Design and Analysis of Algorithms
Lecture 2: Asymptotic Notations and Recurrences


https://www.ccf.org.cn/xspj/gyml/

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Outline

- Asymptotic Notations（渐近记号）
  - Big-Oh
  - Big-Omega
  - Big-Theta
  - Algorithm Design and Algorithm Turing
- Solving Recurrences
  - Recursion-tree Method（递归树法）
  - Substitution Method（代入法/替代法）
  - Master Method and Master Theorem (主方法)

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Big-Oh

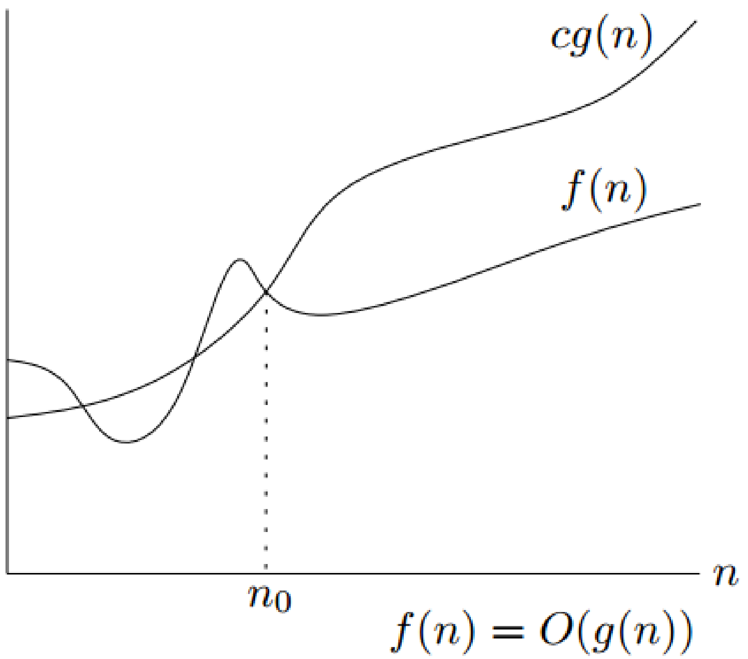
Asymptotic upper bound

Definition (big-Oh)

$f(n) = O(g(n))$: There exists constant $c > 0$ and n_0 such that $f(n) \leq c \cdot g(n)$ for $n \geq n_0$

When estimating the growth rate of $T(n)$ using big-Oh:

- ignore the low order terms
- ignore the constant coefficient of the most significant term



Big-Oh: Example

Definition (big-Oh)

$f(n) = O(g(n))$: There exists constant $c > 0$ and n_0 such that $f(n) \leq c \cdot g(n)$ for $n \geq n_0$

Example

Let $T(n) = 3n^2 + 4n + 5$. Prove that $T(n) = O(n^2)$.

Big-Oh: Example

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Example

Let $T(n) = 3n^2 + 4n + 5$. Prove that $T(n) = O(n^2)$.

Proof.

$$\begin{aligned} T(n) &= 3n^2 + 4n + 5 \\ &\leq 3n^2 + 4n^2 + 5n^2 \\ &= 12n^2. \end{aligned}$$

Thus, $T(n) \leq 12n^2$ for all $n \geq 1$. Setting $n_0 = 1$ and $c = 12$ in the definition, we have that $T(n) = O(n^2)$. $\square$

Big-Oh: More Examples

- $\frac{n^2}{2} - 3n =$

Big-Oh: More Examples

- $\frac{n^2}{2} - 3n = O\left(n^2\right)$
- $1 + 4n =$

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- $\log_{10} n =$

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- $\sum_{i=1}^{n} i^2$

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- $\sum_{i=1}^{n} i^2 \leq n \cdot n^2 = O\left(n^3\right)$
- $\sum_{i=1}^{n} i$

Big-Oh: More Examples

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- $\sum_{i=1}^{n} i^2 \leq n \cdot n^2 = O\left(n^3\right)$
- $\sum_{i=1}^{n} i \leq n \cdot n = O\left(n^2\right)$

Big-Oh: More Examples

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- $\log(n!) = \log(n) + \cdots + log 1 = O(n \log n)$

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- $\log(n!) = \log(n) + \cdots + log 1 = O(n \log n)$
- $\sum_{i=1}^{n} \frac{1}{i} = O(\log n)$ (Harmonic Series, 调和级数)

Big-Oh: More Examples

The Asymptotic Upper Bound of Harmonic Series:

$$\sum_{i=1}^{n} \frac{1}{i} = O(\log n)$$

Big-Oh: More Examples

Proof: Assume that n is a power of two, then

$$\sum_{\mathrm{i}=1}^{n} \frac{1}{i}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{n}$$

Big-Oh: More Examples

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$$< \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{\mathrm{n}/2} + \frac{1}{\mathrm{n}}$$

Big-Oh: More Examples

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$$= \frac{1}{1} + 2 \cdot \left(\frac{1}{2}\right) + 4 \cdot \left(\frac{1}{4}\right) + 8 \cdot \left(\frac{1}{8}\right) + \cdots + \frac{n}{2}\left(\frac{1}{n/2}\right) + \frac{1}{n}$$

Big-Oh: More Examples

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$$= 1/n + \sum_{j=0}^{\log n - 1} 1$$

Big-Oh: More Examples

Proof: Assume that n is a power of two, then

$$
\begin{aligned}
&\sum_{\mathrm{i}=1}^{n} \frac{1}{i} \\
&= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{n} \\
&< \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{\mathrm{n}/2} + \frac{1}{\mathrm{n}} \\
&= \frac{1}{1} + 2 \cdot \left(\frac{1}{2}\right) + 4 \cdot \left(\frac{1}{4}\right) + 8 \cdot \left(\frac{1}{8}\right) + \cdots + \frac{n}{2}\left(\frac{1}{n/2}\right) + \frac{1}{n} \\
&= 1/n + \sum_{j=0}^{\log n - 1} 1 \\
&= \log n + \frac{1}{n}
\end{aligned}
$$

Big-Oh: More Examples

Proof: Assume that n is a power of two, then

$$\sum_{\mathrm{i=1}}^{n} \frac{1}{i}$$
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$$= 1/n + \sum_{j=0}^{\log n - 1} 1$$
$$= \log n + \frac{1}{n}$$

Thus, $\sum_{i=1}^{n} \frac{1}{i} = O(\log n)$

Big-Oh: Examples of Complexity Analysis

algorithm scan(v)

1. **for** $i = 1$ **to** n **do**
2.   **if** $S[i] = v$ **then**
3.   **return** *yes*
4. **return** *no*

Big-Oh: Examples of Complexity Analysis

algorithm scan(v)

for $i = 1$ **to** n **do**

$$\left.\begin{array}{l} \textbf{if } S[i] = v \textbf{ then} \\ \quad \textbf{return } \textit{yes} \end{array}\right\} O(1)$$

return *no*

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$$\left.\begin{array}{l} \left.\begin{array}{l} \textbf{if } S[i] = v \textbf{ then} \\ \quad \textbf{return } yes \end{array}\right\} O(1) \end{array}\right\} n \cdot O(1) = O(n)$$

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4. **return** *no*

Although Lines 2-3 may be executed less than n times, we are considering the worst-case complexity

Big-Oh: Examples of Complexity Analysis

algorithm CountingInversedPairs($A[1..n]$)

```
ans = 0
for i = 1 to n do
    for j = i + 1 to n do
        if A[i] > A[j] then
            ans = ans + 1
return ans
```

What’s the worst-case complexity of this program?

Big-Oh: Examples of Complexity Analysis

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1. $ans = 0$
2. **for** $i = 1$ **to** n **do**
3. **for** $j = i + 1$ **to** n **do**
4. **if** $A[i] > A[j]$ **then**
5. $ans = ans + 1$
6. **return** ans

Big-Oh: Examples of Complexity Analysis

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\quad\quad\quad ans = ans + 1
\end{array}\right\} O(1) \\
\textbf{return } ans
\end{array}
$$

Big-Oh: Examples of Complexity Analysis

algorithm CountingInversedPairs($A[1..n]$)

$ans = 0$

for $i = 1$ **to** n **do**

$$\left.\begin{array}{l} \textbf{for } j = i+1 \textbf{ to } n \textbf{ do} \\ \quad \left.\begin{array}{l} \textbf{if } A[i] > A[j] \textbf{ then} \\ \quad ans = ans + 1 \end{array}\right\} O(1) \end{array}\right\} (n-i)O(1)$$

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$$

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\end{array}\right\} ?? \\
\textbf{return } ans
\end{array}$$

$$\begin{aligned}
?? &= (n-1)O(1) + (n-2)O(1) + \cdots + (n-n)O(1) \\
&= n(n-1)O(1) \\
&= O(n^2)
\end{aligned}$$

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Asymptotic lower bound

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$f(n) = \Omega(g(n))$: There exists constant $c > 0$ and n_0 such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$.

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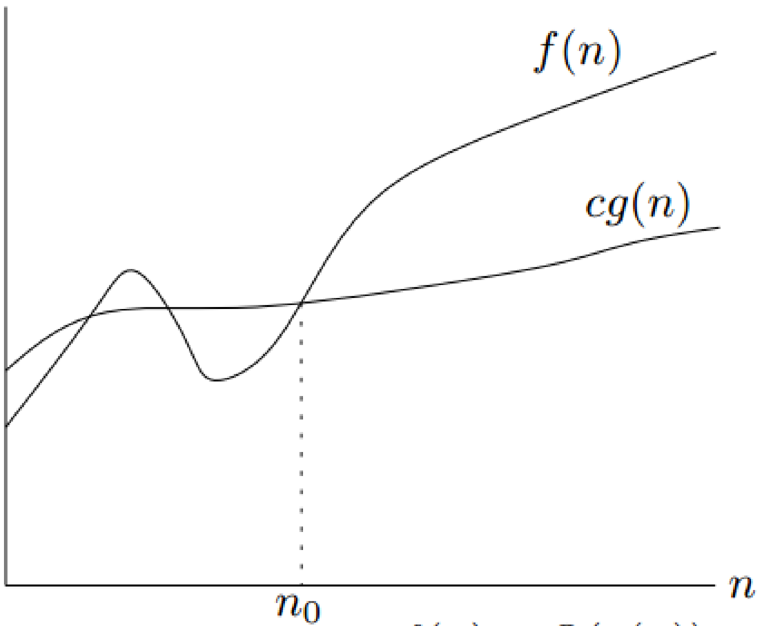
It is easy to show that

$\frac{n^2}{2} - 3n \geq \frac{n^2}{4}$ for all $n \geq 12$.

Thus, $n^2/2 - 3n = \Omega(n^2)$.



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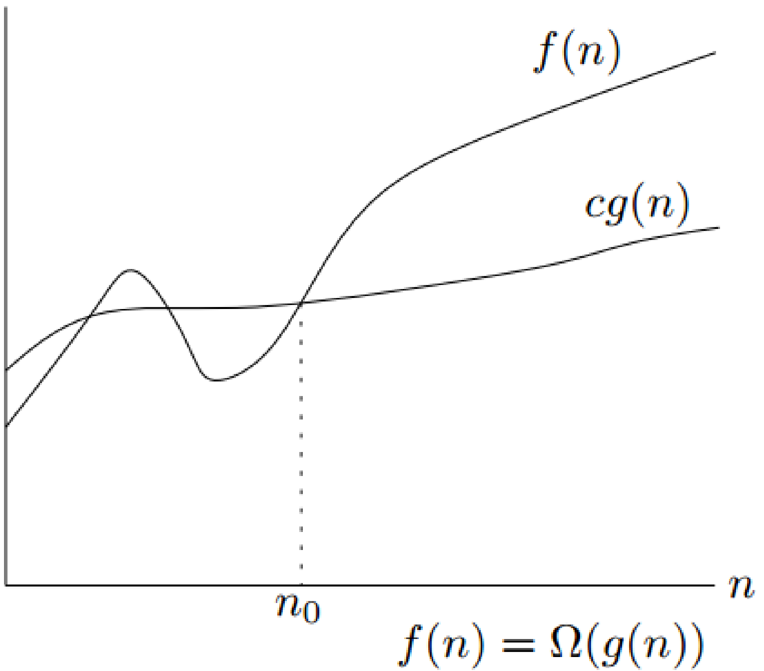

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Example

$$\log(n!) = \log(n) + \log(n-1) + \cdots + \log 1$$

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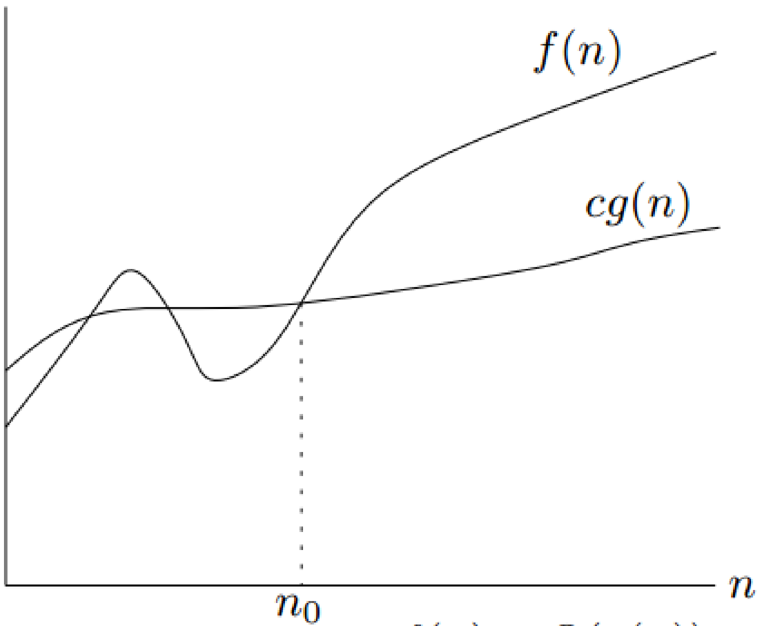

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Example

$$\begin{aligned} \log(n!) &= \log(n) + \log(n-1) + \cdots + \log 1 \\ &\geq \log(n) + \log(n-1) + \cdots + \log(n/2) \end{aligned}$$

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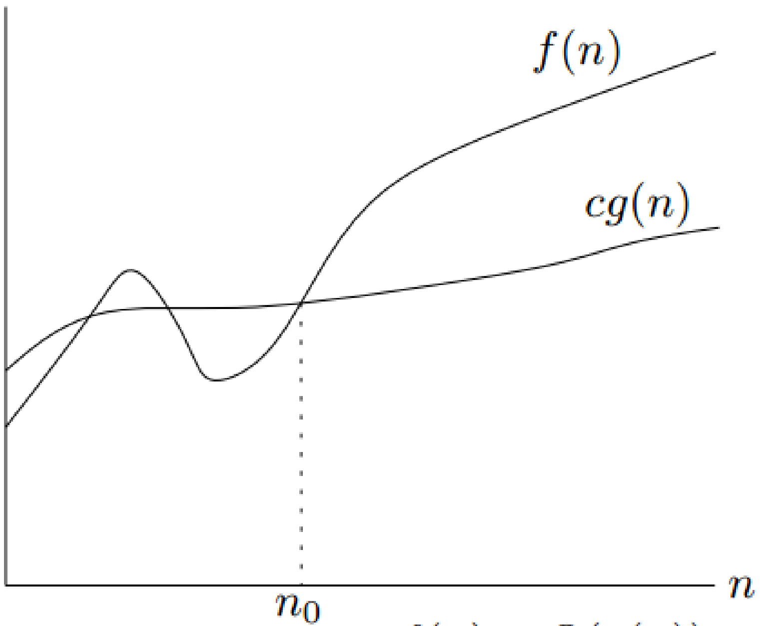

$f(n) = \Omega(g(n))$

It is easy to show that

$$\frac{n^2}{2} - 3n \geq \frac{n^2}{4} \quad \text{for all } n \geq 12.$$

Thus, $n^2/2 - 3n = \Omega(n^2)$.

Example

$$\begin{aligned}
\log(n!) &= \log(n) + \log(n-1) + \cdots + \log 1 \\
&\geq \log(n) + \log(n-1) + \cdots + \log(n/2) \\
&\geq n/2 \cdot \log(n/2)
\end{aligned}$$

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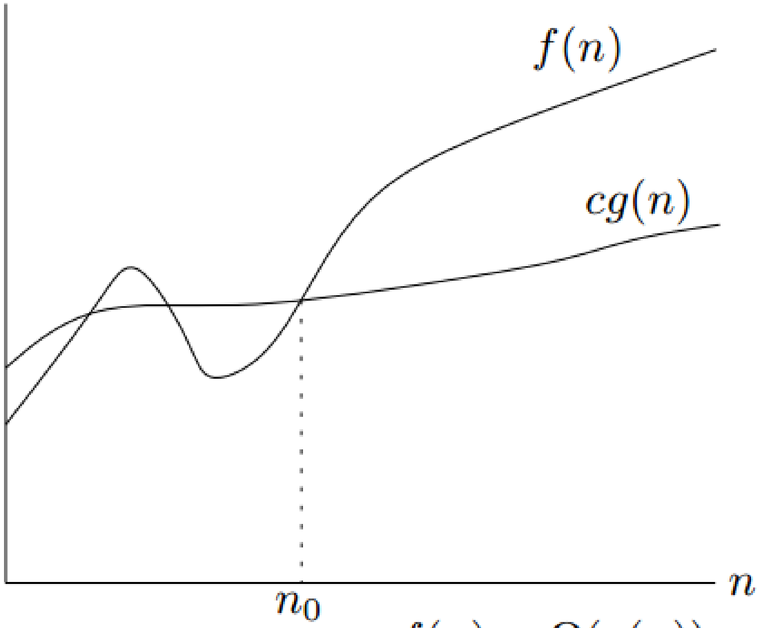

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Thus, $n^2/2 - 3n = \Omega(n^2)$.

Example

$$
\begin{aligned}
\log(n!) &= \log(n) + \log(n-1) + \cdots + \log 1 \\
&\geq \log(n) + \log(n-1) + \cdots + \log(n/2) \\
&\geq n/2 \cdot \log(n/2) \\
&= n/2 \cdot (\log n - 1) = \Omega(n \log n).
\end{aligned}
$$

Big-Omega: Harmonic Series

The Asymptotic Lower Bound of Harmonic Series:

$$\sum_{i=1}^{n} \frac{1}{i} = \Omega(\log n)$$

Big-Omega: Harmonic Series

Proof: Assume that n is a power of two, then

$$\sum_{\mathrm{i}=1}^{n} \frac{1}{i}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{n}$$

Big-Omega: Harmonic Series

Proof: Assume that n is a power of two, then

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$$= 1 + \frac{1}{2} \log n$$

Thus, $\sum_{i=1}^{n} \frac{1}{i} = \Omega(\log n)$

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Big-Theta

Asymptotic tight bound

Definition (big-Theta)

$f(n) = \Theta(g(n))$: $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Big-Theta

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We have shown that

$$n^2/2 - 3n = O(n^2),$$

and

$$n^2/2 - 3n = \Omega(n^2).$$

Therefore, we have that $n^2/2 - 3n = \Theta(n^2)$.

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Usually (and in this course), it is sufficient to show only upper bounds (big-Oh), though we should try to make these as tight as we can.

Asymptotic Notations

Upper bounds. T(n)=O(f(n)) if there exist constants c > 0 and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $T(n) \leq c \cdot f(n)$.

Equivalent definition: $\lim_{n \longrightarrow \infty} \frac{T(n)}{f(n)} < \infty$.

Lower bounds. T(n)=Ω(f(n)) if there exist constants c> 0 and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $T(n) \geq c \cdot f(n)$.

Equivalent definition: $\lim_{n \longrightarrow \infty} \frac{T(n)}{f(n)} > 0$.

Tight bounds. $T(n) = \Theta\big(f(n)\big)$ if $T(n) = O\big(f(n)\big)$ and $T(n) = \Omega(f(n))$.

Note: Here "=" means "is", not equal. The more mathematically correct way should be $T(n) \in O(f(n))$.

For example, for the harmonic series,
we have: $\sum_{i=1}^{n} \frac{1}{\mathrm{i}} = \mathrm{O}(\log n) = \Omega(\log n) = \Theta(\log n)$

Examples

- $100n^2 = O(n^3)$?

- $100n^2 = \Omega(n^3)$?

- $10n^2 - 100n = O(n^2)$?
- $10n^2 - 100n = \Omega(n^2)$?
- $10n^2 - 100n = \Theta(n^2)$?
- $\log(2n) = O(\log n)$?
- $(2n)^{10} = O(n^{10})$?
- $2^{2n} = O(2^n)$?

Examples

- $100n^2 = O(n^3)$?
  Answer: Yes. $C = 1$ and $n_0 = 100$. Then $\forall n \geq n_0$, $100n^2 \leq C \cdot n^3$.
- $100n^2 = \Omega(n^3)$?

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- $100n^2 = \Omega(n^3)$?
  Answer: No. $\forall\ C > 0$, $n_0 > 0$, there exists $n > n_0$ ($n = n_0 + 100/C$) such that $100n^2 < C \cdot n^3$.
- $10n^2 - 100n = O(n^2)$?
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Solutions

- $10n^2 - 100n = O(n^2)$?

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Solutions

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  Answer: **Yes**. $\forall\ n > 0$, $10n^2 - 100n \leq 10n^2$.

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Solutions

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  Answer: **Yes**. $\forall\ n > 0,\ 10n^2 - 100n \le 10n^2$.

- $10n^2 - 100n = \Omega(n^2)$?
  Answer: **Yes**. $\forall\ n \ge 20,\ 10n^2 - 100n \ge 5n^2$.

- $10n^2 - 100n = \Theta(n^2)$?

Solutions

- $10n^2 - 100n = O(n^2)$?
  Answer: **Yes**. $\forall\, n > 0,\ 10n^2 - 100n \le 10n^2$.

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  Answer: **Yes**. $\forall\, n \ge 20,\ 10n^2 - 100n \ge 5n^2$.

- $10n^2 - 100n = \Theta(n^2)$?
  Answer: **Yes**. Because

$$10n^2 - 100n = O(n^2)$$

and

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Solutions

- $\log(2n) = O(\log n)$?
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Solutions

- $\log(2n) = O(\log n)$?
  Answer: **Yes**. $\forall\, n \geq 2$, $\log(2n) = \log n + 1 \leq 2 \log n$.

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- $\log(2n) = O(\log n)$?
  Answer: Yes. $\forall$ $n \geq 2$, $\log(2n) = \log n + 1 \leq 2 \log n$.

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  Answer: **Yes.** $\forall$ $n > 0$, $(2n)^{10} = (2^{10})(n^{10})$.

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Solutions

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  Answer: **Yes.** $\forall\, n \geq 2$, $\log(2n) = \log n + 1 \leq 2 \log n$.

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  Answer: **Yes.** $\forall\, n > 0$, $(2n)^{10} = (2^{10})(n^{10})$.

- $2^{2n} = O(2^n)$?
  Answer: **No.** $\forall\, C > 0$, $n_0 > 0$, let $n = n_0 + \log C > n_0$.
  Then $2^{2n} = 2^n \cdot 2^n > C \cdot 2^n$.

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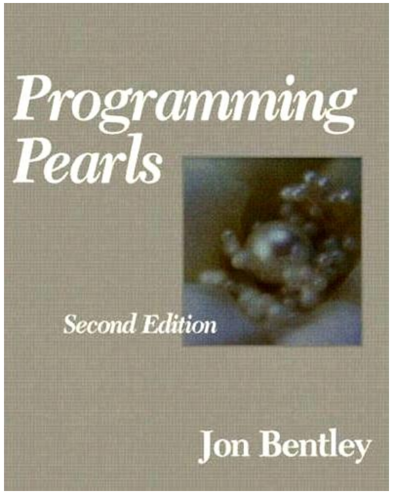
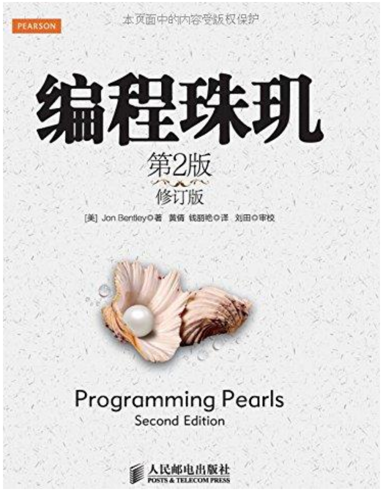
Some Thoughts on Algorithm Design

- Algorithm Design, as taught in this class, is mainly about designing algorithms that have big-Oh running times.
- As n gets larger and larger, O(nlogn) algorithms will run faster than O(n^2) ones and O(n) algorithms will beat O(nlogn) ones.
- Good algorithm design & analysis allows you to identify the hard parts of your problem and deal with them effectively.
- Too often, programmers try to solve problems using brute force techniques and end up with slow complicated code!
- A few hours of abstract thought devoted to algorithm design often results in faster, simpler, and more general solutions.

Algorithm Tuning

- After algorithm design one can continue on to Algorithm tuning
  - concentrate on improving algorithms by cutting down on the constants in the big O() bounds.
  - needs a good understanding of both algorithm design principles and efficient use of data structures.
- In this course we will not go further into algorithm tuning
  - For a good introduction, see chapter 9 in Programming Pearls, 2nd ed by Jon Bentley





An interesting fact about logarithm

$$\boldsymbol{log_{b_1} n = O(log_{b_2} n)}$$

For any constant $b_1 > 1$ and $b_2 > 1$.

For example, let us verify $\log_2 n = O(\log_3 n)$.

Notice that

$$\log_3 n = \frac{\log_2 n}{\log_2 3} \Longrightarrow \log_2 n = \log_2 3 \cdot \log_3 n$$

Hence, we can set $c_1 = \log_2 3$ and $c_2 = 1$, which makes

$$\log_2 n \leq c_1 \log_3 n$$

Hold for all $n \geq c_2$.

An interesting fact about logarithm

$$\log_{b_1} n = O(\log_{b_2} n)$$

For any constant $b_1 > 1$ and $b_2 > 1$.

Because of the above, in computer science, we omit all the constant logarithm bases in big-O. For example, instead of $O(\log_2 n)$, we will simply write $O(\log n)$

- Essentially, this says that "you are welcome to put any constant base there, and it will be the same asymptotically".
- Obviously, Ω, Θ also have this property.

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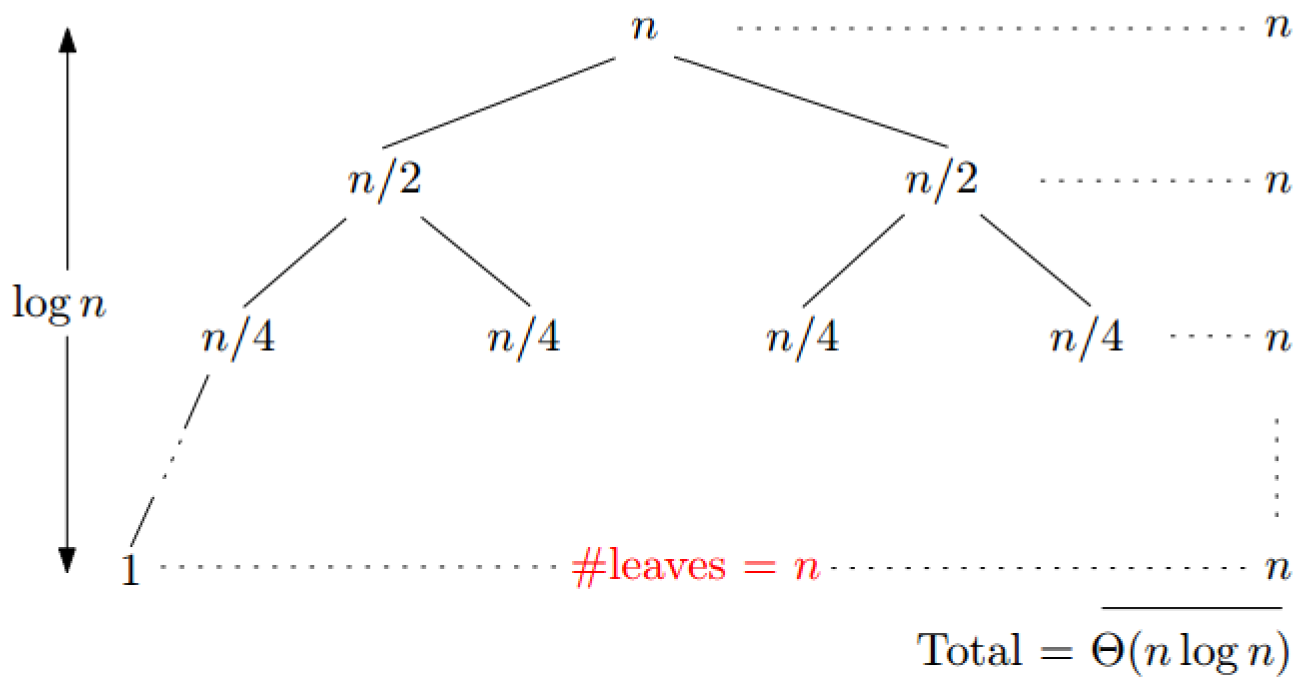
Solving recurrences: Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
  - Each node represents the cost of a single subproblem.

$$T(n) = \begin{cases} 2T(n/2) + n, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$



Recursion-tree method: Example

$$T(n) = \begin{cases} 3T(n/4) + n^2, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$

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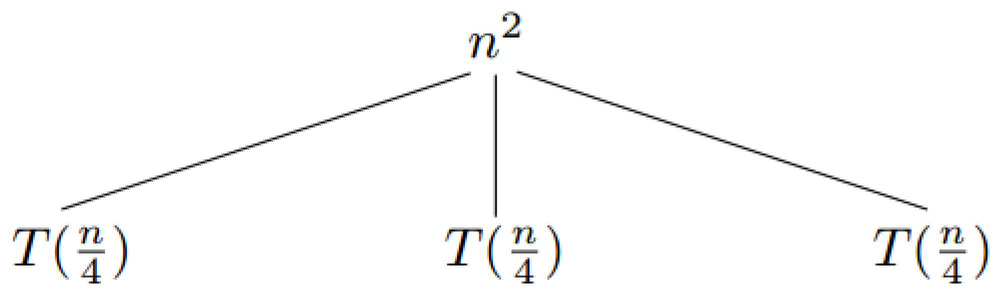

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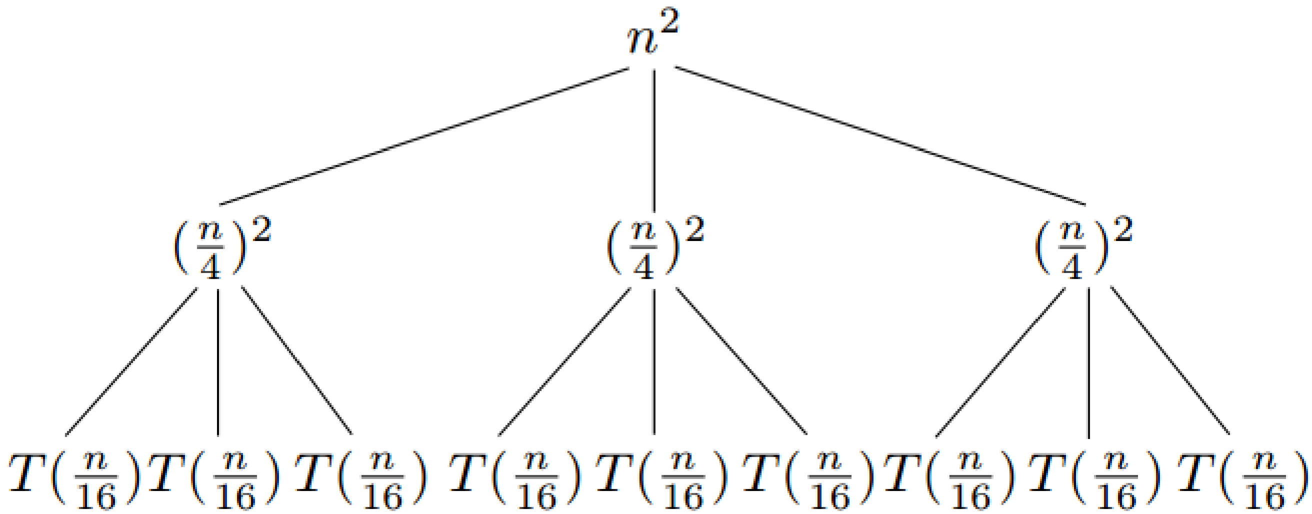

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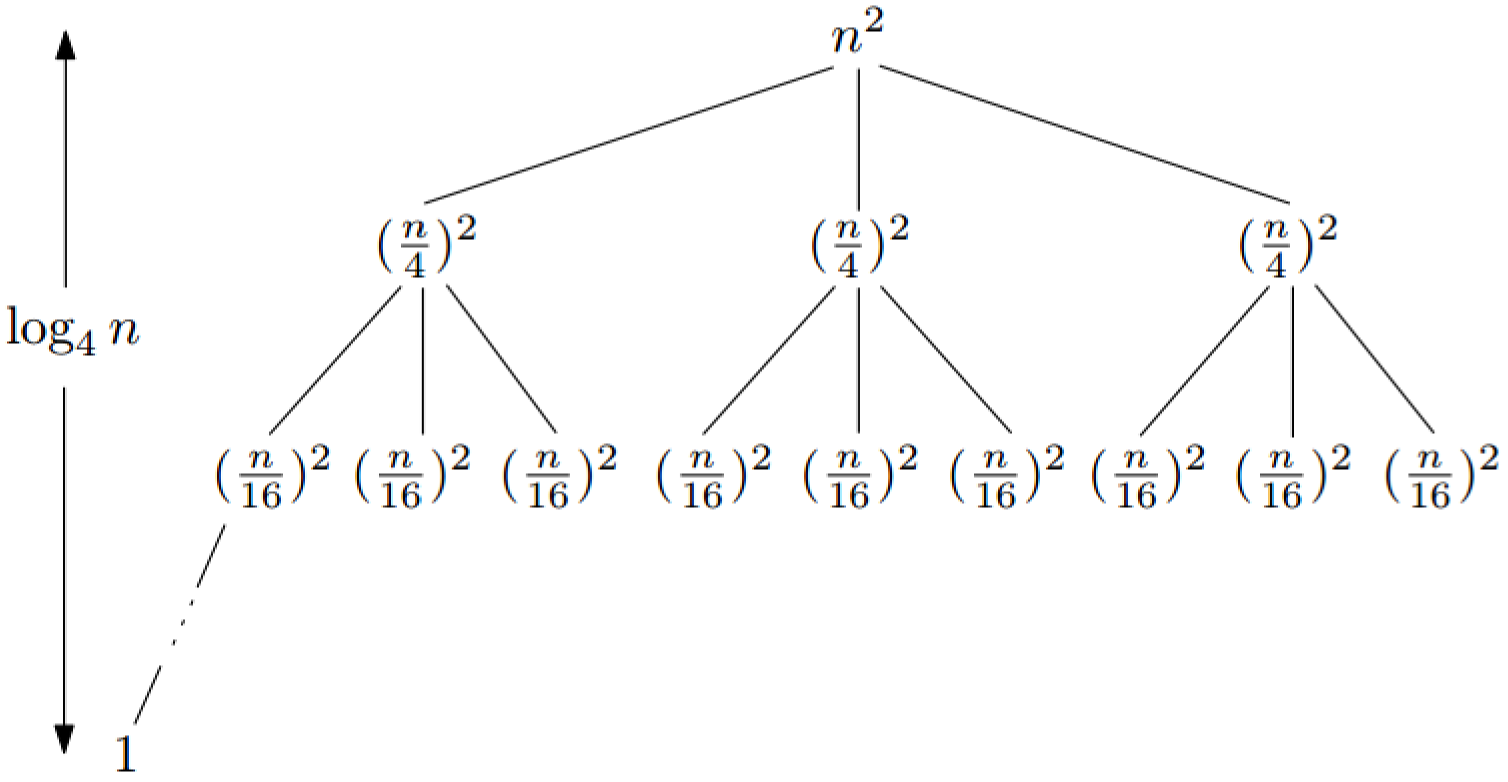

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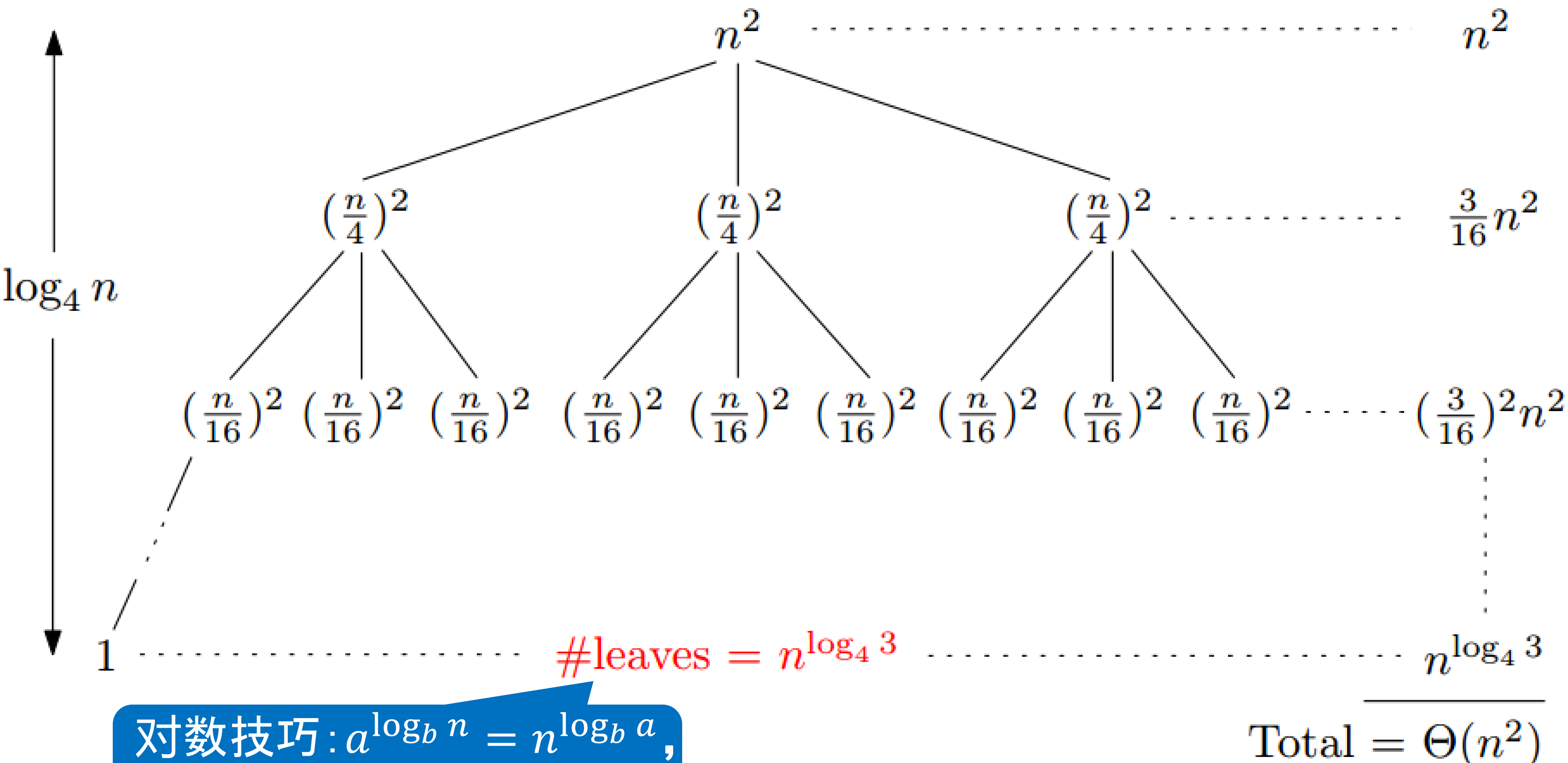

对数技巧: $a^{\log_b n} = n^{\log_b a}$,
$3^{\log_4 n} = n^{\log_4 3}$

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$$\begin{aligned} T(n) &\le n^2 + \frac{3}{16}n^2 + \left(\frac{3}{16}\right)^2 n^2 + \cdots \\ &= O(n^2). \end{aligned}$$

geometric series

几何级数（又称为等比级数）

Recursion-tree method: Example

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- Since $T(n) = 3T(n/4) + n^2$, it follows that $T(n) \geq n^2$

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- Since $T(n) = 3T(n/4) + n^2$, it follows that $T(n) \geq n^2$
- So, $T(n) = \Omega(n^2)$.
- Thus, $T(n) = \Theta(n^2)$

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Prove $T(n) \leq cn^2$ by induction, where c is a large constant.

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Proof.

- Base (n=1) : obviously holds for any $c \geq 1$

$\square$

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Proof.

- Base (n=1) : obviously holds for any $c \geq 1$
- Induction:

$$\begin{aligned} T(n) &= 3T(n/4) + n^2 \\ &\leq 3c(n/4)^2 + n^2 \\ &= cn^2 - (13c/16 - 1)n^2 \\ &\leq cn^2, \end{aligned}$$

whenever $13c/16 - 1 \geq 0$, or $c \geq 16/13$.

$\square$

Substitution method: Example 2

$$T(n) = \begin{cases} T(n/3) + T(2n/3) + n, & \text{if } n > 2, \\ 1, & \text{if } n = 1, 2. \end{cases}$$

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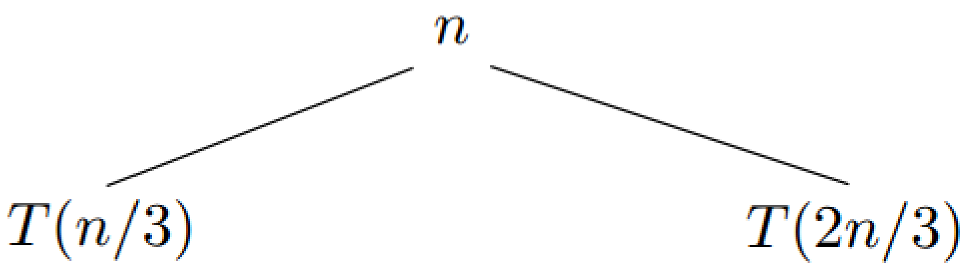

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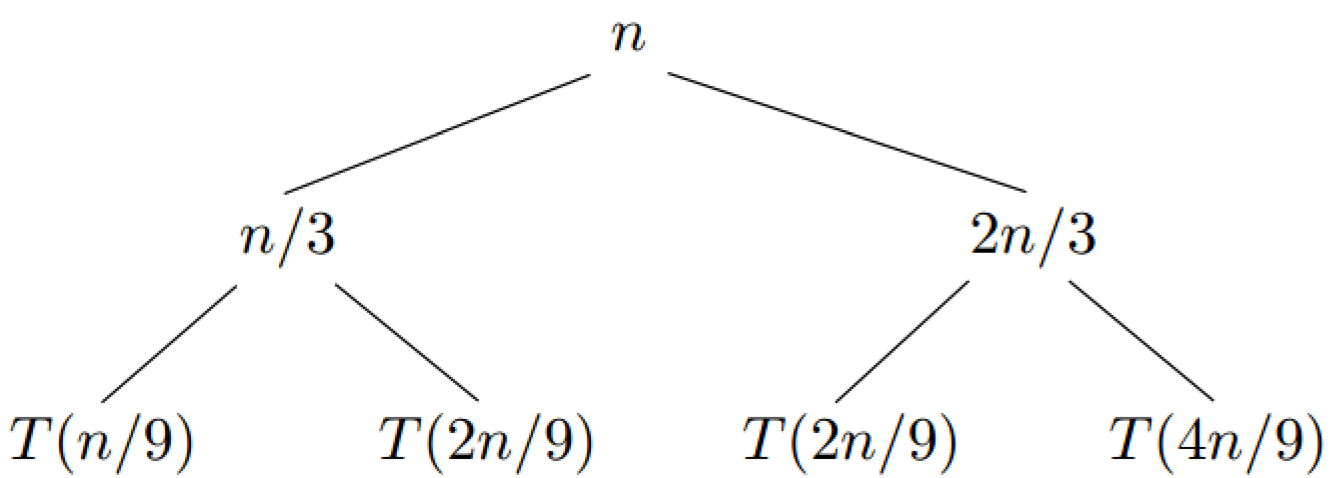

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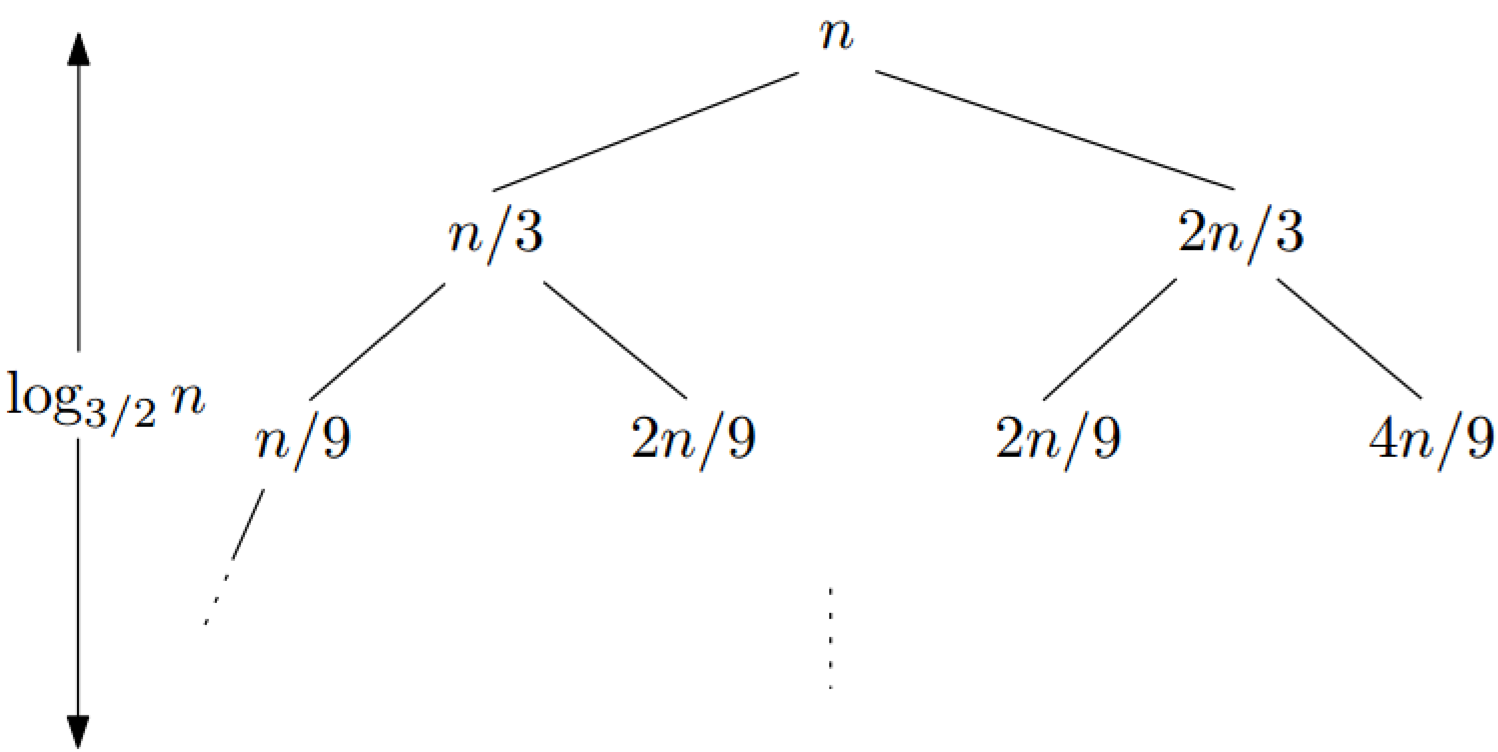

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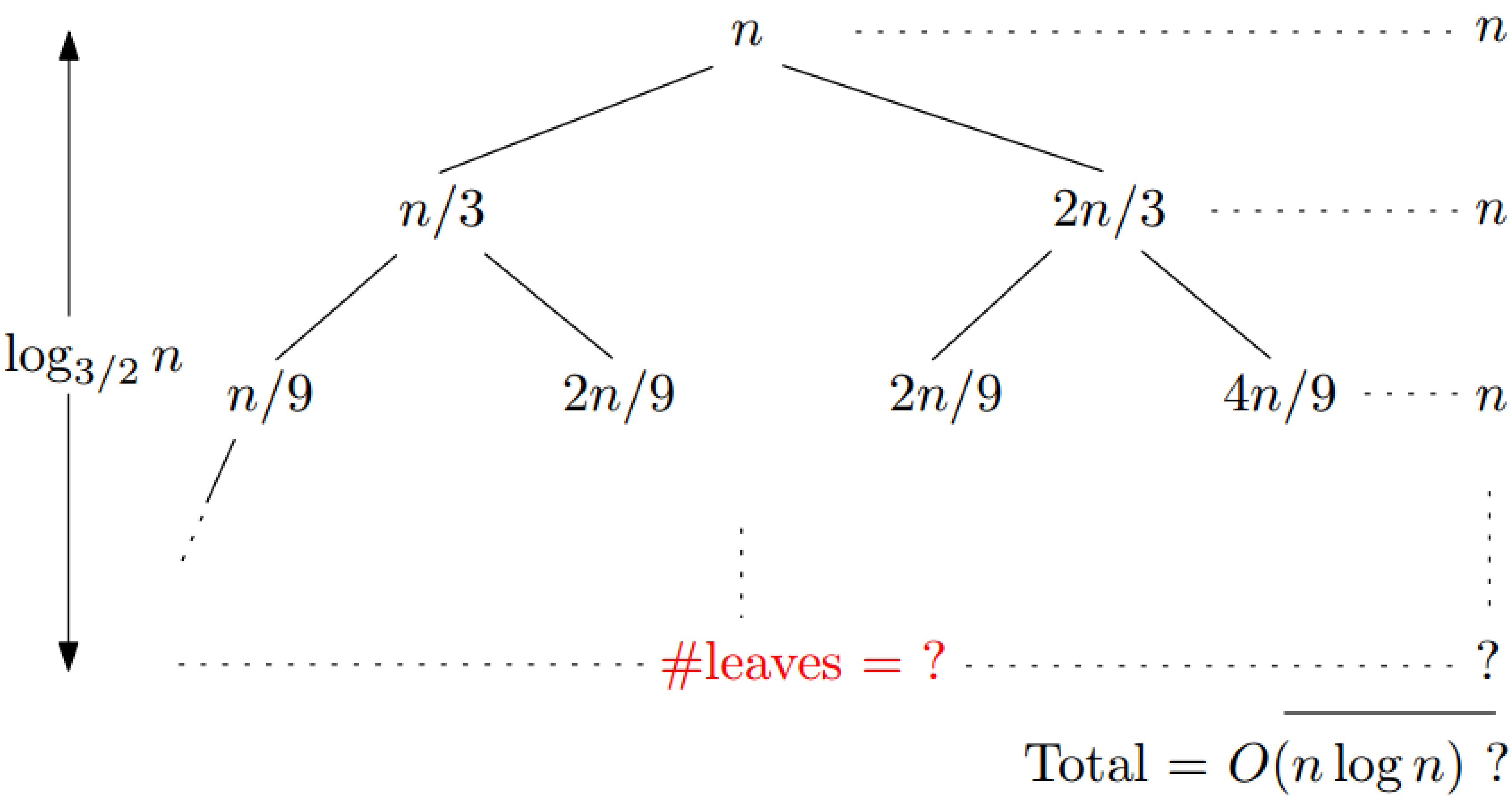

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- Base (n=2) : obviously holds for any $c \geq 1/2$
- Induction:

$$\begin{aligned} T(n) &= T(n/3) + T(2n/3) + n \\ &\leq c(n/3)\log(n/3) + c(2n/3)\log(2n/3) + n \\ &= cn\log n - c((n/3)\log 3 + (2n/3)\log(3/2)) + n \\ &= cn\log n - cn(\log 3 - 2/3) + n \\ &\leq cn\log n, \end{aligned}$$

as long as $c \geq 1/(\log 3 - 2/3)$.

$\square$

Outline

- Asymptotic Notations（渐近记号）
  - Big-Oh
  - Big-Omega
  - Big-Theta
  - Algorithm Design and Algorithm Turing
- Solving Recurrences
  - Recursion-tree Method（递归树法）
  - Substitution Method（代入法/替代法）
  - Master Method and Master Theorem (主方法)

Master Theorem

If $T(n) = aT\left(\left\lceil\frac{n}{b}\right\rceil\right) + O(n^d)$ for some constant a > 0, b > 1 and d ≥ 0, then

$$T(n) = \begin{cases} O\left(n^d\right), & if\ d > \log_b a \\ O\left(n^d \log n\right), & if\ d = \log_b a \\ O(n^{\log_b a}), & if\ d < \log_b a \end{cases}$$

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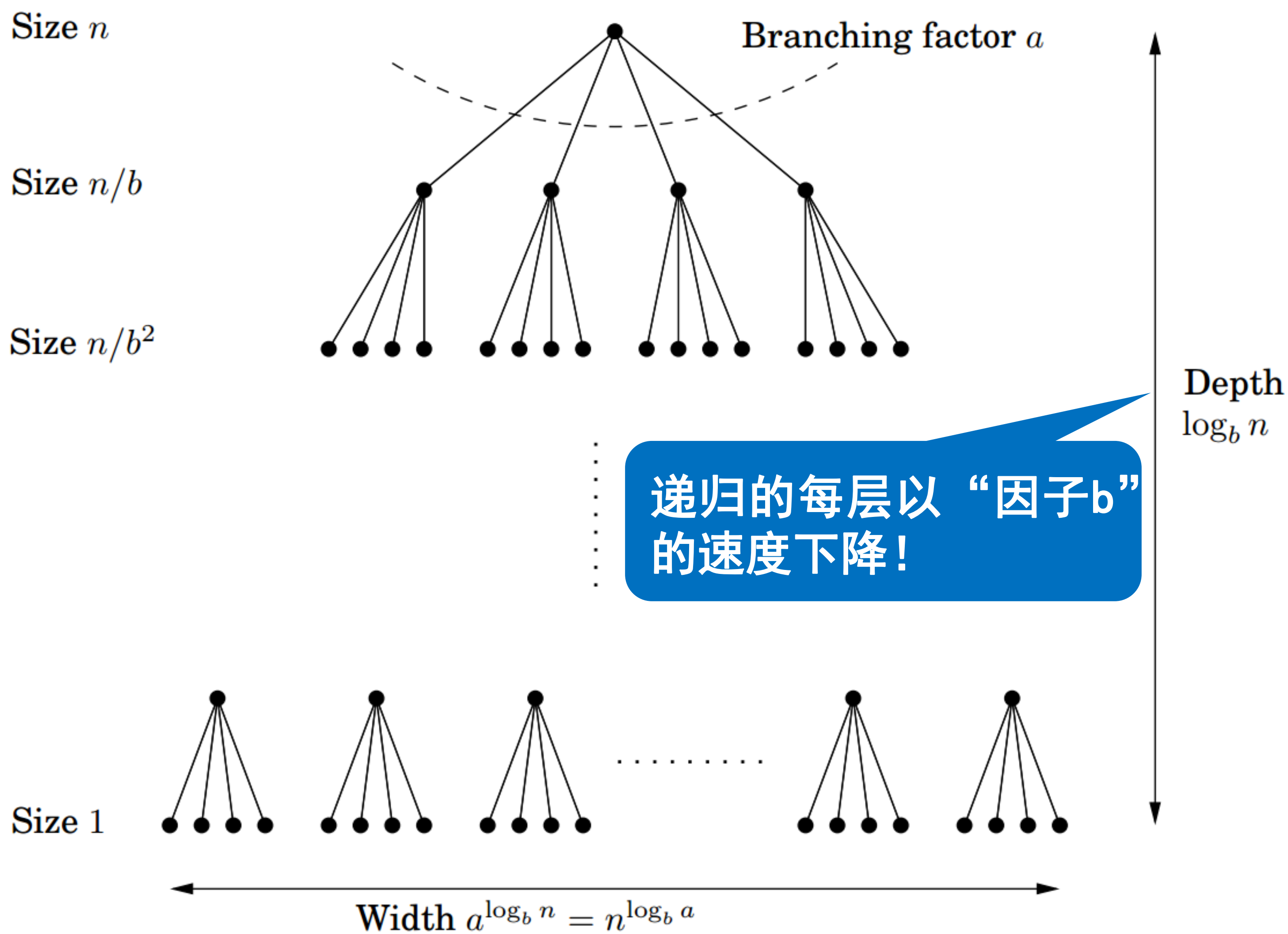
For the sake of convenience, we assume that *n* is a power of *b*. This will not influence the final bound in any important way—*n* is at most a multiplicative factor of b away from some power of *b*—and it will allow us to ignore the rounding effect in $\lceil \frac{n}{b} \rceil$.

Proof of the Master Theorem



递归的每层以“因子b”的速度下降！

$$a^k \times O\left(\frac{n}{b^k}\right)^d = O(n^d) \times \left(\frac{a}{b^d}\right)^k$$

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- The size of the subproblems decreases by a factor of b with each level of recursion, and therefore reaches the base case after $\log_b n$ levels. This is the height of the recursion tree.
- The k-th level of the tree is made up of a^k subproblems, each of size n/b^k
- The total work done at this level is

$$a^k \times O\left(\frac{n}{b^k}\right)^d = O\left(n^d\right) \times \left(\frac{a}{b^d}\right)^k$$

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It comes down to the following three cases.

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- **The ratio a/b^d is exactly 1 (a/b^d =1)**.
  In this case all O(log n) terms of the series are equal to $O(n^d)$.

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谢谢

BDA

Beihang University