Design and Analysis of Algorithms

Part I: Divide and Conquer

Lecture 7: Quicksort

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Outline

- Review to Divide-and-Conquer Paradigm

- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort

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Review to Divide-and-Conquer Paradigm

- Divide-and-conquer (D&C) is an important algorithm design paradigm.
  - **Divide**
    Dividing a given problem into two or more subproblems (ideally of approximately equal size)
  - **Conquer**
    Solving each subproblem (directly if small enough or recursively)
  - **Combine**
    Combining the solutions of the subproblems into a global solution

Review to Divide-and-Conquer Paradigm

- In Part I, we will illustrate Divide-and-Conquer using several examples:
  - Maximum Contiguous Subarray (最大子数组)
  - Counting Inversions (逆序计数)
  - Polynomial Multiplication (多项式乘法)
  - QuickSort and Partition (快速排序与划分)
  - Randomized Selection (随机化选择)
  - Lower Bound for Sorting (基于比较的排序下界)

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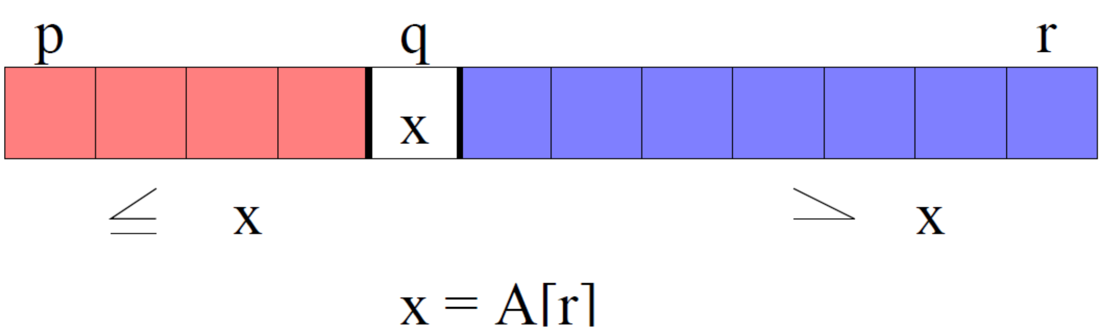
Partition

- Partition
  - Given: An array of numbers
  - Partition: Rearrange the array A[p..r] in place into two (possibly empty) subarrays A[p..q-1] and A[q+1..r] such that

$$A[u] < A[q] < A[v] \ \ for\ any\ p \le u \le q-1\ and\ q+1 \le v \le r$$



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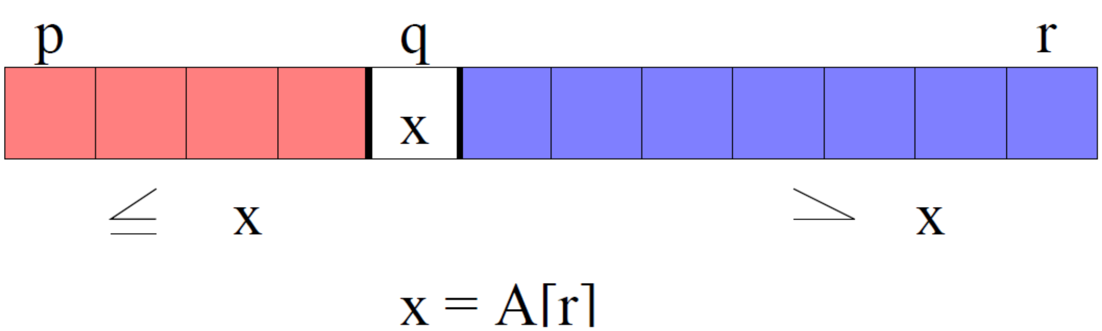

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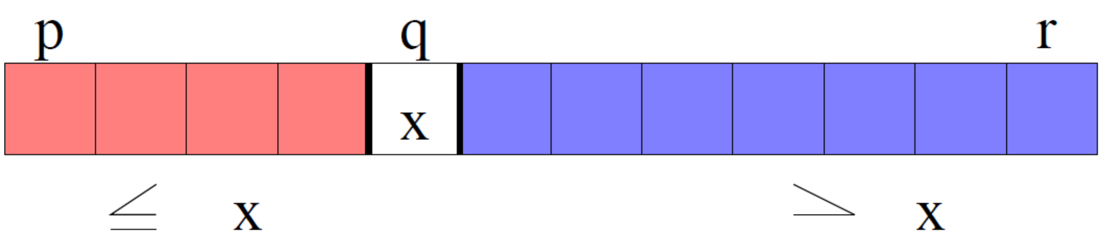

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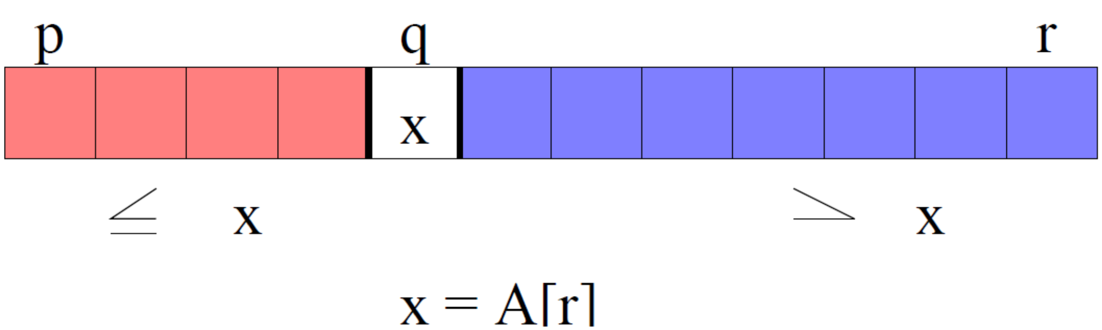

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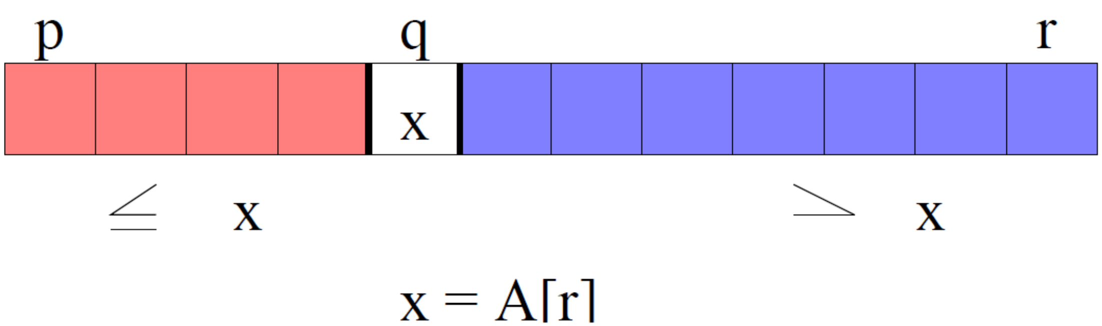

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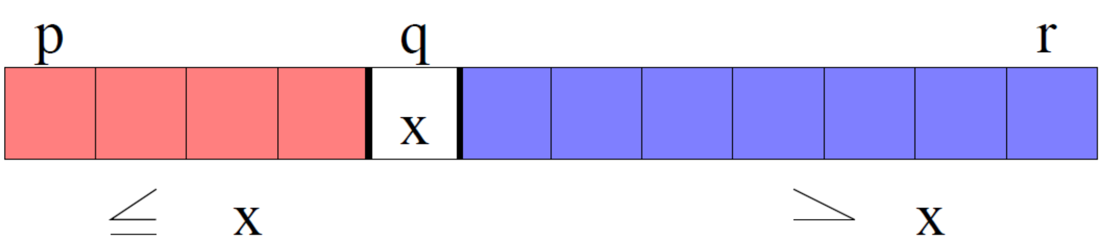

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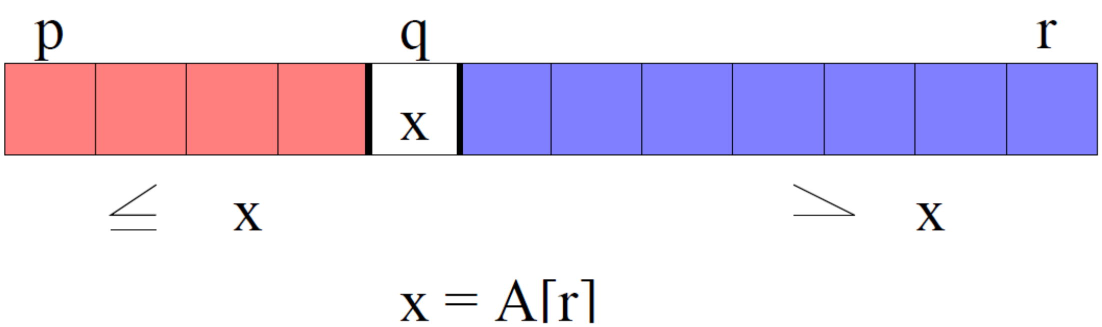

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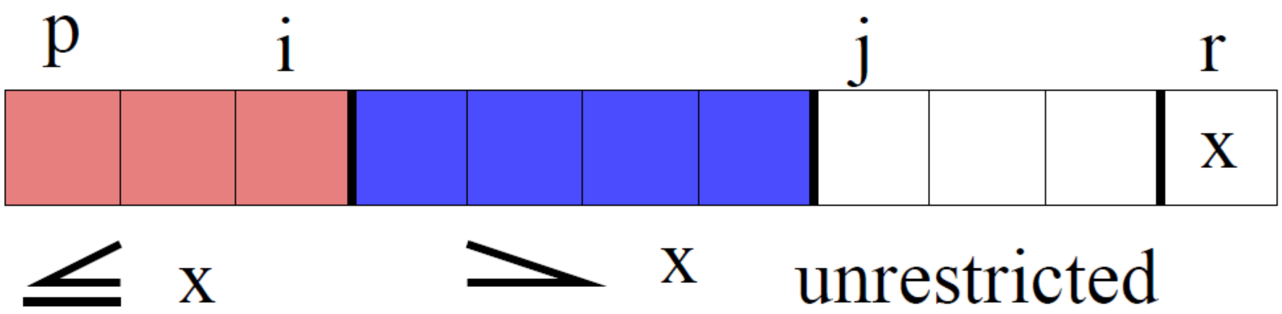
Partition

- The idea of Partition(A, p, r)
  - Use A[r] as the pivot, and grow partition from left to right



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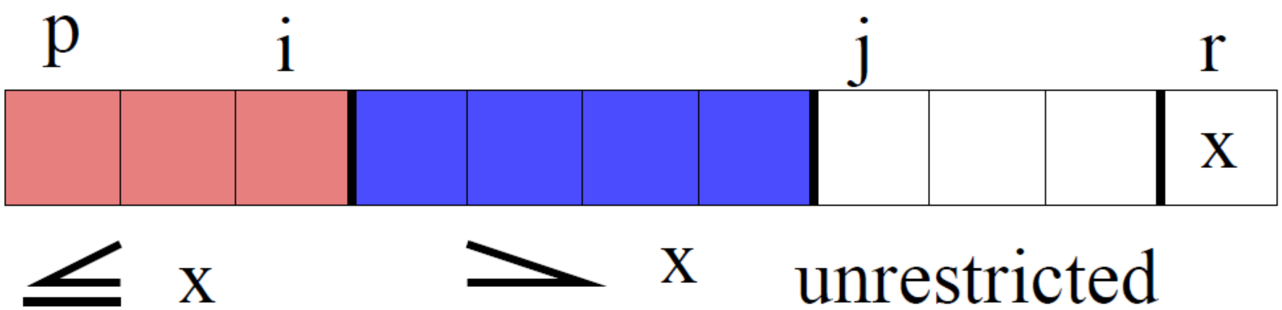

  - Initially (i, j) = (p–1, p)
  - Increase j by 1 each time to find a place for A[j]

    At the same time increase i when necessary
  - Stops when j = r

Partition

- One Iteration of the Procedure Partition
  - Increase j by 1 each time to find a place for A[j]

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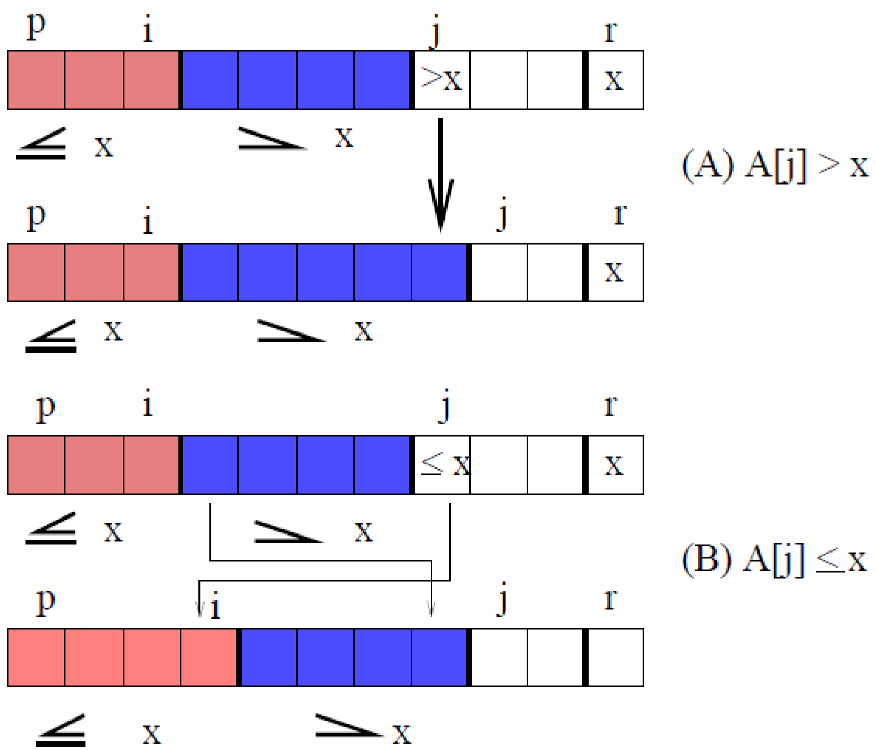

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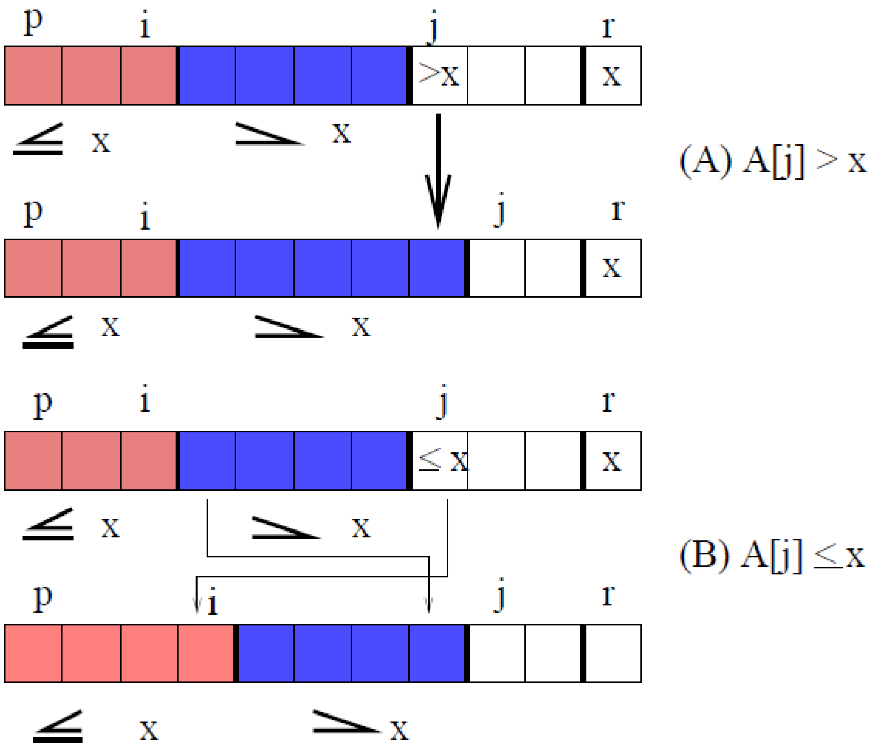

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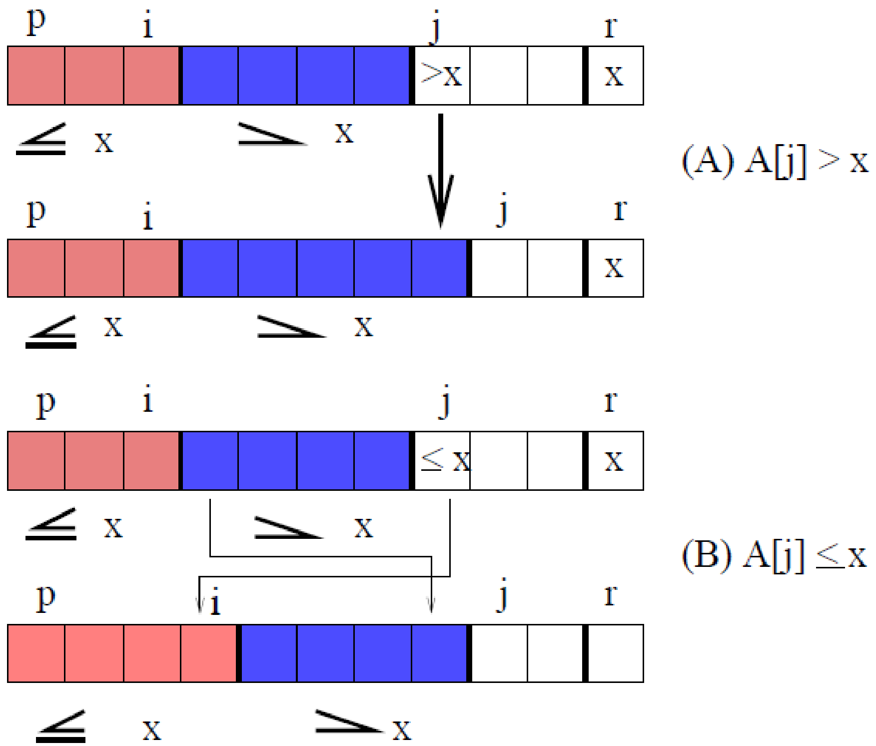
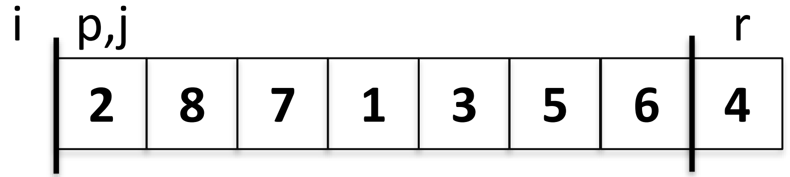


  - Case (A): Only increase j by 1
  - Case (B): i = i + 1; A[i] ↔ A[j]; j = j + 1.

Partition-Example

- The Operation of Partition(A, p, r)



Partition-Example

- The Operation of Partition(A, p, r)

i	p,j							r
	2	**8**	**7**	**1**	**3**	**5**	**6**	**4**

$$A[j] < A[r]$$

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- The Operation of Partition(A, p, r)

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$$i = i + 1, A[i] \leftrightarrow A[j], j = j + 1$$

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Partition-Example

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$$A[i+1] \leftrightarrow A[r]$$

Partition - Pseudocode

Partition(A,p,r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Index of the pivot after partition
$x \leftarrow A[r]$;//$A[r]$ is the pivot element

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end

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    if A[j] ≤ x then

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- Running time is O(   )

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- Running time is O(r – p)
  - linear in the length of the array A[p..r]

Quicksort

Quicksort(A,p,r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Sorted array $\boldsymbol{A}$
if $p < r$ **then**
  $q \leftarrow$Partition(A, p, r);
  Quicksort(A,);
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end
return A;

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A Divide-and-Conquer Framework

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- If we could always partition the array into halves, then we have the recurrence $T(n) \leq 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$.

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A Divide-and-Conquer Framework

- If we could always partition the array into halves, then we have the recurrence $T(n) \leq 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$.
- However, if we always get unlucky with very unbalanced partitions, then $T(n) \leq T(n-1) + O(n)$, hence $T(n) = O(n^2)$.

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  - Analysis of the randomized quicksort

Randomized-Partition(A, p, r)

- Idea
  - In the algorithm Partition(A, p, r), A[r] is always used as the pivot *x* to partition the array A[p..r].

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  - In the algorithm Partition(A, p, r), A[r] is always used as the pivot x to partition the array A[p..r].
  - In the algorithm Randomized-Partition(A, p, r), we randomly choose an j , p ≤ j ≤ r , and use A[j] as pivot.
  - Idea is that if we choose randomly, then the chance that we get unlucky every time is extremely low.

Randomized-Partition(A, p, r)

- Pseudocode of Randomized-Partition
  - Let random(p, r) be a pseudorandom-number generator that returns a random number between p and r.

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Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: A random index in $[p..j]$

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Output: A random index in $[p..j]$
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exchange $A[r]$ and $A[j]$;
Partition(A, p, r);
return j;

Randomized-Partition(A, p, r)

- Pseudocode of Randomized-Quicksort
  - We make use of the Randomized-Partition idea to develop a new version of quicksort.

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Randomized-Quicksort(A,p,r)

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Input: An array A waiting to be sorted,the range of index p,r
Output: Sorted array A
if p < r then
  | q ← Randomized-Partition(A, p, r);
  | Randomized-Quicksort(A,          );
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end
return A;
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if $p < r$ **then**
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  Randomized-Quicksort($A, p, q-1$);
  Randomized-Quicksort($A, q+1, r$);
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Quicksort - Example

2	**8**	**7**	**1**	**3**	**5**	**6**	**4**

Quicksort - Example

Divide

2	8	7	1	3	5	6	4

Quicksort - Example

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2	8	7	1	3	5	6	4

2	1	3	4	7	5	6	8

Quicksort - Example

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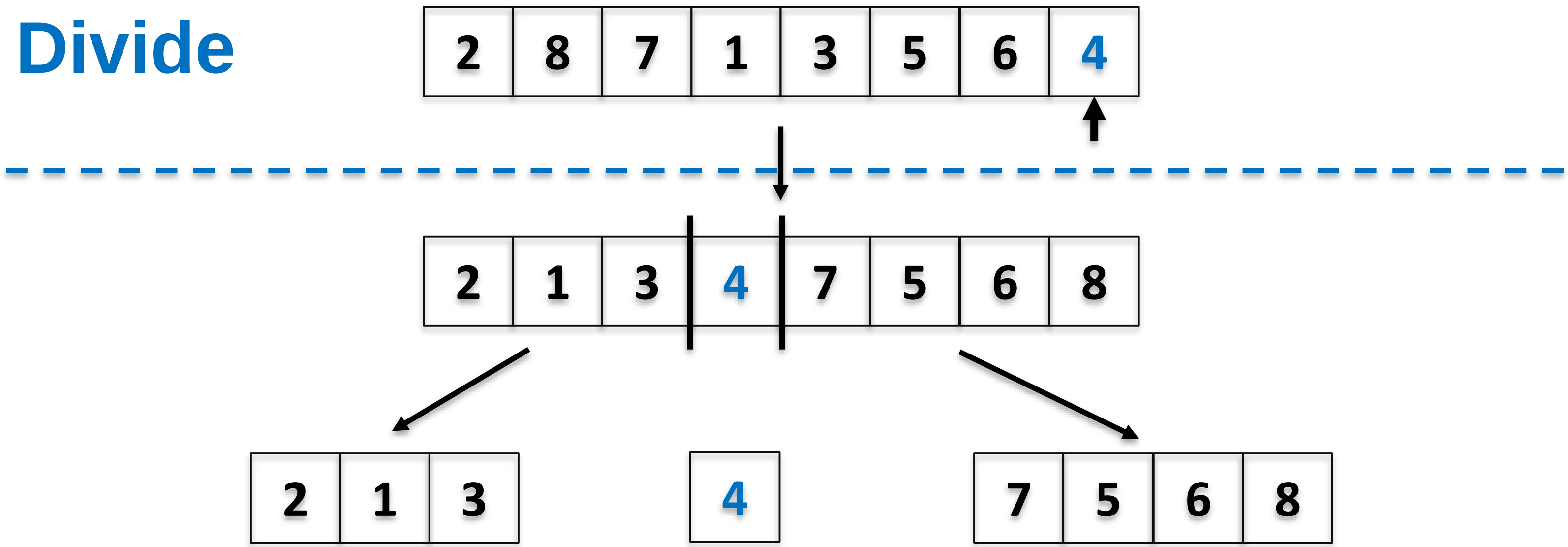

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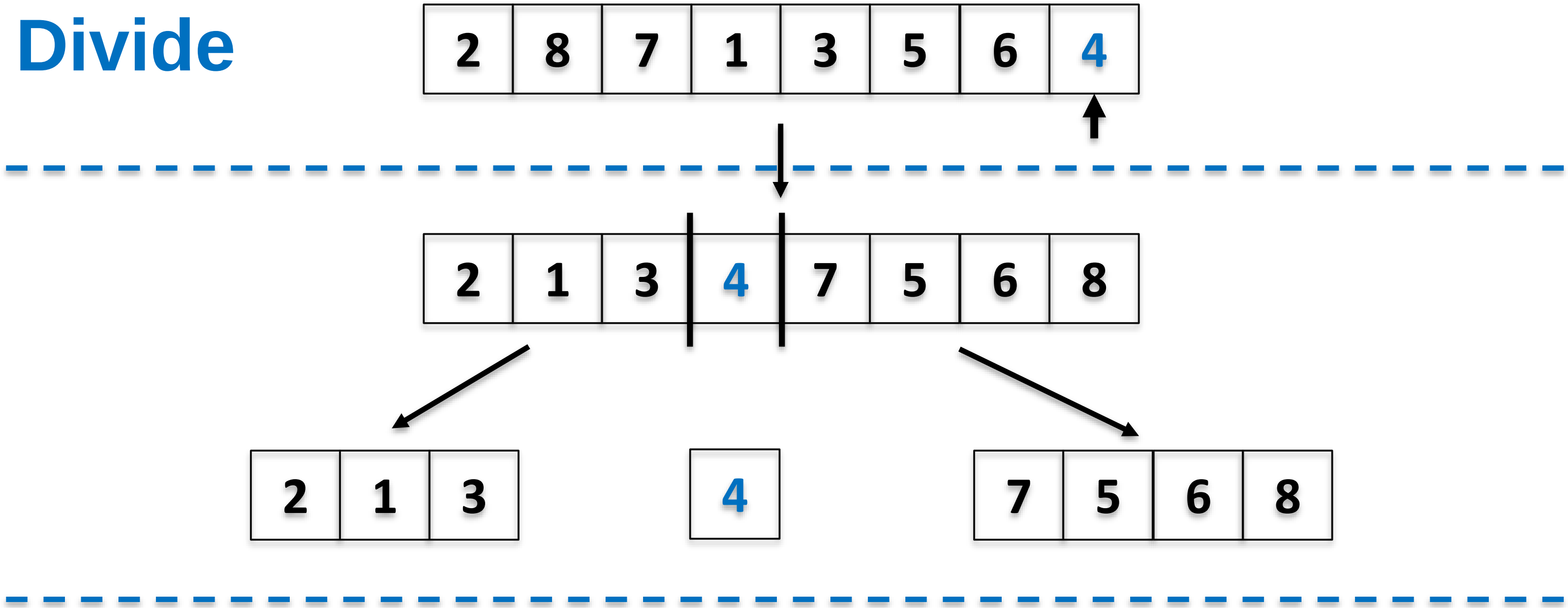

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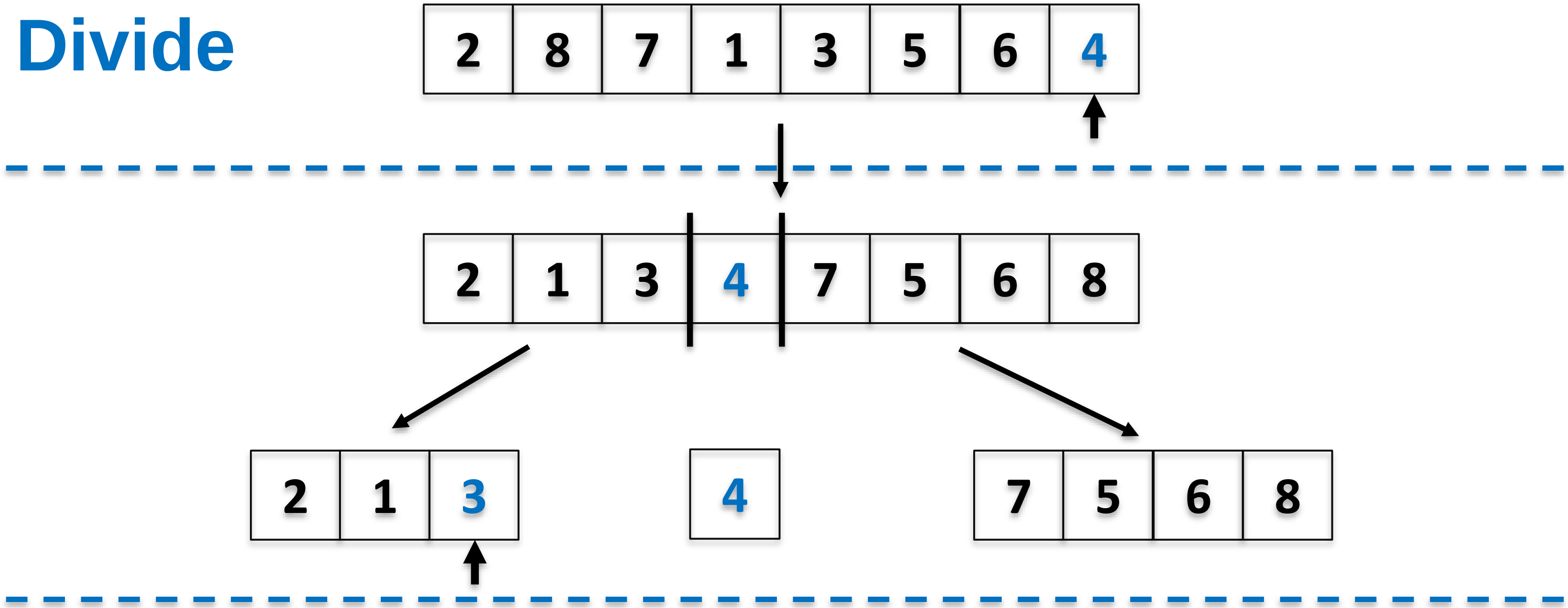

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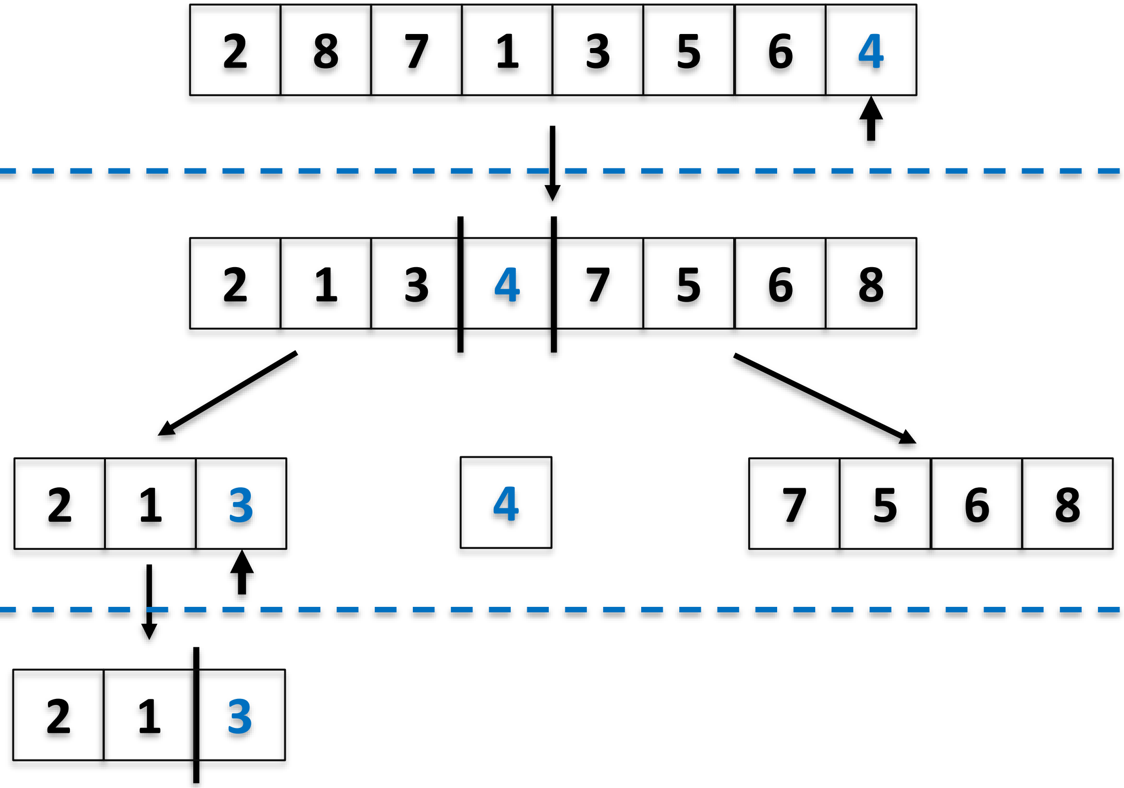

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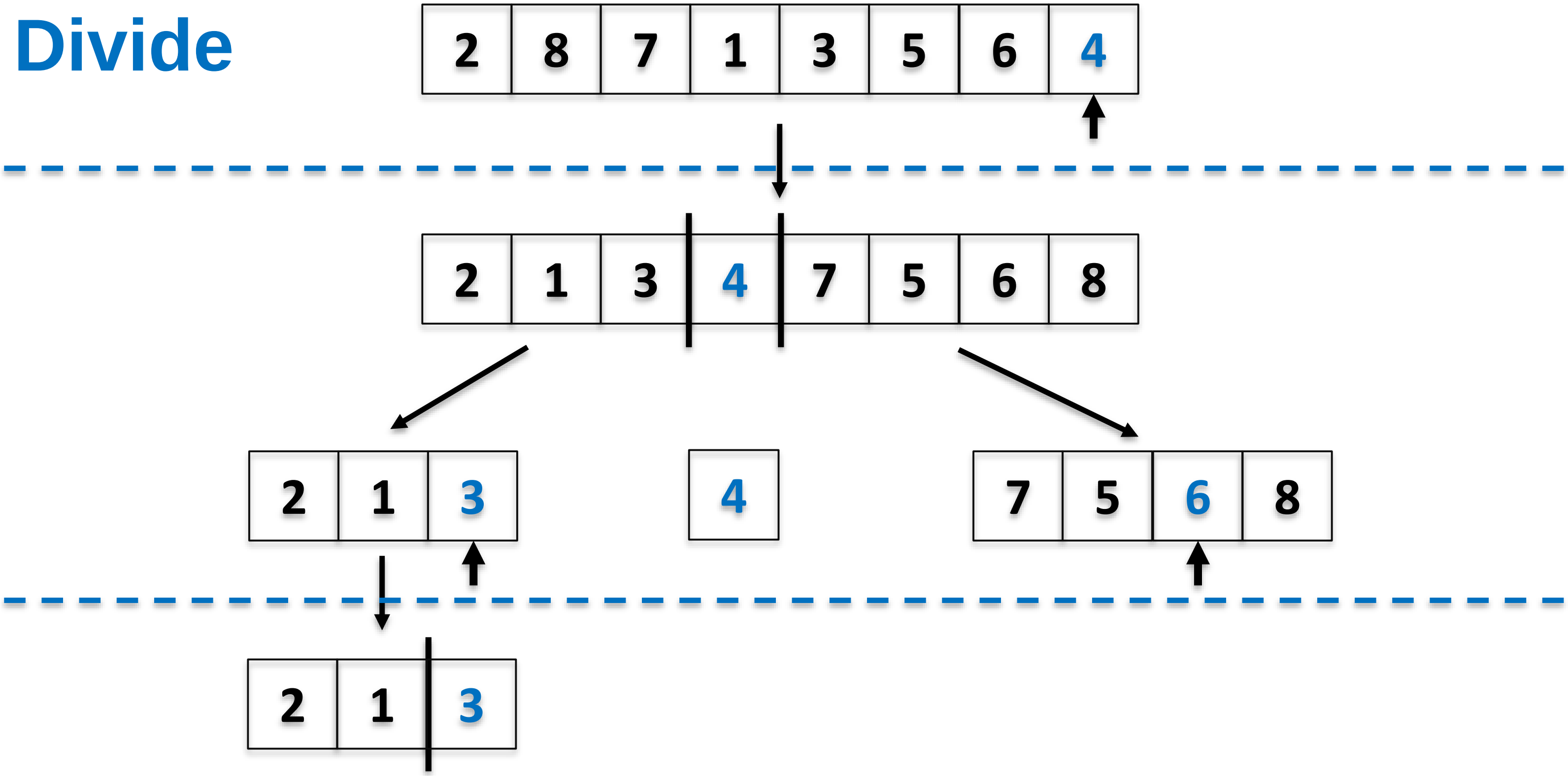

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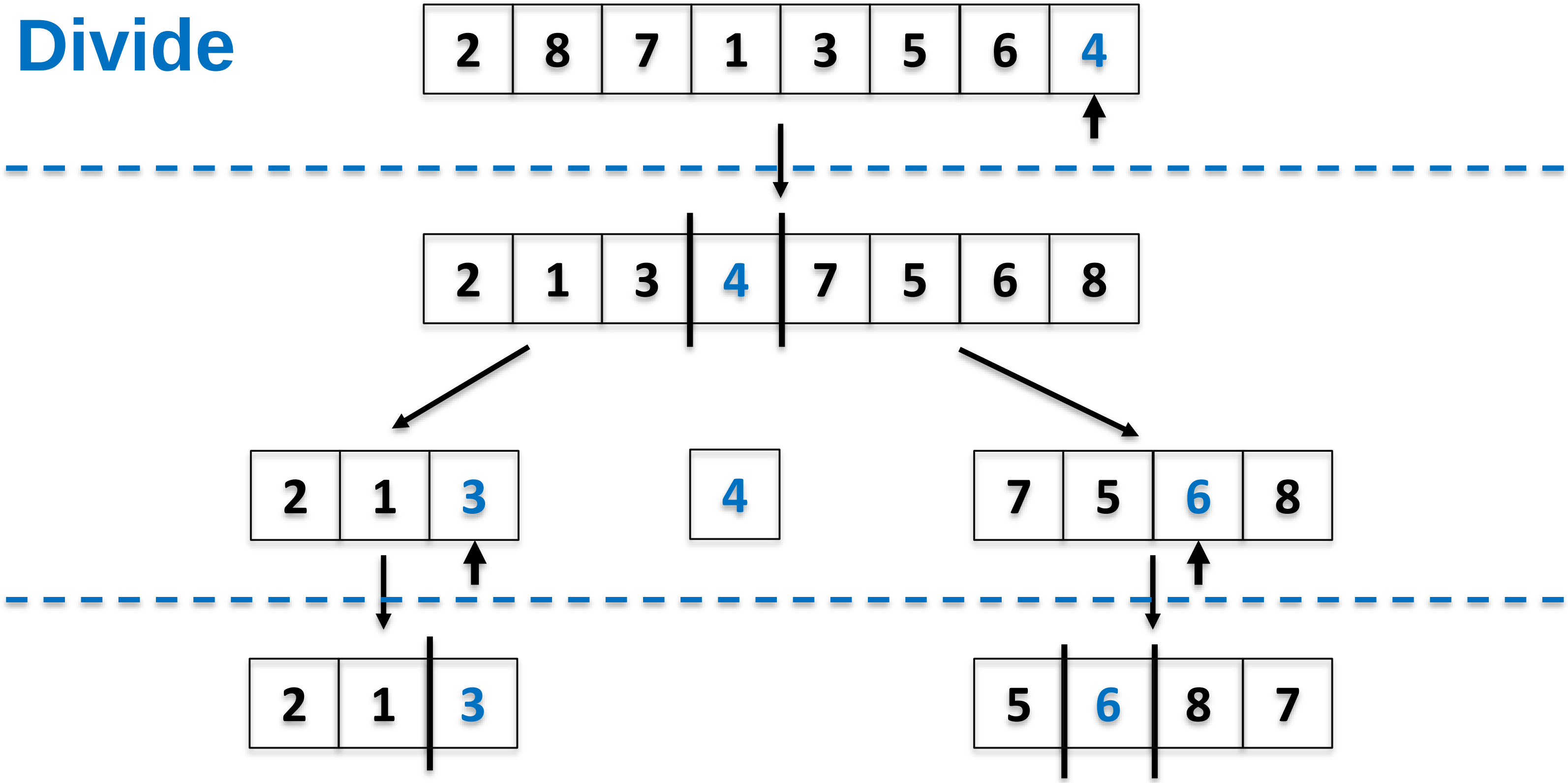

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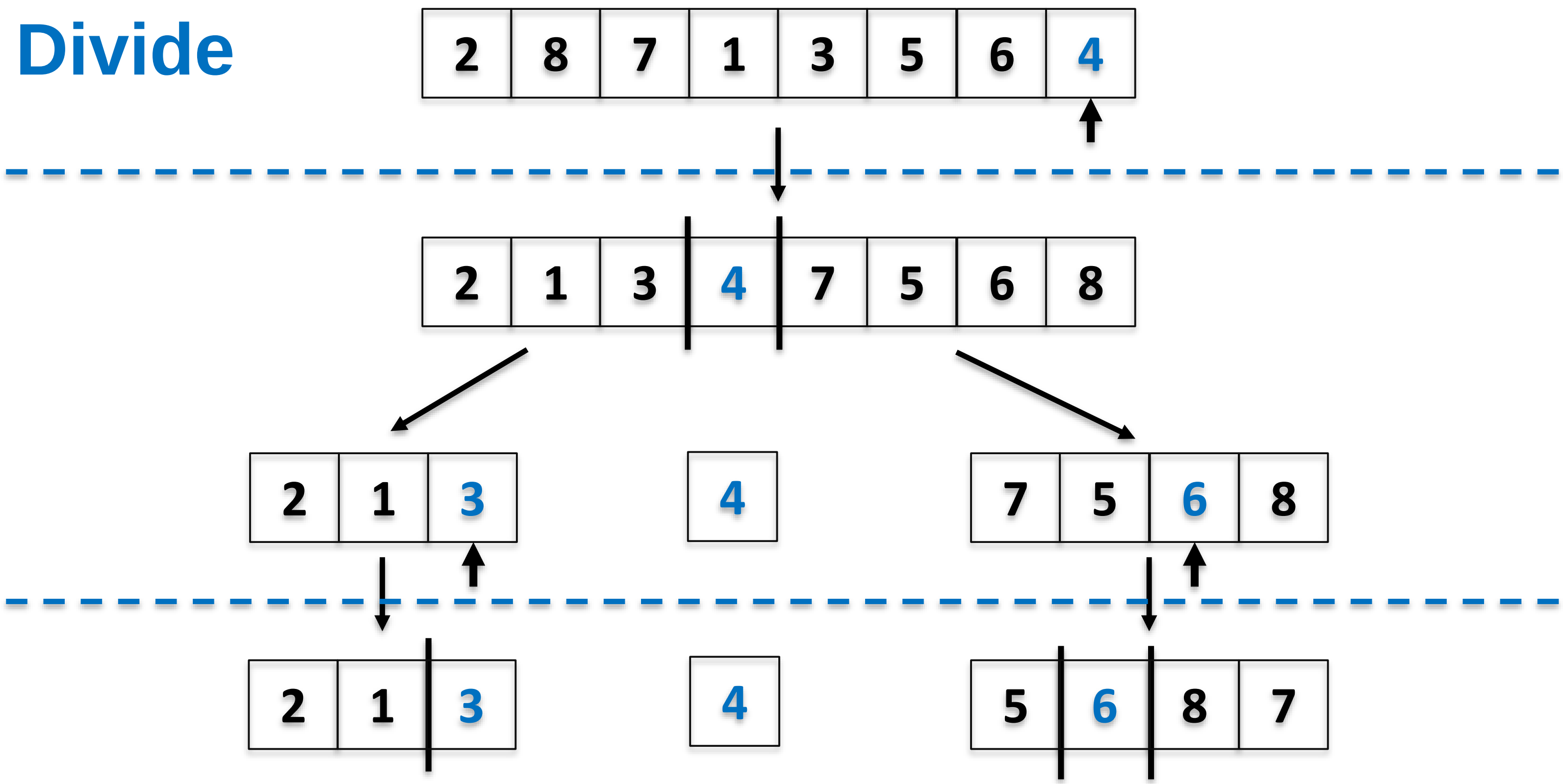

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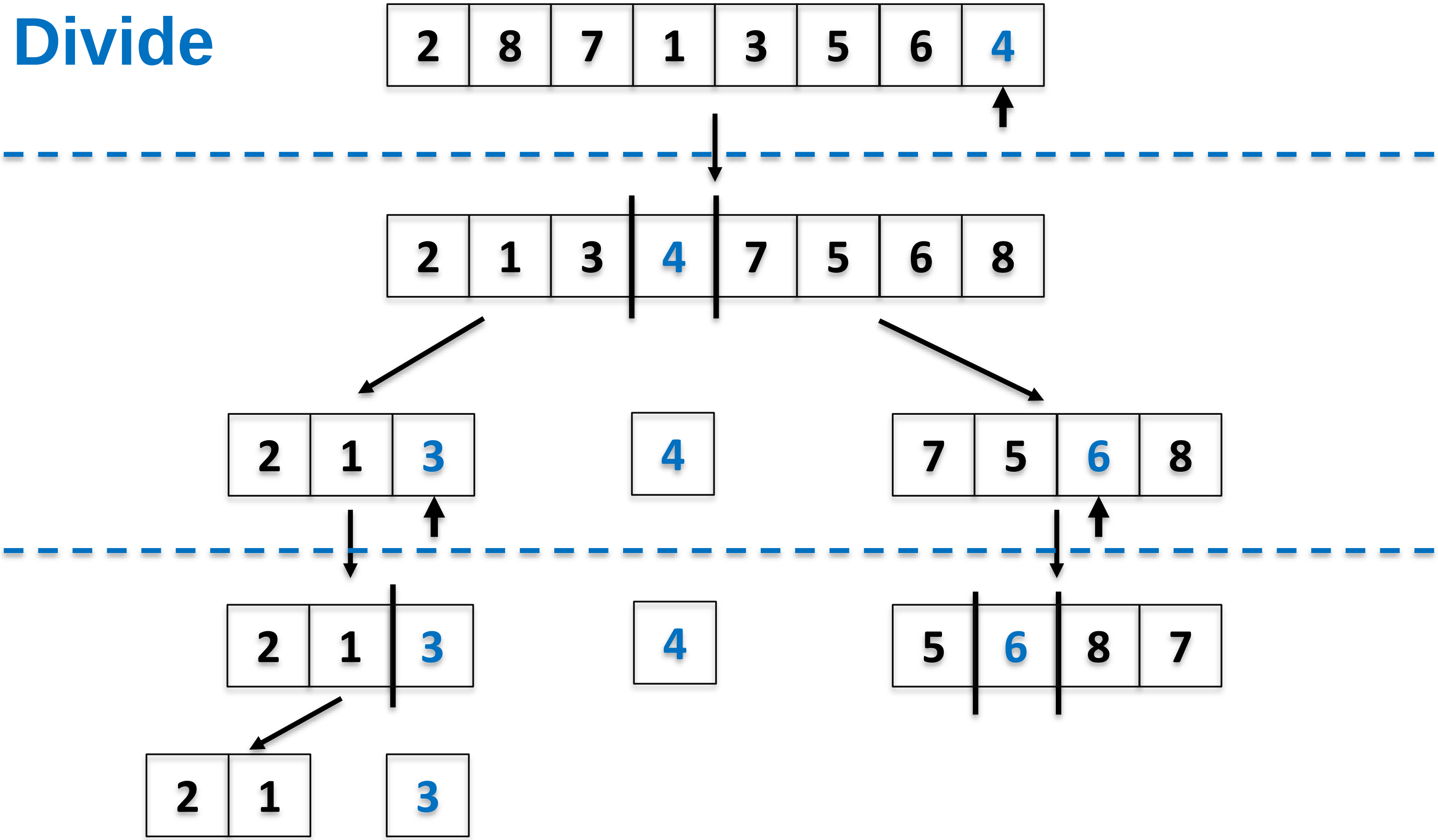

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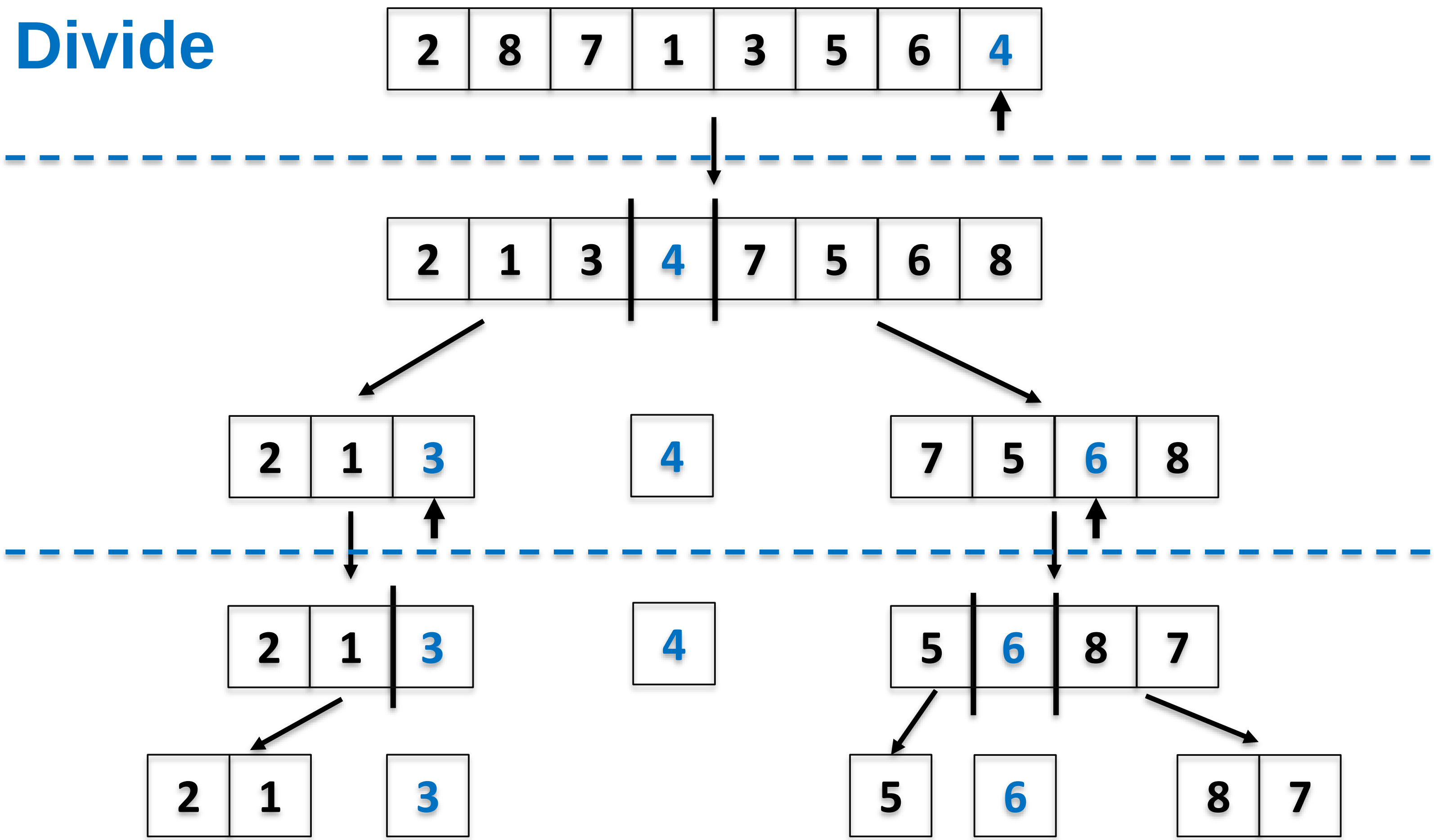

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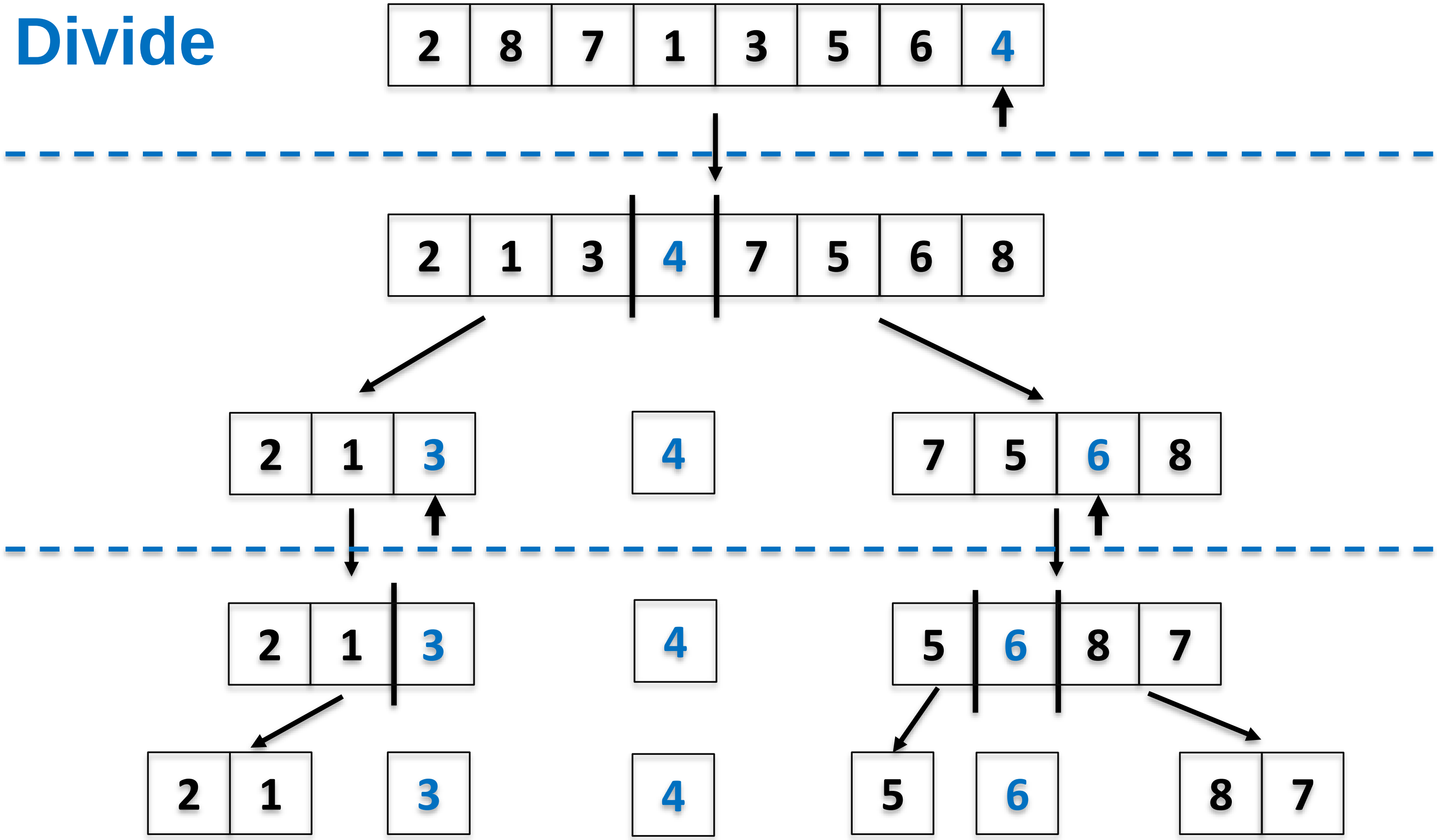

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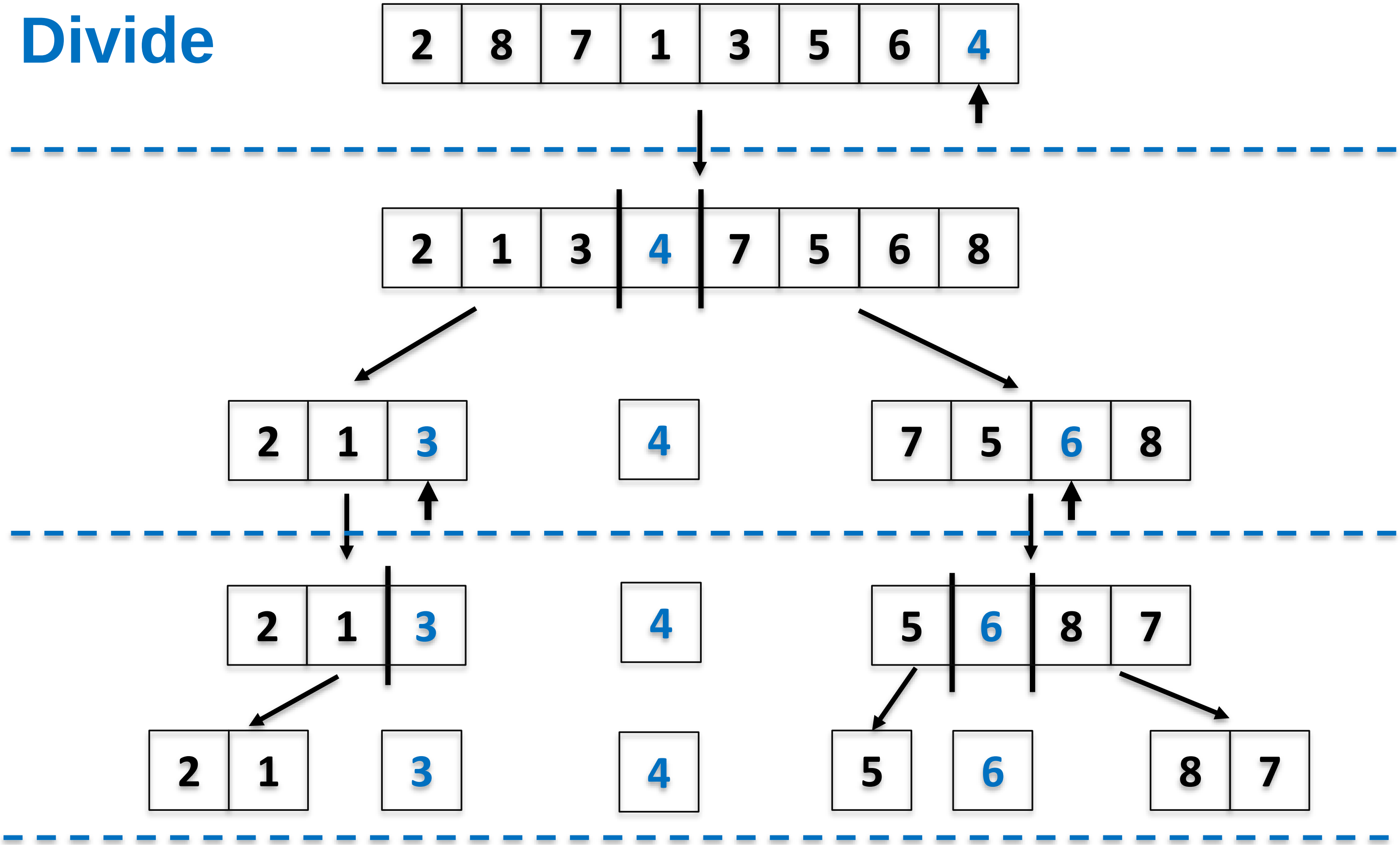

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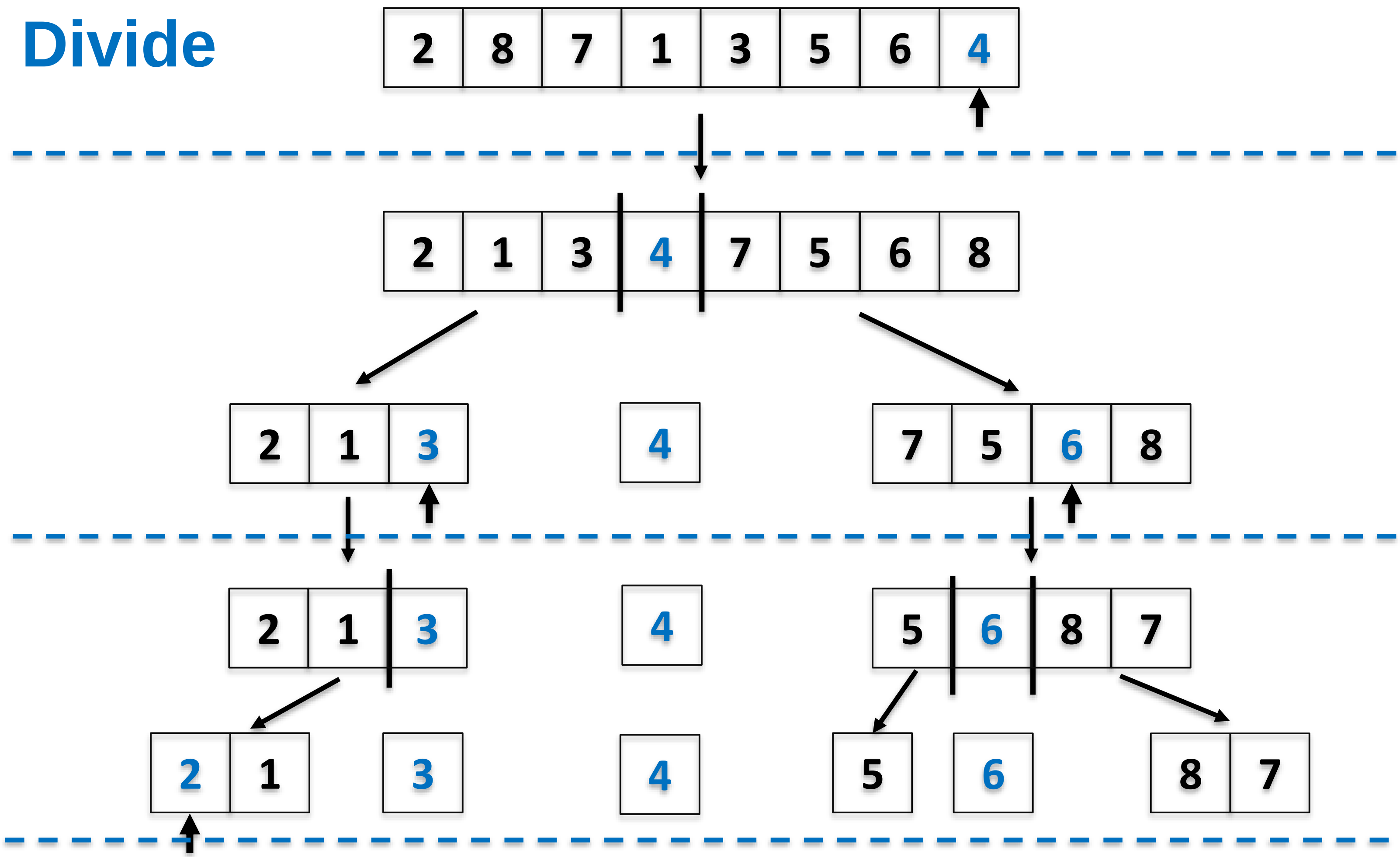

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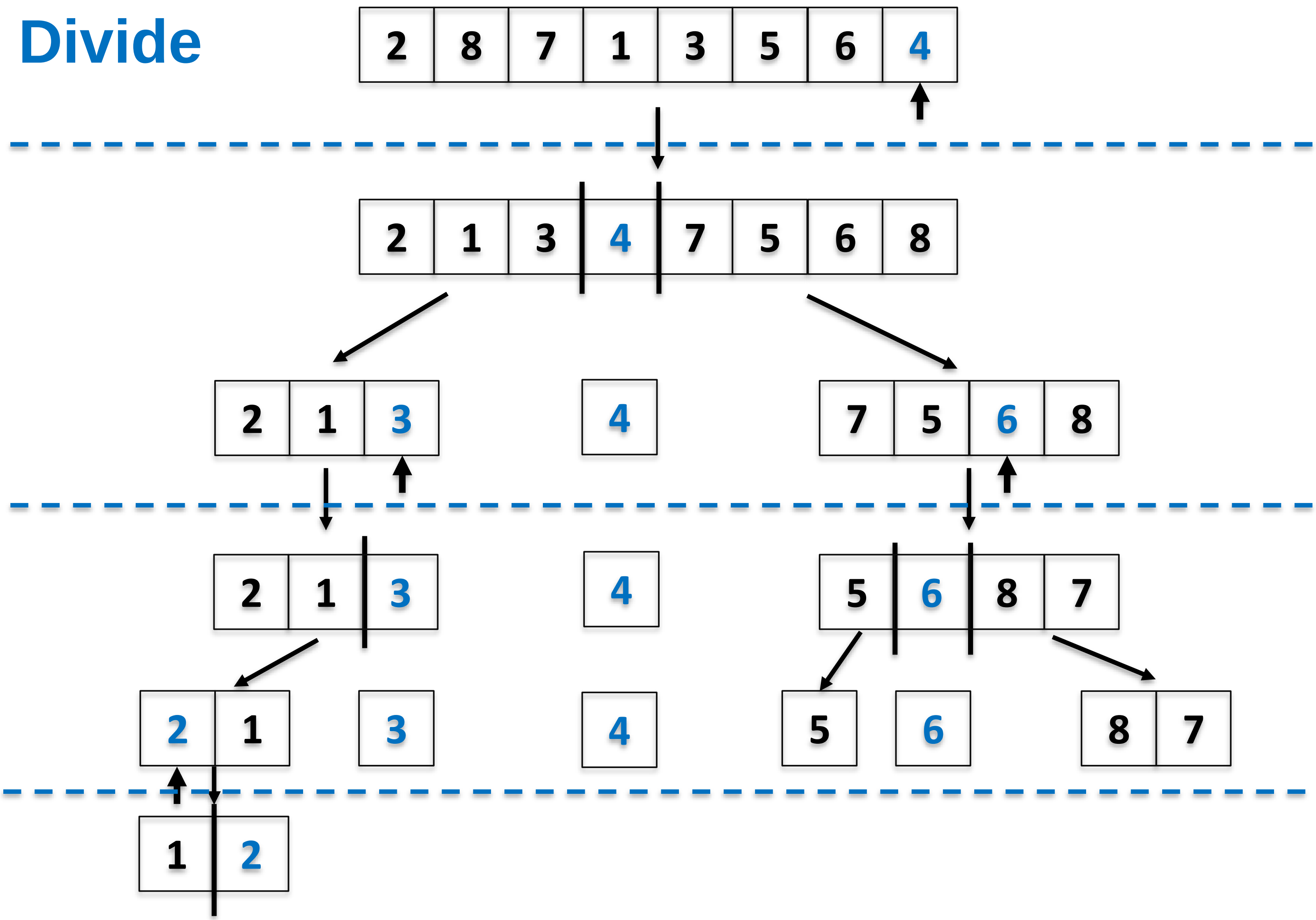

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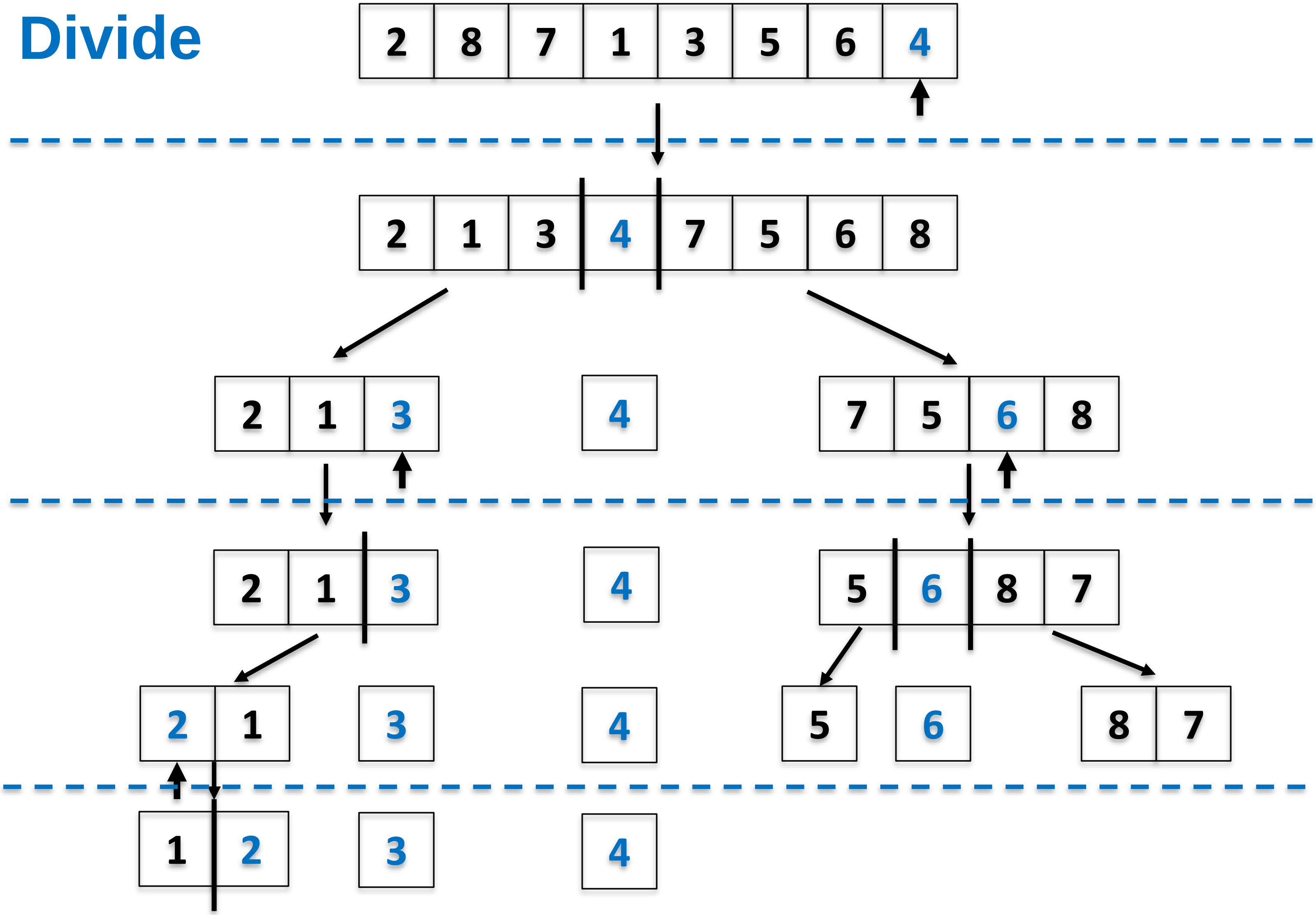

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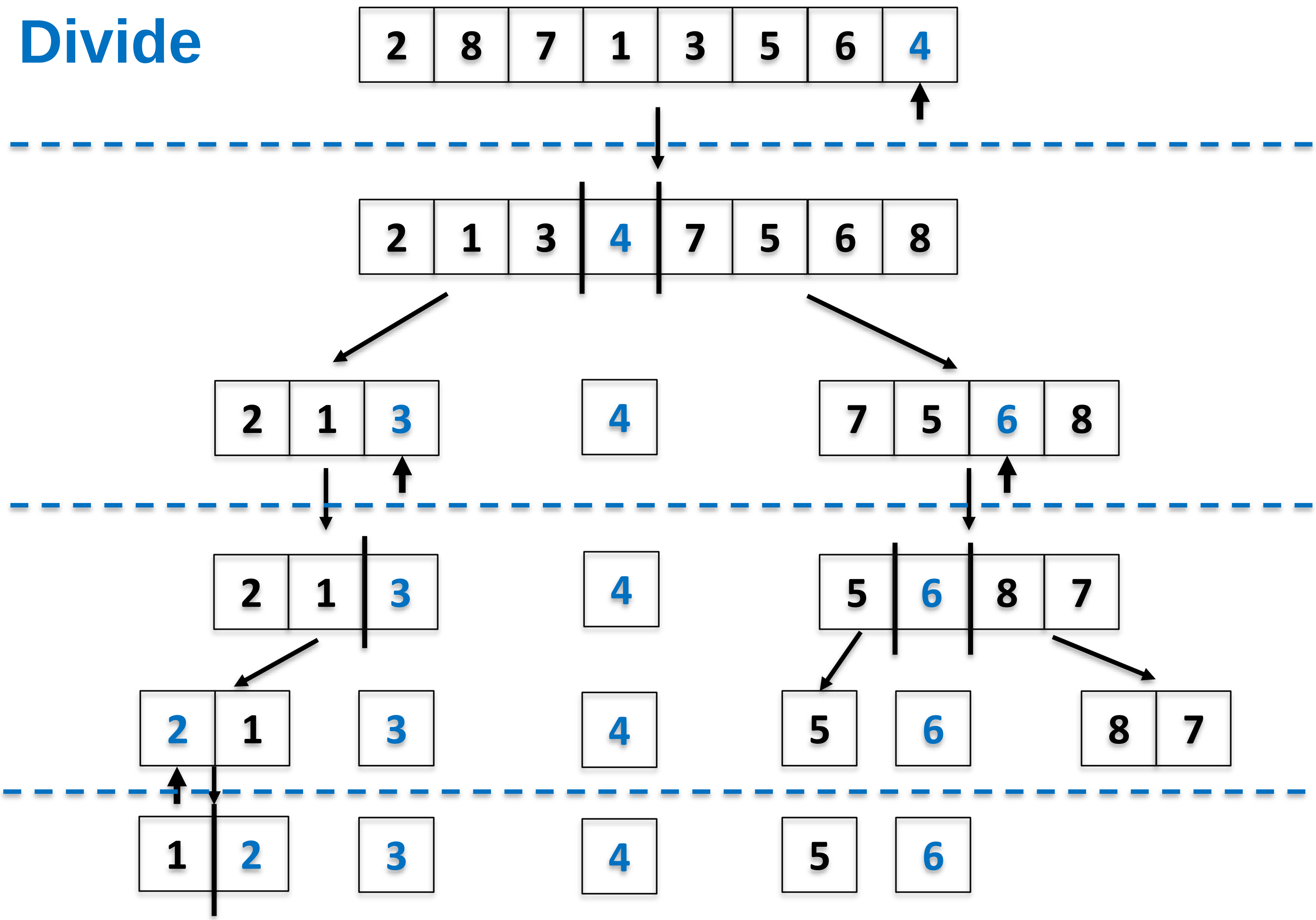

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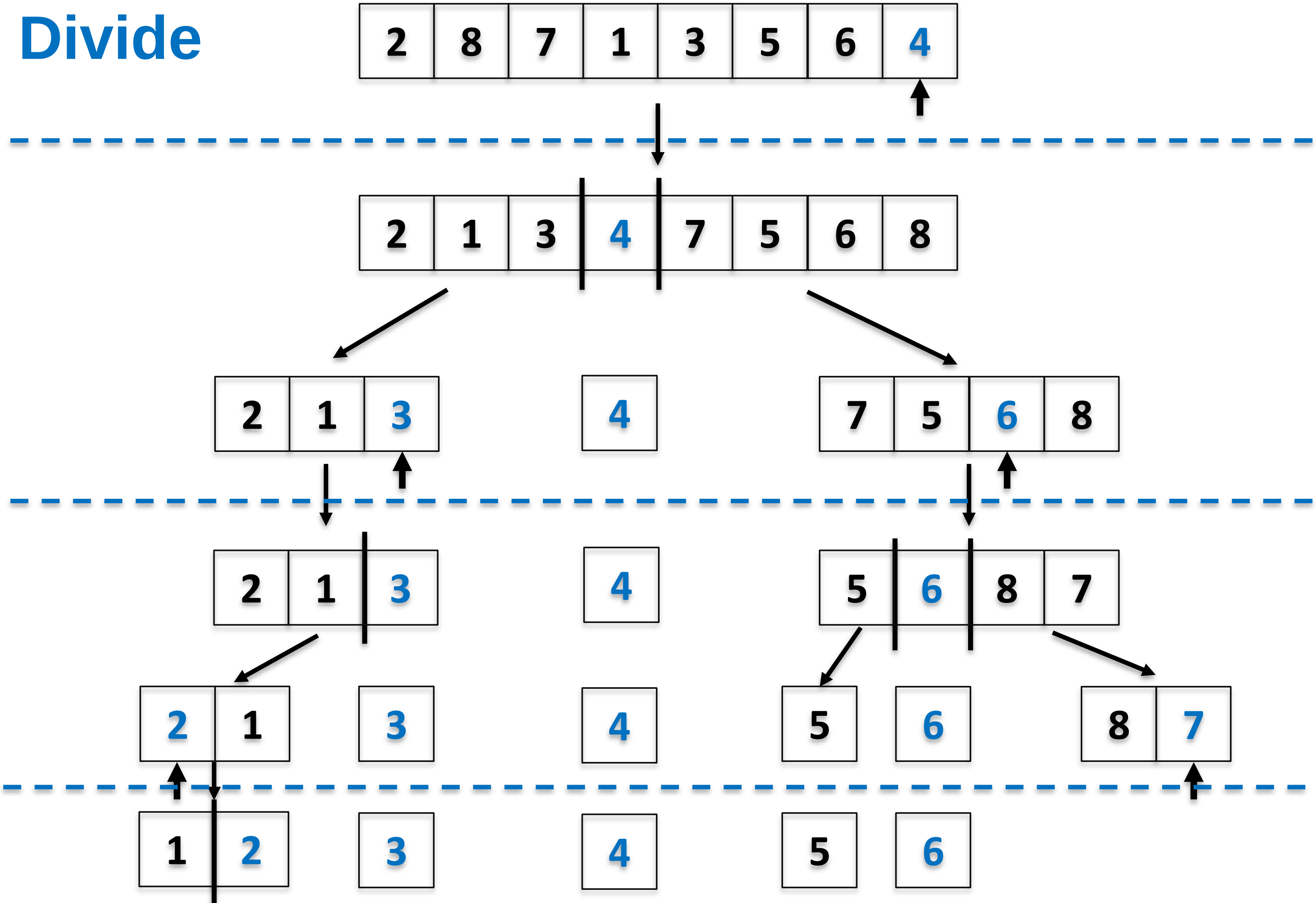

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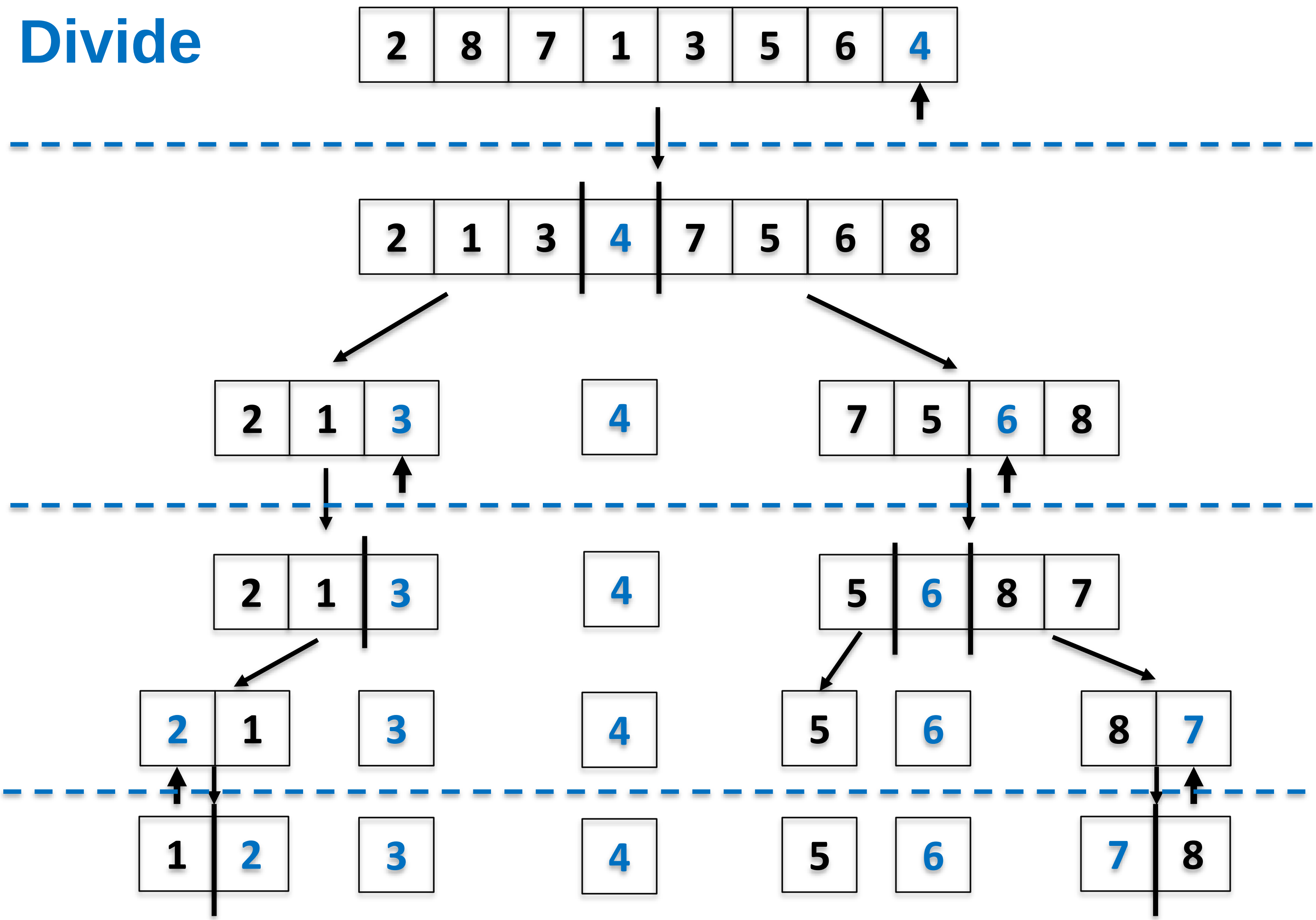

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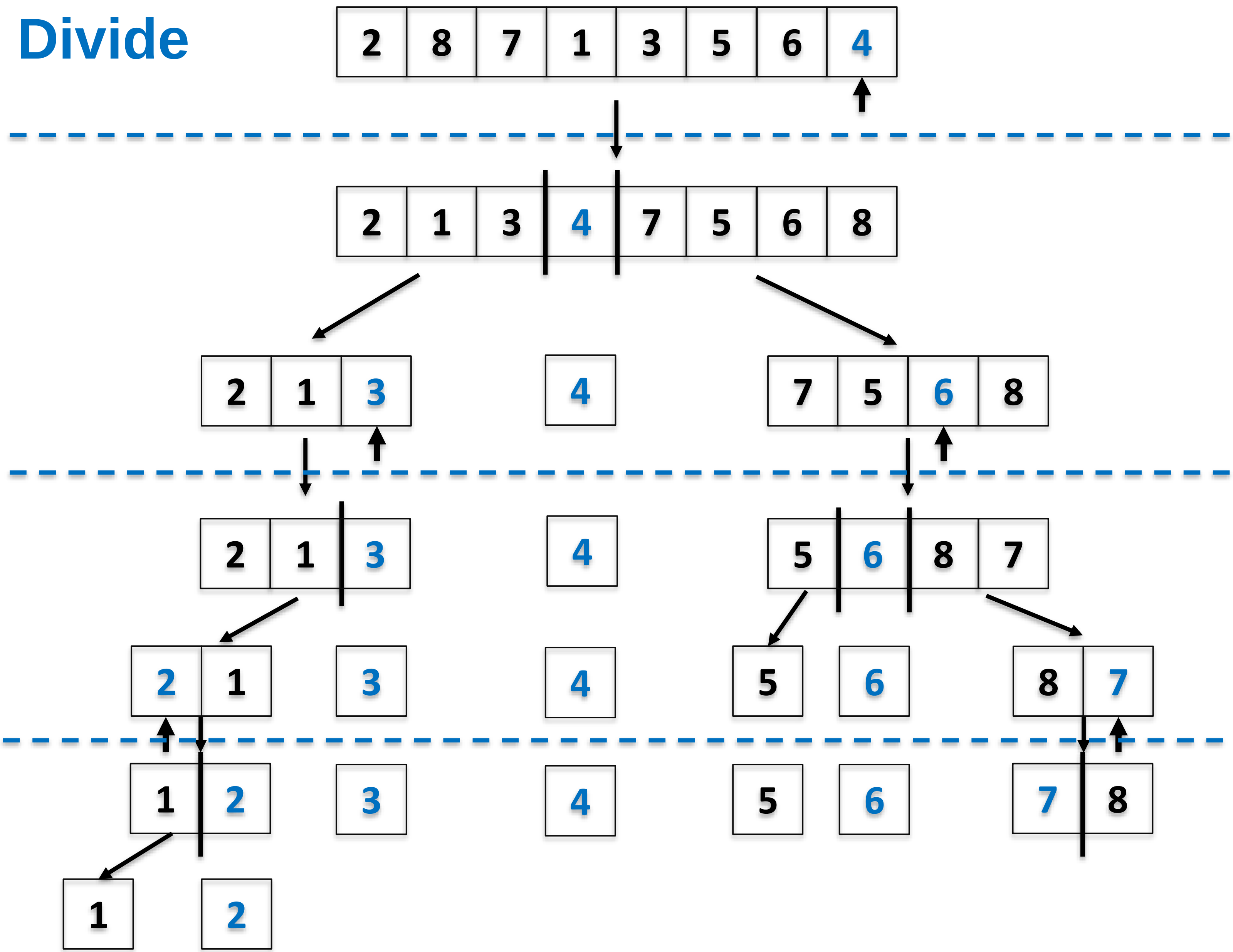

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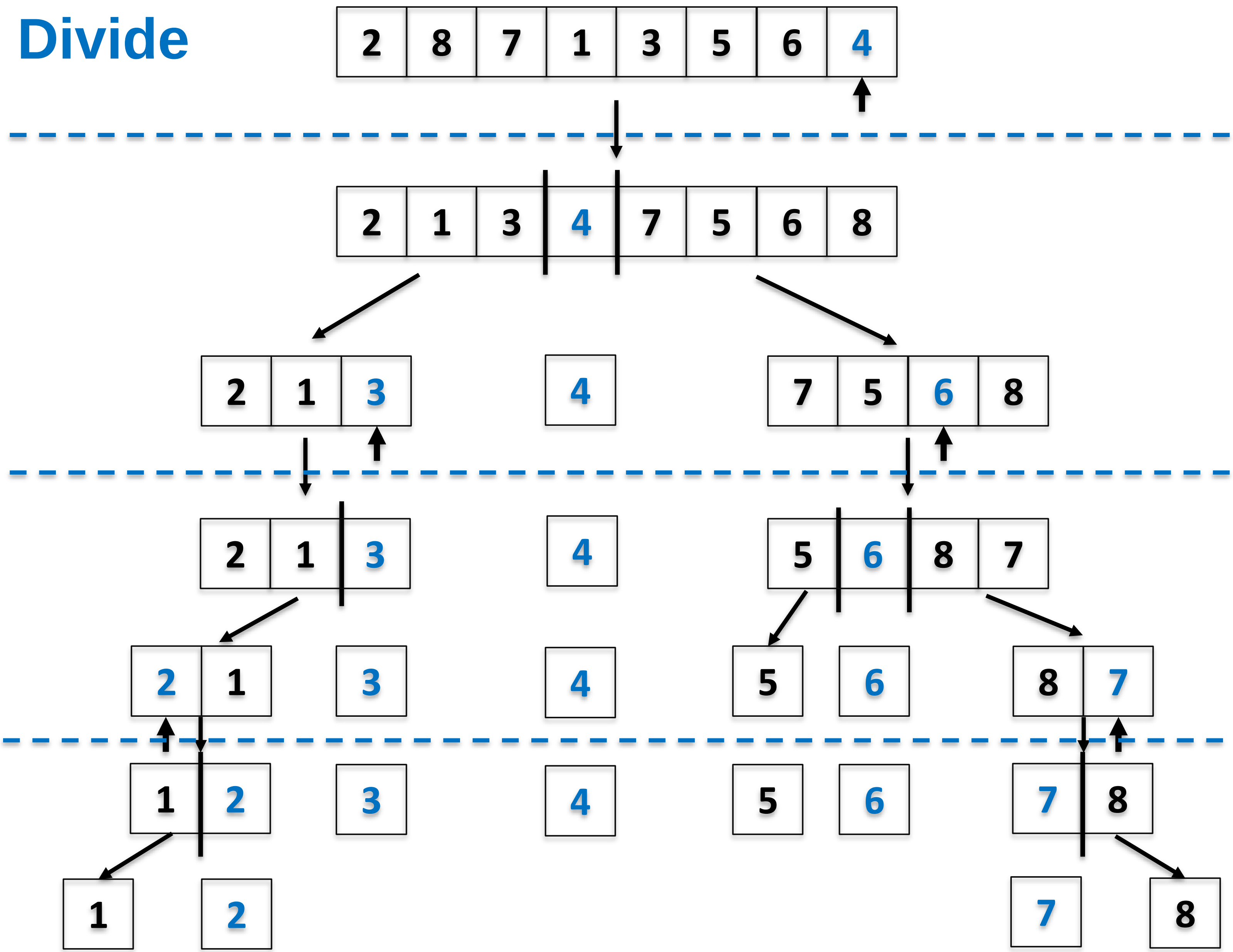

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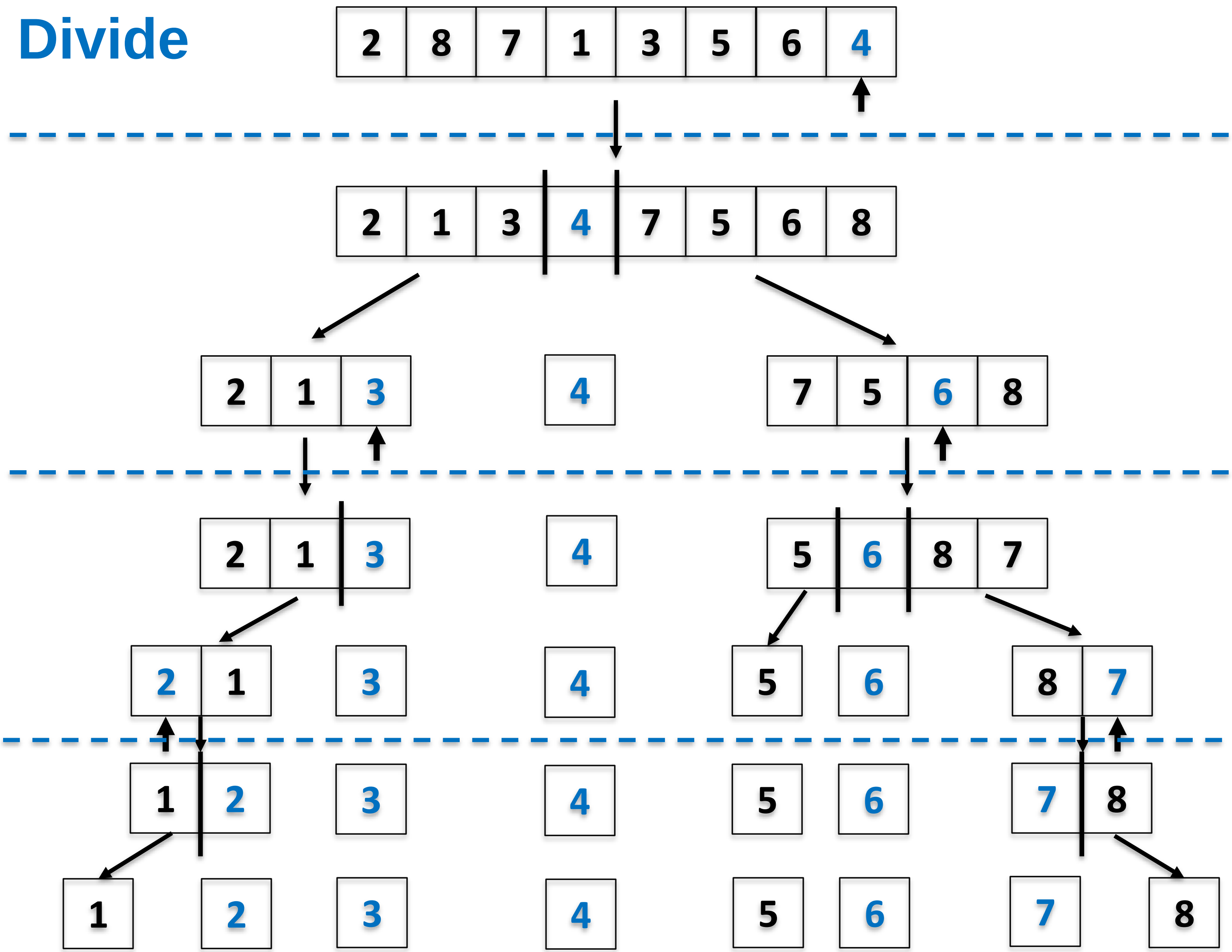

Quicksort - Example

Conquer

1	2	3	4	5	6	7	8

Quicksort - Example

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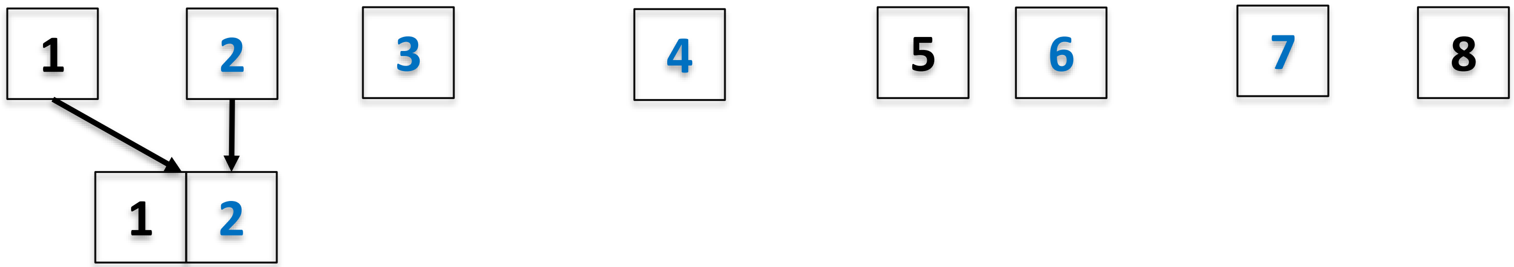

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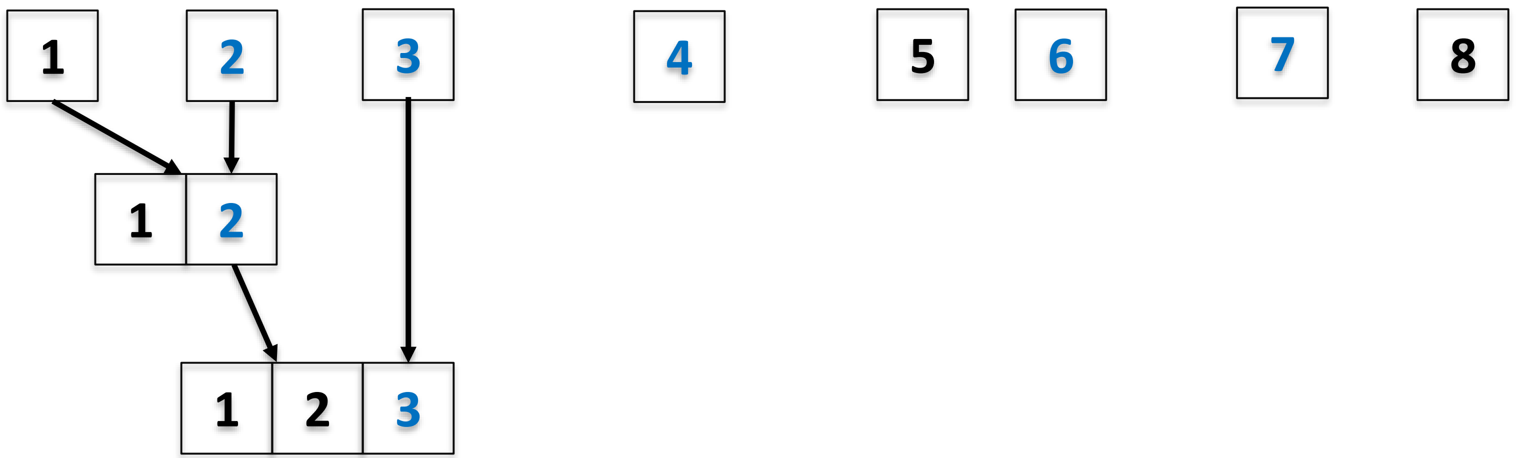

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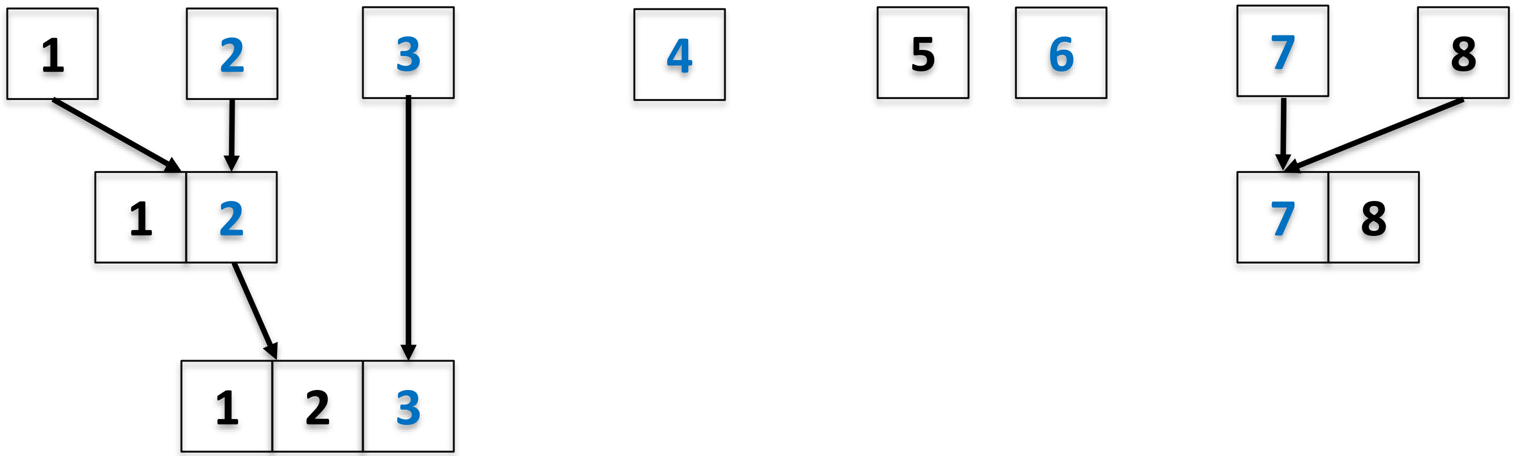

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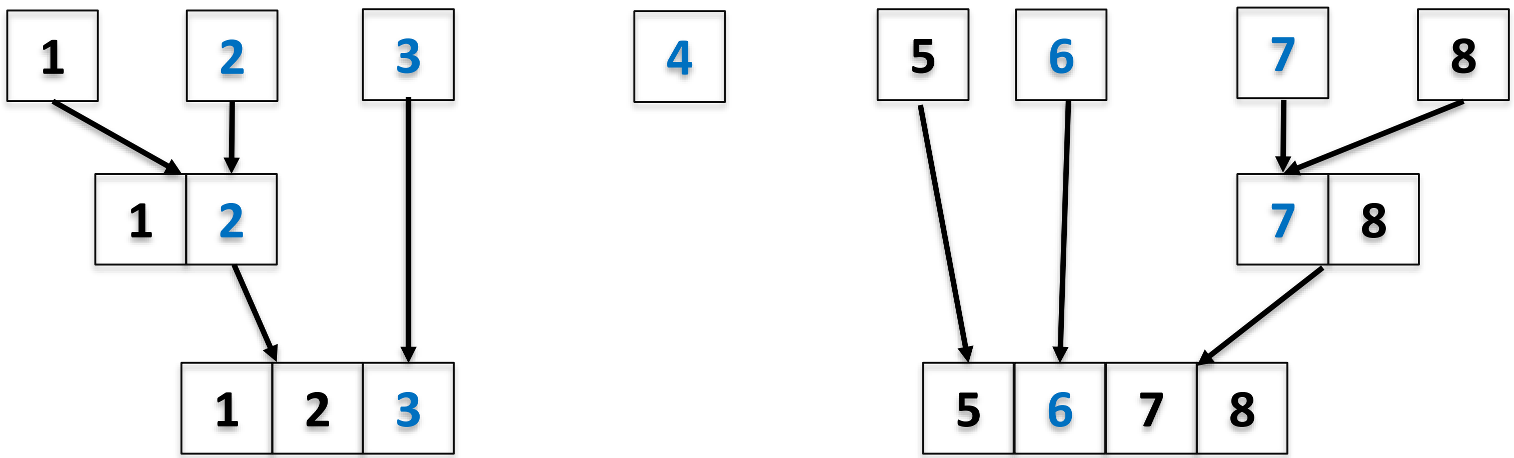

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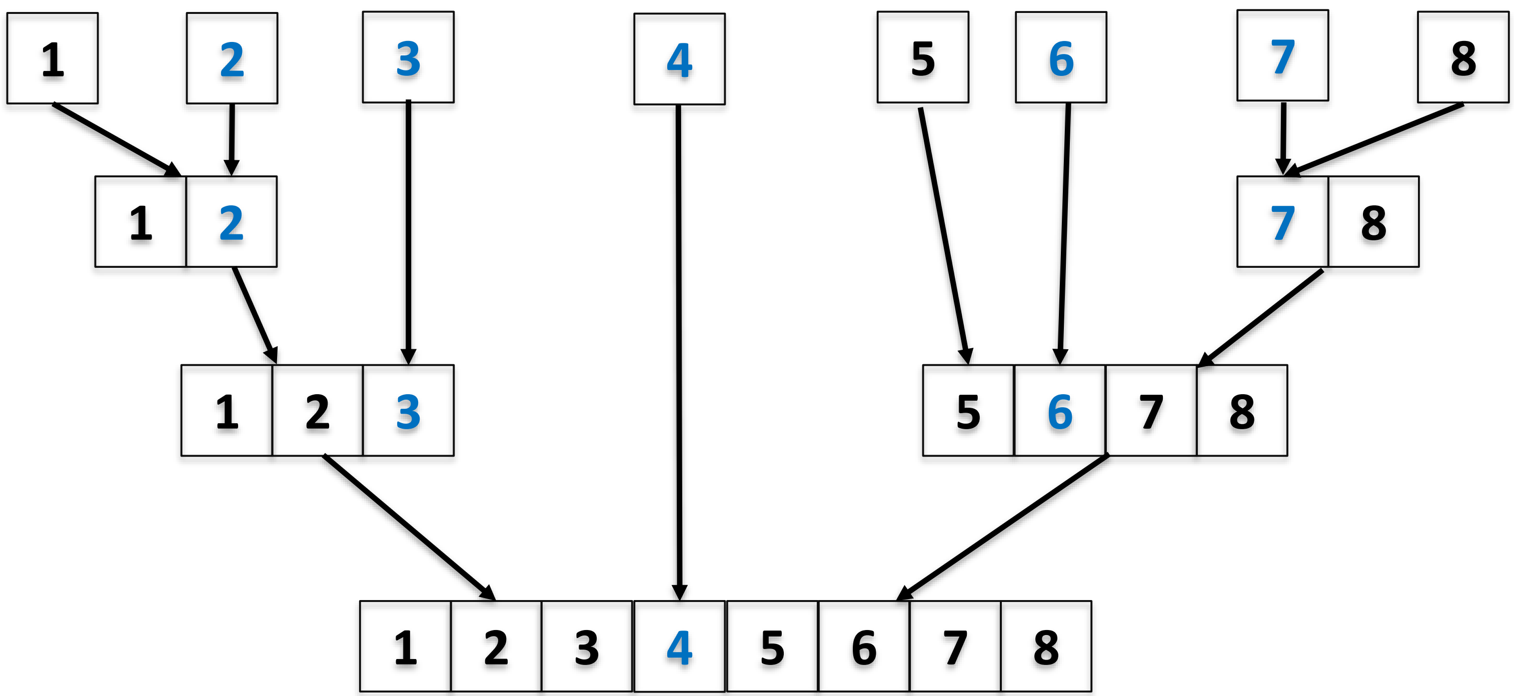

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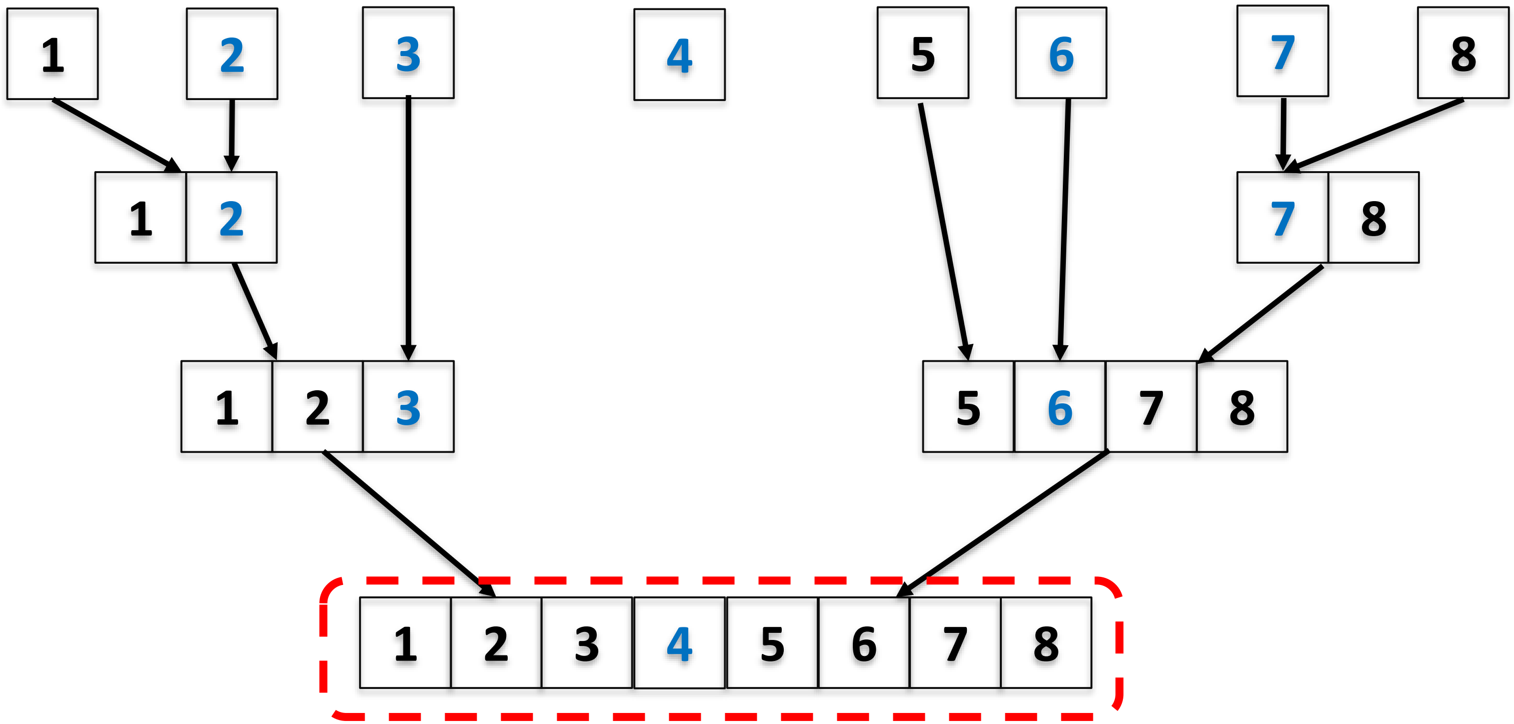

Outline

- Review to Divide-and-Conquer Paradigm

- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort

Running Time of the Quicksort

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  - New: Indicator variables.
    —— Simple and elegant, but needs practice to master.

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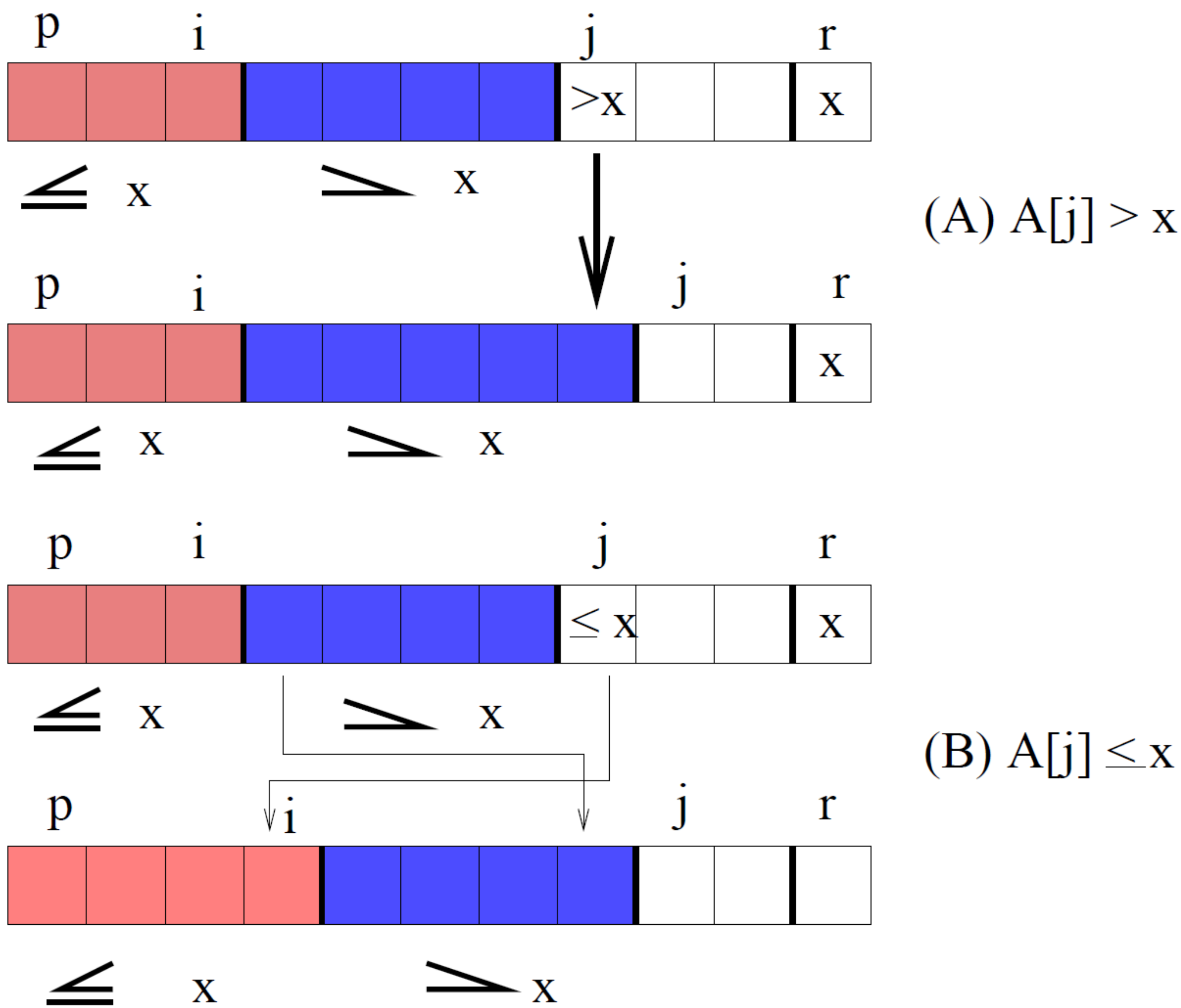

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- If the pivot is any element in Z_{ij} other than z_i or z_j
  - z_i and z_j are not compared with each other in all randomized-partition calls

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i\ is\ compared\ with\ z_j\}$$

$$=$$

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$$= \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\}$$

$$=$$

How to Find Pr{z_i is compared with z_j}?

$\Pr\{z_i \textit{ is compared with } z_j\}$

$= \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\}$

$= \Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\}$

$+$

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \textit{ is compared with } z_j\}$$

$$= \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\}$$

$$= \Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\}$$

$$+ \Pr\{z_j \text{ is the first pivot chosen from } Z_{ij}\}$$

$$= \frac{1}{j-i+1} +$$

How to Find Pr{z_i is compared with z_j}?

$$
\begin{aligned}
&\Pr\{z_i \textit{ is compared with } z_j\} \\
&\quad = \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&\quad = \Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\} \\
&\qquad + \Pr\{z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&\quad = \frac{1}{j-i+1} + \frac{1}{j-i+1} =
\end{aligned}
$$

How to Find Pr{z_i is compared with z_j}?

$$
\begin{aligned}
&\Pr\{z_i \textit{ is compared with } z_j\} \\
&\quad = \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&\quad = \Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\} \\
&\qquad + \Pr\{z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&\quad = \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}
\end{aligned}
$$

How to Find Pr{z_i is compared with z_j}?

$$
\begin{aligned}
&\Pr\{z_i \textit{ is compared with } z_j\} \\
&\quad = \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
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&\quad = \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}
\end{aligned}
$$

$$
E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \textit{ is compared with } z_j\} =
$$

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \ is \ compared \ with \ z_j\}$$
$$= \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\}$$
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$$= \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \ is \ compared \ with \ z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$=$$

How to Find Pr{z_i is compared with z_j}?

$$
\begin{aligned}
&\Pr\{z_i \textit{ is compared with } z_j\} \\
&\quad = \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
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\end{aligned}
$$

$$
E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \textit{ is compared with } z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}
$$

$$
= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}
$$

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \textit{ is compared with } z_j\}$$
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$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \textit{ is compared with } z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k}$$

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \text{ is compared with } z_j\}$$
$$= \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\}$$
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$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared with } z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$
$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} = \sum_{i=1}^{n-1} O(\log n)$$

Note: $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n)$

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \text{ is compared with } z_j\}$$
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$$= \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared with } z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$
$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} = \sum_{i=1}^{n-1} O(\log n) = O(n \log n)$$

Note: $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n)$

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \textit{ is compared with } z_j\}$$
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$$= \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \textit{ is compared with } z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$
$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} = \sum_{i=1}^{n-1} O(\log n) = O(n \log n)$$

Note: $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n)$

Hence, the expected number of comparisons is $O(n \log n)$, which is the expected running time of Randomized-Quicksort

谢谢

BDA
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