Design and Analysis of Algorithms

Part I: Divide and Conquer

Lecture 9: Heapsort, Lower Bound for Sorting and Sorting in Linear Time

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Outline

- Review to Divide-and-Conquer Paradigm
- Heapsort
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Comparison-based Sorting
  - Objective
  - Decision Tree Model
- Sorting in Linear Time
  - Counting Sort

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Review to Divide-and-Conquer Paradigm

- Divide-and-conquer (D&C) is an important algorithm design paradigm.
  - **Divide**

    Dividing a given problem into two or more subproblems (ideally of approximately equal size)
  - **Conquer**

    Solving each subproblem (directly if small enough or recursively)
  - **Combine**

    Combining the solutions of the subproblems into a global solution

Review to Divide-and-Conquer Paradigm

- In Part I, we will illustrate Divide-and-Conquer using several examples:
  - Maximum Contiguous Subarray (最大子数组)
  - Counting Inversions (逆序计数)
  - Polynomial Multiplication (多项式乘法)
  - QuickSort and Partition (快速排序与划分)
  - Randomized Selection (随机化选择)
  - Supplement Topic of Sorting (排序问题补充主题)
    - Heapsort (堆排序)
    - Lower Bound for Sorting (基于比较的排序下界)
    - Sorting in Linear Time (线性时间排序)

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Priority Queue: Motivating Example

3 jobs have been submitted to a printer in the order A, B, C. Consider the printing pool at this moment.

Sizes: Job A — 100 pages

Job B — 10 pages

Job C — 1 page

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Average finish time with FIFO service:

$$(100 + 110 + 111)\ /\ 3\ =\ 107\ time\ units$$

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Average finish time with FIFO service:

$$(100 + 110 + 111)\ /\ 3\ =\ 107\ time\ units$$

Average finish time for shortest-job-first service:

$$(1 + 11 + 111)\ /\ 3\ =\ 41\ time\ units$$

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- The elements in the queue are printing jobs, each with the associated number of pages that serves as its priority
- Processing the shortest job first corresponds to extracting the smallest element from the queue
- Insert new printing jobs as they arrive

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A queue is capable of supporting two operations: Insert and Extract-Min?

Priority Queue

Priority queue is an abstract data structure that supports two operations

- Insert: inserts the new element into the queue
- Extract-Min: removes and returns the smallest element from the queue

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in O(1) time
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Question

Is there any data structure that supports both these priority queue operations in $O(\log n)$ time?

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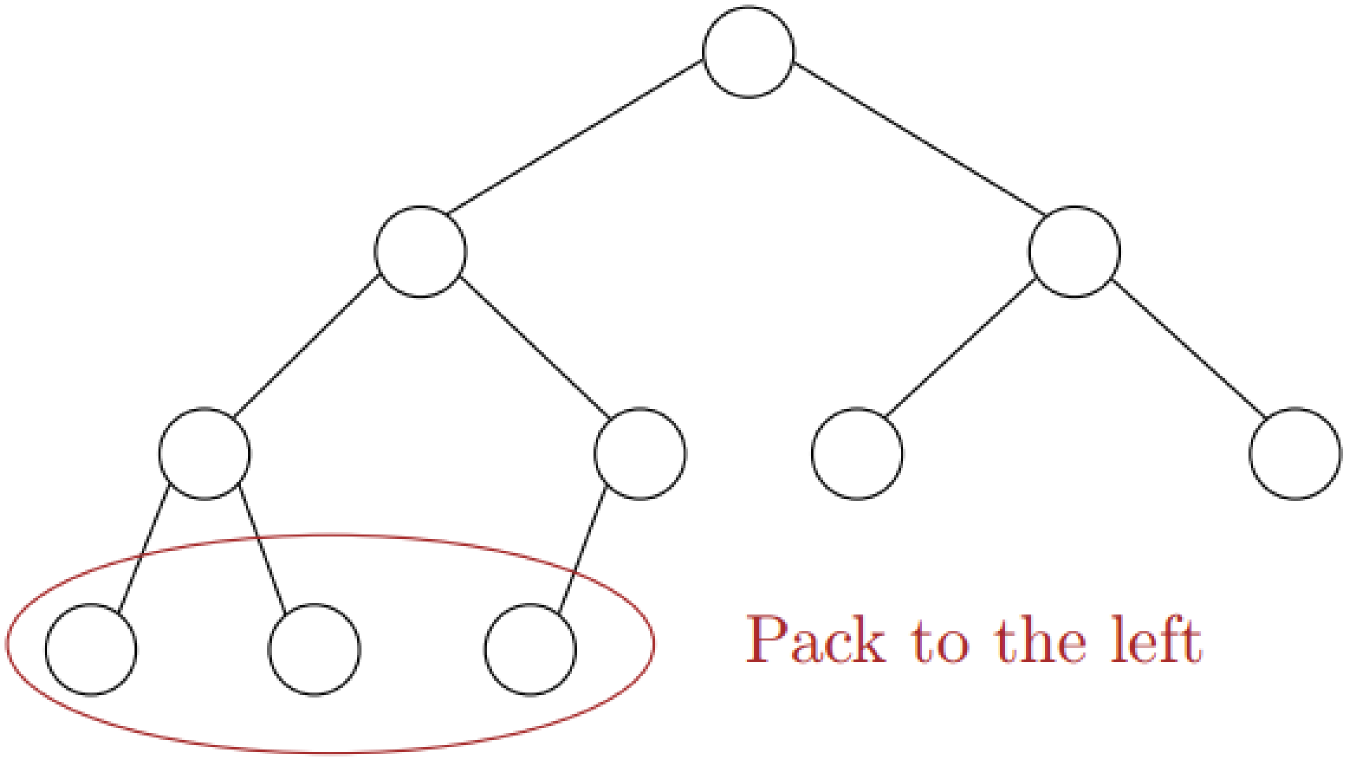
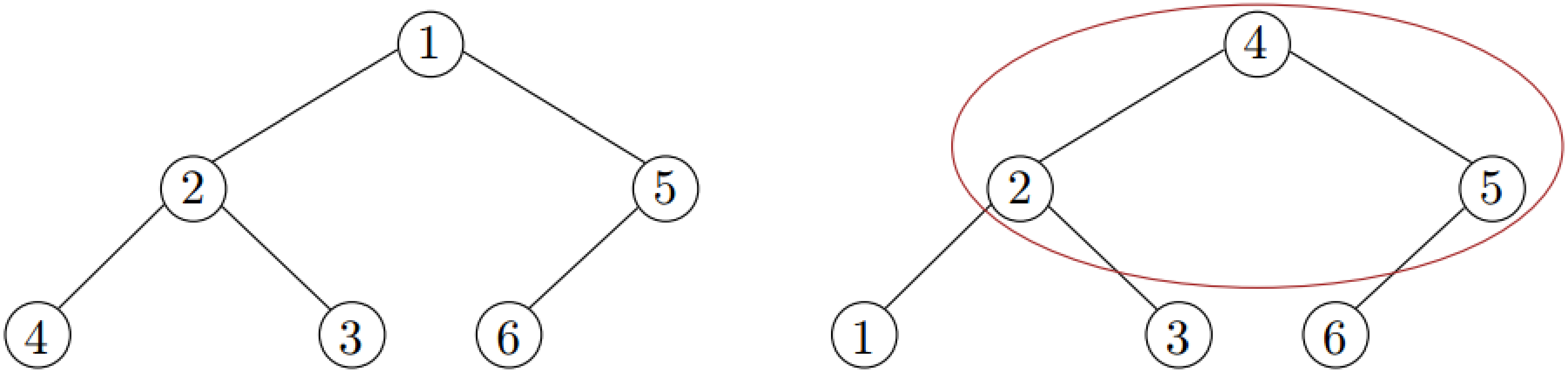
(Binary) Heap



Heaps are "almost complete binary trees"

- All levels are full except possibly the lowest level.
- If the lowest level is not full, then nodes must be packed to the left.

Heap-order Property



A min-heap   Not a heap

Heap-order property (*Min-heap*):

The value of a node is at least the value of its parent.

A[Parent(i)] ≤ A[i]

Heap Properties

- If the heap-order property is maintained, heaps support the following operations efficiently (assume there are n elements in the heap)
  - Insert in O(log n) time
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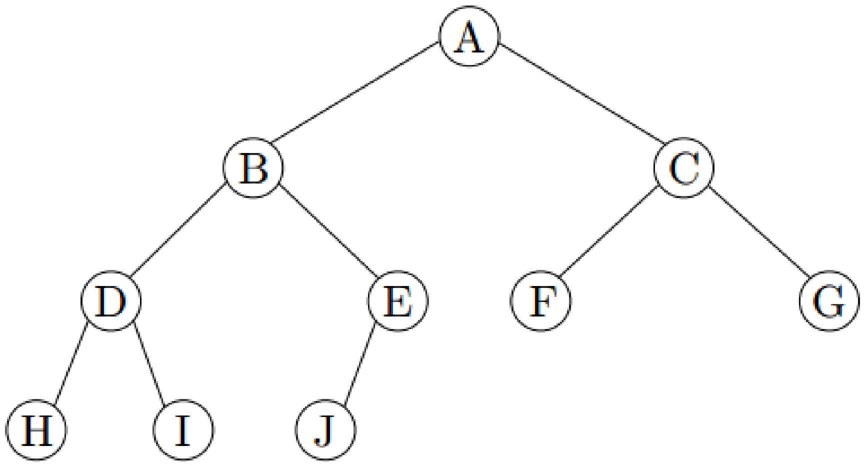
  - The structure is so regular, it can be represented in an array and no links are necessary !

Array Implementation of Heap

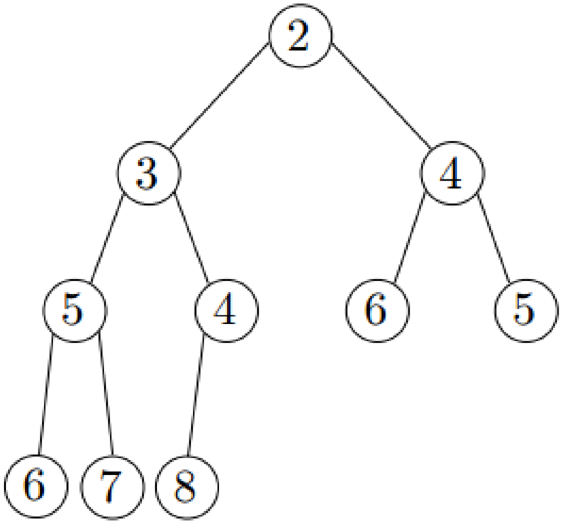


1	2	3	4	5	6	7	8	9	10
A	B	C	D	E	F	G	H	I	J

- The root is in array position 1.
- For any element in array position i,
  - The left child is in position 2i.
  - The right child is in position 2i+1.
  - The parent is in position $\lfloor i/2 \rfloor$.
- We will draw the heaps as trees, with the understanding that an actual implementation will use simple arrays.

Insertion

- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



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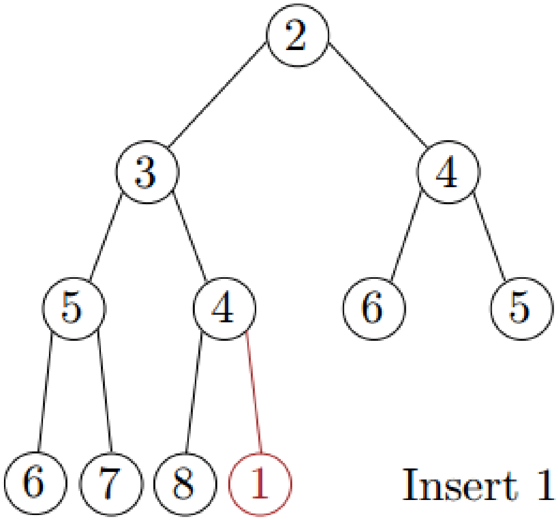

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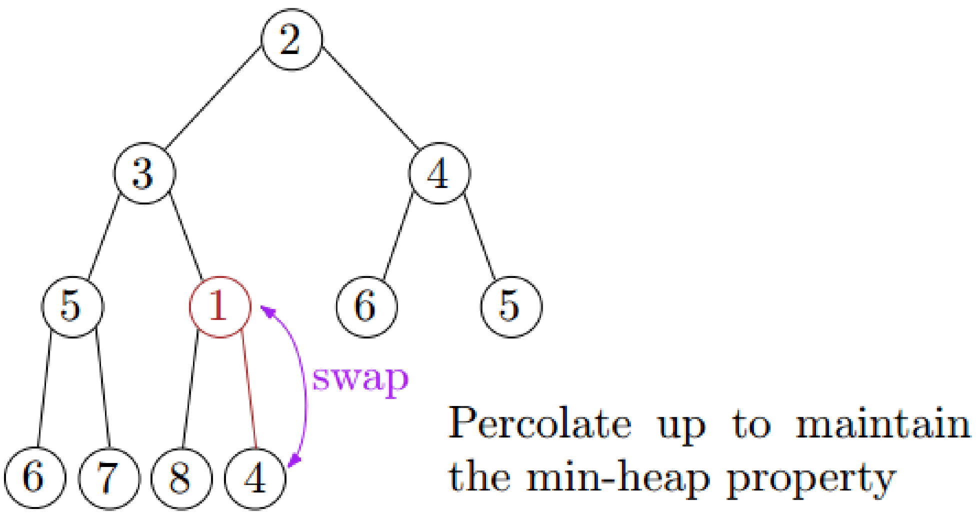

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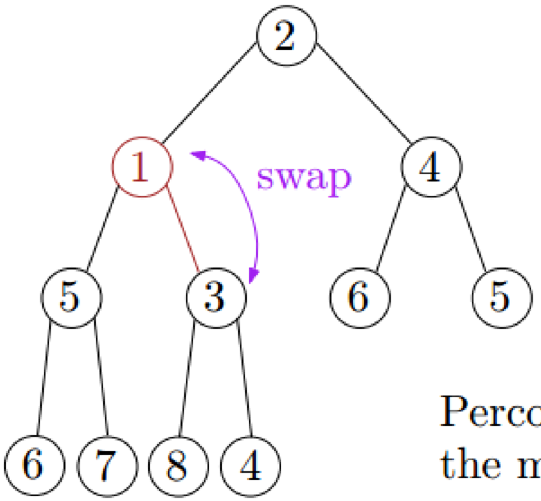

Percolate up to maintain the min-heap property

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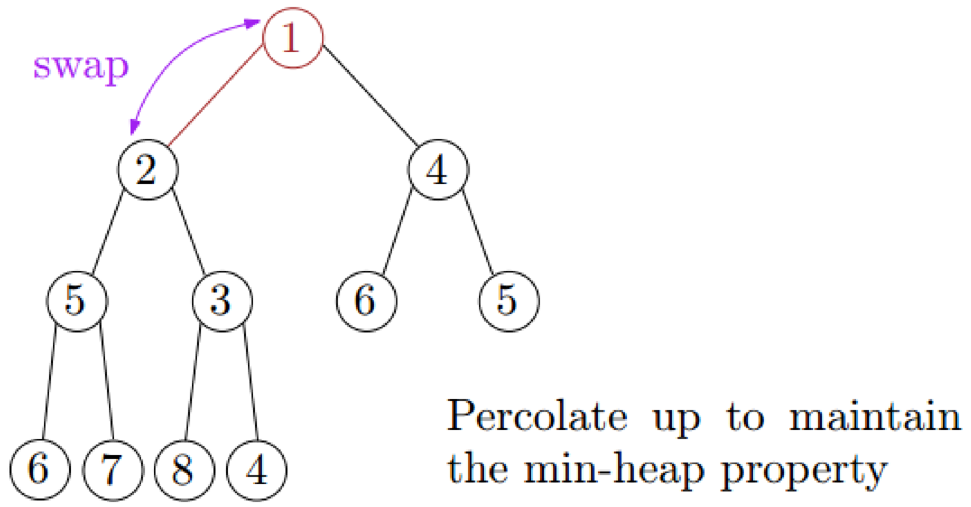

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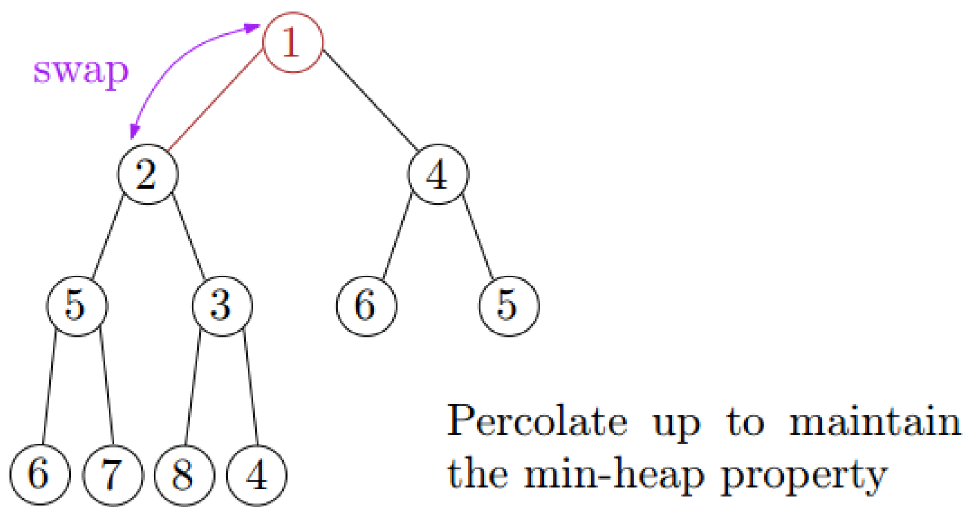

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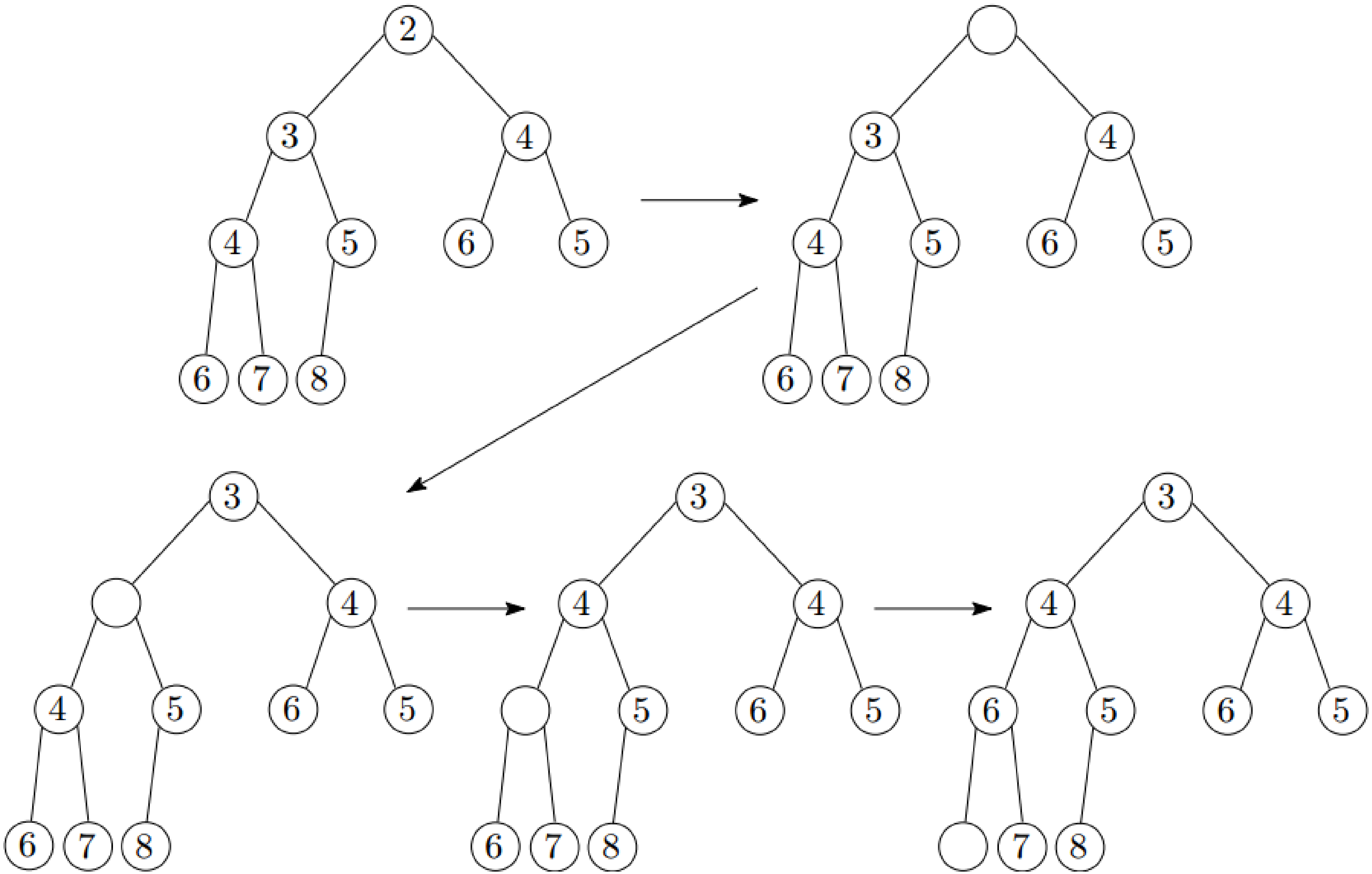
- Time complexity = O(height) = O(log n)

Extract-Min: First Attempt



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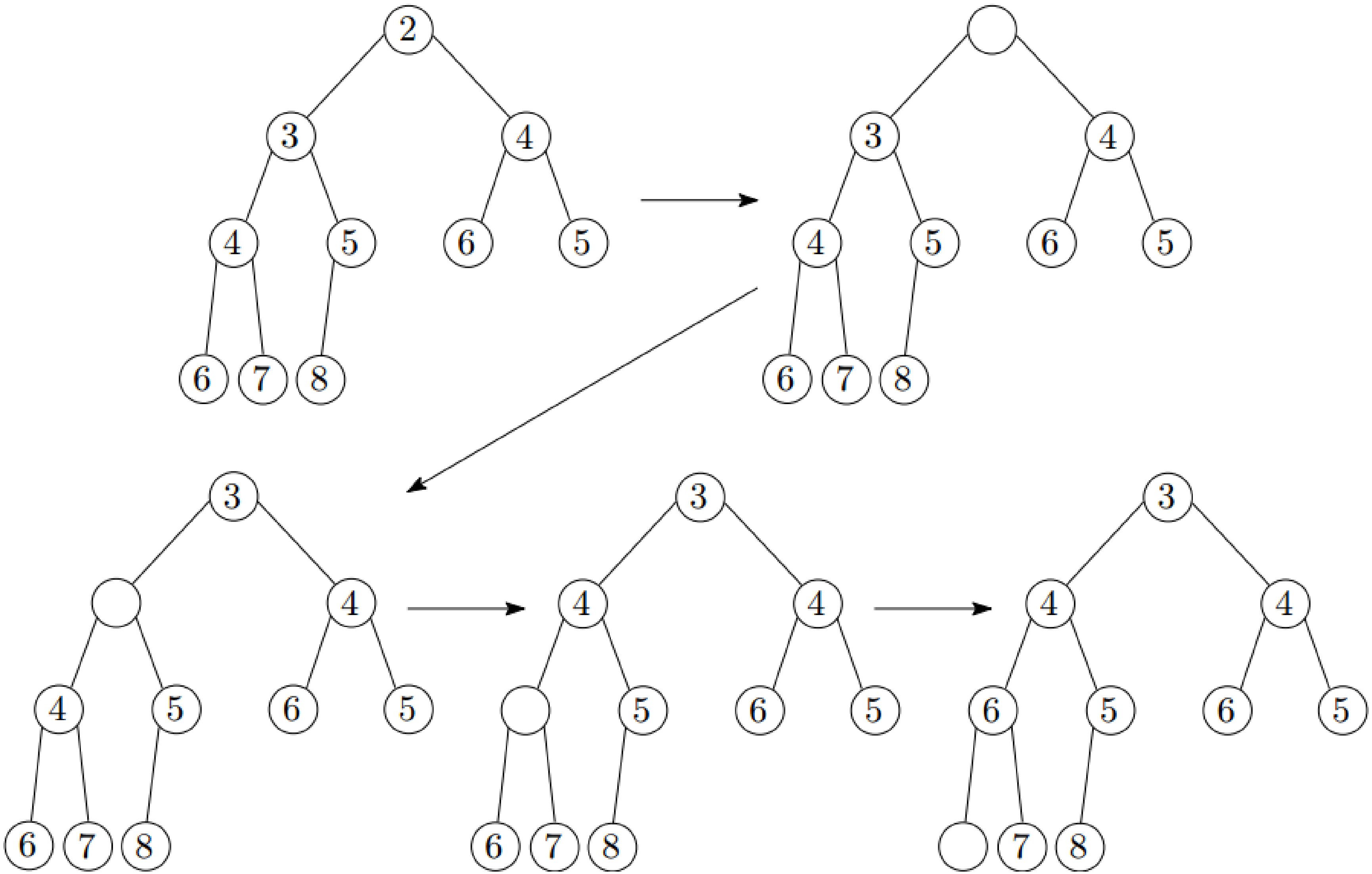
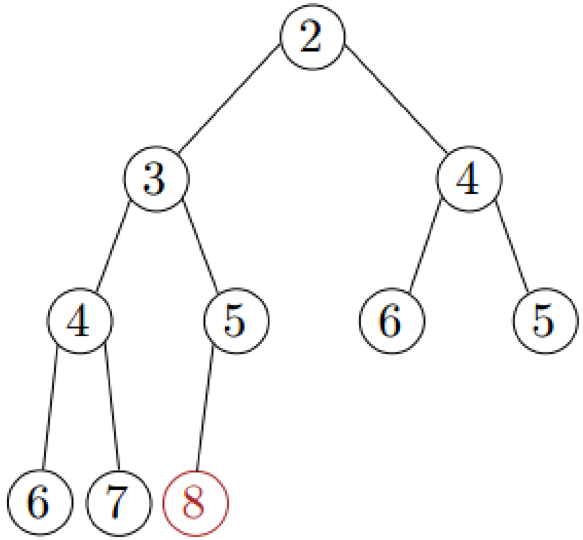


Min-heap property preserved, but completeness not preserved!

Extract-Min

- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



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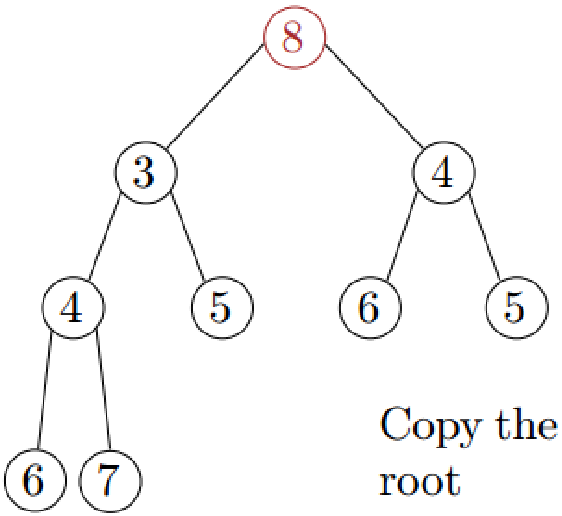

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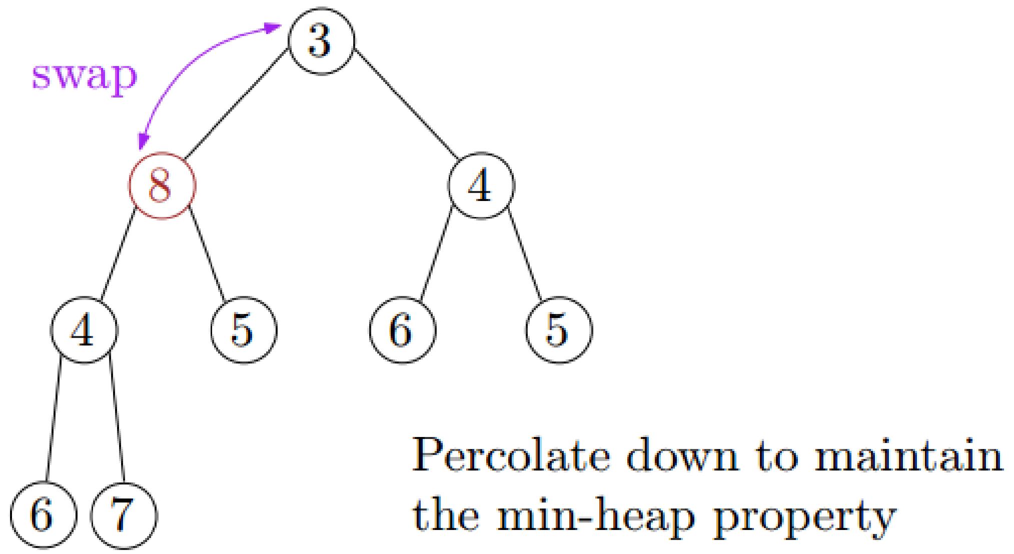

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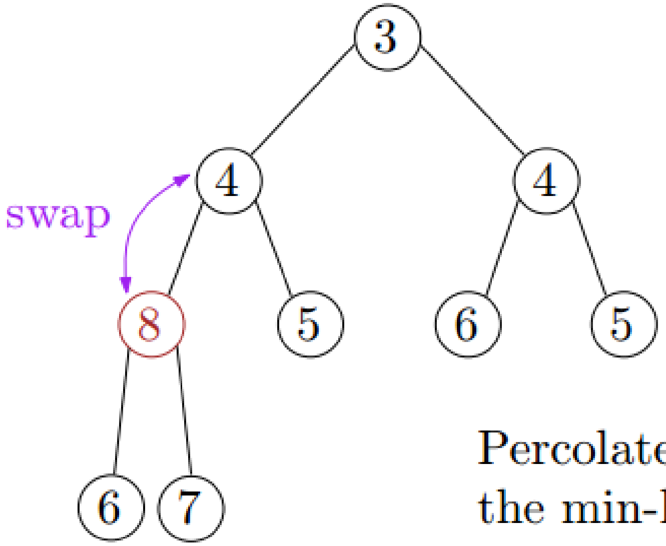

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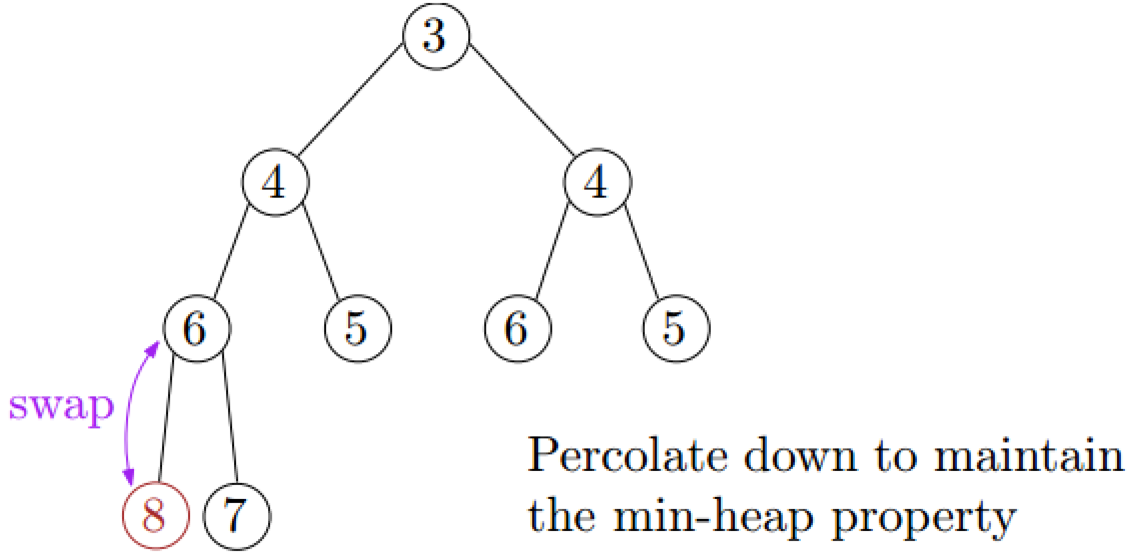

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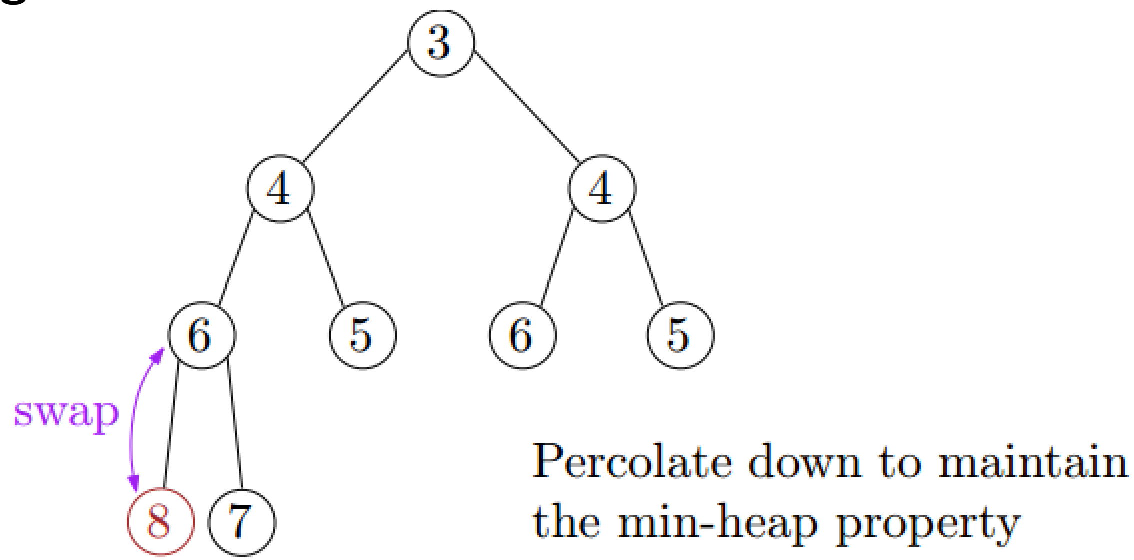

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- Total time complexity: O(n log n)

Heapsort - Example

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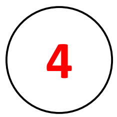

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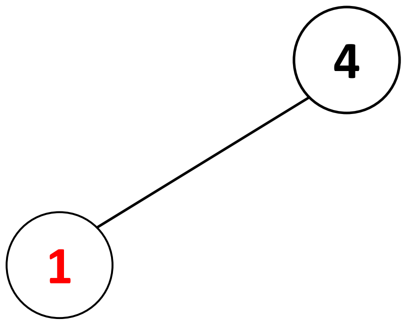

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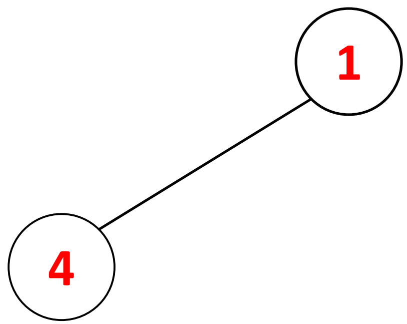

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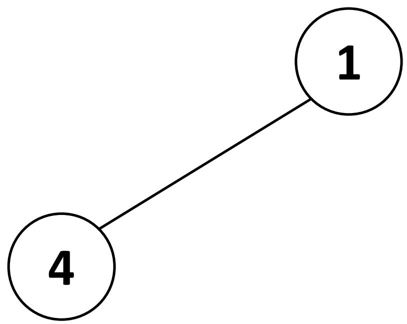

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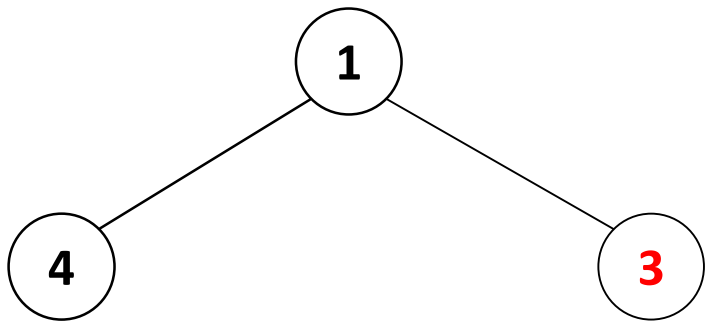

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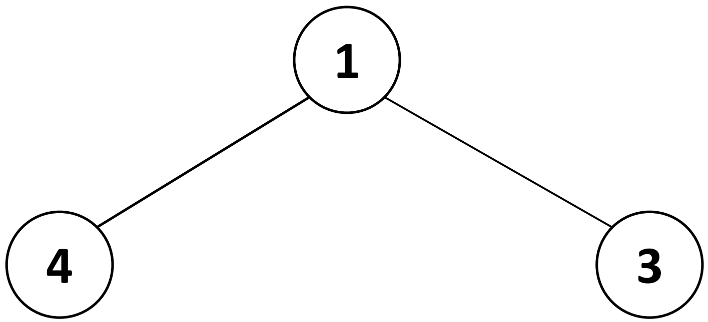

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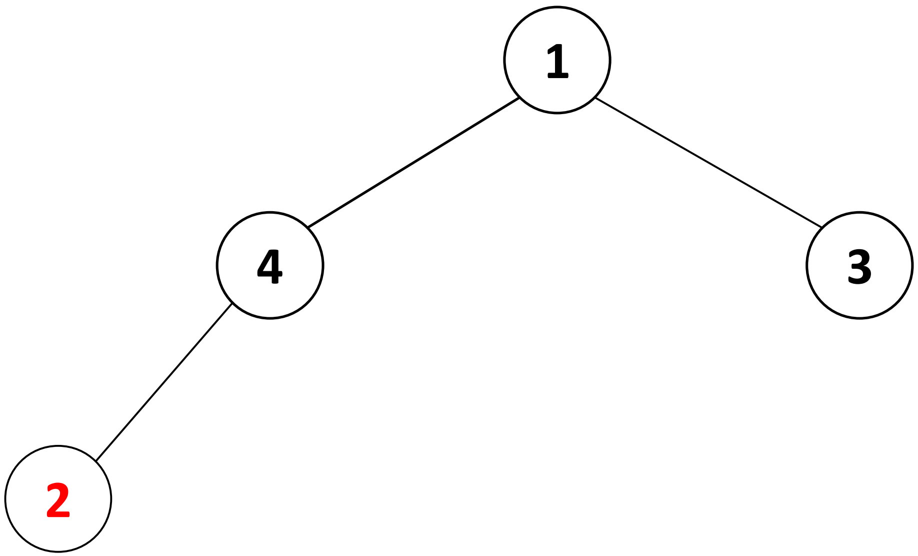

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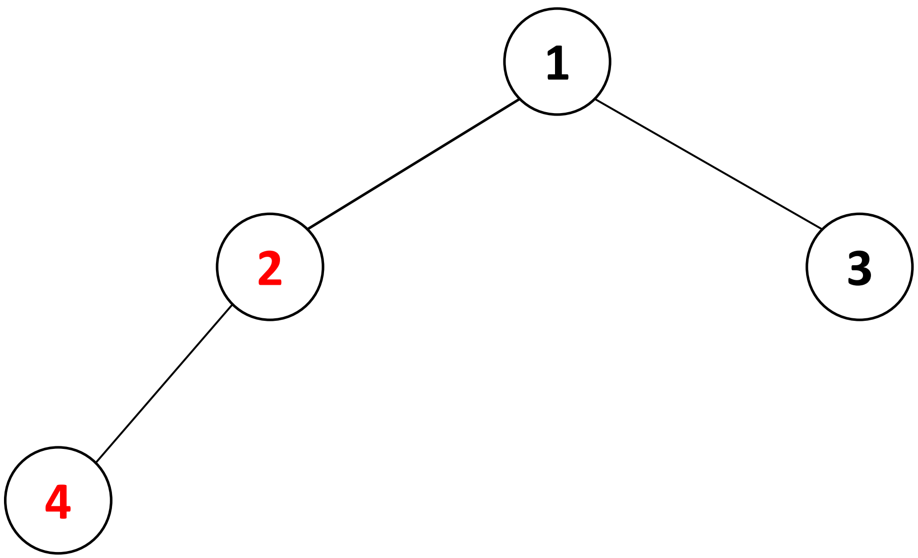

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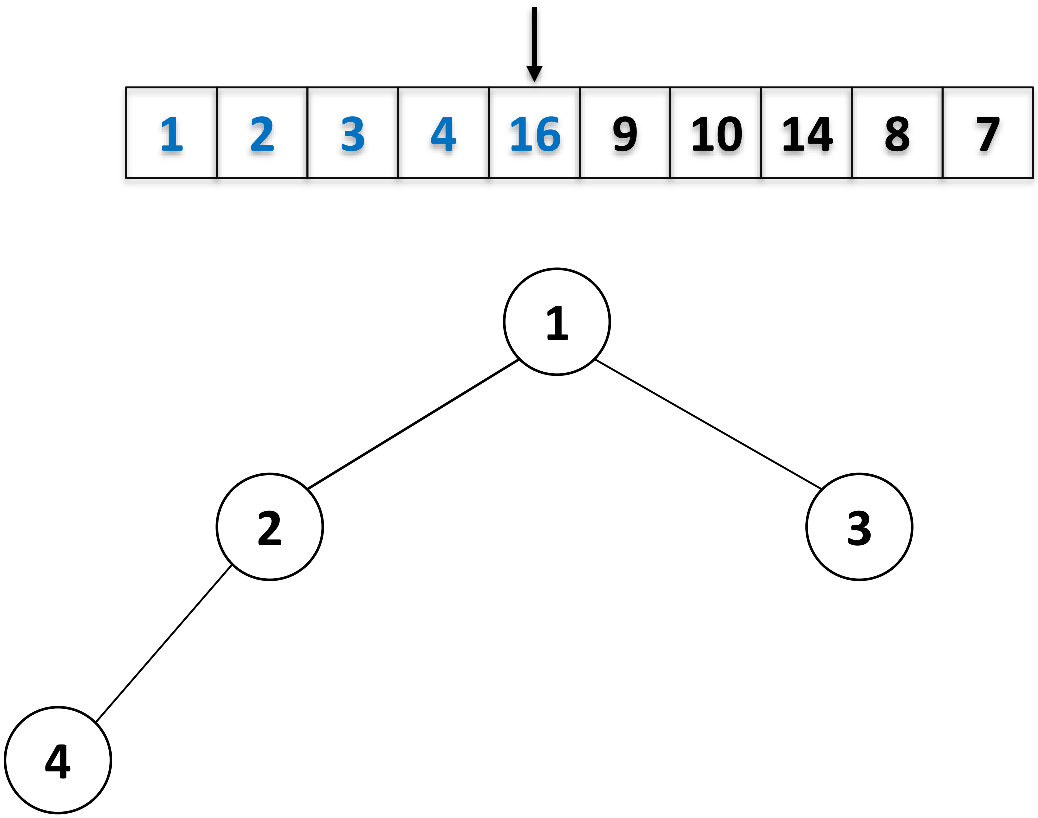

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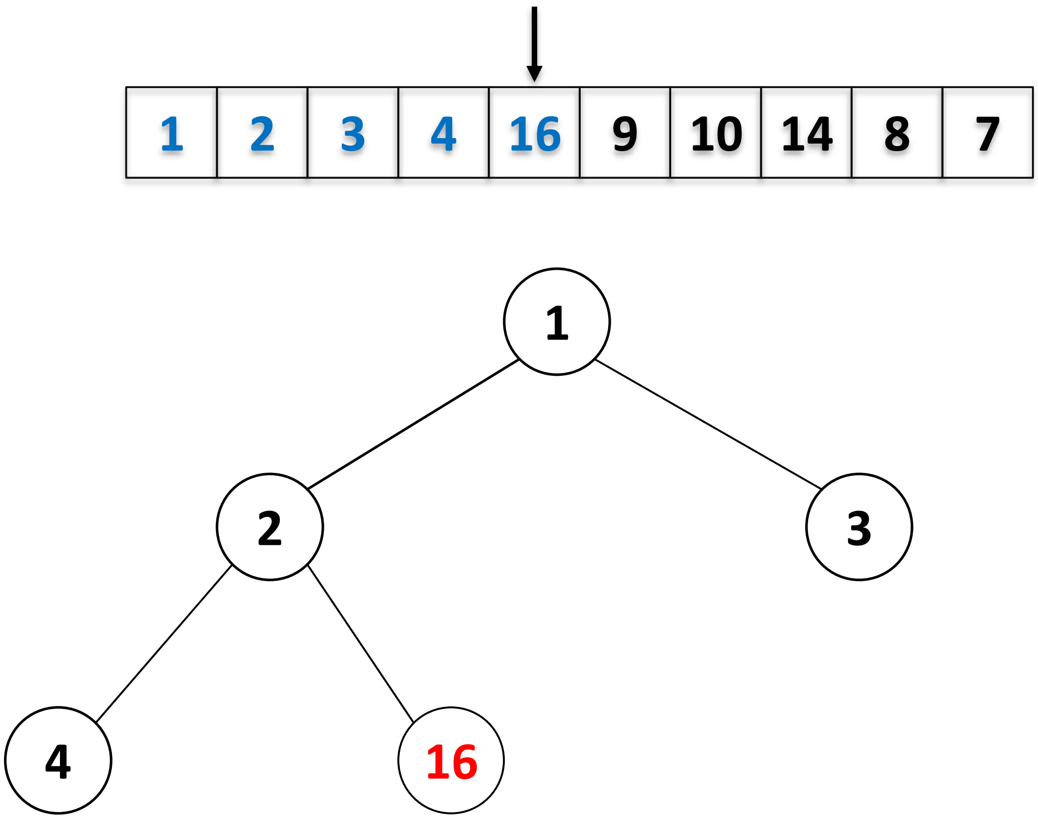

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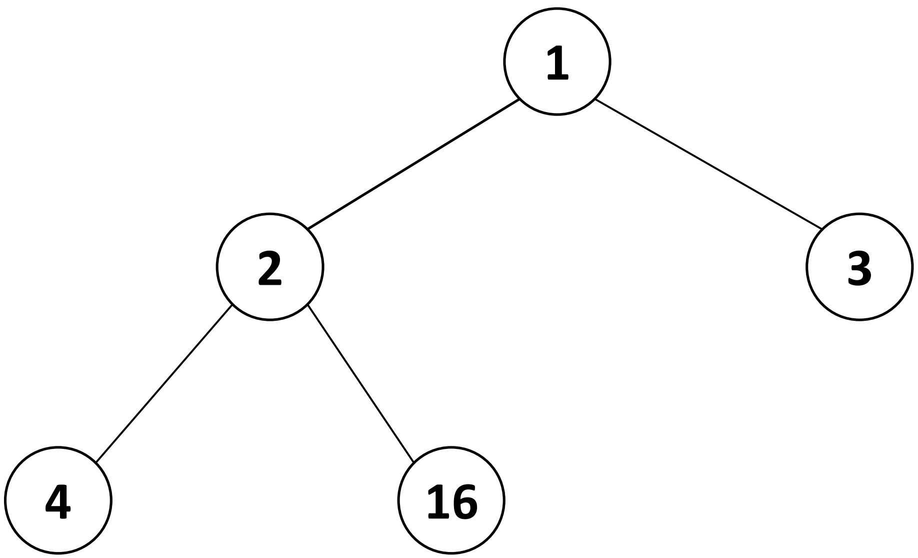

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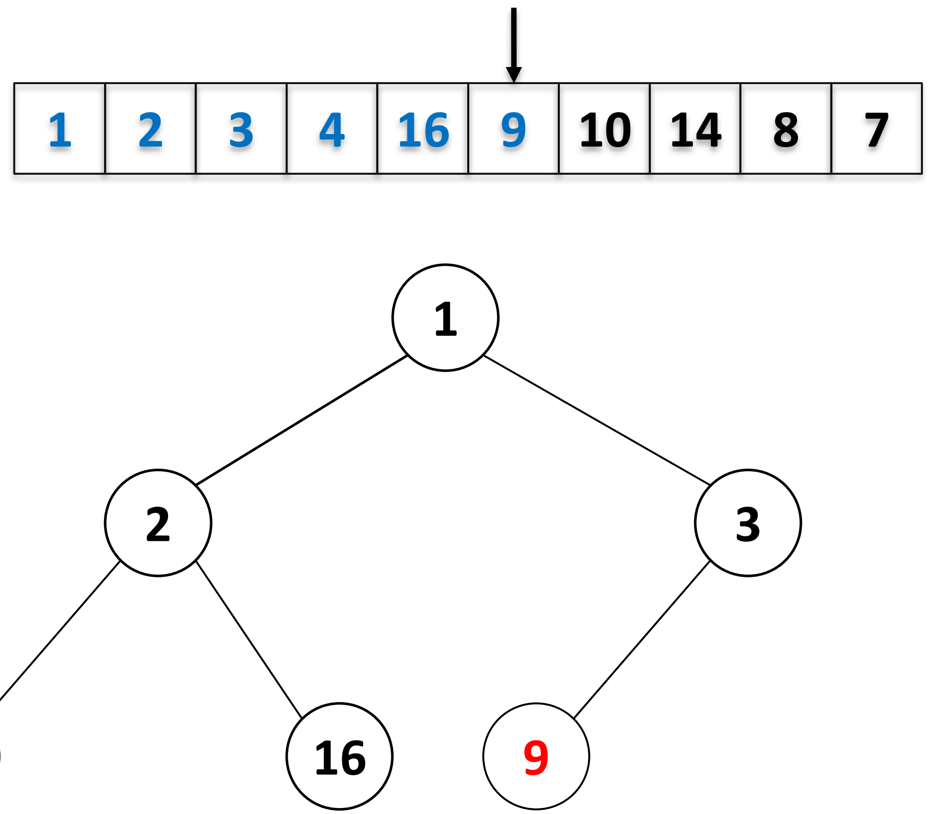

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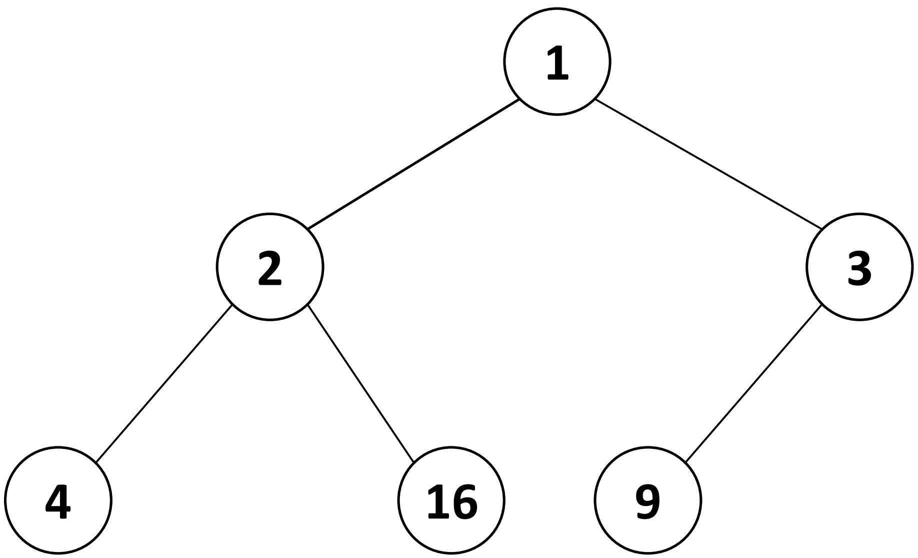

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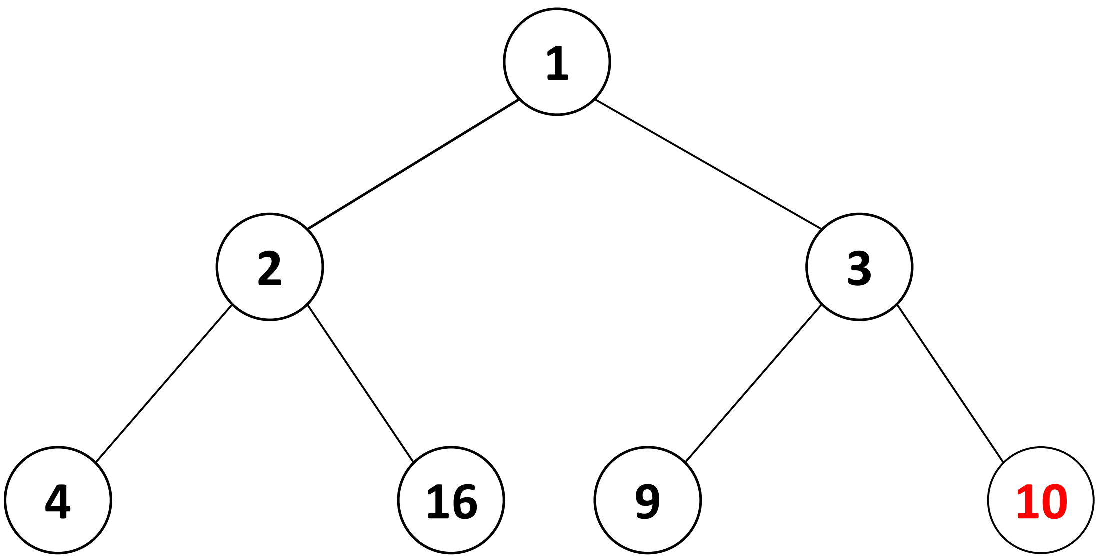

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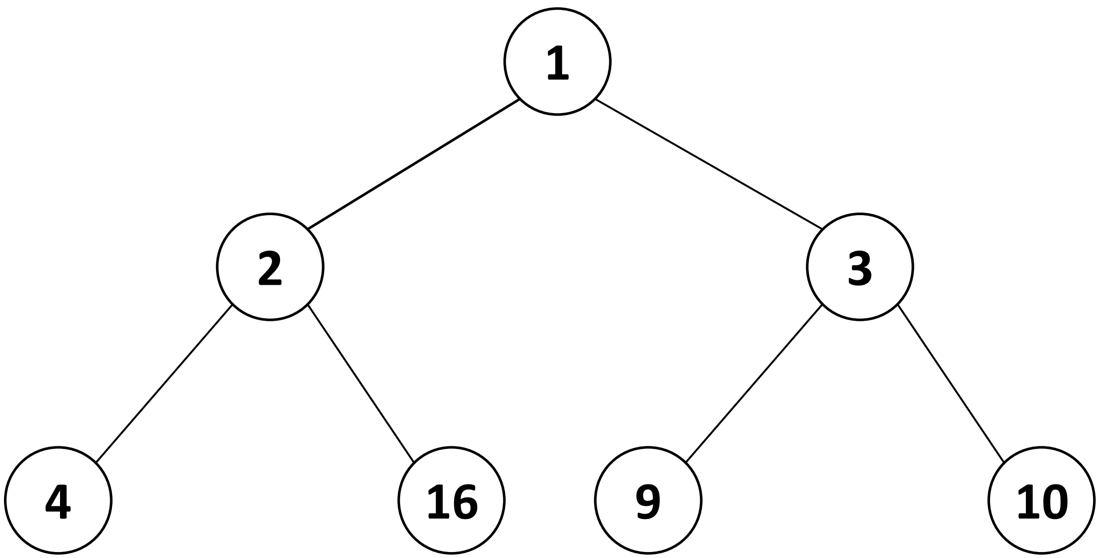

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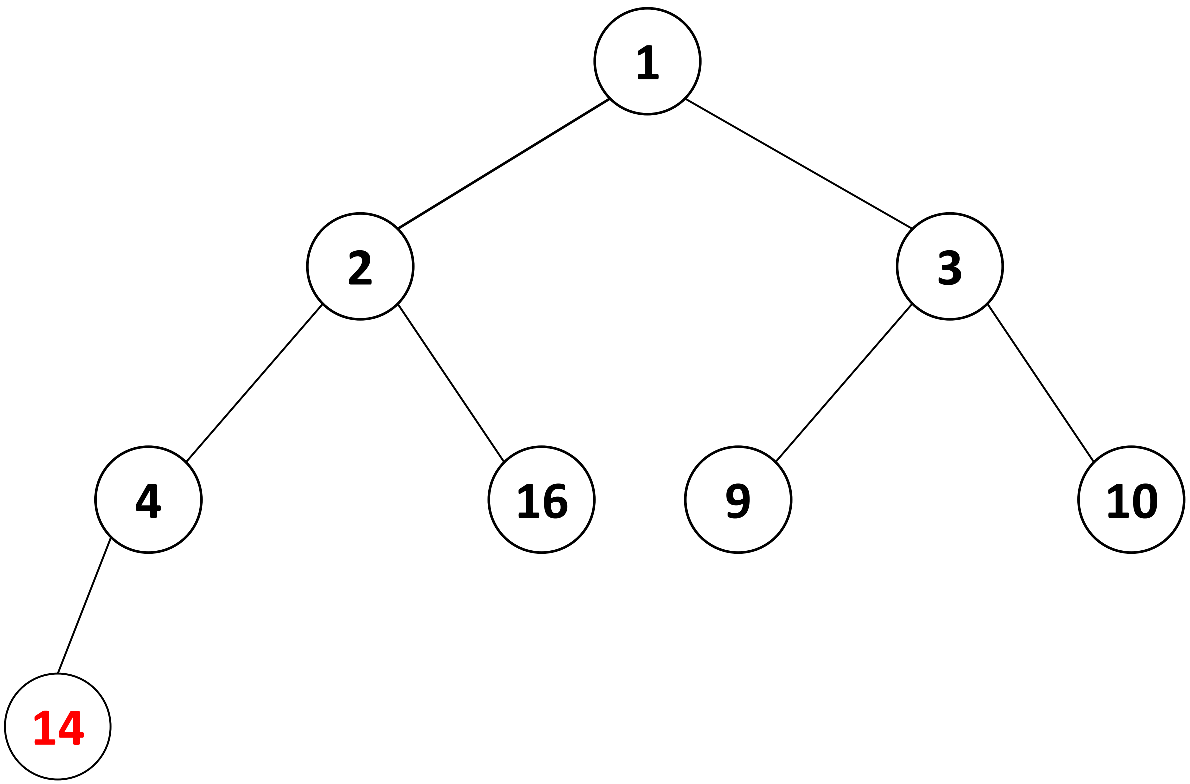

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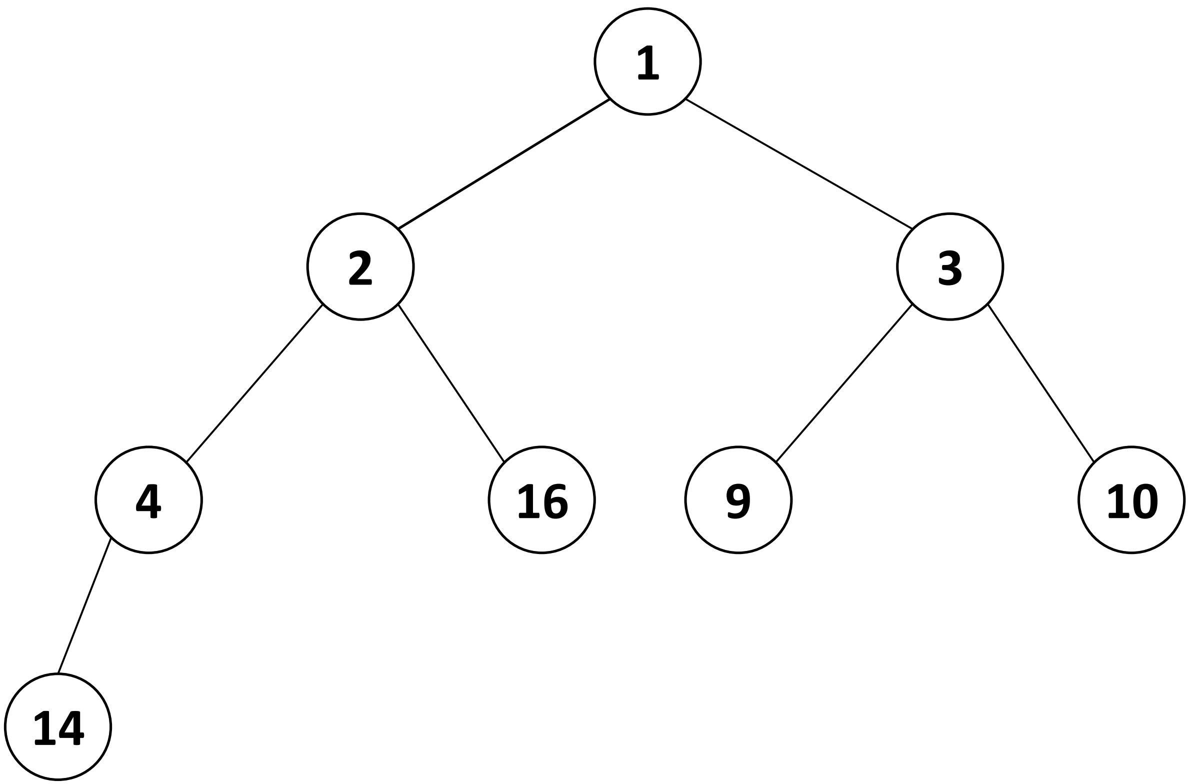

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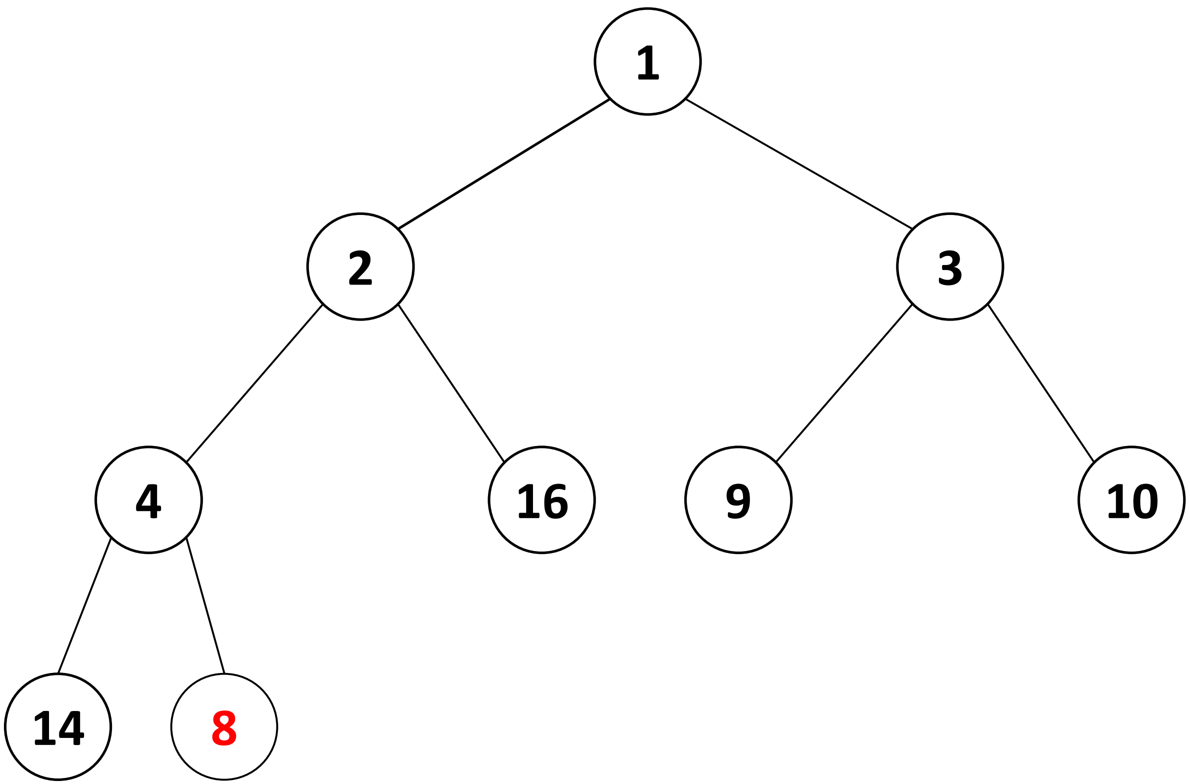

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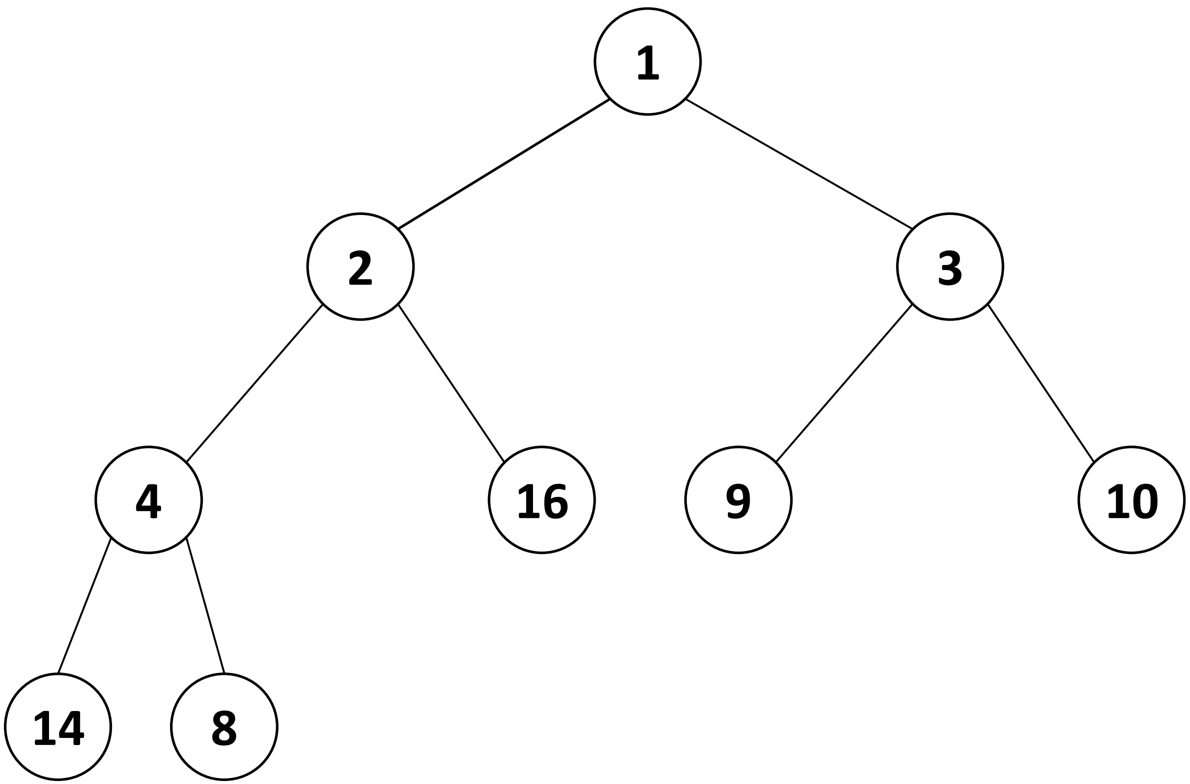

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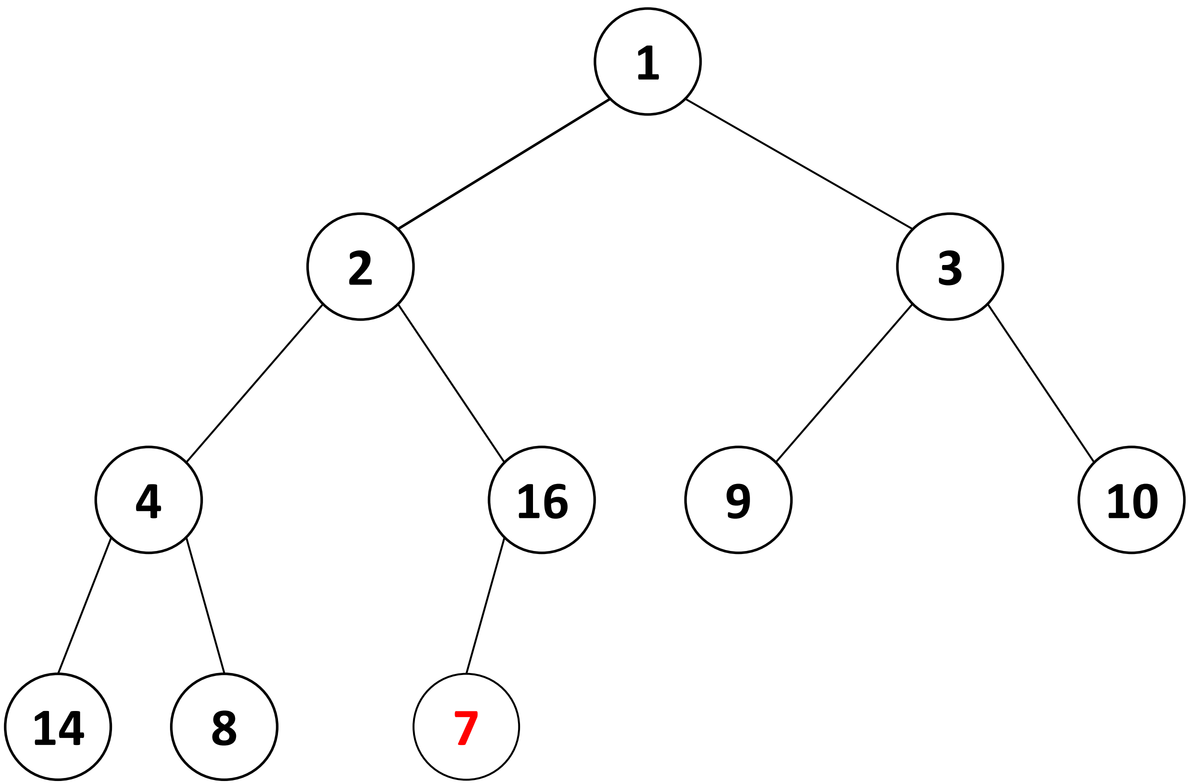
1	2	3	4	16	9	10	14	8	7



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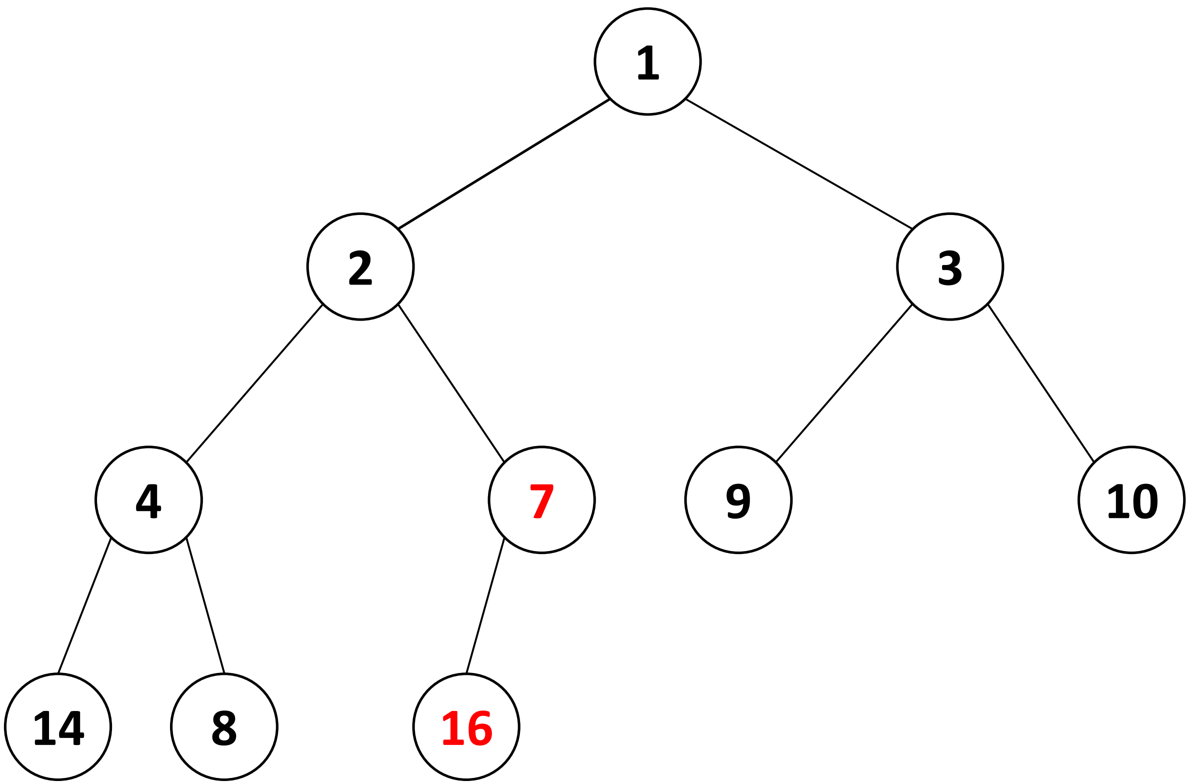

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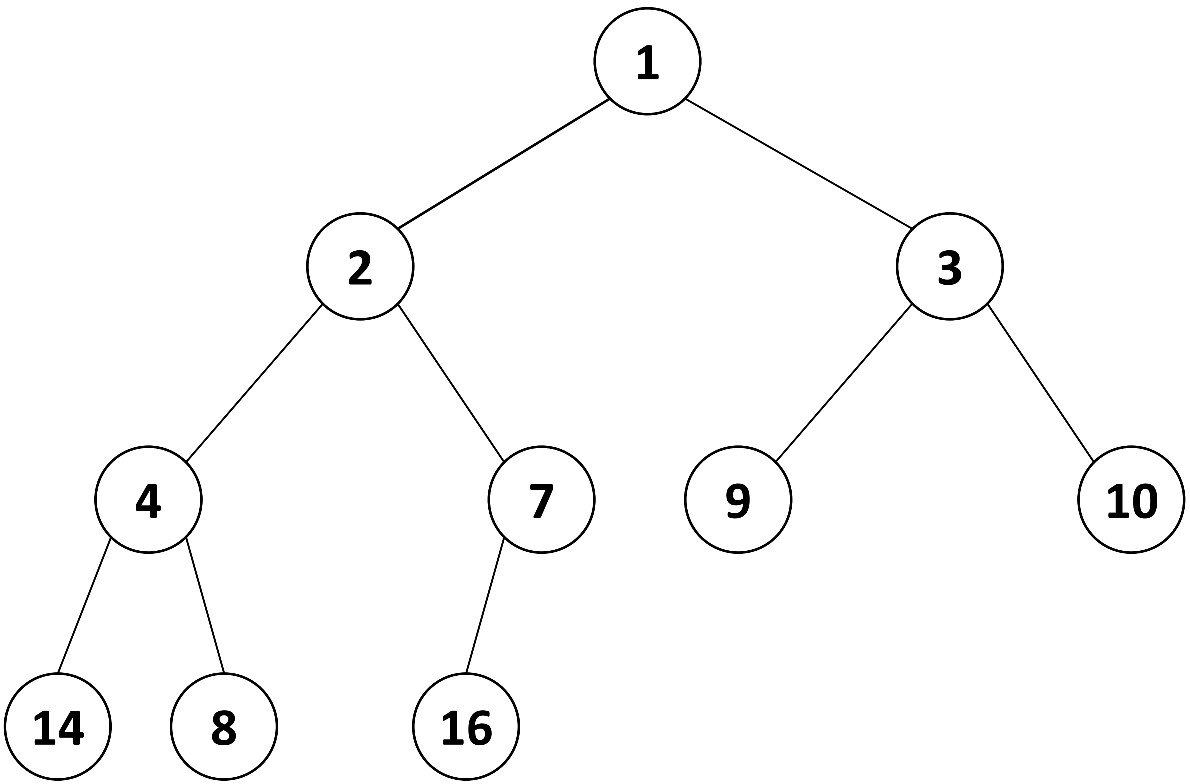
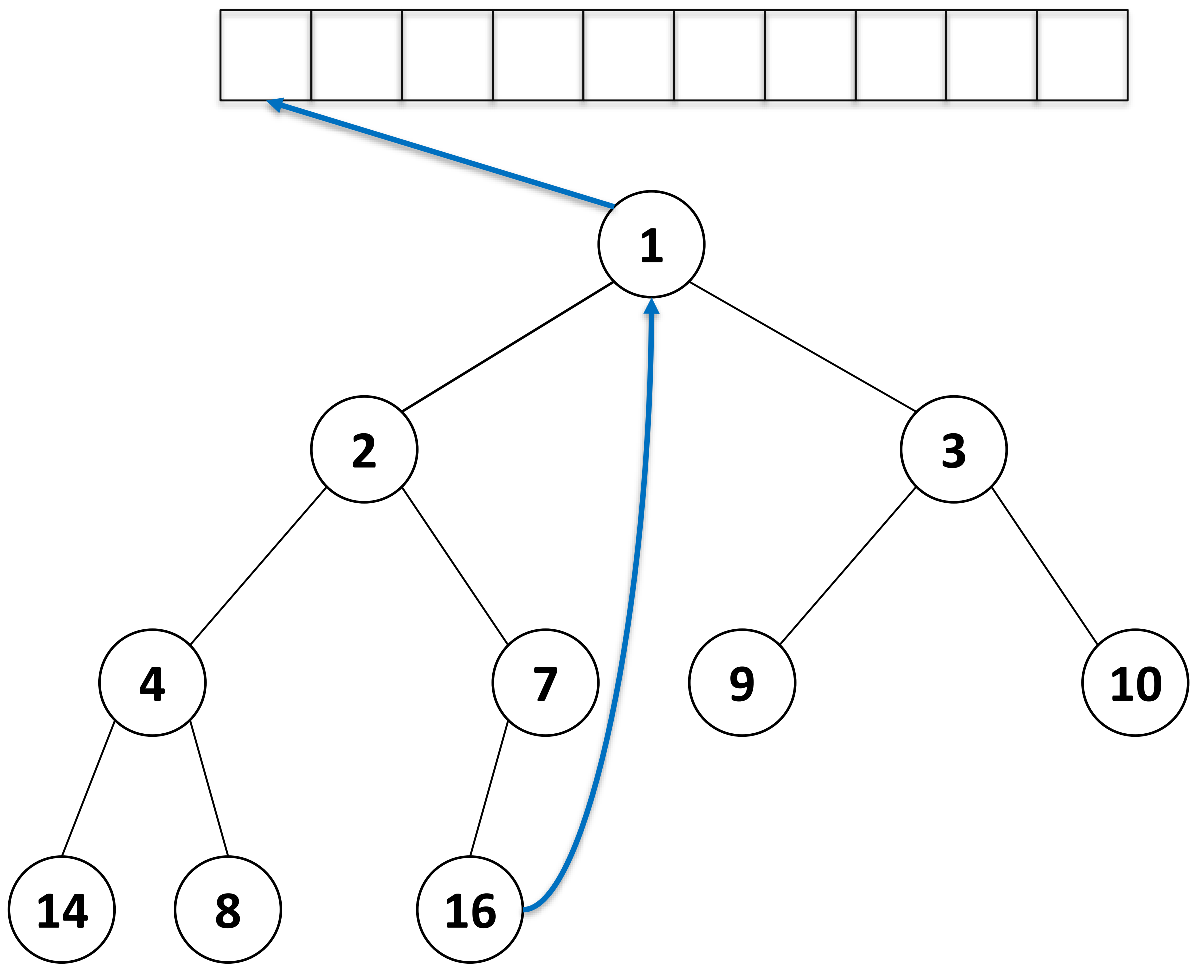


Heapsort - Example

- Perform n Extract-Min operations



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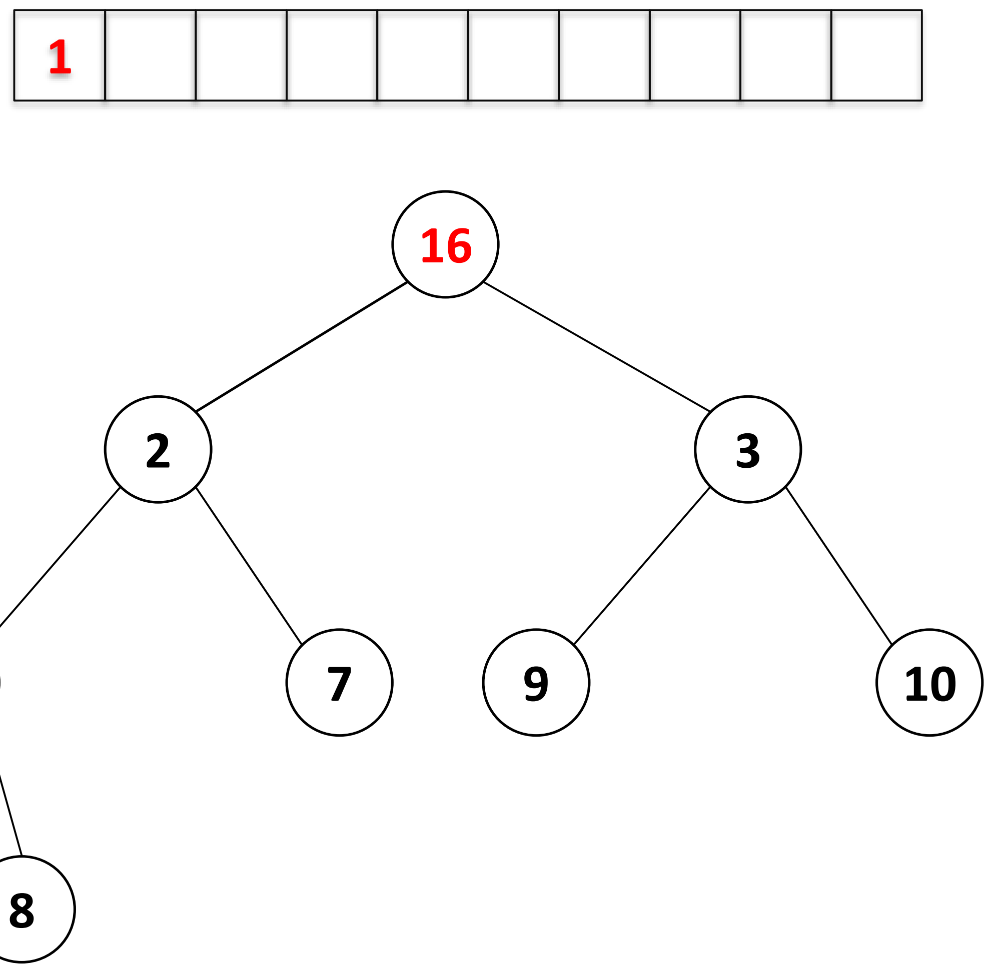

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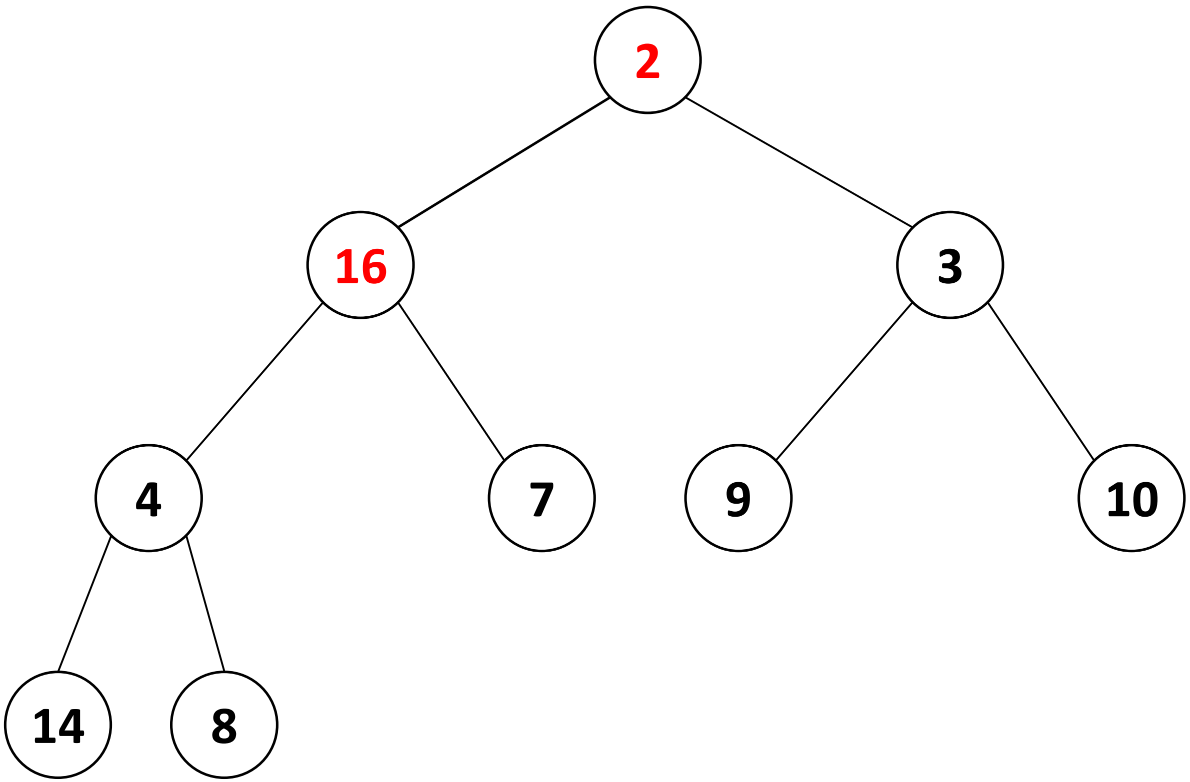
1									



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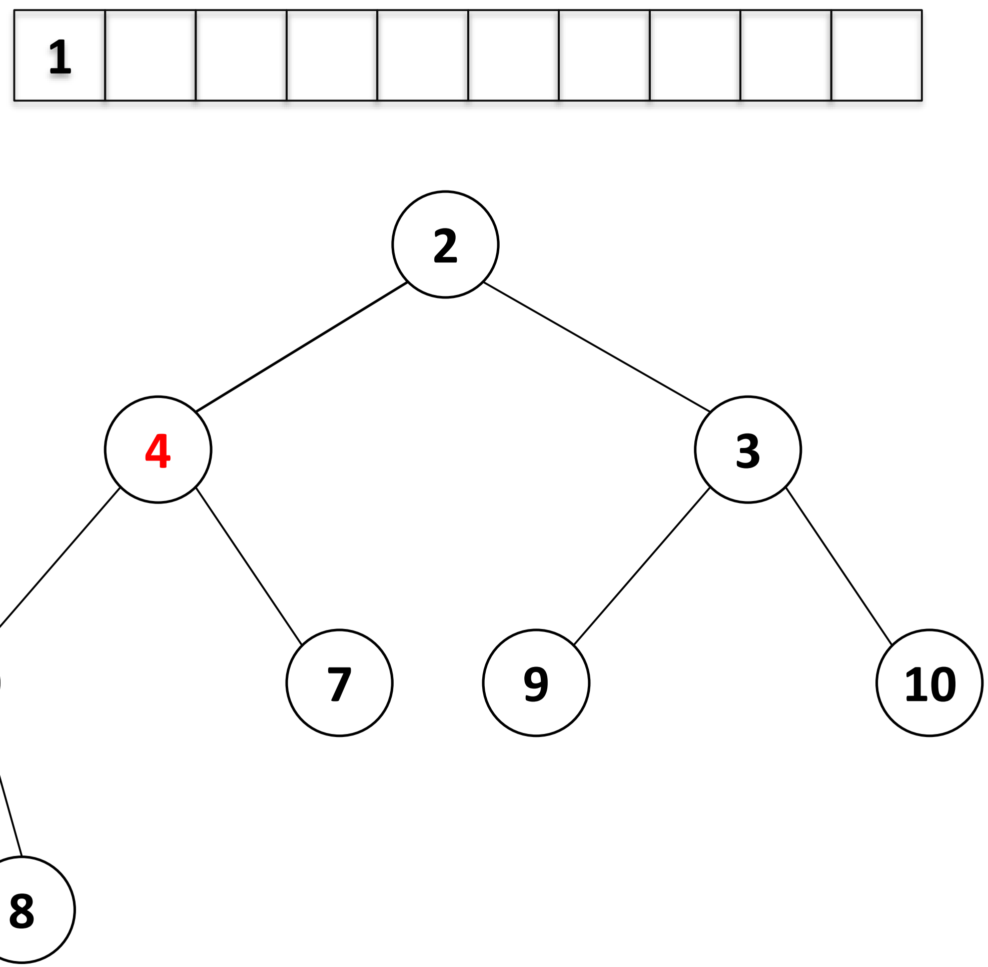

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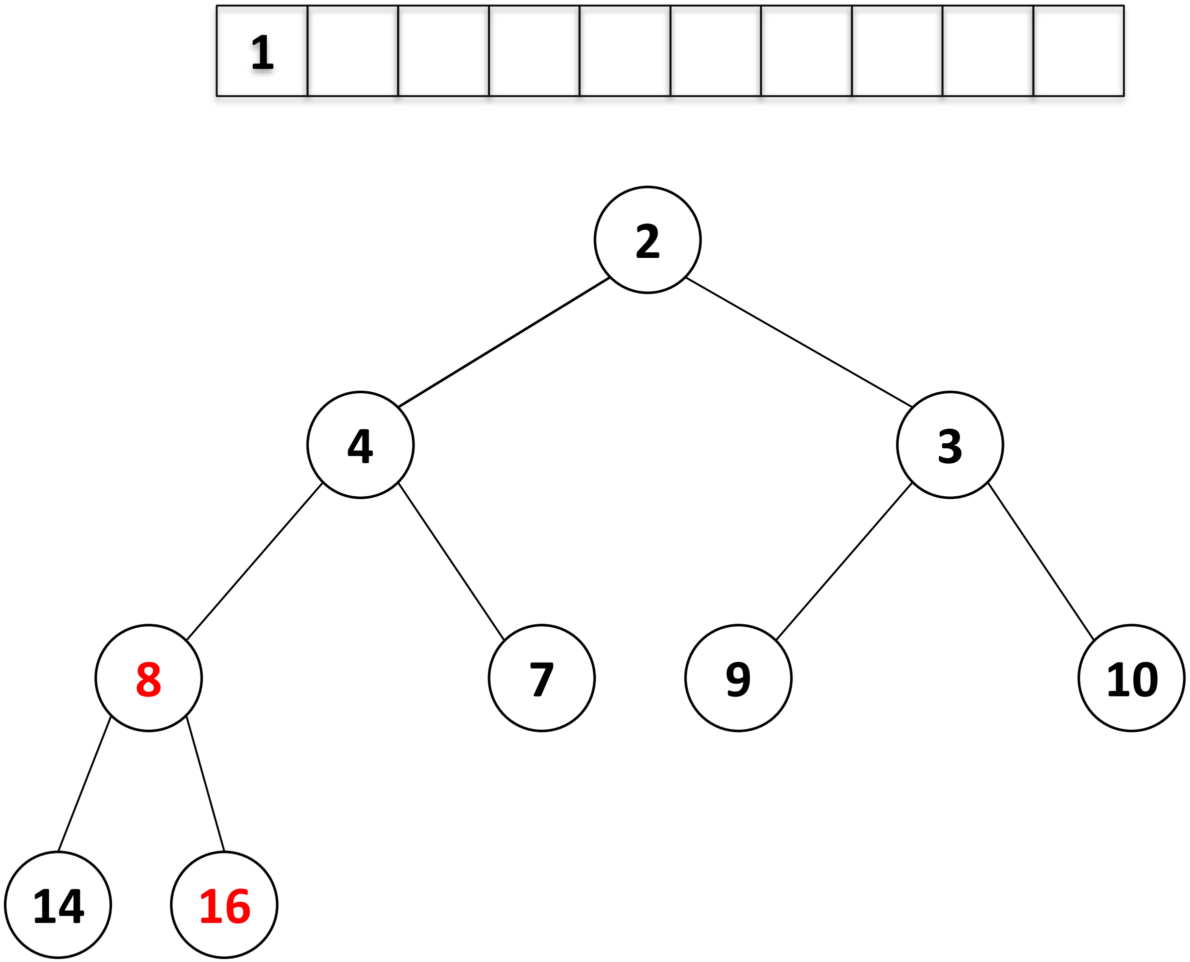

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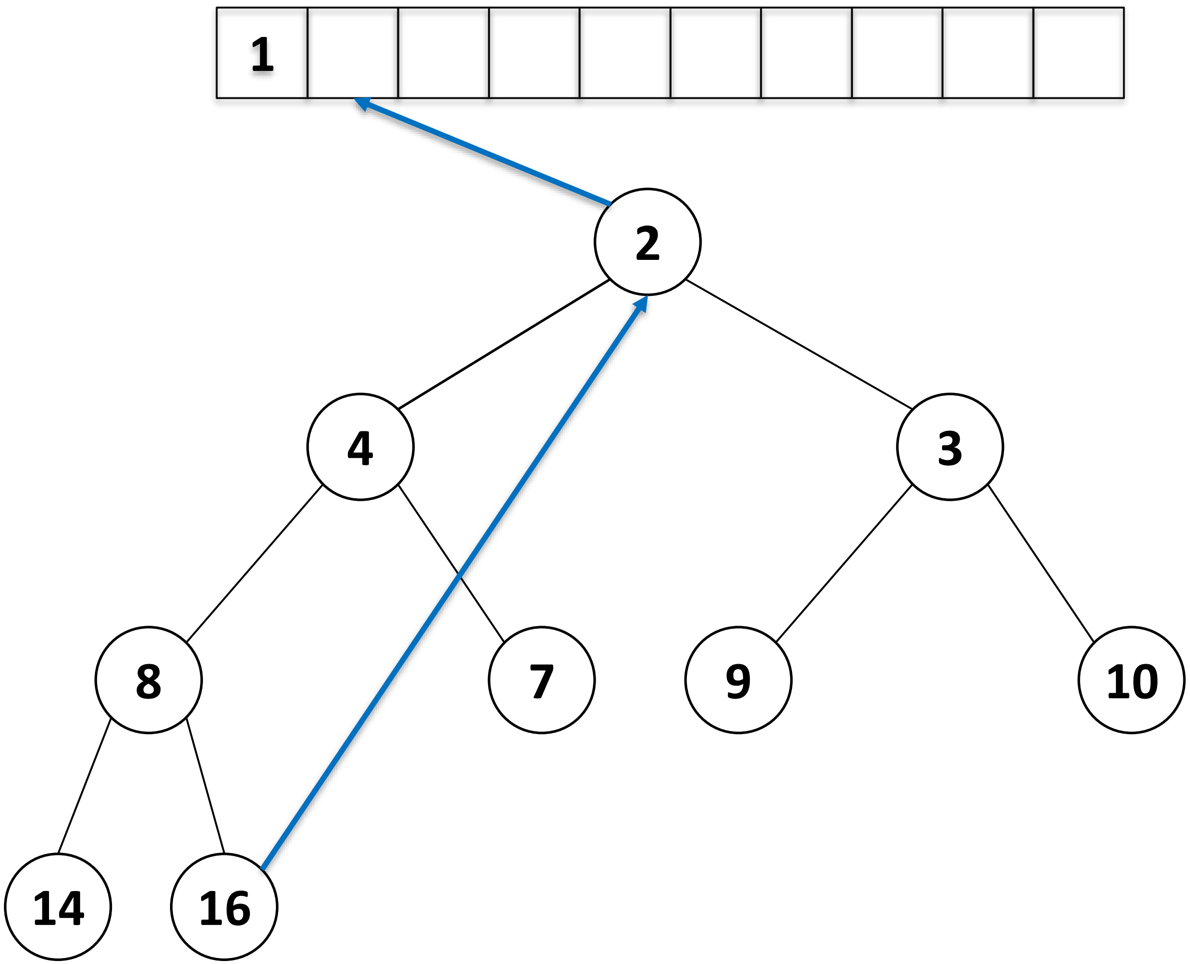

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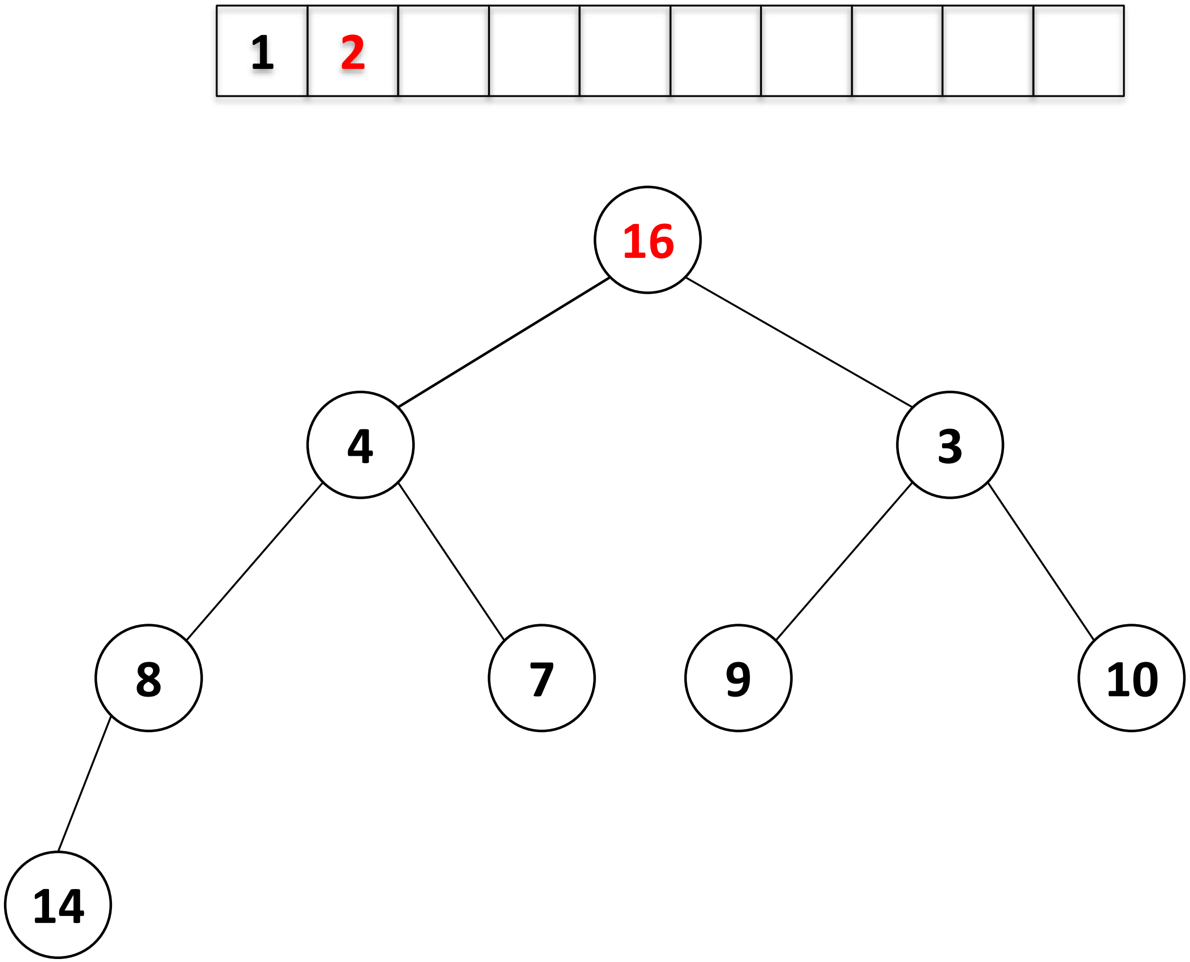

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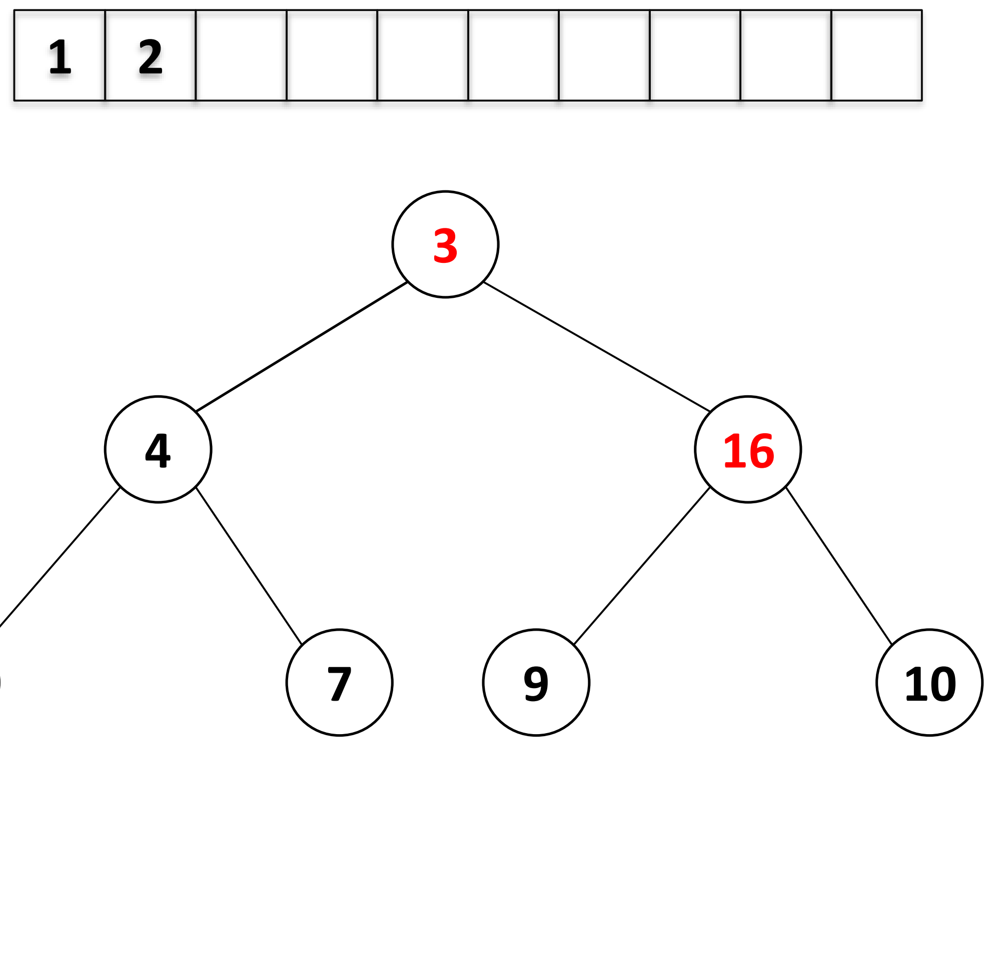
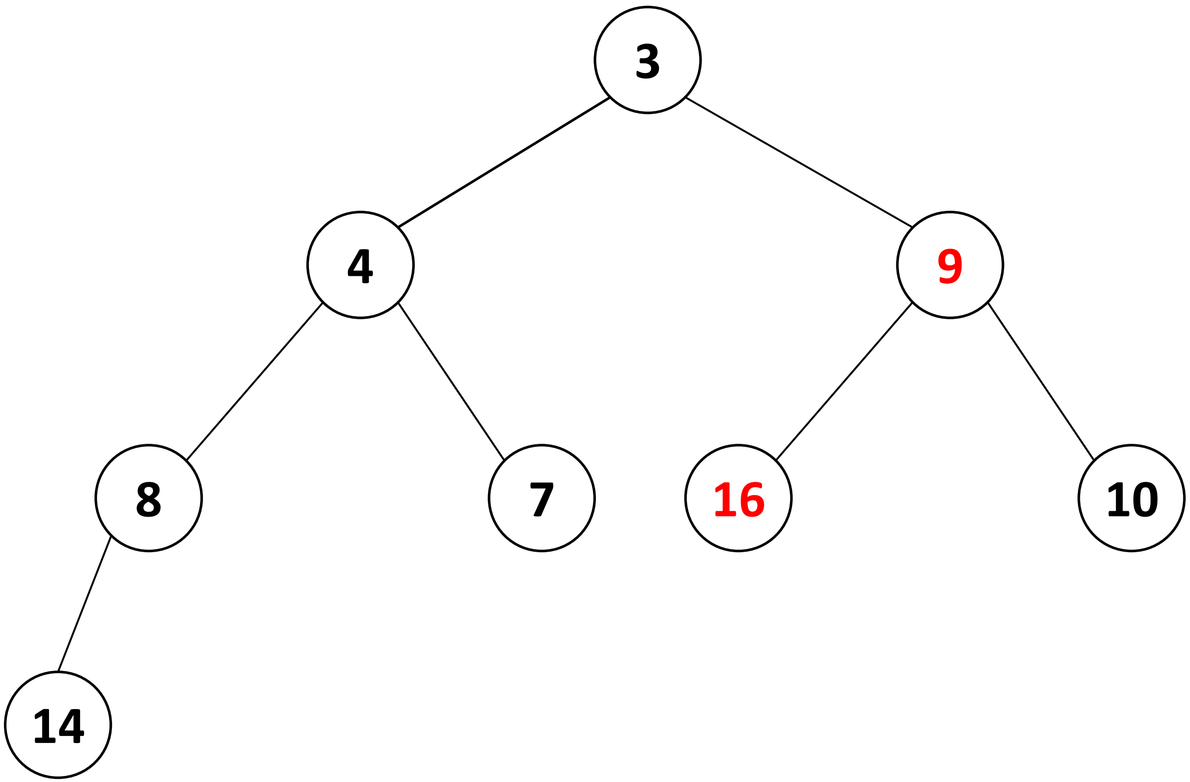


Heapsort - Example

- Perform n Extract-Min operations

1	2								



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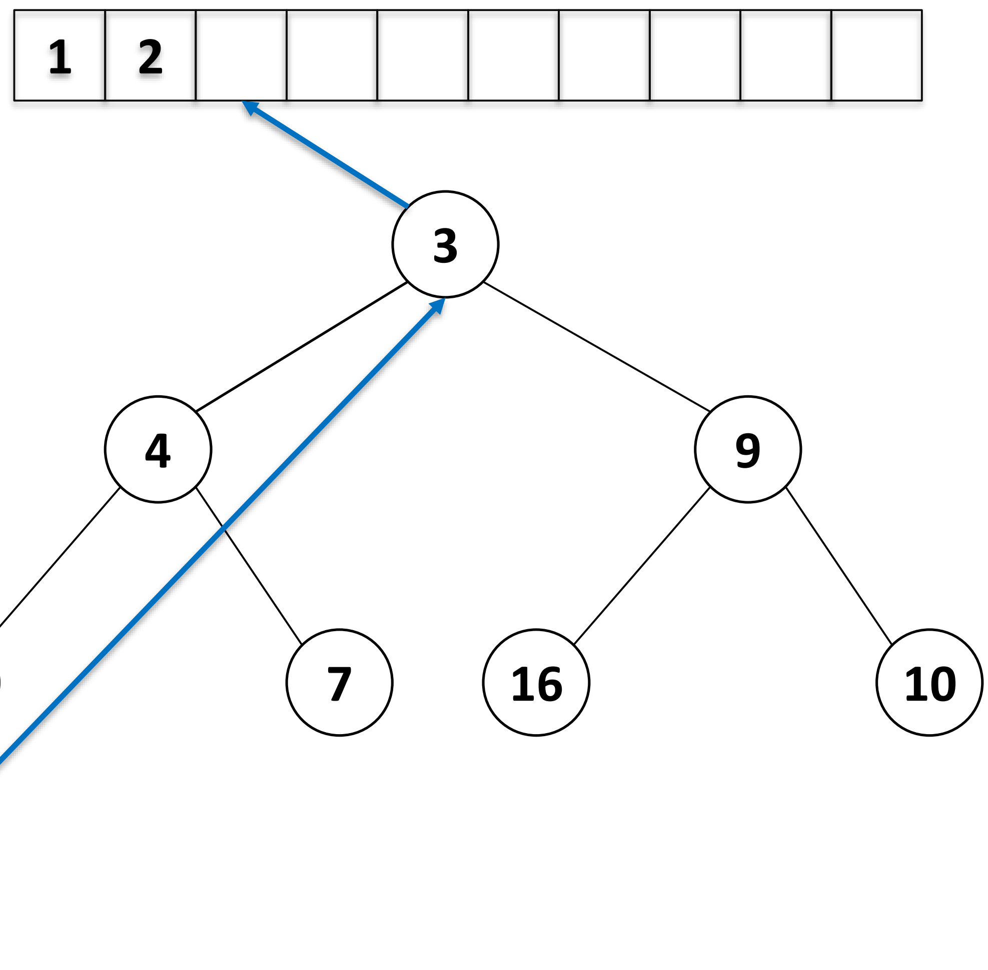

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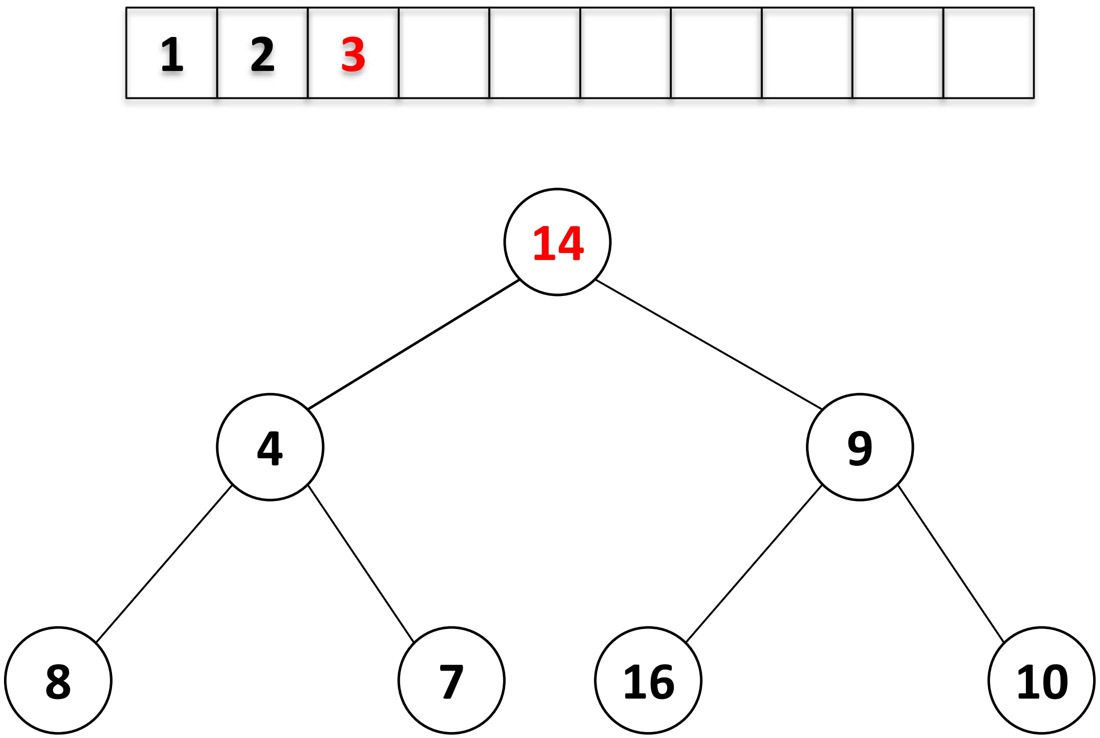

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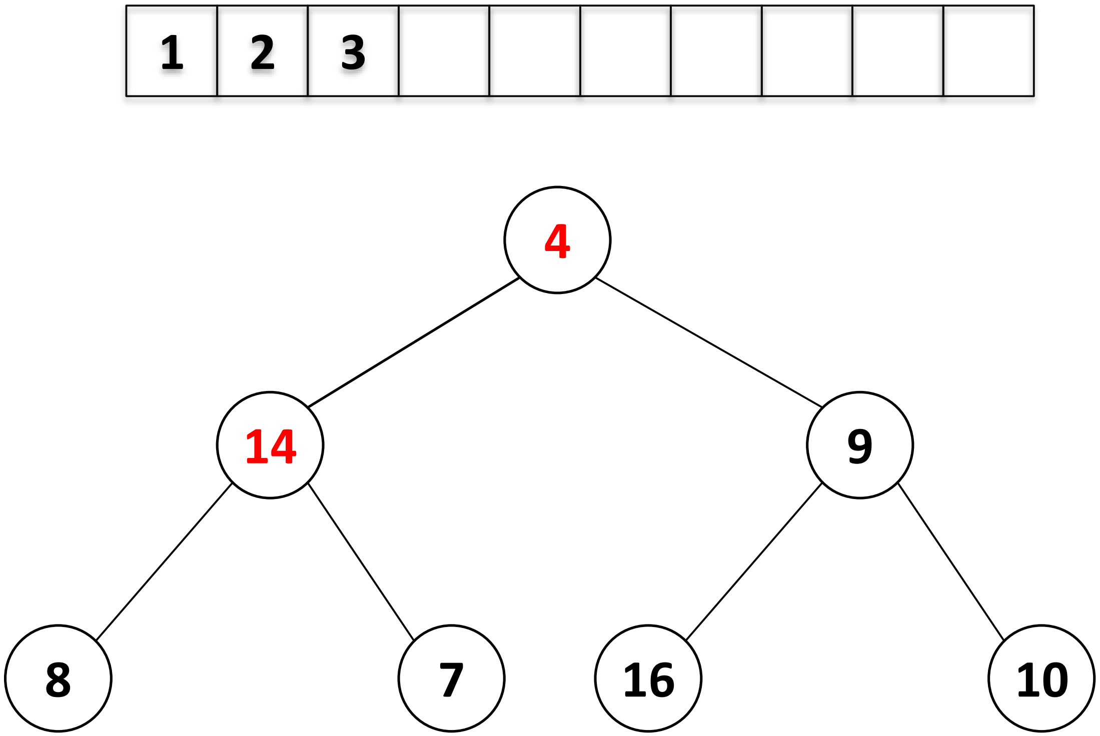

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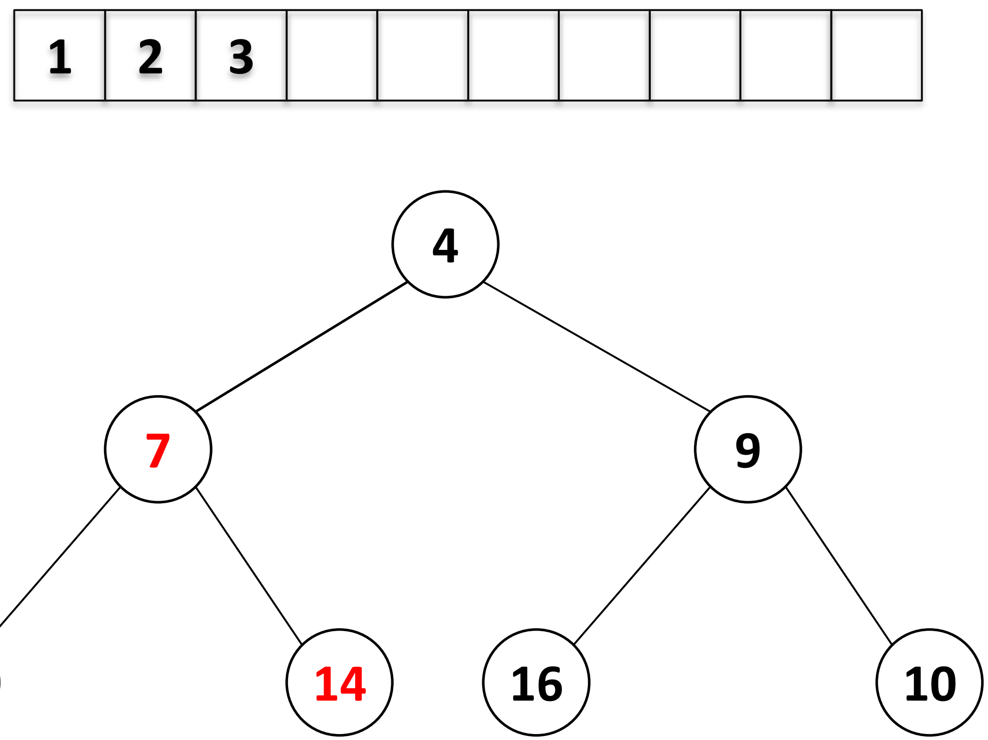

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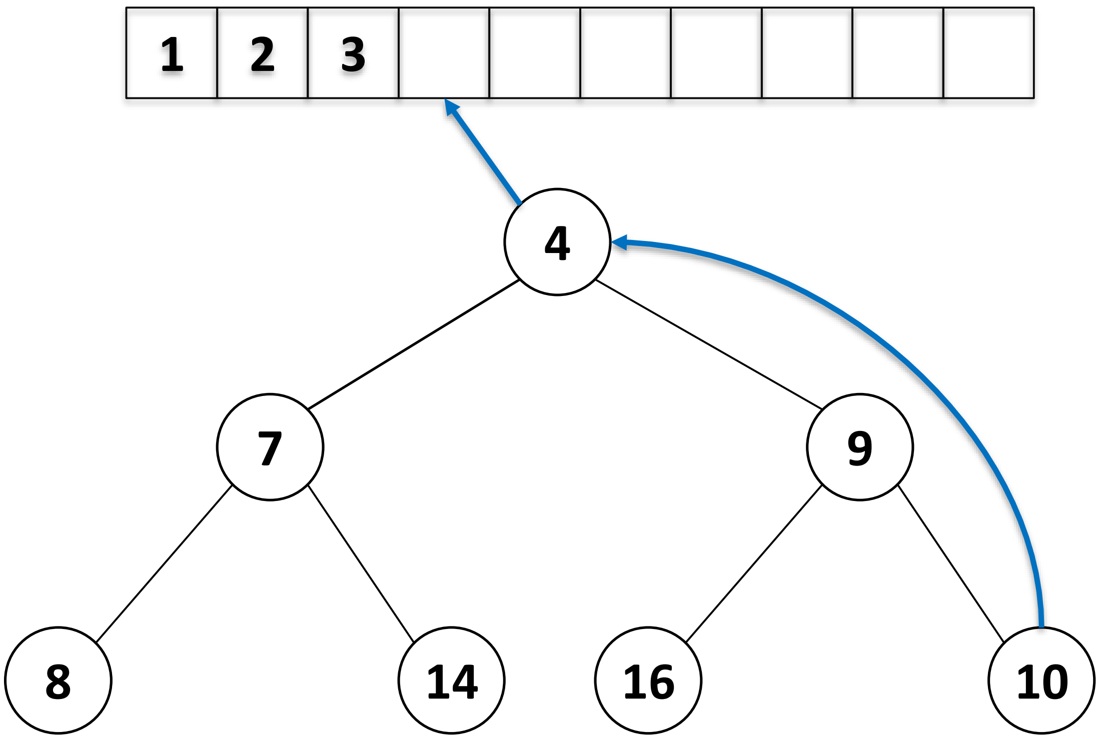

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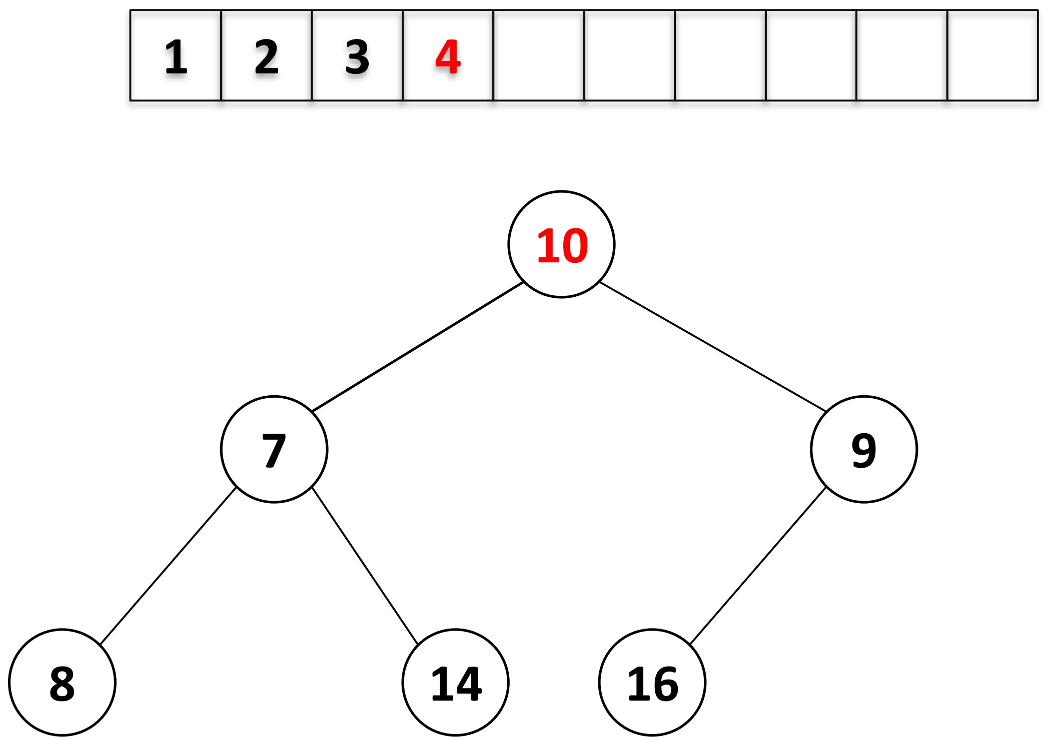

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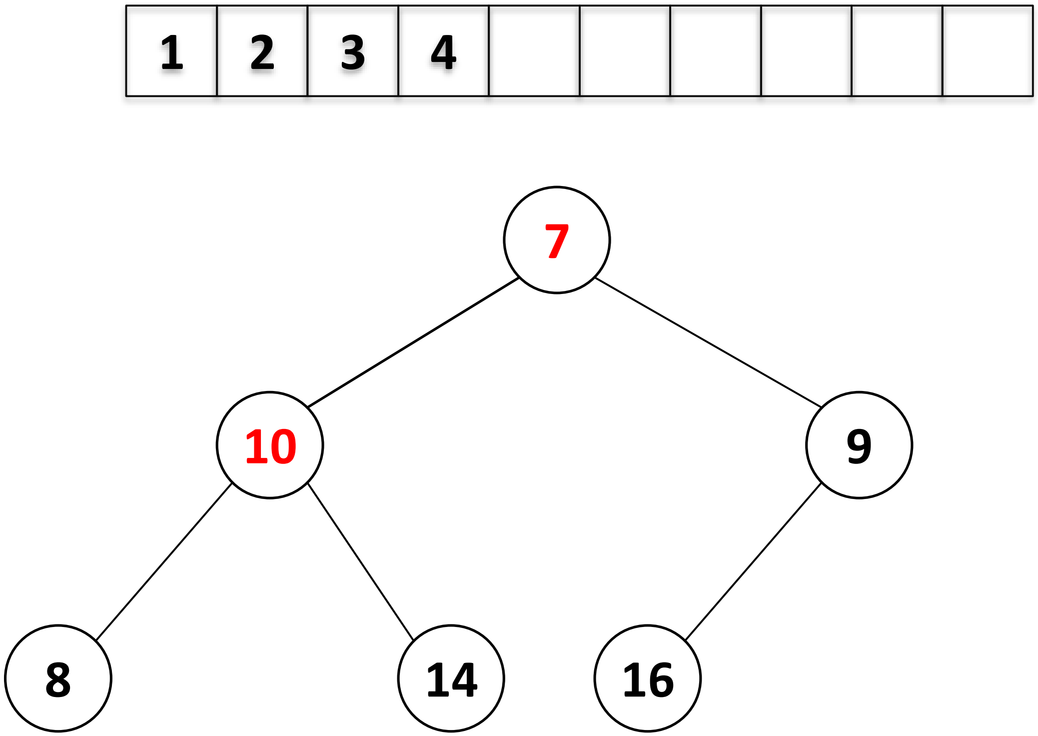

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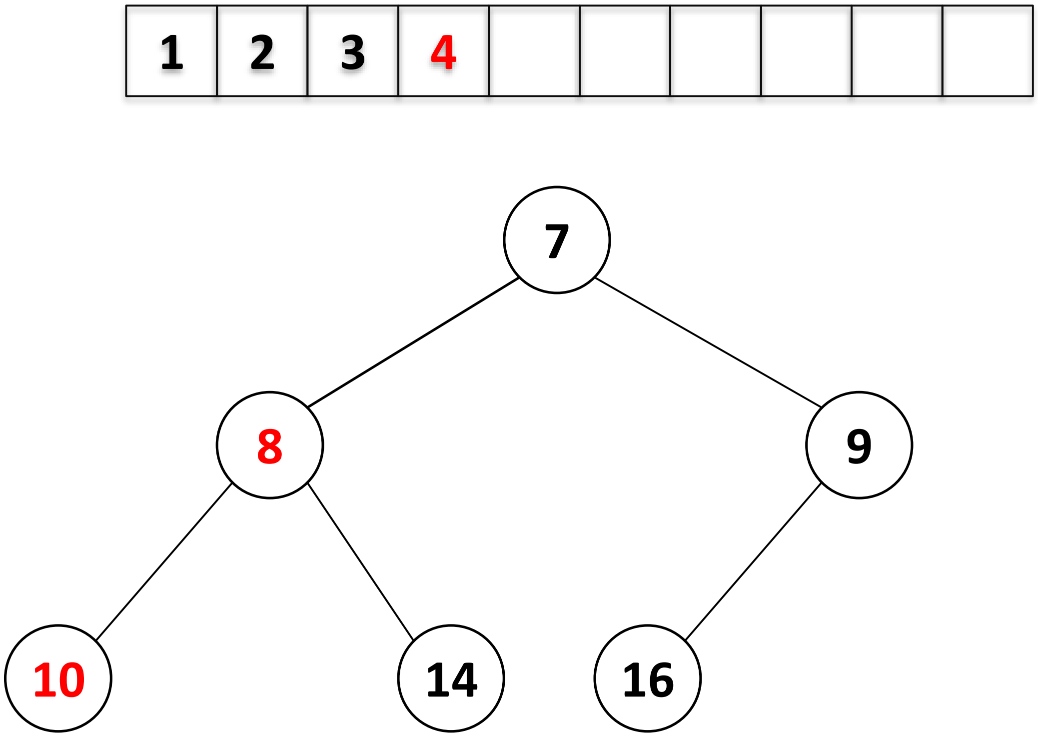

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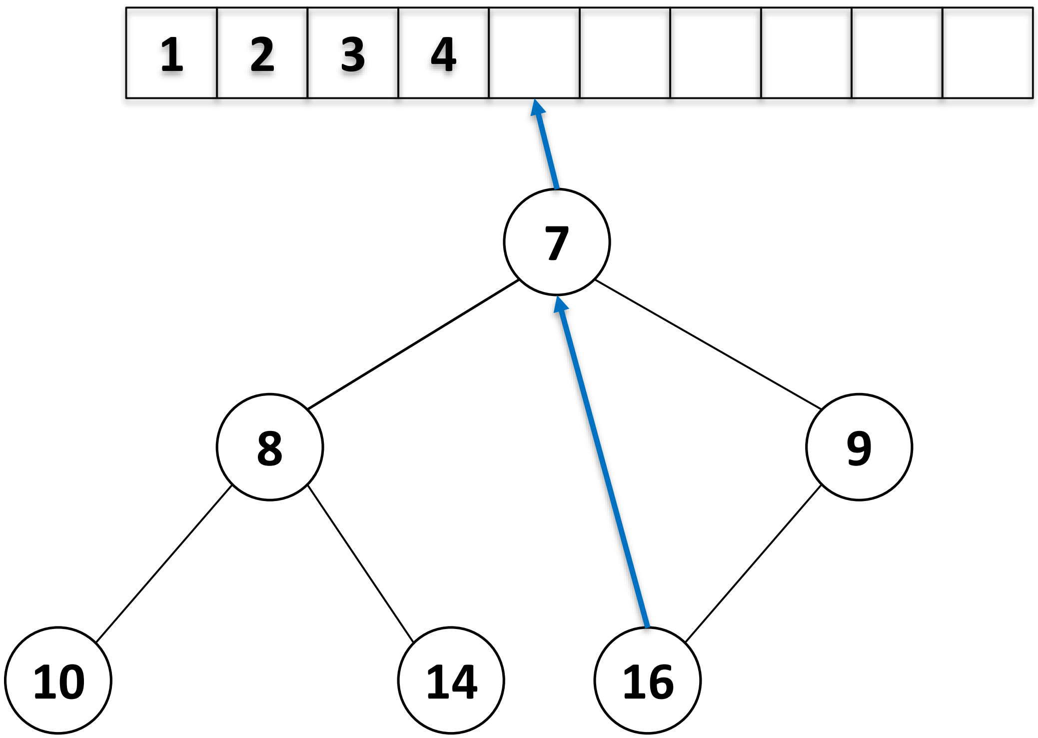

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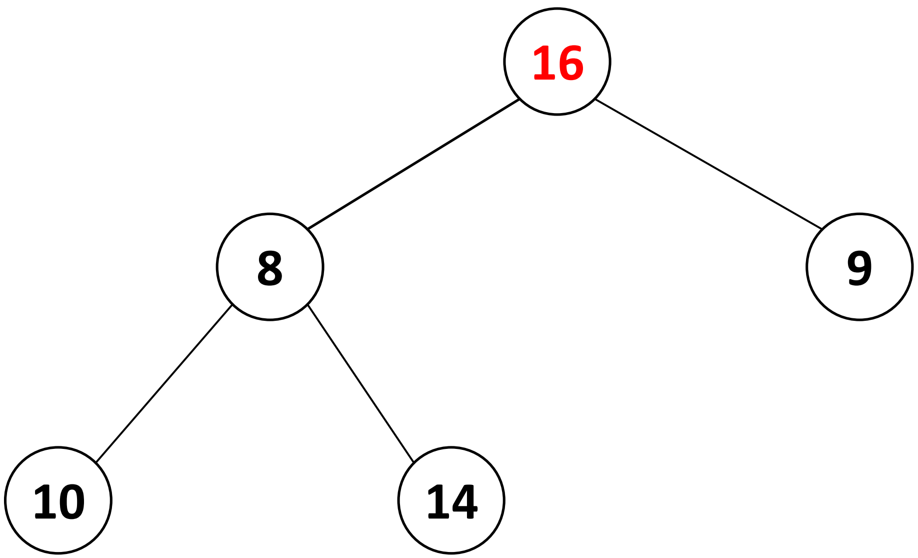
1	2	3	4	7					



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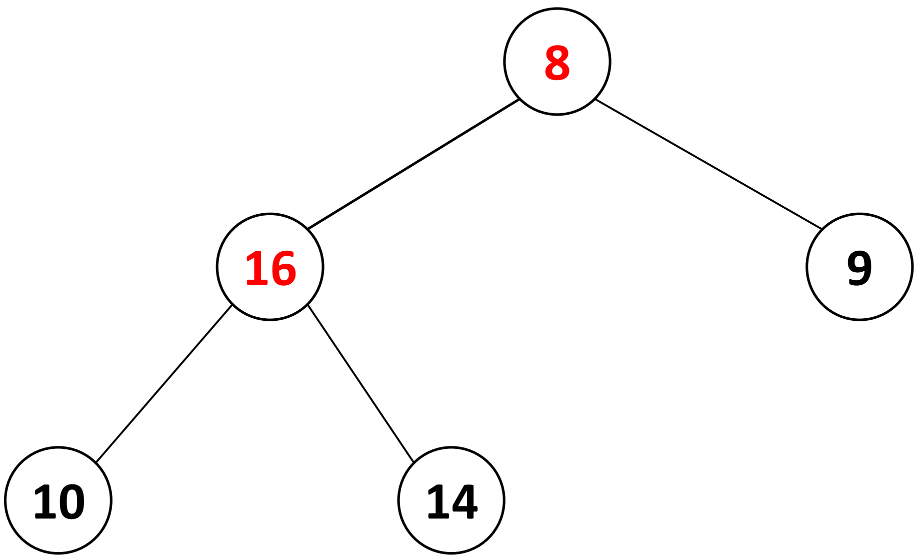

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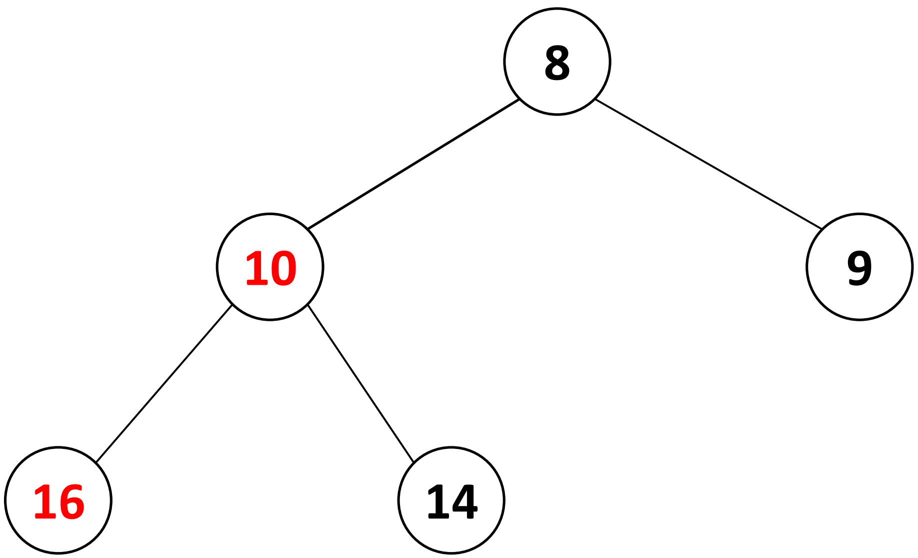

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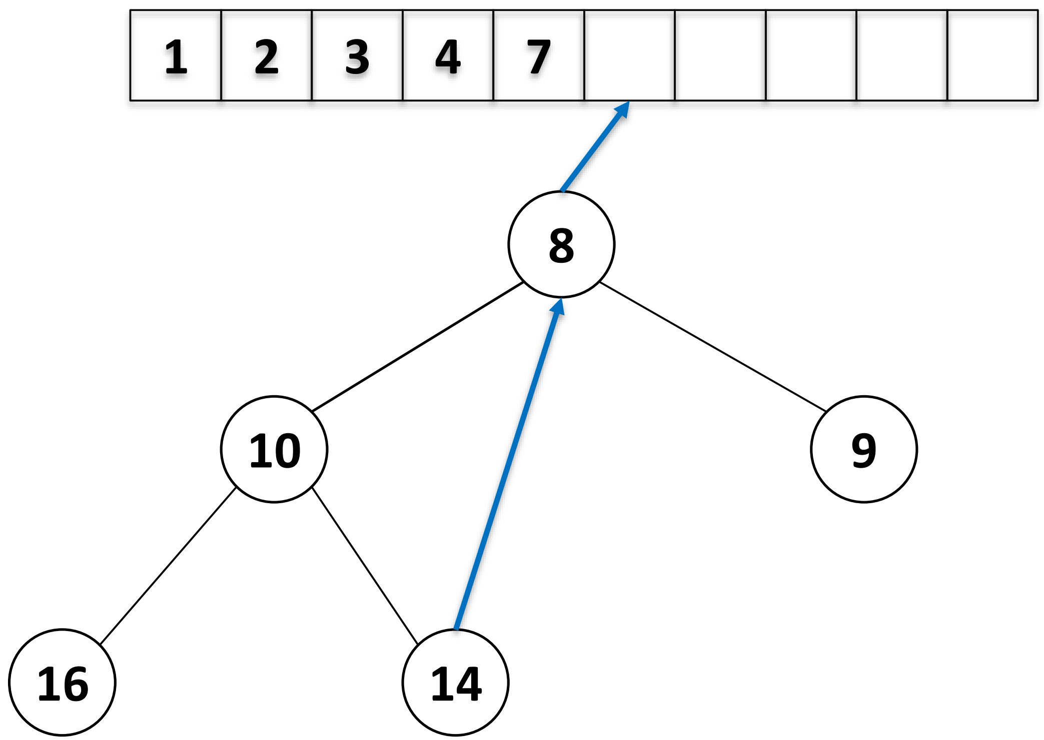
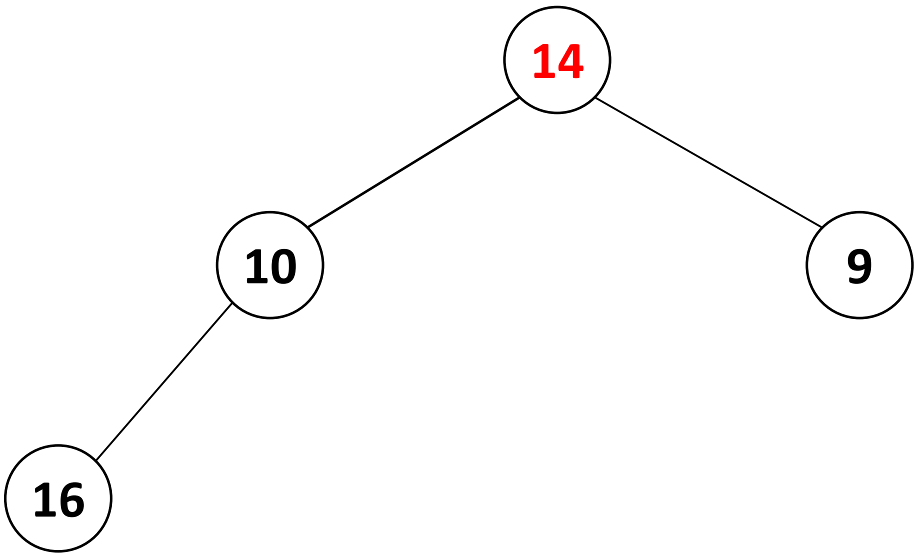


Heapsort - Example

- Perform n Extract-Min operations

1	2	3	4	7	8				



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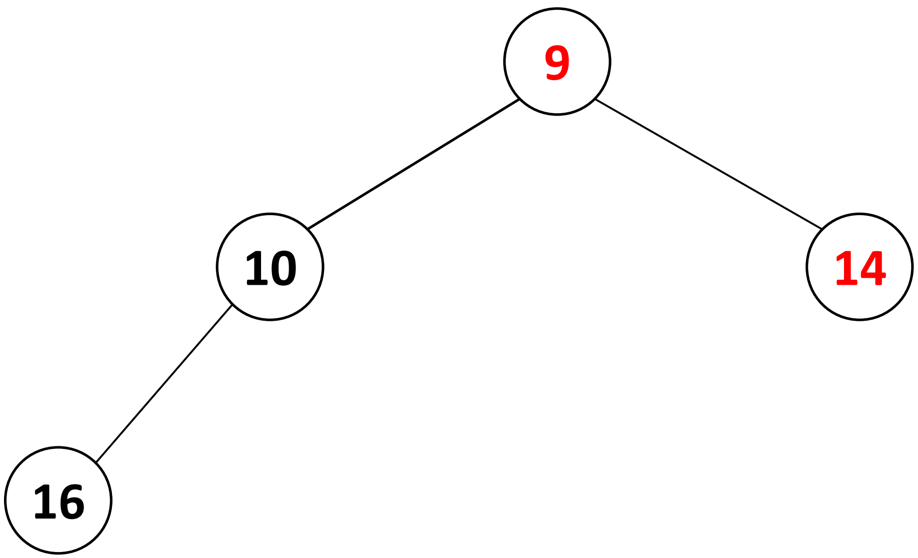
1	2	3	4	7	8				



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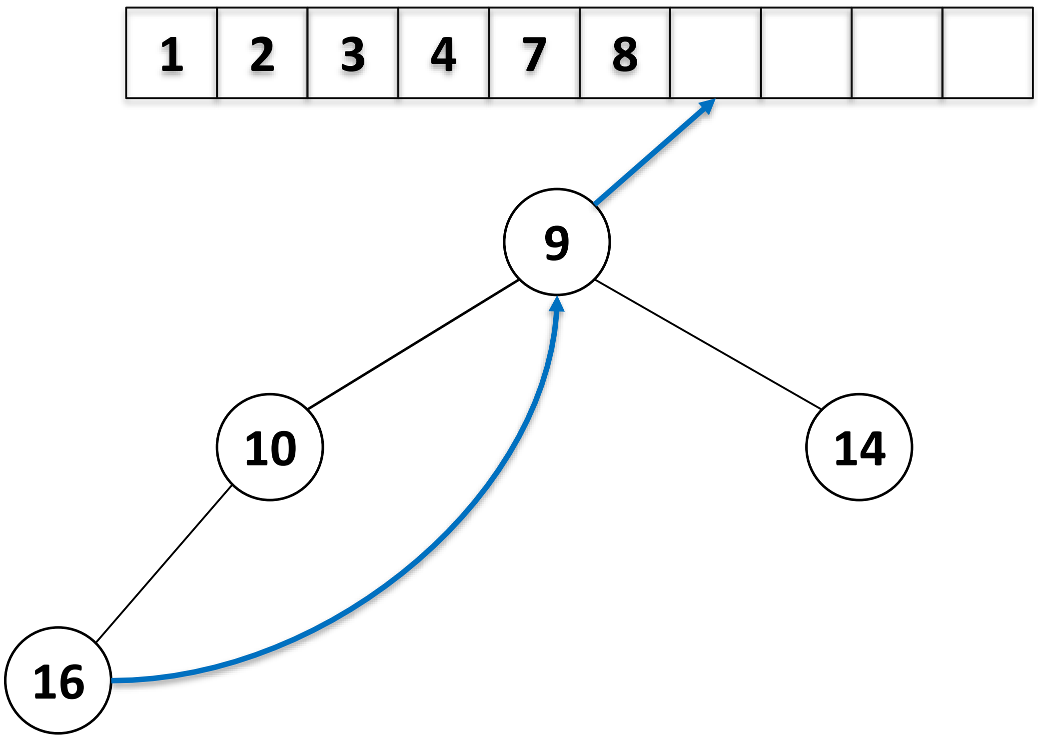
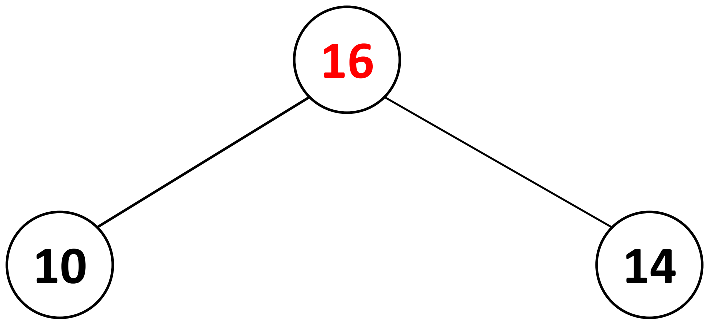


Heapsort - Example

- Perform n Extract-Min operations

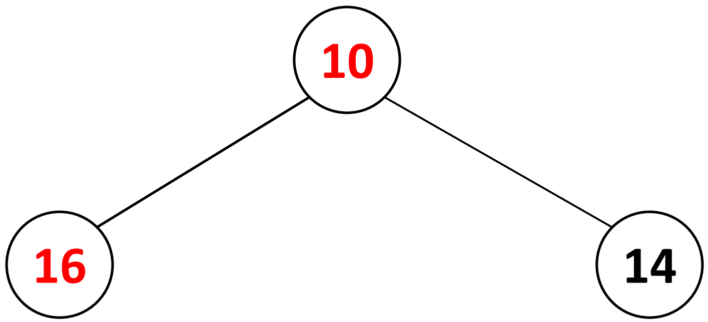
1	2	3	4	7	8	9			



Heapsort - Example

- Perform n Extract-Min operations

1	2	3	4	7	8	9			



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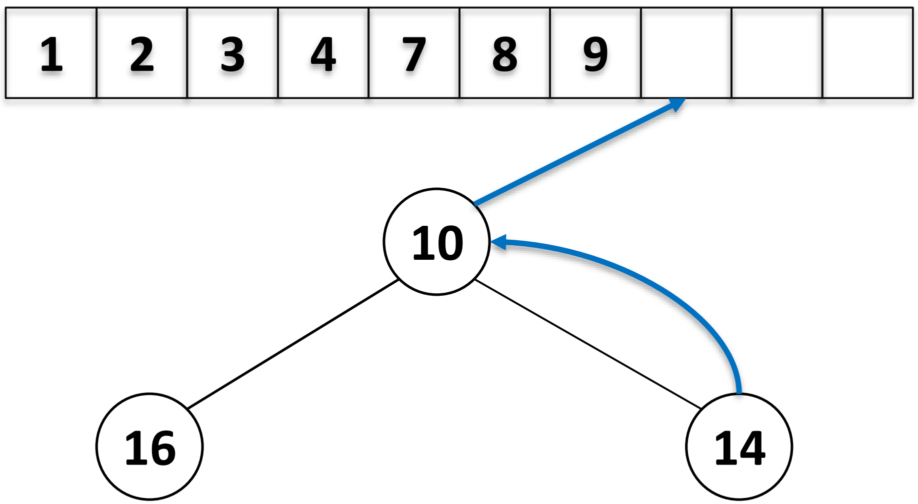
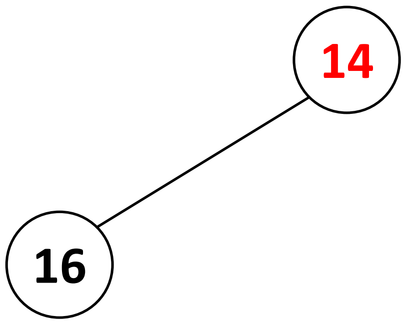


Heapsort - Example

- Perform n Extract-Min operations

1	2	3	4	7	8	9	10		



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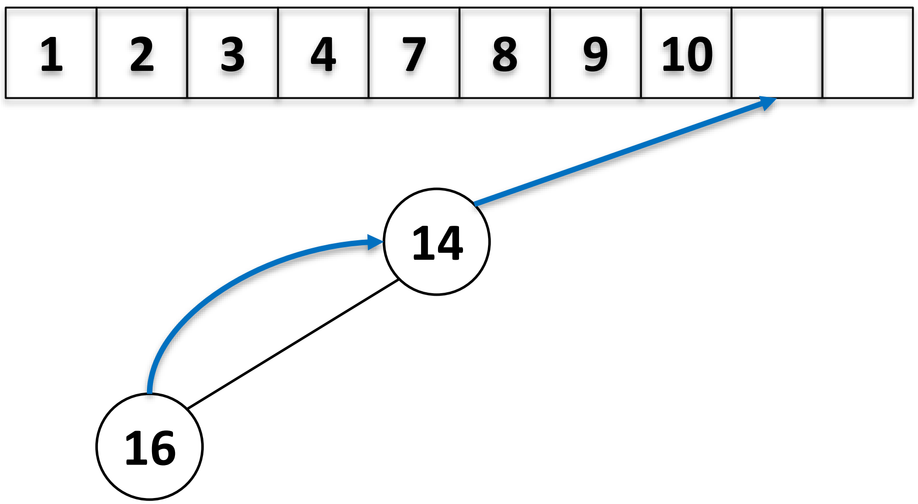

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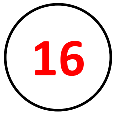

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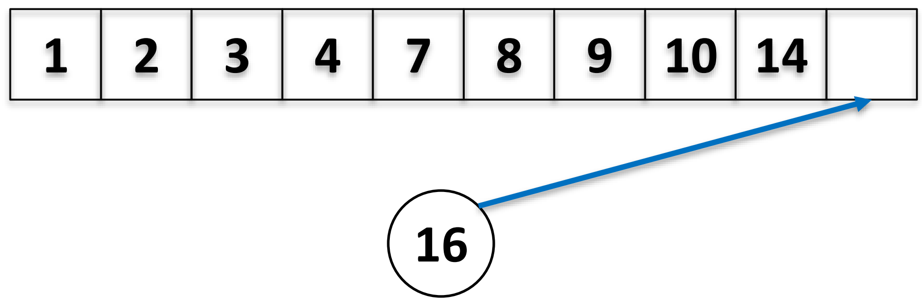

Heapsort - Example

- Perform n Extract-Min operations

1	**2**	**3**	**4**	**7**	**8**	**9**	**10**	**14**	**16**

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- If priority queues are implemented using heaps, then these two operations are supported in O(log n) time.

- Heapsort takes O(n log n) time, which is as efficient as merge sort and quicksort.

Outline

- Review to Divide-and-Conquer Paradigm
- Heapsort
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Comparison-based Sorting
  - Objective
  - Decision Tree Model
- Sorting in Linear Time
  - Counting Sort

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Can we do better?

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Goal

We will prove that any comparison-based sorting algorithm has a worst-case running time $\Omega(n \log n)$.

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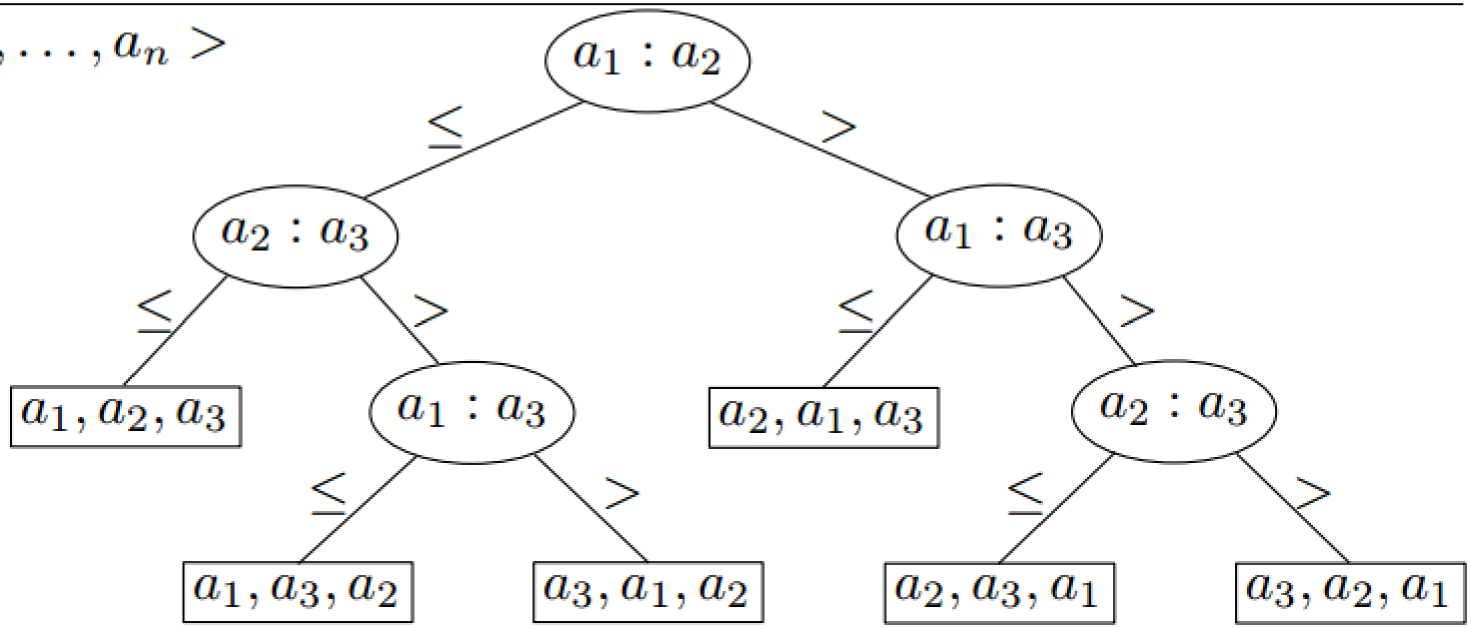
Decision-tree Model

Sort $< a_1, a_2, \ldots, a_n >$



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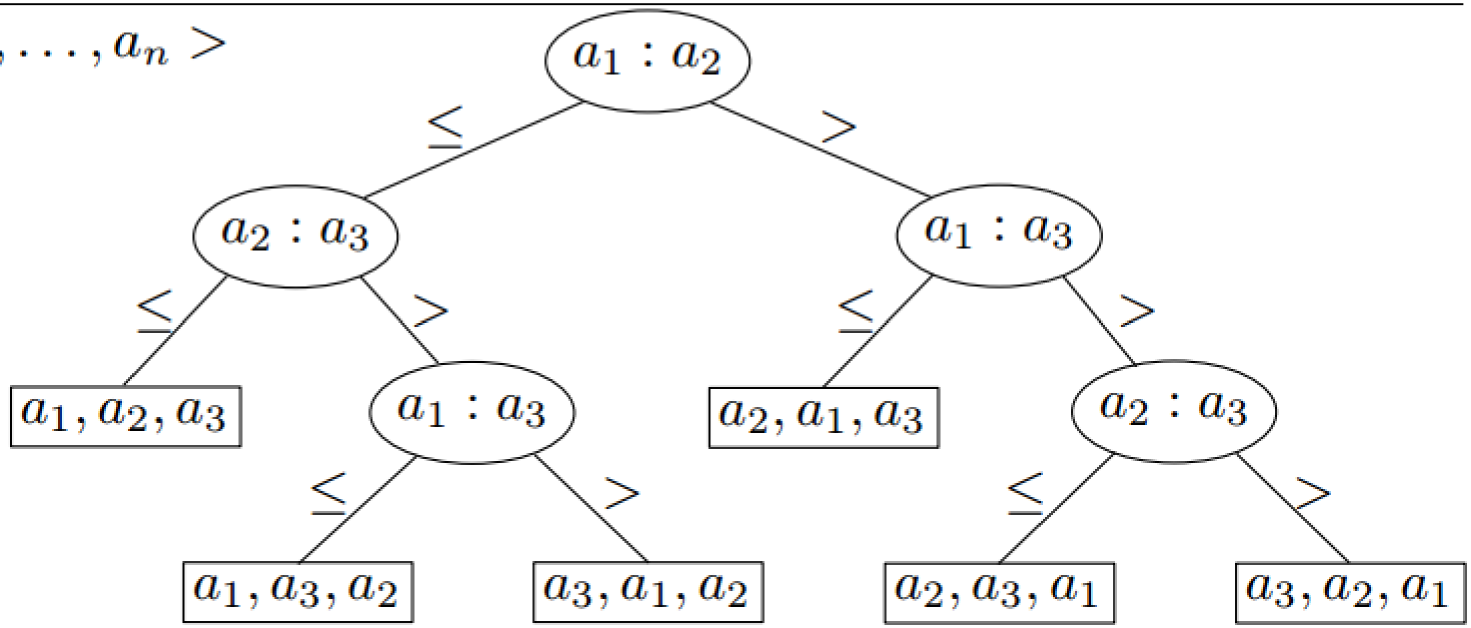

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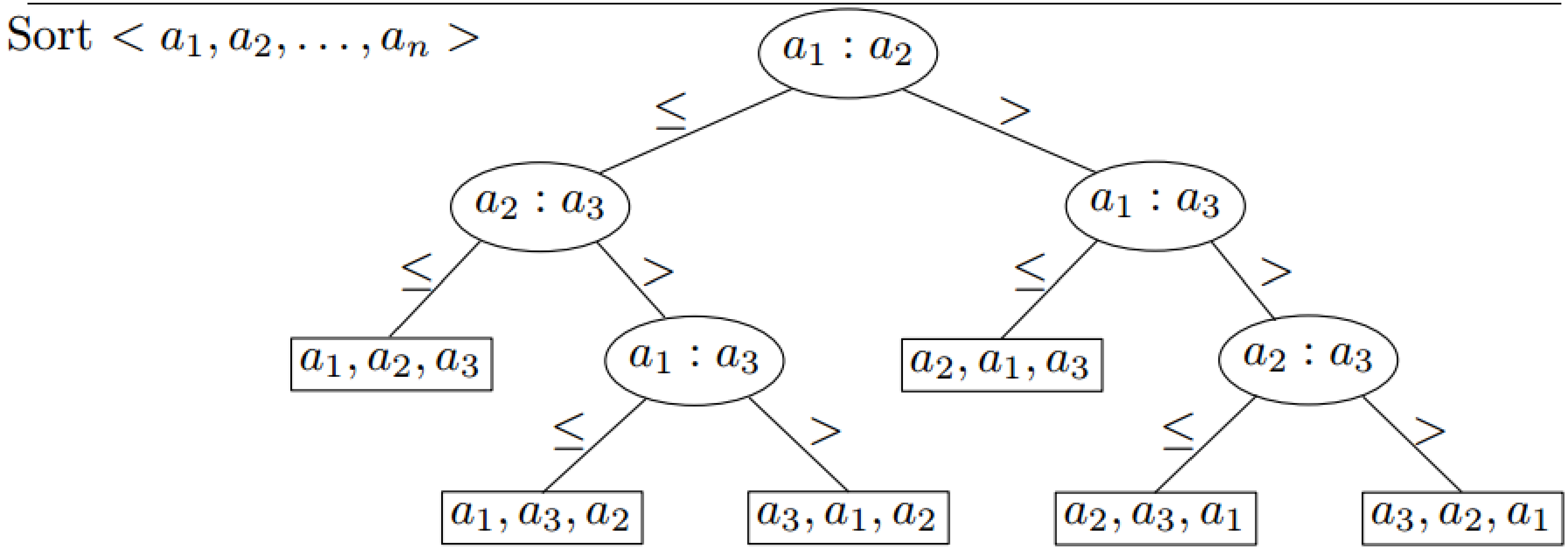

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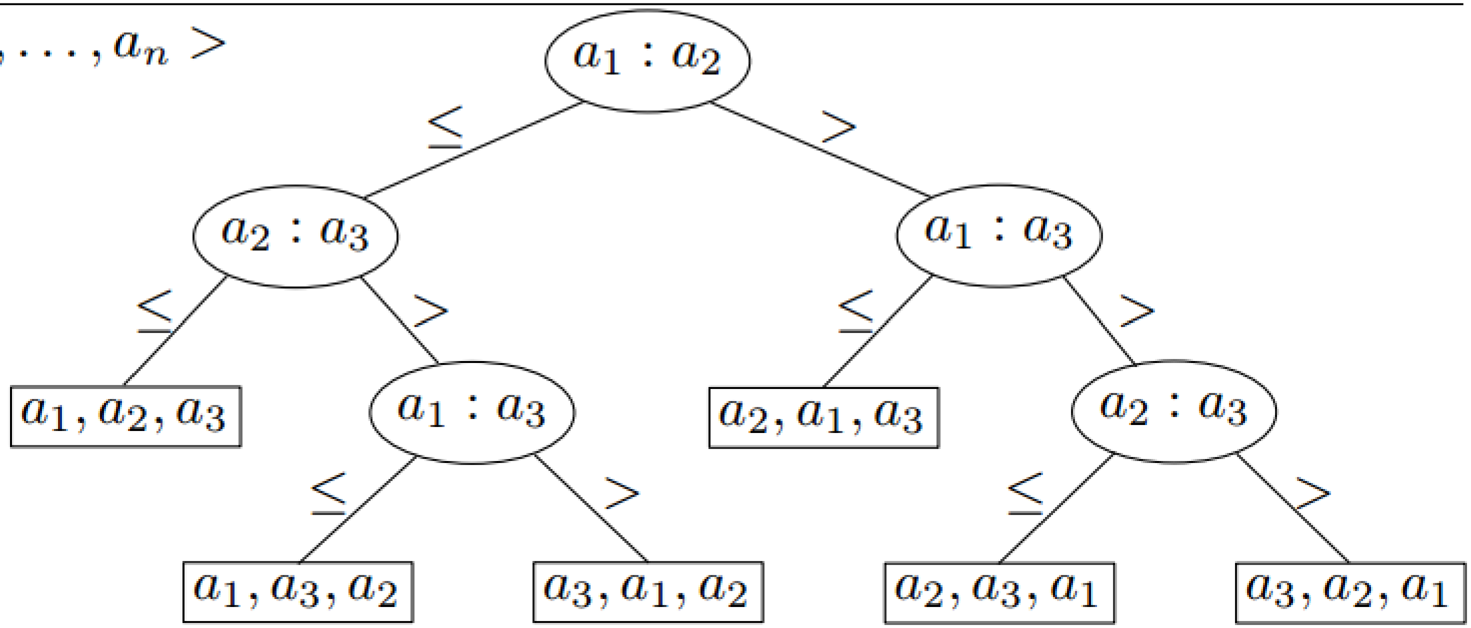

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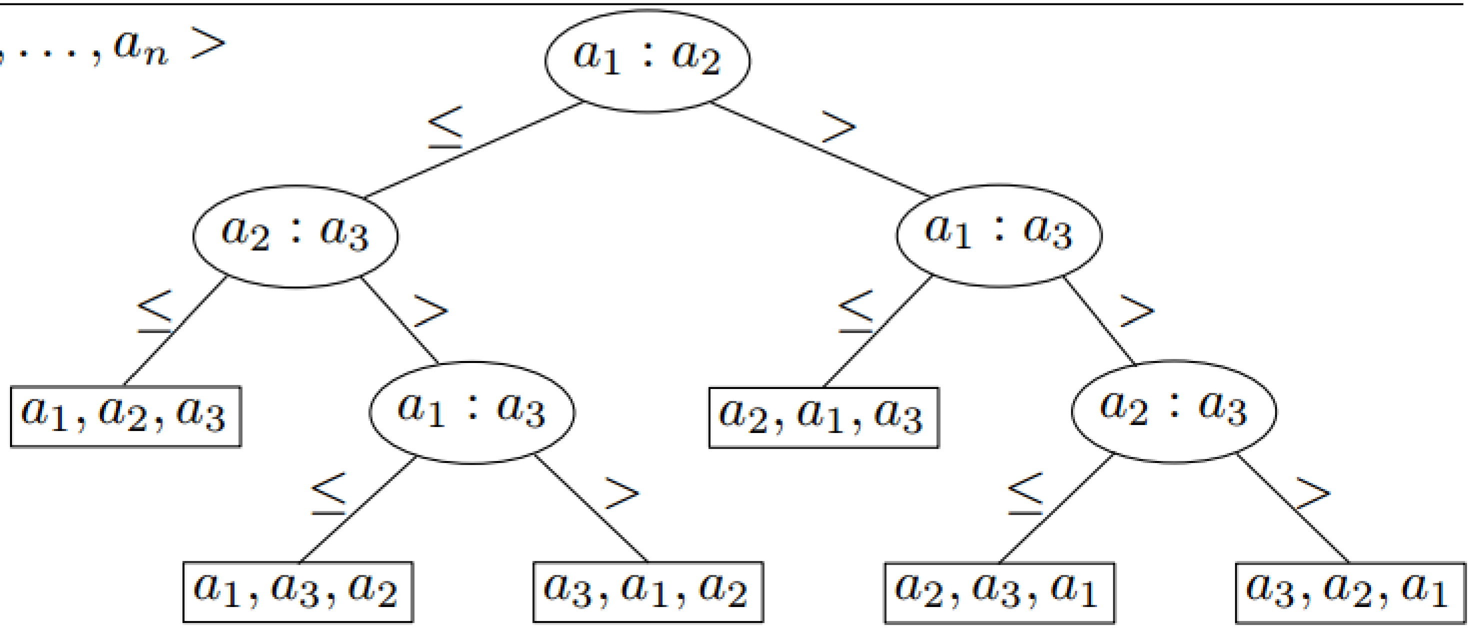

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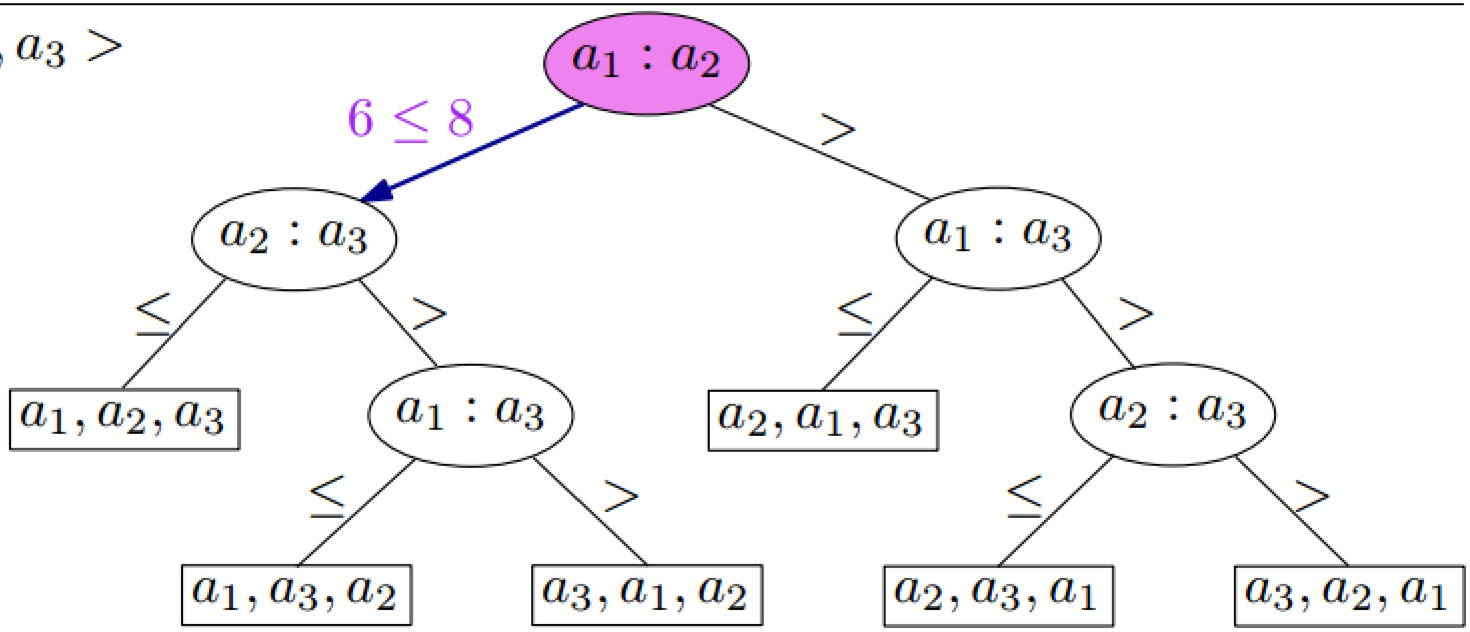

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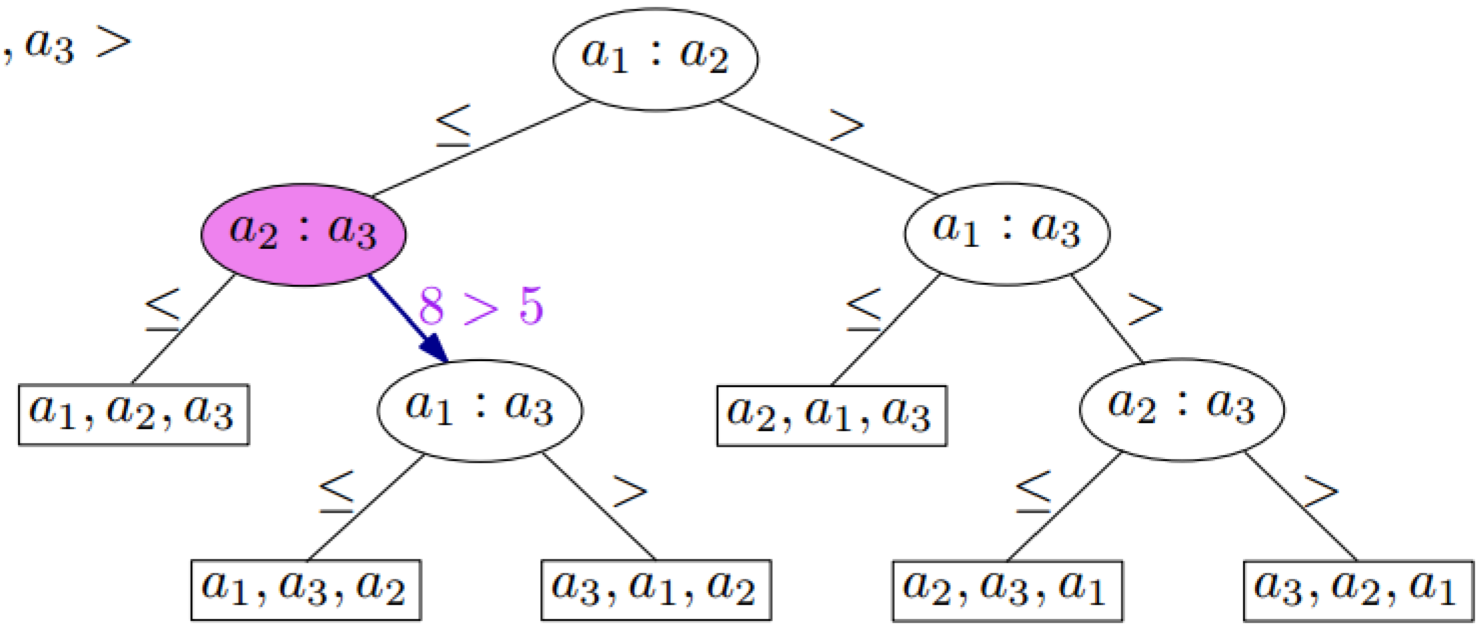

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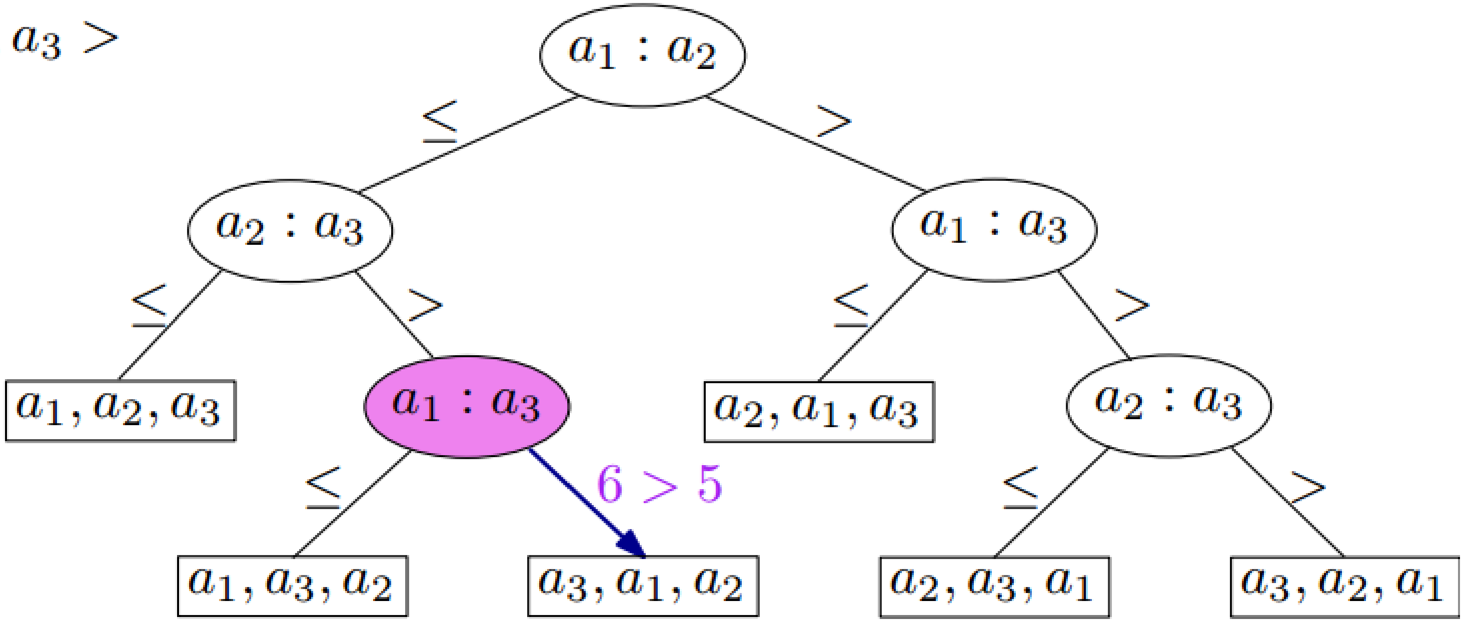

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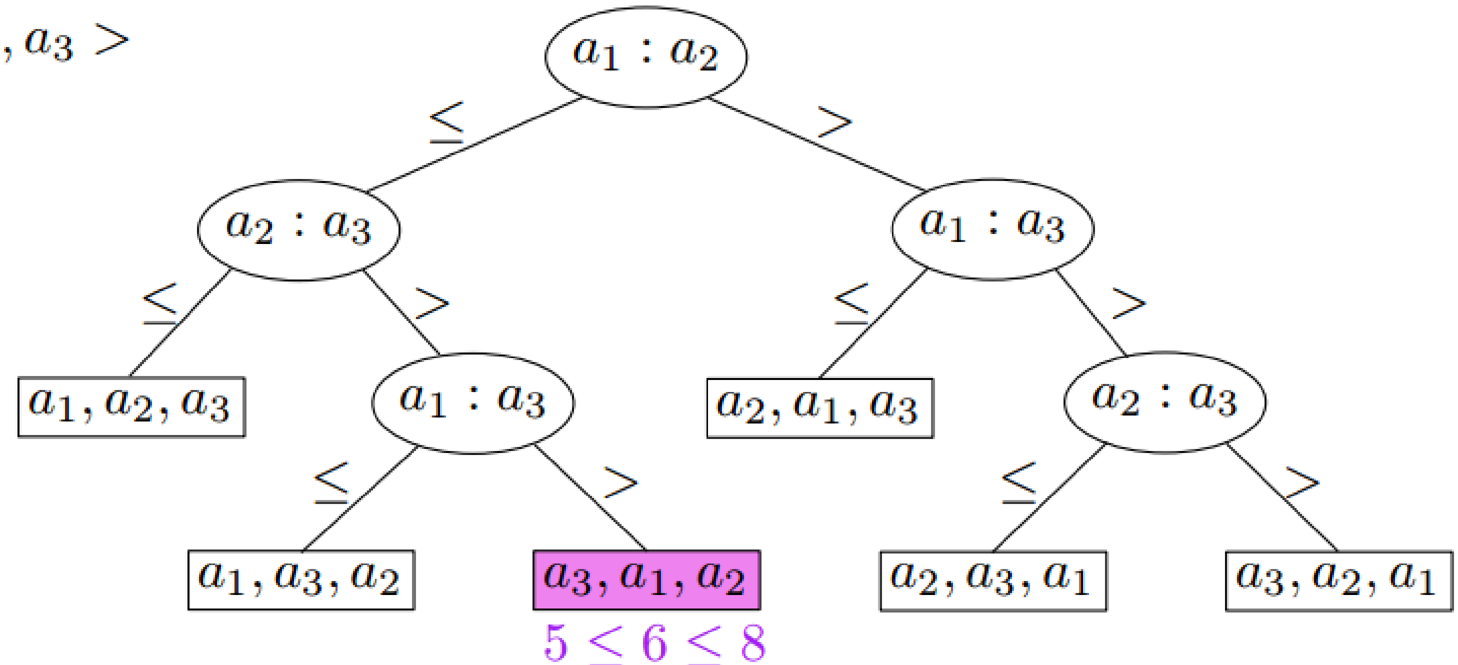

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- One tree for each input size n
- Worst-case running time = height of tree

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Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ *comparisons.*

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$\square$

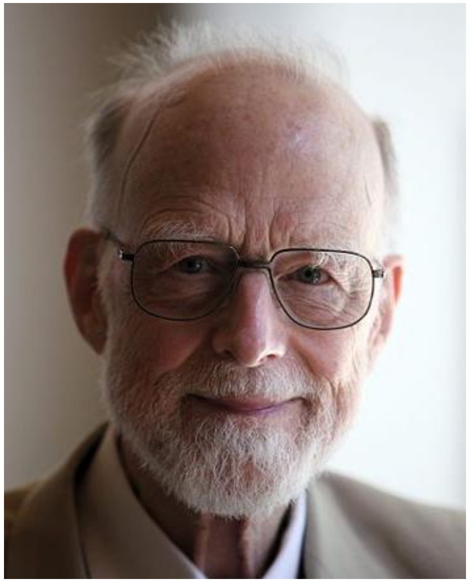
Corollary

Heapsort and merge sort are asymptotically optimal comparison-based sorting algorithms.

Summary

John von Neumann

Merge Sort Algorithm was invented in 1945



Tony Hoare

Quicksort Algorithm was invented in 1959

J. W. J. Williams

Heapsort Algorithm was invented in 1964

Can we do better for the sorting problem?

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- It uses this information to place element *x* directly into its position in the output array.
    - For example, if 17 elements are less than *x*, then *x* belongs in output position 18.

Counting Sort

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
| $C[i] \leftarrow 0$;
end
for $j \leftarrow 1$ *to* n **do**
| $C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$
end
for $i \leftarrow 2$ *to* k **do**
| $C[i] \leftarrow C[i] + C[i-1]; // C[i] = |\{key \leq i\}|$
end
for $j \leftarrow n$ *to* 1 **do**
| $B[C[A[j]]] \leftarrow A[j]$;
| $C[A[j]] \leftarrow C[A[j]] - 1$;
end
return B;

Counting Sort

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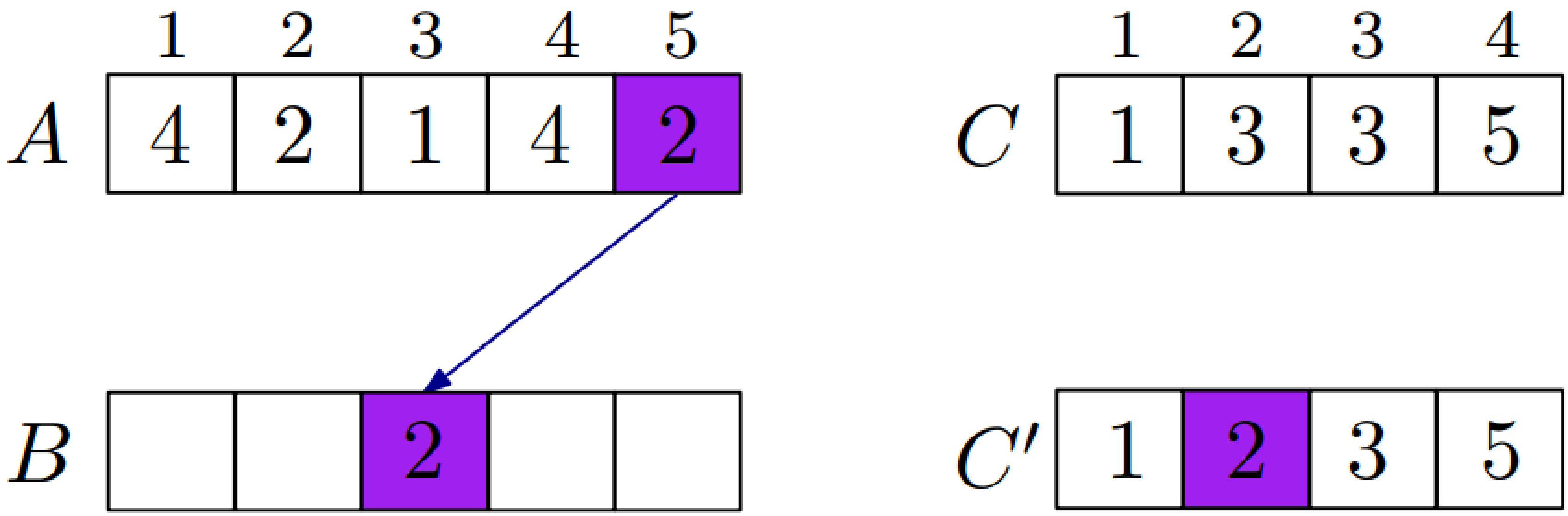

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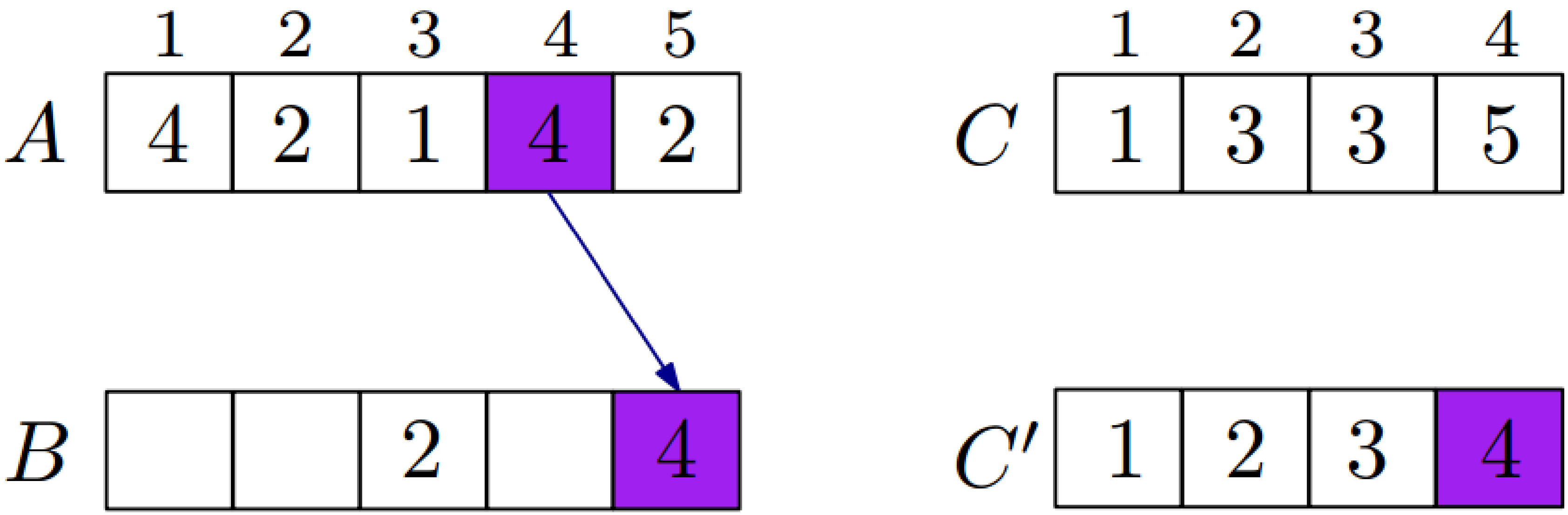

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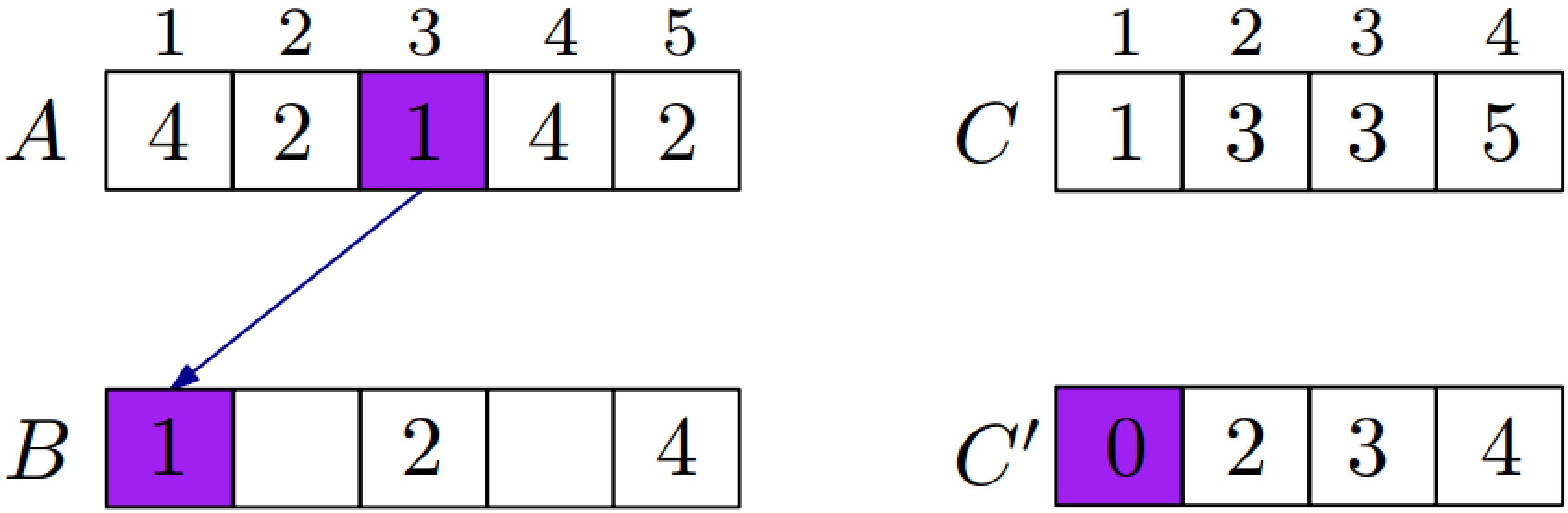

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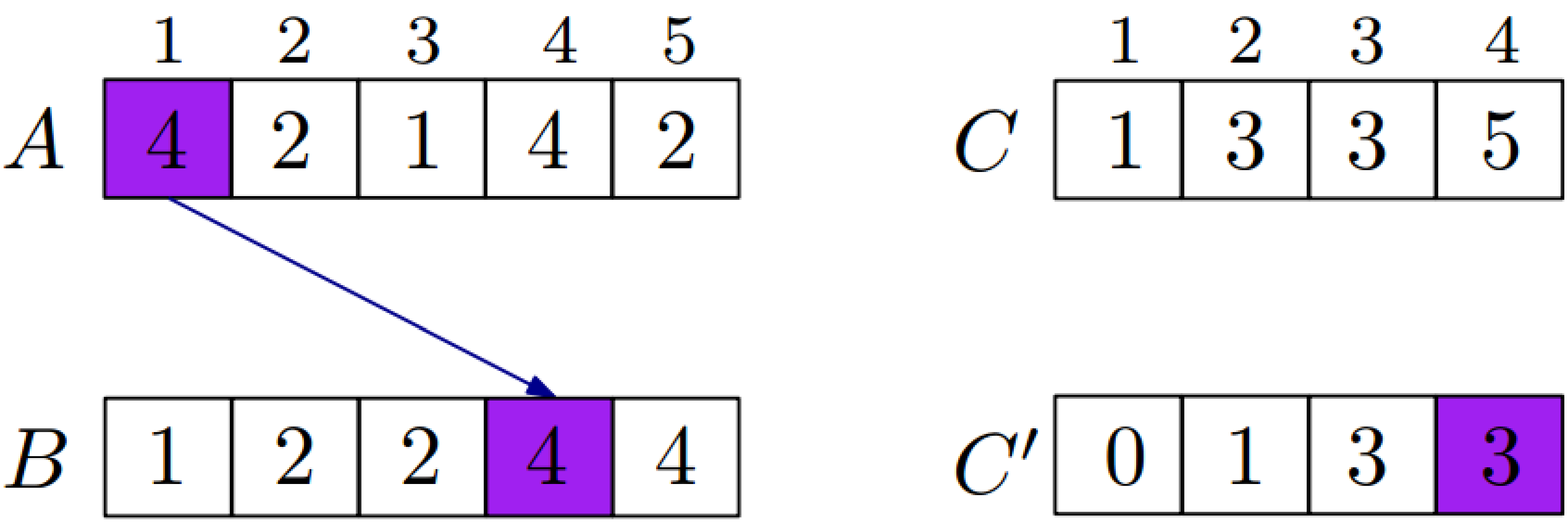

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Total: $O(n+k)$

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If $k = O(n)$, then counting sort takes $O(n)$ time.

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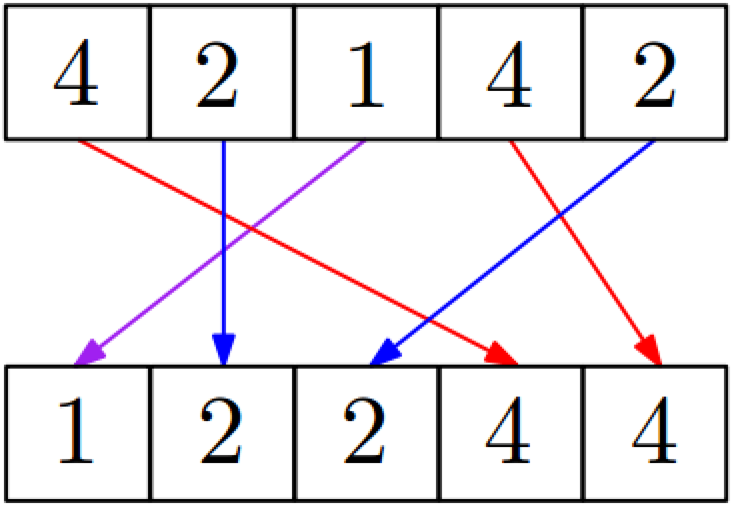
- In fact, it makes no comparison at all!

Stable Sorting

Counting sort is a stable sort

- it preserves the input order among equal elements.



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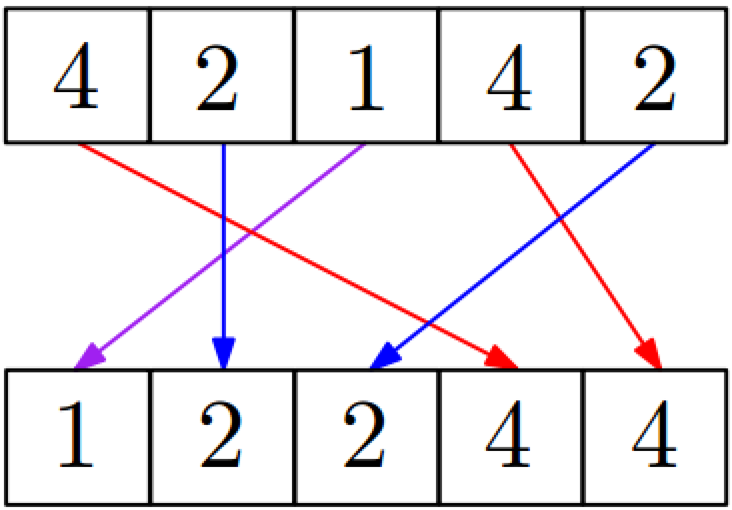

Exercise

What other sorts have this property?

谢谢

BDA
Beihang University