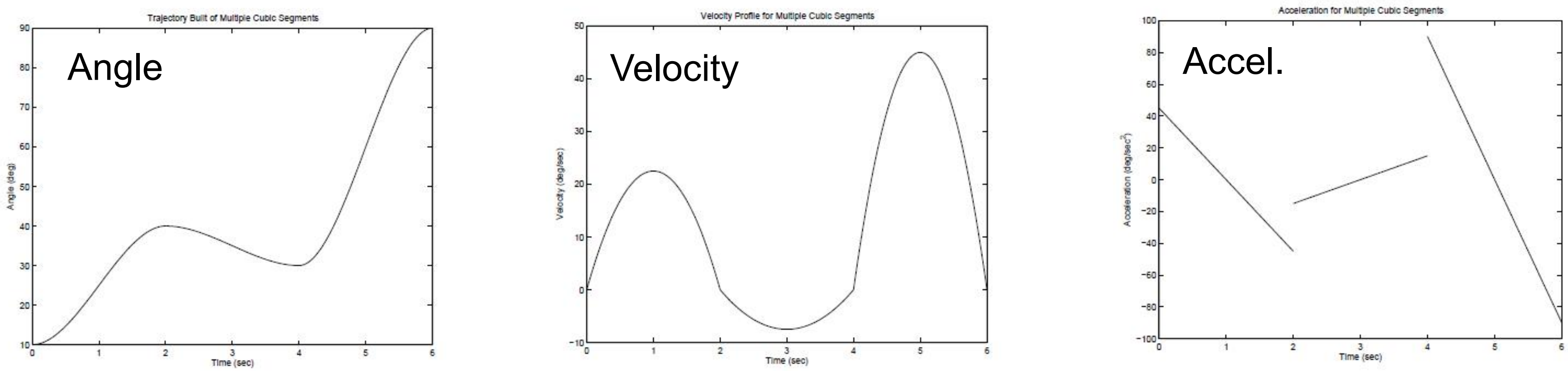


Figure 5.13: (a) Cubic polynomial trajectory. (b) Velocity profile for cubic polynomial trajectory. (c) Acceleration profile for cubic polynomial trajectory.

A sequence of moves can be planned using the above formula by using the end conditions q_f, v_f of the i-th moves as initial conditions for the i+1-th move.



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2. Quintic Polynomial Trajectories

A cubic trajectory gives continuous positions and velocities at the start and finish points times but **discontinuities in the acceleration**.
A discontinuity in acceleration leads to an **impulsive jerk**, which may excite vibrational modes in the manipulator and reduce tracing accuracy.
For this reason, one may wish to specify additional acceleration constraints.
Therefore, we require a **5th order polynomial**

We can obtain the following equations

$$q_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 + a_4 t_0^4 + a_5 t_0^5$$
$$v_0 = a_1 + 2a_2 t_0 + 3a_3 t_0^2 + 4a_4 t_0^3 + 5a_5 t_0^4$$
$$\alpha_0 = 2a_2 + 6a_3 t_0 + 12a_4 t_0^2 + 20a_5 t_0^3$$
$$q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$$
$$v_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4$$
$$\alpha_f = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$$

which can be written as

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} q_0 \\ v_0 \\ \alpha_0 \\ q_f \\ v_f \\ \alpha_f \end{bmatrix}$$

The following figure shows a quantic polynomial trajectory with $q(0)=0, \quad q(2)=20$ with **zero initial and final velocities and accelerations**.

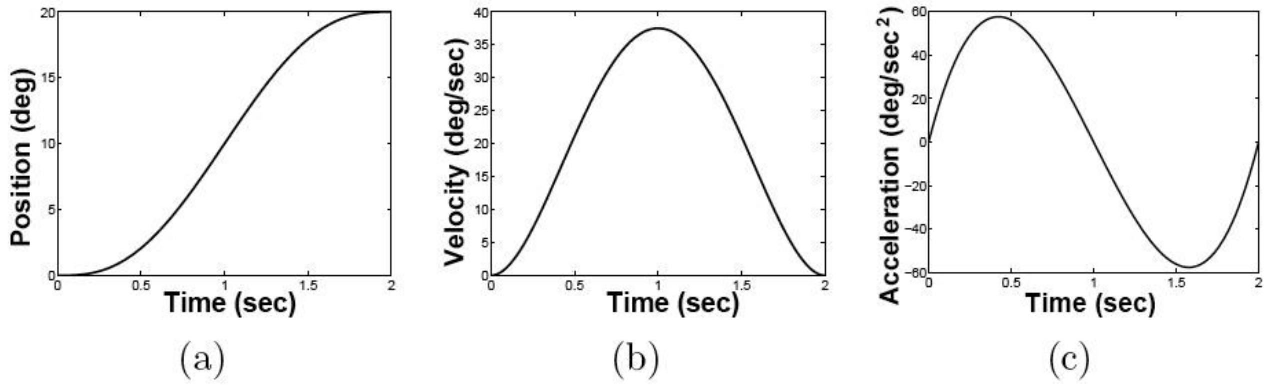

Figure 5.14: (a) Quintic polynomial trajectory, (b) its velocity profile, and (c) its acceleration profile.

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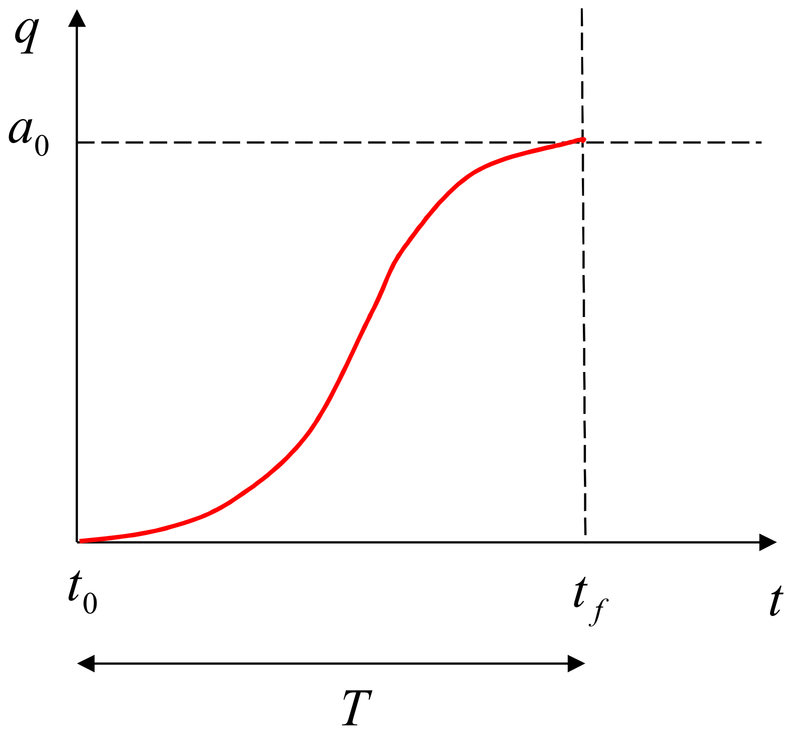
3. Sinusoidal Trajectories

We consider a sinusoidal trajectory of the form

where, a_0 : traveling distance, $T = t_f - t_0$: traveling time

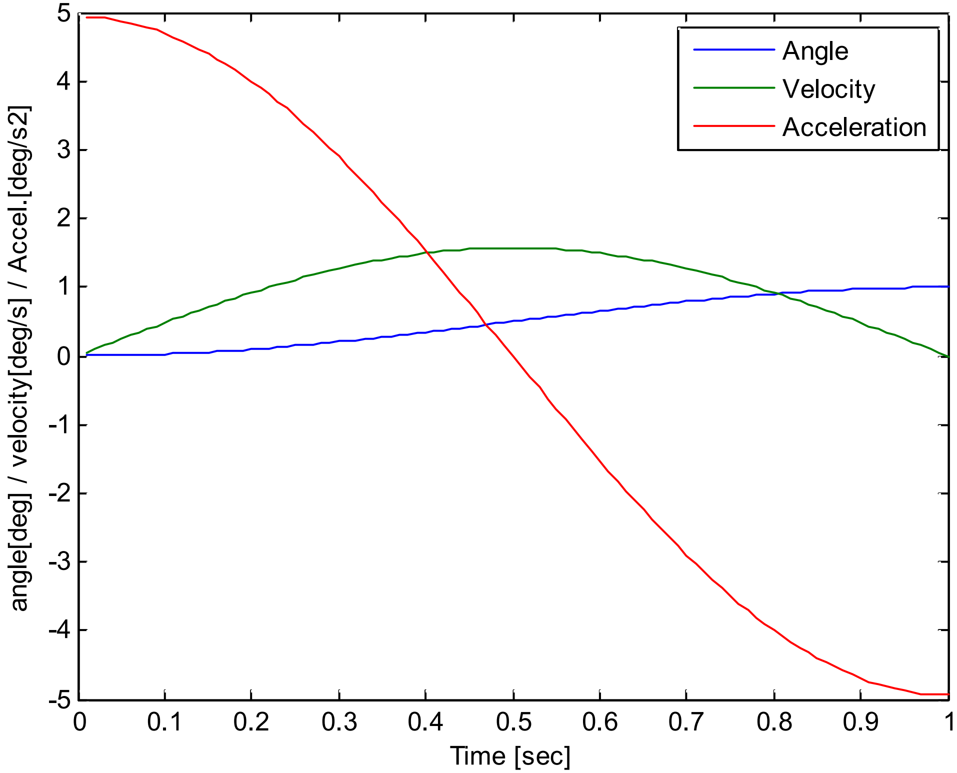


If we differentiate the equation,

$$\dot{q}(t) = a_0 \cdot 0.5 \cdot \left(\frac{\pi}{T}\right)\left(\sin\left(\frac{\pi}{T}(t - t_0)\right)\right)$$

$$\ddot{q}(t) = a_0 \cdot 0.5 \cdot \left(\frac{\pi}{T}\right)^2\left(\cos\left(\frac{\pi}{T}(t - t_0)\right)\right)$$

When $a_0 = 1$, $T = 1$ sec ($t_f = 1$, $t_0 = 0$) ,



4. Linear Segments with Parabolic Blends (LSPB)

This type of trajectory has a **Trapezoidal Velocity Profile** and is appropriate when a constant velocity is desired along a portion of the path.

To achieve this, we specify the desired trajectory in three parts.

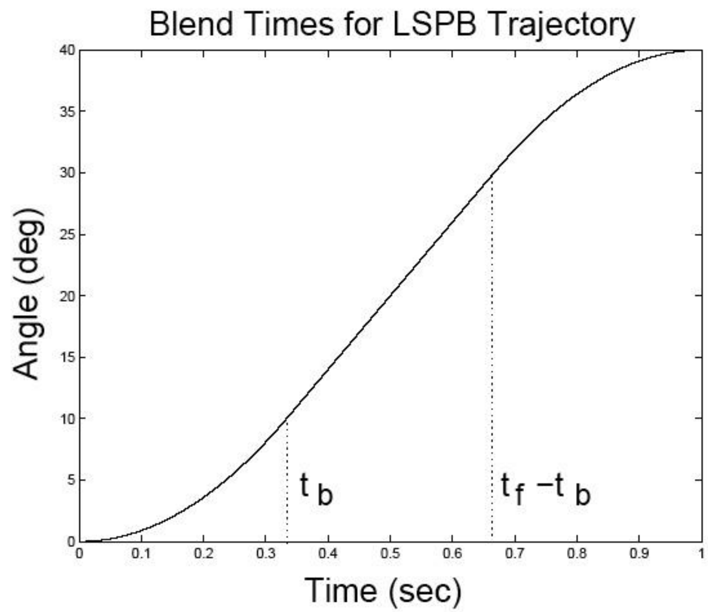


Figure 5.15: Blend times for LSPB trajectory.

We choose the blend time t_b so that the position curve is symmetric.

5.2 Trajectories for Point to Point Motion

Suppose that t_0 = 0 and $\dot{q}(t_f) = \dot{q}(0) = 0$. Then, between times 0 and t_b, we have

so that the velocity is

$$\dot{q}(t) = a_1 + 2a_2 t$$

The constraints $q(0) = q_0$ and $\dot{q}(0) = 0$ imply that

$$a_0 = q_0$$
$$a_1 = 0$$

At time t_b we want the velocity to equal a given constant, say V. Thus,

$$\dot{q}(t_b) = 2a_2 t_b = V$$

$$\therefore a_2 = \frac{V}{2t_b}$$

5.2 Trajectories for Point to Point Motion

Therefore, the required trajectory between 0 and t_b is given as

$$q(t) = q_0 + \frac{V}{2t_b}t^2 = q_0 + \frac{\alpha}{2}t^2$$

$$\dot{q}(t) = \frac{V}{t_b}t = \alpha t$$

$$\ddot{q} = \frac{V}{t_b} = \alpha$$

where, α denotes the acceleration.

Now, between time t_b and $t_f - t_b$, the trajectory is a linear segment with velocity V

Since, by symmetry

$$q\left(\frac{t_f}{2}\right) = \frac{q_0 + q_f}{2}$$

We have

$$\frac{q_0 + q_f}{2} = q(t_b) + V\left(\frac{t_f}{2} - t_b\right)$$

$$\therefore q(t_b) = \frac{q_0 + q_f}{2} - V\left(\frac{t_f}{2} - t_b\right)$$

Since the two segments must “blend” at time t_b we require

$$q_0 + \frac{V}{2} t_b = \frac{q_0 + q_f - Vt_f}{2} + Vt_b$$

$$\therefore t_b = \frac{q_0 - q_f + Vt_f}{V}$$

Note that we have the constraints $0 < t_b \le \frac{t_f}{2}$. This leads to the inequality

$$\frac{q_f - q_0}{V} < t_f \le \frac{2(q_f - q_0)}{V}$$

The portion of the trajectory between $t_f - t_b$ and t_f is now found by symmetry considerations. The complete LSPB trajectory is given by

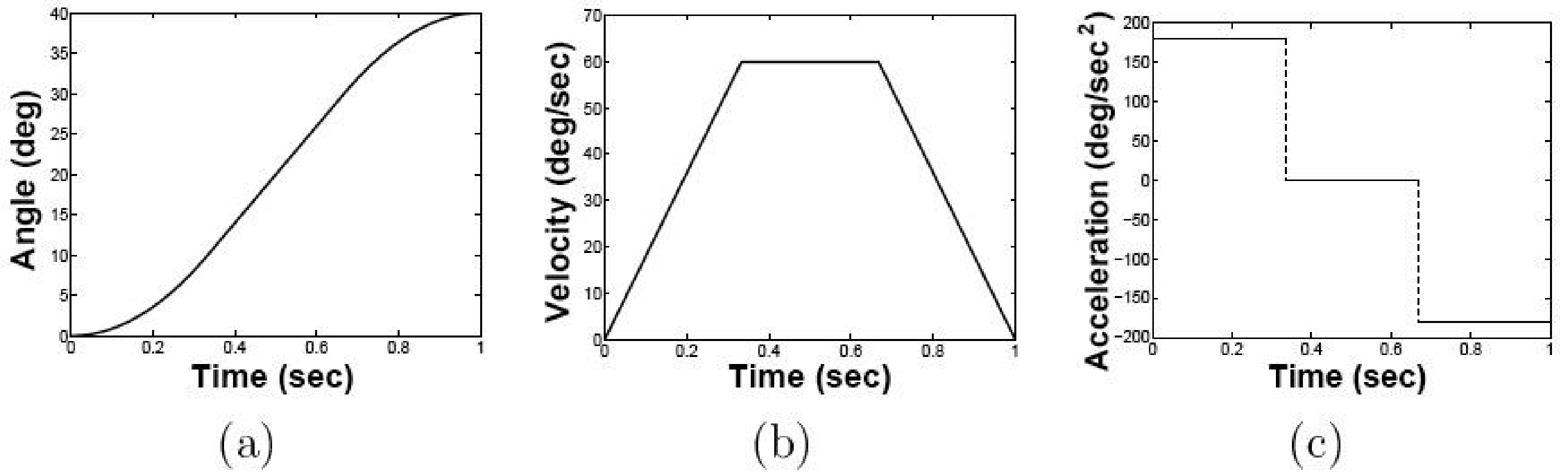


Figure 5.16: (a) LSPB trajectory. (b) Velocity profile for LSPB trajectory. (c) Acceleration profile for LSPB trajectory.

We generalize our approach to the case of planning a trajectory that passes through a sequence of configurations, called **via points**.

Consider the simple example of a path specified by three points, q_0, q_1, and q_2, such that the via points are reached at times t_0, t_1 and t_2, respectively.

In addition, we impose constraints on the initial and final velocities and accel.

$$
\begin{aligned}
q(t_0) &= q_0 \\
\dot{q}(t_0) &= v_0 \\
\ddot{q}(t_0) &= \alpha_0 \\
q(t_1) &= q_1 \\
q(t_2) &= q_2 \\
\dot{q}(t_2) &= v_2 \\
\ddot{q}(t_2) &= \alpha_2
\end{aligned}
$$

The above constraints could be satisfied by generating a trajectory using the 6^{th} order polynomial

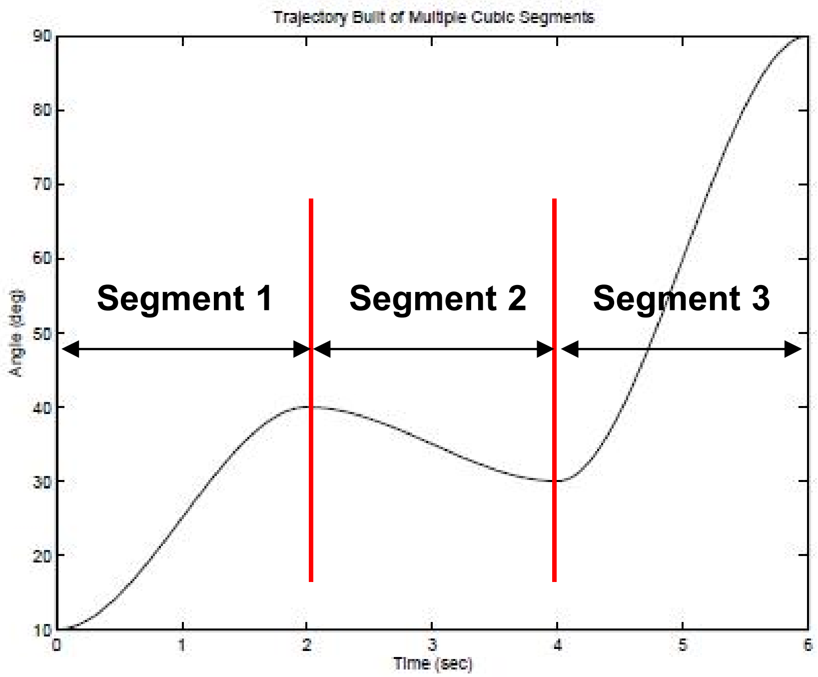
$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6$$

5.3 Trajectories for Paths Specified by Via Points

To determine the coefficients for the above polynomial, we must solve a linear system of dimension 7.

The clear disadvantage is that as the number of via points increases, the dimension of the corresponding linear system also increases, making the method intractable when many via points are used.

An alternative method is to use



For the first segment of the trajectory, suppose that the initial and final times are t_0 and t_f, respectively, and the constraints on initial and final velocities are given by

$$q(t_0) = q_0 \quad ; \quad q(t_f) = q_1$$
$$\dot{q}(t_0) = v_0 \quad ; \quad \dot{q}(t_f) = v_1$$

The required cubic polynomial for this segment of the trajectory can be computed from

$$q(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + a_3(t - t_0)^3$$
$$\dot{q}(t) = a_1 t + 2a_2(t - t_0) + 3a_3(t - t_0)^2$$

where

5.3 Trajectories for Paths Specified by Via Points

A sequence of moves can be planned using the above formula by using the end conditions q_f, v_f of the i^th move as initial conditions for the subsequent move.

The same via points

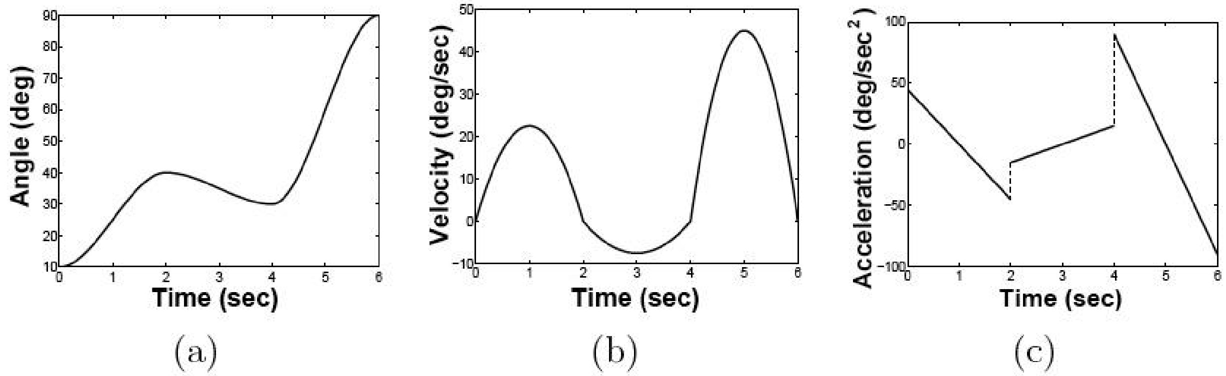


Figure 5.18: (a) Cubic spline trajectory made from three cubic polynomials. (b) Velocity profile for multiple cubic polynomial trajectory. (c) Acceleration profile for multiple cubic polynomial trajectory.

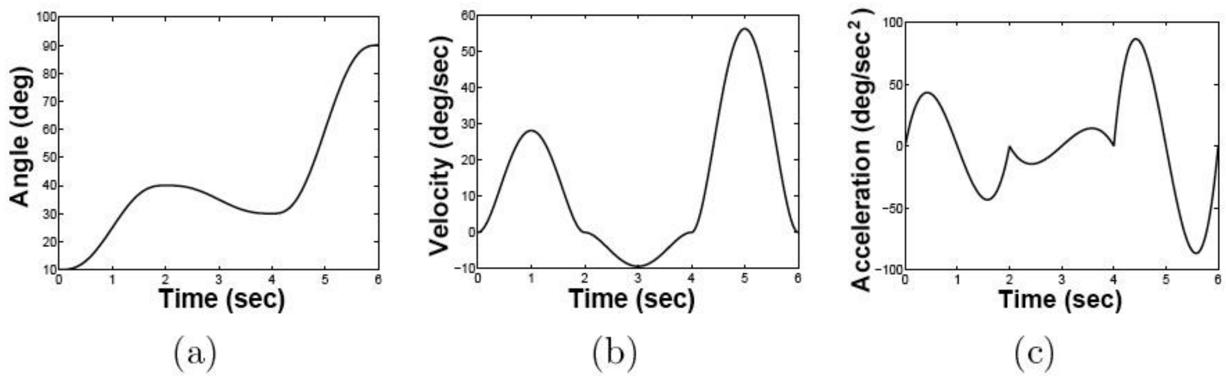


Figure 5.19: (a) Trajectory with multiple quintic segments. (b) Velocity profile for multiple quintic segments. (c) Acceleration profile for multiple quintic segments.