# FUNDAMENTAL THEOREM OF THE LOCAL THEORY OF CURVES

KAIXIN WANG

Abstract. In this expository paper, we present the fundamental theorem of the local theory of curves along with a detailed proof. We first prove the local existence and uniqueness of solutions to ordinary differential equations by applying contraction mapping theorem. Then, we use the local existence and uniqueness of solutions to ordinary differential equations to find a curve that satisfies the requirement and verify the uniqueness of the curve.

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## 1. Introduction

Classical differential geometry is the study of the local properties of curves and surfaces. This paper will focus on the regular curves in $\mathbb{R}^3$. We will first introduce several fundamental concepts that characterize the behavior of a curve in the neighborhood of a point. The question then arises naturally: how much knowledge is required to describe the local behavior of a curve completely? The fundamental theorem of the local theory of curves answers this question; it reveals the fact that every regular curve in $\mathbb{R}^3$ is completely determined by a positive differentiable function $\kappa(s)$ and a differentiable function $\tau(s)$.

This paper is devoted to presenting a detailed proof to the fundamental theorem of the local theory of curves. Since the local behavior of a curve is related to a system of ordinary differential equations, in order to find a required curve, it is necessary to first solve the specific ordinary differential equation. Thus, we will first study the space of the "putative solutions", and then apply the contraction mapping theorem to prove the existence of a unique solution to the differential equation in that space. Once we find the required curve, finally, we will prove its uniqueness up to a rigid motion.

## 2. Curves in $\mathbb{R}^3$

In $\mathbb{R}^3$, we define the norm as follows.

**Definitions 2.1.** Let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$. Define its *norm* by

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Since $\mathbb{R}^3$ is an inner product space, we also have the following operations.

**Definitions 2.2.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and let $\theta \in [0, \pi]$ be the angle formed by the segments $0\mathbf{u}$, $0\mathbf{v}$. The *inner product* of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos \theta.$$

**Definitions 2.3.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The *vector product* of $\mathbf{u}$ and $\mathbf{v}$ is the unique vector $\mathbf{u} \wedge \mathbf{v} \in \mathbb{R}^3$ such that

$$\forall \mathbf{w} \in \mathbb{R}^3,\ (\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}).$$

It is not hard to verify that

$$\|\mathbf{u} \wedge \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta.$$

Now, we are ready to introduce concepts of differentiable curves in $\mathbb{R}^3$.

**Definition 2.4.** A *parametrized differentiable curve* in $\mathbb{R}^3$ is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b) \subseteq \mathbb{R}$. A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be *regular* if for all $t \in I$, $\alpha'(t) \neq \mathbf{0}$.

From now on, we assume all curves in this paper are regular.

**Definition 2.5.** Given $t \in I$, the *arc length* of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ from $t_0$ is

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| dt$$

where

$$\|\alpha'(t)\| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}$$

is the length of the vector $\alpha'(t)$.

For a regular parametrized curve $\alpha$, since $\alpha'(t) \neq \mathbf{0}$, the arc length $s(t)$ is a differentiable function of $t$, and

$$\frac{ds}{dt} = \|\alpha'(t)\|.$$

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by the arc length $s \in I$. Then, we have

$$\|\alpha'(s)\| = \frac{ds}{ds} \equiv 1.$$

Since the tangent vector $\alpha'(s)$ has unit length, we can define the unit tangent vector as follows.

**Definition 2.6.** The *unit tangent vector* $\mathbf{t}$ of a curve $\alpha$ at $s$ is defined by

$$\mathbf{t}(s) = \alpha'(s).$$

**Definition 2.7.** The *curvature* $\kappa$ of a curve $\alpha$ at $s$ is defined by

$$\kappa(s) = \|\alpha''(s)\|.$$

Geometrically, the curvature gives a measure of how rapidly the curve pulls away from the tangent lines at $s$ in a neighborhood of $s$.

Since $\alpha'(s) \cdot \alpha'(s) = \|\alpha'(s)\|^2 = 1$, we have $\big(\alpha'(s) \cdot \alpha'(s)\big)' = 2\alpha''(s) \cdot \alpha'(s) = 0$. This implies that $\alpha''(s) \perp \alpha'(s)$ and hence motivates the following definition.

**Definition 2.8.** At points where the curvature $\kappa(s) \neq 0$, the *normal vector* $\mathbf{n}(s)$ of a curve $\alpha$ is defined by

$$\mathbf{n}(s) = \frac{\alpha''(s)}{\kappa(s)}.$$

The definition of the normal vector implies

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s).$$

Note that the normal vector $\mathbf{n}$ is orthonormal to the tangent vector $\mathbf{t}$. The plane determined by $\mathbf{t}(s)$ and $\mathbf{n}(s)$ is called the *osculating plane* at $s$.

**Definition 2.9.** The *binormal vector* $\mathbf{b}$ at $s$ of a curve $\alpha$ is defined by

$$\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s).$$

The image of a curve with the tangent, normal, and binormal vectors is represented below in Figure 1.

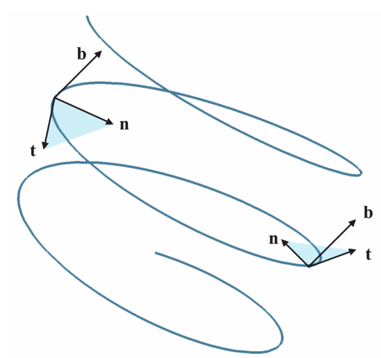


FIGURE 1. A curve in $\mathbb{R}^3$ with its tangent, normal, and binormal vectors.

It is important to note that, since $\mathbf{t}$ and $\mathbf{n}$ are unit vectors, $\mathbf{b}$ is the unit vector normal to the osculating plane. Also, we have

$$\begin{aligned}
\mathbf{b}'(s) &= \big(\mathbf{t}(s) \wedge \mathbf{n}(s)\big)' \\
&= \mathbf{t}'(s) \wedge \mathbf{n}(s) + \mathbf{t}(s) \wedge \mathbf{n}'(s) \\
&= \kappa(s)\mathbf{n}(s) \wedge \mathbf{n}(s) + \mathbf{t}(s) \wedge \mathbf{n}'(s) \\
&= \mathbf{t}(s) \wedge \mathbf{n}'(s).
\end{aligned}$$

It follows that the derivative of the binormal vector, $\mathbf{b}'$, is orthogonal to the tangent vector $\mathbf{t}$. Hence $\mathbf{b}'$ is parallel to the normal vector $\mathbf{n}$. Then, we have the following definition.

**Definition 2.10.** The *torsion* $\tau$ of a curve $\alpha$ at $s$ is defined by

$$\mathbf{b}'(s) = \tau(s)\mathbf{n}(s).$$

By definition, the torsion measures the change in direction of the binormal vector $\mathbf{b}$. Since $\mathbf{b}$ is orthogonal to the osculating plane, $\tau$ also measures how the osculating plane behaves in $\mathbb{R}^3$ space.

The derivatives of $\mathbf{t}(s)$ and $\mathbf{n}(s)$ give us the curvature $\kappa(s)$ and the torsion $\tau(s)$, which describe the geometric properties of $\alpha$ in a neighborhood of a point. It is natural to compute the derivative of the normal vector $\mathbf{n}'(s)$.

Since $\mathbf{b} \wedge \mathbf{t} = \mathbf{n}$, we have

$$\begin{aligned}\mathbf{n}'(s) &= \big(\mathbf{b}(s) \wedge \mathbf{t}(s)\big)' \\ &= \mathbf{b}'(s) \wedge \mathbf{t}(s) + \mathbf{b}(s) \wedge \mathbf{t}'(s) \\ &= \tau(s)\mathbf{n}(s) \wedge \mathbf{t}(s) + \mathbf{b}(s) \wedge \kappa(s)\mathbf{n}(s) \\ &= -\tau(s)\mathbf{b}(s) - \kappa(s)\mathbf{t}(s).\end{aligned}$$

**Lemma 2.11.** *The torsion $\tau$ of a curve $\alpha$ is given by*

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{\|\kappa(s)\|^2}.$$

*Proof.* By definition of $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$, we have,

$$-\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{\|\kappa(s)\|^2} = -\frac{\mathbf{t}(s) \wedge \mathbf{n}(s) \cdot \alpha'''(s)}{\kappa(s)} = -\frac{\mathbf{b}(s) \cdot \alpha'''(s)}{\kappa(s)}.$$

Note that

$$\alpha'''(s) = \big(\kappa(s)\mathbf{n}(s)\big)' = \kappa'(s)\mathbf{n}(s) + \kappa(s)\mathbf{n}'(s).$$

Since $\mathbf{b}$ is orthogonal to $\mathbf{n}$, $\mathbf{b}(s) \cdot \mathbf{n}(s) = 0$. Then,

$$\begin{aligned}-\frac{\mathbf{b}(s) \cdot \alpha'''(s)}{\kappa(s)} &= -\mathbf{b}(s) \cdot \mathbf{n}'(s) \\ &= -\mathbf{b}(s) \cdot \big(-\tau\mathbf{b}(s) - \kappa\mathbf{t}(s)\big) \\ &= \mathbf{b}(s) \cdot \tau(s)\mathbf{b}(s) + \mathbf{b}(s) \cdot \kappa(s)\mathbf{t}(s).\end{aligned}$$

Since $\mathbf{b}$ is orthonormal to the osculating plane,

$$\mathbf{b}(s) \cdot \tau(s)\mathbf{b}(s) + \mathbf{b}(s) \cdot \kappa(s)\mathbf{t}(s) = \tau(s)\|\mathbf{b}(s)\|^2 = \tau(s).$$

□

Let us summarize the concepts we have introduced. For a regular differentiable curve $\alpha : I \to \mathbb{R}^3$, we associate three orthogonal unit vectors $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ for all values of the parameter $s \in I$. The trihedron formed by these three vectors is called the *Frenet trihedron*. The relationship among these three vectors, which describes the local properties of a curve, can be summarized by the *Frenet formulas*:

$$\begin{cases}\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \\ \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) = \tau(s)\mathbf{n}(s)\end{cases}$$

One objective of this paper is to find, given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, a curve $\alpha(s)$ with the arc length $s$, curvature $\kappa(s)$ and torsion $\tau(s)$, satisfying these differential equations.

Let $\xi : I \to \mathbb{R}^9$ be a map such that $\xi(s) = \big(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\big)$. Then, we can rewrite Frenet's equations as a linear system

$$\xi'(s) = \begin{pmatrix} 0 & \kappa(s)I_3 & 0 \\ -\kappa(s)I_3 & 0 & -\tau(s)I_3 \\ 0 & \tau(s)I_3 & 0 \end{pmatrix} \xi(s),$$

where $I_3$ is the identity of the space of $3 \times 3$ matrices.

Let $M(s)$ be the corresponding matrix. We define a map $F : I \times \mathbb{R}^9 \to \mathbb{R}^9$ by

$$F\big(s, \xi(s)\big) = M(s)\xi(s) = \xi'(s).$$

Let $s_0 \in I$ and $\xi_0 = (\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0) \in \mathbb{R}^9$ be the initial conditions. Now, we obtain an initial value problem

$$\xi'(s) = F\big(s, \xi(s)\big), \quad \xi(s_0) = \xi_0. \tag{2.12}$$

Once this ordinary differential equation is solved, we can obtain a family of differentiable maps $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ satisfying Frenet's equations such that $\mathbf{t}(s_0) = \mathbf{t}_0, \mathbf{n}(s_0) = \mathbf{n}_0, \mathbf{b}(s_0) = \mathbf{b}_0$. Based on the discussion above, our first step is to prove the existence of a solution to the given initial value problem.

## 3. The Space of Continuous Functions

Before proving the existence and uniqueness of the solution to (2.12), we will first consider solutions to the general initial value problem:

$$x'(t) = F\big(x(t)\big), \quad x(0) = x_0, \tag{3.1}$$

where $x(t)$ is a function from $I \subset \mathbb{R}$ to $\mathbb{R}^d$. To do this, we first study the spaces in which a putative solution to (3.1) can be found. Clearly, the solution $x$ is a differentiable map.

**Definition 3.2.** If $V \subset \mathbb{R}$, then

$$C(V, \mathbb{R}^d) = \{f : V \to \mathbb{R}^d \mid f \text{ is continuous}\},$$
$$C^1(V, \mathbb{R}^d) = \{f : V \to \mathbb{R}^d \mid f' \text{ is continuous}\},$$
$$C_b(V, \mathbb{R}^d) = \{f \in C(V, \mathbb{R}^d) \mid f \text{ is bounded}\}.$$

**Definition 3.3.** For $f \in C_b(V, \mathbb{R}^d)$, we define the norm $\|\cdot\|_\infty$ by

$$\|f\|_\infty = \sup_V \|f\|.$$

**Theorem 3.4.** *The space $C_b(V, \mathbb{R}^d)$ with the norm $\|\cdot\|_\infty$ is complete.*

*Proof.* Let $\{f_n\}$ be a Cauchy sequence in $C_b(V, \mathbb{R}^d)$. Since $\mathbb{R}^d$ is complete, for each $a \in V$ the sequence $\{f_n(a)\}$ converges in $\mathbb{R}^d$. Then, for each $a \in V$, we may define

$$f(a) = \lim_{n\to\infty} f_n(a).$$

We need to show that $f_n$ converges uniformly to $f$ and $f$ is in $C_b(V, \mathbb{R}^d)$. First, we show that $f_n$ converges uniformly to $f$; that is $\|f - f_n\|_\infty \to 0$ as $n \to \infty$.

Take arbitrary $\epsilon > 0$. Choose $N$ such that for all $n, m \geq N$, $\|f_n - f_m\|_\infty < \frac{\epsilon}{2}$. Then, we have

$$\|f - f_n\|_\infty \leq \|f - f_N\|_\infty + \|f_N - f_n\|.$$

For $n \geq N$, we have $\|f_n - f_N\|_\infty < \frac{\epsilon}{2}$. Hence,

$$\|f - f_N\|_\infty = \lim_{n\to\infty} \|f_n - f_N\|_\infty \leq \frac{\epsilon}{2} \implies \|f - f_n\|_\infty \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon$ is arbitrary, we conclude that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$.

Now, we need to show that $f$ is in $C_b(V, \mathbb{R}^d)$; that is $f$ is continuous and bounded. Choose an arbitrary $a$ in $V$. Then, we have

$$\|f(a)\| \leq \|f(a) - f_N(a)\| + \|f_N(a)\| \leq \|f - f_N\|_\infty + \|f_N\|_\infty < \infty.$$

Thus, $f$ is bounded.

Finally, we show $f$ is continuous. We need to show that

$$\forall \epsilon > 0,\ \exists \delta,\ s.t.\ \forall x, y \in V, \|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon.$$

Note that

$$\|f(x) - f(y)\| \leq \|f(x) - f_n(x)\| + \|f_n(x) - f_n(y)\| + \|f_n(y) - f(y)\|.$$

Take arbitrary $\epsilon > 0$. Since $f_n$ converges uniformly to $f$, we can choose $N$ such that for all $n \geq N$, $\|f(x) - f_n(x)\| \leq \frac{\epsilon}{3}$ for all $x \in V$. Since $f_n$ is continuous, we can find $\delta > 0$ such that if $\|x - y\| \leq \delta$ then $\|f_n(x) - f_n(y)\| \leq \frac{\epsilon}{3}$. Thus, if $\|x - y\| < \delta$, then

$$\begin{aligned}\|f(x) - f(y)\| &\leq \|f(x) - f_N(x)\| + \|f_N(x) - f_N(y)\| + \|f_N(y) - f(y)\| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Hence, $f$ is continuous. Therefore, $C_b(V, \mathbb{R}^d)$ is complete. □

From now on, we assume $V = [a, b] \subset \mathbb{R}$. Since $V$ is compact in $\mathbb{R}$, for any $f \in C(V, \mathbb{R}^d)$, $f$ is bounded. This implies that $C(V, \mathbb{R}^d) = C_b(V, \mathbb{R}^d)$. If (3.1) has a solution $x(t) \in C(V, \mathbb{R}^d)$, then by the fundamental theorem of calculus, we have

$$x(t) = x_0 + \int_a^t F(x(s))ds.$$

We can thus define a map $A : C(V, \mathbb{R}^d) \to C(V, \mathbb{R}^d)$ such that

$$Ax(t) = x_0 + \int_a^t F(x(s))ds.$$

Note that $Ax(t)$ is also differentiable. So, solving (3.1) is the same as solving

$$Ax(t) = x(t).$$

**Definition 3.5.** Let $A : X \to X$ be a mapping of a metric space $(X, d)$ to itself. A point $x \in X$ is called a *fixed point* of $A$ if $Ax = x$.

Note that the space of "putative solutions", $C_b([a, b], \mathbb{R}^d)$, is a complete space. In a complete space, we can apply some theorems to show whether a map has a fixed point and whether the fixed point is unique.

**Definition 3.6.** Let $A : X \to X$ be a mapping of a metric space $(X, d)$ to itself. $A$ is a *contraction mapping* if

$$\exists \lambda \in (0, 1),\ \forall x, y \in X,\ d(Ax, Ay) \leq \lambda d(x, y).$$

**Theorem 3.7.** *(Contraction Mapping Theorem) Let $(X, d)$ be a complete metric space and $A : X \to X$ be a contraction mapping. Then, $A$ has a unique fixed point $x \in X$.*

*Proof.* Let $d(x, Ax) = d$. Then, there exists $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(x, y) \leq \lambda d(Ax, Ay).$$

It follows that

$$d(A^n x, A^{n+1} x) \leq \lambda^n d.$$

Since the series

$$\sum_{n=1}^{\infty} \lambda^n$$

converges, the sequence $\{A^n x\}$ is a Cauchy sequnece. We know that $X$ is complete, so for some $x_0 \in X$, $A^n x \to x_0$ as $n \to \infty$. Then, we have

$$Ax_0 = A(\lim_{n\to\infty} A^n x) = \lim_{n\to\infty} A^{n+1} x = x_0.$$

Thus, $x_0$ is a fixed point of $A$. Now, suppose $x_0$ and $x_1$ are two fixed point of $A$. Then, $d(x_0, x_1) = d(Ax_0, Ax_1) \leq \lambda d(x_0, x_1)$. This implies $x_0 = x_1$. So, $A$ has a unique fixed point. $\square$

We can extend the contraction mapping theorem to get the following corollary.

**Corollary 3.8.** *Let $(X, d)$ be a complete metrix space and $A : X \to X$ be a map. Assume that there is $n \in \mathbb{N}$ such that $A^n$ is a contraction. Then, there is a unique fixed point of $A$.*

*Proof.* Since $A^n$ is a contraction, $A^n x_0 = x_0$ for some unique $x_0$. Then, we have

$$A(A^n x_0) = Ax_0$$

However, this implies $A^n(Ax_0) = Ax_0$ and thus $Ax_0$ is a fixed point of $A^n$. By the uniqueness of fixed point of a contraction mapping, we have $Ax_0 = x_0$. Therefore, $x_0$ is a unique fixed point of $A$. $\square$

## 4. Local Existence and Uniqueness of Solutions to ODEs

In the previous section, we argued that solving (3.1) is the same as finding a unique fixed point for the operator $A$. We also showed that if $A^n$ is a contraction mapping, then $A$ has a unique fixed point. If the function $F$ in (3.1) exhibits some special properties such that $A$ is a contraction mapping, then we are done. We first need to introduce the following properties.

**Definition 4.1.** A function $F : \mathbb{R}^d \to \mathbb{R}^d$ is *globally Lipschitz* if

$$\exists L > 0,\ \forall y, z \in \mathbb{R}^d,\ \|F(y) - F(z)\| \leq L\|y - z\|.$$

**Definition 4.2.** A function $F : \mathbb{R}^d \to \mathbb{R}^d$ is *locally Lipschitz* if

$$\forall x \in \mathbb{R}^d,\ \exists L > 0,\ \forall y, z \in B(x, 1),\ \|F(y) - F(z)\| \leq L\|y - z\|.$$

Using the triangle inequality, we can equivalently define $F$ is locally Lipschitz if

$$\forall x \in \mathbb{R}^d,\ \forall R > 0,\ \exists L = L(R),\ \forall y, z \in B(x, R),\ \|F(y) - F(z)\| \leq L\|y - z\|.$$

We can now state the following existence and uniqueness result for solutions of ordinary differential equations.

**Theorem 4.3.** *Let $F : I \times \mathbb{R}^d \to \mathbb{R}^d$ be a function, $I \subset \mathbb{R}$ be an open interval. Choose $T_1, T_2$ in $\mathbb{R}$ such that $t_0 \in [T_1, T_2] \subset I$. Assume*

*(1) $F$ is continuous;*
*(2) $F(t, x)$ is globally Lipschitz in second variable:*

$$L(t) = \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{\|F(t,x) - F(t,y)\|}{\|x - y\|}.$$

*(3) The definite integral of $L(t)$ on $[T_1, T_2]$ is finite:*

$$\int_{T_1}^{T_2} L(t) dt < \infty.$$

*Then, there is a unique function $x \in C^1([T_1, T_2], \mathbb{R}^d)$ that solves the initial value problem*

$$x'(t) = F\big(t, x(t)\big), \quad x(t_0) = x_0.$$

*Proof.* We will prove this for $t_0 = T_1 < T_2$ and get a solution in $C^1([t_0, T_2], \mathbb{R}^d)$. The same argument will work for $T_1 < T_2 = t_0$. Then, glue the two solutions together to get a solution in $C^1([T_1, T_2], \mathbb{R}^d)$.

Suppose $t_0 = T_1 < T_2$. Define the operator $A : C([t_0, T_2], \mathbb{R}^d) \to C([t_0, T_2], \mathbb{R}^d)$ by

$$Ax(t) = x_0 + \int_{t_0}^{t} F\big(s, x(s)\big) ds.$$

Since $[t_0, T_2]$ is compact and functions defined on a compact set are bounded, the space $C([t_0, T_2], \mathbb{R}^d)$ with the norm $\|\cdot\|_\infty$ is complete. Note that the fixed points of $A$ are exactly the solutions to the initial value problem. If $A$ is a contraction mapping on the complete space $C([t_0, T_2], \mathbb{R}^d)$, then we are done. Therefore, our goal is to prove $A$ is a contraction mapping.

We claim that

$$\|A^m x(t) - A^m y(t)\| \le \frac{\big(\int_{t_0}^{t} L(s) ds\big)^m}{m!} \|x - y\|_\infty.$$

We show this by induction. For the base case, let $m = 1$. By the definition of $A$, we know

$$\begin{aligned} Ax(t) - Ay(t) &= \int_{t_0}^{t} F\big(s, x(s)\big) ds - \int_{t_0}^{t} F\big(s, y(s)\big) ds \\ &= \int_{t_0}^{t} F\big(s, x(s)\big) - F\big(s, y(s)\big) ds. \end{aligned}$$

Since $F$ is globally Lipschitz in the second variable, we have

$$\|F\big(s, x(s)\big) - F\big(s, y(s)\big)\| \le L(s) \|x(s) - y(s)\|$$

for some $L(s)$.

It follows that for any $t \in [t_0, T_2]$,

$$\begin{aligned}
\|Ax(t) - Ay(t)\| &= \|\int_{t_0}^{t} F\big(s, x(s)\big) - F\big(s, y(s)\big) ds\| \\
&\leq \int_{t_0}^{t} \|F\big(s, x(s)\big) - F\big(s, y(s)\big)\| ds \\
&\leq \int_{t_0}^{t} L(s)\|x(s) - y(s)\| ds \\
&\leq \int_{t_0}^{t} L(s)\|x - y\|_\infty ds \\
&= \Big(\int_{t_0}^{t} L(s) ds\Big)\|x - y\|_\infty .
\end{aligned}$$

Hence, the claim is true for $m = 1$.

Now, suppose our claim is true for $m$. For $m+1$, by the same argument in the base case, we have

$$\begin{aligned}
\|A^{m+1}x(t) - A^{m+1}y(t)\| &\leq \int_{t_0}^{t} \|F\big(s, A^m x(s)\big) - F\big(s, A^m y(s)\big)\| ds \\
&\leq \int_{t_0}^{t} L(s)\|A^m x(s) - A^m y(s)\| ds \\
&\leq \int_{t_0}^{t} L(s)\Big(\int_{t_0}^{s} L(r) dr\Big)^m \frac{\|x - y\|_\infty}{m!} ds \\
&= \frac{\|x - y\|_\infty}{m!} \int_{t_0}^{t} L(s)\Big(\int_{t_0}^{s} L(r) dr\Big)^m ds
\end{aligned}$$

for some $L(s)$.

Let $L_1(s) = \int_{t_0}^{s} L(r) dr$. Then, we have $L_1'(s) = L(s)$. So,

$$\begin{aligned}
\|A^{m+1}x(t) - A^{m+1}y(t)\| &\leq \frac{\|x - y\|_\infty}{m!} \int_{t_0}^{t} L_1'(s)\big(L_1(s)\big)^m ds \\
&= \frac{\|x - y\|_\infty}{m!} \int_{t_0}^{t} \big(L_1(s)\big)^m dL_1(s) \\
&= \frac{\big(L_1(s)\big)^{m+1}}{(m+1)!}\|x - y\|_\infty .
\end{aligned}$$

By induction, the statement is true for all $m \in \mathbb{N}$.

Hence, for sufficiently large $m$,

$$\|A^m x - A^m y\|_\infty = \lambda\|x - y\|_\infty$$

for some $\lambda \in (0, 1)$. Thus, we conclude that $A^m$ is a contraction, and by the contraction mapping theorem, $A^m$ has a unique fixed point. It follows that $A$ has a unique fixed point $x \in C([T_1, T_2], \mathbb{R}^d)$. By construction, $x$ is also in $C^1([t_0, T_2], \mathbb{R}^d)$. Therefore, it is the required solution to the initial value problem. □

**Corollary 4.4.** *If for every $[T_1, T_2] \subset I$ containing $t_0$, the hypotheses of Theorem 4.3 hold, then there is a unique function $x \in C^1(I, \mathbb{R}^n)$ that solves the initial value problem.*

## 5. Fundamental Theorem of the Local Theory of Curves

Before we prove the theorem, we will first introduce the concept of rigid motion and prove that the arc length, the curvature, and the torsion of a parametrized curve are invariant under rigid motion.

**Definition 5.1.** A *translation* by a vector $\mathbf{v}$ in $\mathbb{R}^3$ is a map $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $A\mathbf{u} = \mathbf{u} + \mathbf{v}$.

**Definition 5.2.** A linear map $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an *orthogonal transformation* if $\rho\mathbf{u} \cdot \rho\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

**Definition 5.3.** A *rigid motion* in $\mathbb{R}^3$ is the result of composing a translation with an orthogonal transformation with positive determinant; that is, if $A$ is a rigid motion, then there exists an orthogonal linear map $\rho$ of $\mathbb{R}^3$, with positive determinant, and a vector $\mathbf{c}$ such that $A\mathbf{v} = \rho\mathbf{v} + \mathbf{c}$.

It is notable that rigid motion preserves the arc length, curvature, and torsion of a curve. To see this, we need the following lemmas.

**Lemma 5.4.** *The norm of a vector and the angle $\theta$ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.*

*Proof.* Let $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation with positive determinant. Choose an arbitrary $\mathbf{v}$ in $\mathbb{R}^3$. Then, we have

$$\|\rho\mathbf{v}\|^2 = \rho\mathbf{v} \cdot \rho\mathbf{v} = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$

Thus, the norm is invariant under $\rho$. Choose arbitrary $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$ and let $\theta$ be the angle between them. The angle $\bar{\theta}$ between $\rho\mathbf{u}$ and $\rho\mathbf{v}$ is given by

$$\cos\bar{\theta} = \frac{\rho\mathbf{u} \cdot \rho\mathbf{v}}{\|\rho\mathbf{u}\|\|\rho\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos\theta$$

Hence, the angle between two vectors is also invariant under $\rho$. $\square$

**Lemma 5.5.** *The vector product of two vectors is invariant under orthogonal transformations with positive determinant.*

*Proof.* Let $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation with positive determinant. Choose arbitrary $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$ and let $\theta$ be the angle between them. Since $\det(\rho) > 0$, for all $\mathbf{w} \in \mathbb{R}^3$, there is $\bar{\mathbf{w}} \in \mathbb{R}^3$ such that $\rho\bar{\mathbf{w}} = \mathbf{w}$. Thus,

$$(\rho\mathbf{u} \wedge \rho\mathbf{v}) \cdot \mathbf{w} = \det(\rho\mathbf{u}, \rho\mathbf{v}, \rho\bar{\mathbf{w}}) = \det(\rho)\det(\mathbf{u}, \mathbf{v}, \bar{\mathbf{w}}) = \det(\rho)(\mathbf{u} \wedge \mathbf{v}) \cdot \bar{\mathbf{w}}$$

Since $\rho$ is an orthogonal transformation, $(\mathbf{u} \wedge \mathbf{v}) \cdot \bar{\mathbf{w}} = \rho(\mathbf{u} \wedge \mathbf{v}) \cdot \rho\bar{\mathbf{w}} = \rho(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}$. Then

$$(\rho\mathbf{u} \wedge \rho\mathbf{v}) \cdot \mathbf{w} = \det(\rho)\rho(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} \implies \rho\mathbf{u} \wedge \rho\mathbf{v} = \det(\rho)\rho(\mathbf{u} \wedge \mathbf{v})$$

Also, we have

$$\|\rho\mathbf{u} \wedge \rho\mathbf{v}\|^2 = \|\rho\mathbf{u}\|^2\|\rho\mathbf{v}\|^2 \sin^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2\theta = \|\mathbf{u} \wedge \mathbf{v}\|^2 = \big(\det(\rho)\big)^2\|\mathbf{u} \wedge \mathbf{v}\|^2$$

Since $\det(\rho) > 0$, we have $\det(\rho) = 1$ and hence $\rho\mathbf{u} \wedge \rho\mathbf{v} = \rho(\mathbf{u} \wedge \mathbf{v})$. $\square$

**Theorem 5.6.** *The arc length, the curvature, and the torsion of a parametrized curve are invariant under rigid motion.*

*Proof.* Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and $A : \mathbb{R}^3 \to \mathbb{R}^3$ a rigid motion. Let $s$ be its arc length, $\kappa$ its curvature, $\tau$ its torsion. Since translations and orthogonal transformations preserve norm, $A$ preserves norm. Thus, the arc length under $A$,

$$s_A(t) = \int_{t_0}^{t} \|A\alpha(t)\| dt = \int_{t_0}^{t} \|\alpha(t)\| = s(t)$$

is preserved. For curvature, we have $\kappa(s) = \|\alpha''(s)\|$. Since $M$ is a linear transformation, we have $\big(A\alpha(s)\big)'' = A\alpha''(s)$. Then, the curvature under A

$$\kappa_A(s) = \|\big(A\alpha(s)\big)''\| = \|A\alpha''(s)\| = \|\alpha''(s)\| = \kappa(s)$$

is preserved. For torsion, we have

$$\begin{aligned}\tau(s)\mathbf{n}(s) = \mathbf{b}'(s) &= \big(\mathbf{t}(s) \wedge \mathbf{n}(s)\big)' \\ &= \mathbf{t}'(s) \wedge \mathbf{n}(s) + \mathbf{t}(s) \wedge \mathbf{n}'(s) \\ &= \kappa(s)\mathbf{n}(s) \wedge \mathbf{n}(s) + \alpha'(s) \wedge \mathbf{n}'(s) \\ &= \alpha'(s) \wedge \mathbf{n}'(s)\end{aligned}$$

Then, $\tau_A(s)A\mathbf{n}(s) = A\alpha'(s) \wedge A\mathbf{n}'(s) = A\big(\alpha'(s) \wedge \mathbf{n}'(s)\big) = A\big(\tau(s)\mathbf{n}(s)\big) = \tau(s)A\mathbf{n}(s)$. Thus, $\tau_A(s) = \tau(s)$. So, the torsion under $A$ is also preserved. $\square$

We are now ready to prove the fundamental theorem of the local theory of curves.

**Theorem 5.7.** *(Fundamental Theorem of the Local Theory of Curves) Given differentiable functions $\kappa(s) > 0$ and $\tau(s), s \in I$, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that $s$ is the arc length, $\kappa(s)$ is the curvature, and $\tau(s)$ is the torsion of $\alpha$. Moreover, any other curve $\tilde{\alpha}$, satisfying the same conditions, differs from $\alpha$ by a rigid motion.*

*Proof.* First, we show existence. We have already showed that the Frenet's equations

$$\begin{cases}\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \\ \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s) \\ \mathbf{b}(s) = \tau(s)\mathbf{n}(s)\end{cases}$$

can be rewritten as a linear system

$$\xi'(s) = \begin{pmatrix} 0 & \kappa(s)I_3 & 0 \\ -\kappa(s)I_3 & 0 & -\tau(s)I_3 \\ 0 & \tau(s)I_3 & 0 \end{pmatrix} \xi(s) = M(s)\xi(s)$$

for some matrix $M(s)$. Given initial conditions $\mathbf{t}_0$, $\mathbf{n}_0$, $\mathbf{b}_0$ and $s_0 \in I$, this system is equivalent to the initial value problem

$$\xi'(s) = F\big(s, \xi(s)\big), \quad \xi(s_0) = \xi_0,$$

where $\xi : I \to \mathbb{R}^9$ is a map such that $\xi(s) = \big(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\big)$, $\xi_0 = (\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0)$, and $F : I \times \mathbb{R}^9 \to \mathbb{R}^9$ is a map such that $F\big(s, \xi(s)\big) = M(s)\xi(s)$.

Since $F$ is a linear transformation, it is globally Lipschitz in the second variable. $F$ is differentiable as $\kappa$ and $\tau$ are differentiable. Moreover, $L$ is bounded on compact subsets of $I$. We can therefore apply the local existence and uniqueness of the solution to ODEs.

So, if we choose $s_0$ in $I$ and take $\{\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0\}$ to be an orthonormal, positively oriented triple of vectors in $\mathbb{R}^3$, then there exist unique functions $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s) : I \to \mathbb{R}^3$ solving Frenet equations with $\mathbf{t}(s_0) = \mathbf{t}_0$, $\mathbf{n}(s_0) = \mathbf{n}_0$, $\mathbf{b}(s_0) = \mathbf{b}_0$.

By Frenet's equations, we have

$$\begin{cases} (\mathbf{t}\cdot\mathbf{n})' = \mathbf{t}'\cdot\mathbf{n} + \mathbf{t}\cdot\mathbf{n}' = \kappa\mathbf{n}\cdot\mathbf{n} - \mathbf{t}\cdot\kappa\mathbf{t} - \mathbf{t}\cdot\tau\mathbf{b} \\ (\mathbf{t}\cdot\mathbf{b})' = \mathbf{t}'\cdot\mathbf{b} + \mathbf{t}\cdot\mathbf{b}' = \kappa\mathbf{n}\cdot\mathbf{b} + \mathbf{t}\cdot\tau\mathbf{n} \\ (\mathbf{n}\cdot\mathbf{b})' = \mathbf{n}'\cdot\mathbf{b} + \mathbf{n}\cdot\mathbf{b}' = -\tau\mathbf{b}\cdot\mathbf{b} - \kappa\mathbf{t}\cdot\mathbf{b} + \mathbf{n}\cdot\tau\mathbf{n} \\ (\mathbf{t}\cdot\mathbf{t})' = 2\mathbf{t}'\cdot\mathbf{t} = 2\kappa\mathbf{n}\cdot\mathbf{t} \\ (\mathbf{n}\cdot\mathbf{n})' = 2\mathbf{n}'\cdot\mathbf{n} = -2\kappa\mathbf{t}\cdot\mathbf{n} - 2\tau\mathbf{b}\cdot\mathbf{n} \\ (\mathbf{b}\cdot\mathbf{b})' = 2\mathbf{b}'\cdot\mathbf{b} = 2\tau\mathbf{n}\cdot\mathbf{b} \end{cases}$$

We can check that

$$\mathbf{t}\cdot\mathbf{n} \equiv 0, \quad \mathbf{t}\cdot\mathbf{b} \equiv 0, \quad \mathbf{n}\cdot\mathbf{b} \equiv 0, \quad \mathbf{t}\cdot\mathbf{t} \equiv 1, \quad \mathbf{n}\cdot\mathbf{n} \equiv 1, \quad \mathbf{b}\cdot\mathbf{t} \equiv 1$$

is a solution to the above system with initial conditions $0, 0, 0, 1, 1, 1$. By uniqueness, $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is orthonormal for every $s \in I$. We can thus obtain a curve by setting

$$\alpha(s) = \int_{s\in I} \mathbf{t}(s) ds.$$

It is clear that $\alpha'(s) = \mathbf{t}(s)$ and $\alpha''(s) = \kappa(s)\mathbf{n}(s)$. Since

$$\begin{aligned} \alpha'''(s) &= \big(\kappa(s)\mathbf{n}(s)\big)' = \kappa'(s)\mathbf{n}(s) + \kappa(s)\mathbf{n}'(s) \\ &= \kappa'(s)\mathbf{n}(s) - \kappa(s)\tau(s)\mathbf{b}(s) - \kappa^2(s)\mathbf{t}(s), \end{aligned}$$

for the torsion of $\alpha$, we have

$$\begin{aligned} -\frac{\alpha'(s)\wedge\alpha''(s)\cdot\alpha'''(s)}{|\kappa(s)|^2} &= -\frac{\mathbf{t}(s)\wedge\kappa(s)\mathbf{n}(s)\cdot\big(\kappa'(s)\mathbf{n}(s) - \kappa(s)\tau(s)\mathbf{b}(s) - \kappa^2(s)\mathbf{t}(s)\big)}{\kappa^2(s)} \\ &= \mathbf{t}(s)\wedge\mathbf{n}(s)\cdot\tau(s)\mathbf{b}(s) \\ &= \mathbf{b}(s)\cdot\tau(s)\mathbf{b}(s) = \tau(s). \end{aligned}$$

Therefore, $\alpha$ is the required curve.

Now, we show uniqueness. We have proved that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

Suppose that two curves $\alpha = \alpha(s)$, $\bar{\alpha} = \bar{\alpha}(s)$ satisfy the conditions $\kappa(s) = \bar{\kappa}(s)$ and $\tau(s) = \bar{\tau}(s)$ for $s \in I$. Let $\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0$ and $\bar{\mathbf{t}}_0, \bar{\mathbf{n}}_0, \bar{\mathbf{b}}_0$ be the Frenet trihedrons at $s = s_0 \in I$ of $\alpha$ and $\bar{\alpha}$ respectively. There is a rigid motion which takes $\bar{\alpha}(s_0) = \alpha(s_0)$ and $\bar{\mathbf{t}}_0, \bar{\mathbf{b}}_0, \bar{\mathbf{n}}_0$ to $\mathbf{t}_0, \mathbf{b}_0, \mathbf{n}_0$.

After performing this rigid motion on $\bar{\alpha}$, we have that

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} - \tau\mathbf{b}, \quad \frac{d\mathbf{b}}{ds} = \tau\mathbf{n}$$

$$\frac{d\bar{\mathbf{t}}}{ds} = \kappa\bar{\mathbf{n}}, \quad \frac{d\bar{\mathbf{n}}}{ds} = -\kappa\bar{\mathbf{t}} - \tau\bar{\mathbf{b}}, \quad \frac{d\bar{\mathbf{b}}}{ds} = \tau\bar{\mathbf{n}}$$

with $\mathbf{t}(s_0) = \bar{\mathbf{t}}(s_0)$, $\mathbf{n}(s_0) = \bar{\mathbf{n}}(s_0)$, $\mathbf{b}(s_0) = \bar{\mathbf{b}}(s_0)$.

By Frenet equations, for all $s \in I$, we have

$$\begin{aligned}
&\frac{d}{ds}\big(\|\mathbf{t}-\bar{\mathbf{t}}\|^2 + \|\mathbf{n}-\bar{\mathbf{n}}\|^2 + \|\mathbf{b}-\bar{\mathbf{b}}\|\big) \\
&\quad = 2(\mathbf{t}-\bar{\mathbf{t}})\cdot(\mathbf{t}'-\bar{\mathbf{t}}') + 2(\mathbf{b}-\bar{\mathbf{b}})\cdot(\mathbf{b}'-\bar{\mathbf{b}}') + 2(\mathbf{n}-\bar{\mathbf{n}})\cdot(\mathbf{n}'-\bar{\mathbf{n}}') \\
&\quad = 2\kappa(\mathbf{t}-\bar{\mathbf{t}})\cdot(\mathbf{n}-\bar{\mathbf{n}}) + 2\tau(\mathbf{b}'-\bar{\mathbf{b}}')\cdot(\mathbf{n}-\bar{\mathbf{n}}) \\
&\qquad - 2\kappa(\mathbf{n}-\bar{\mathbf{n}})\cdot(\mathbf{t}-\bar{\mathbf{t}}) - 2\tau\cdot(\mathbf{n}-\bar{\mathbf{n}})\cdot(\mathbf{b}'-\bar{\mathbf{b}}') \\
&\quad = 0.
\end{aligned}$$

Thus, $\|\mathbf{t}-\bar{\mathbf{t}}\|^2 + \|\mathbf{n}-\bar{\mathbf{n}}\|^2 + \|\mathbf{b}-\bar{\mathbf{b}}\|^2$ is constant. Since

$$\|\mathbf{t}(s_0)-\bar{\mathbf{t}}(s_0)\|^2 + \|\mathbf{n}(s_0)-\bar{\mathbf{n}}(s_0)\|^2 + \|\mathbf{b}-\bar{\mathbf{b}}(s_0)\|^2 = 0,$$

$\|\mathbf{t}-\bar{\mathbf{t}}\|^2 + \|\mathbf{n}-\bar{\mathbf{n}}\|^2 + \|\mathbf{b}-\bar{\mathbf{b}}\|^2$ is always 0. Thus, $\mathbf{t}=\bar{\mathbf{t}}$, $\mathbf{n}=\bar{\mathbf{n}}$, $\mathbf{b}=\bar{\mathbf{b}}$. Then,

$$\frac{d\alpha}{ds} = \mathbf{t} = \bar{\mathbf{t}} = \frac{d\bar{\alpha}}{dt} \implies \frac{d}{ds}(\alpha-\bar{\alpha}) = \mathbf{0} \implies \alpha = \bar{\alpha} + \mathbf{v}.$$

where $\mathbf{v}$ is a constant vector. Since $\alpha(s_0) = \bar{\alpha}(s_0)$, we have $\mathbf{v} = \mathbf{0}$. This concludes our proof. □

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## References

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