[see (37)[1]] $e_i^2=(1-b^{-2})/b^{2T}$. [Golay's value is larger than this because of his nonoptimal insistence on unity main-lobe output, whereas the optimal output main lobe is $1-((1-b^{-2})/b^{2T})$ which, interestingly, and valid generally, is also $1-e_i^2$.] Thus the greater the allotted processing delay, the smaller is e_i^2. However, for the transversal feedforward equalizer utilizing the same number of delay cells T, Golay finds that the processing delay can be made $T+1$ cells, and that for this delay the intersymbol variance is $(1-b^{-2})/(b^{2(T+1)}-1)$. Again, this can be further reduced by the use of an overall less than unity gain which reduces his arbitrary unity main lobe and all other side lobes of the output; the resulting minimal variance is $(1-b^{-2})/(b^{2(T+1)}-b^{-2})$. Because of the $T+1$ factor in the exponent of b, there are values of (b, T) for which this variance is less than that of the Wiener equalizer whose exponent factor of b is T. If the Wiener processing delay had also been $T+1$, then its intersymbol variance would be $(1-b^{-2})/(b^{2(T+1)})$, which is less than that of the transversal equalizer for the same $T+1$ processing delay.

Further understanding would be obtained if a set of solutions [using (121)] were found for the optimal tap gains in a transversal equalizer comprising T cells, and where the main-lobe output delay (the processing delay) is assigned the values T, $T-1$, etc. I have begun this, but have not completed it in time for this reply.

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Enumeration of All Circuits of a Graph

Abstract—An algorithm is presented to enumerate all circuits of a linear graph. The method is suitable for computerization and does not require a large computer memory. The application of the procedure to some other enumeration problems is discussed.

A number of methods are given in the literature to enumerate all the circuits of a linear graph [1]–[6]. Of these, the methods applicable to a general graph [1]–[4] require computers with large memories as one must avoid duplications in the output [3], [4] or detect either a disjoint union of circuits [1] or circuits containing nonexistent edges [2]. In this letter, we present an alternate procedure, for a general graph, suitable for use on a computer without a large memory. A feature of this method is that the same principle can be applied to enumerate all trees and all cutsets of a graph.

We start with the fundamental circuit matrix B_f of the graph G with respect to an arbitrary tree T. Let T be regarded as the set of branches $b_1, b_2, \cdots, b_r$ and let the set of chords $c_1, c_2, \cdots, c_\mu$ constitute C, the cotree of G with respect to T. The method is based on the following properties which may be readily proved.

1) Every nonfundamental circuit of G with respect to T is the ring union of two or more distinct fundamental circuits of G with respect to T.
2) Every circuit of G is a fundamental circuit with respect to some tree T^x of G.
3) The ring union of some k fundamental circuits with respect to T is a *circ*, which is either a circuit or an edge-disjoint union of circuits [1].
4) The ring union of *two* fundamental circuits with respect to T *is* a circuit, if their intersection is not a null set.
5) If b_k is in the fundamental circuit defined by c_i for the tree T, then the set of edges $b_1, b_2, \cdots, b_{k-1}, b_{k+1}, \cdots, b_r, c_i$ forms another tree of G.
6) Let $C_s \cup T_s$ be a circuit where $C_s \subseteq C$ and $T_s \subseteq T$. Then
   a) for every proper subset C_{sk} of C_s there exists a tree T^k of G such that $T^k = C_{sk} \cup T_{sk}$, where $T_s \subseteq T_{sk} \subset T$, and further,
   b) if $C_s \oplus C_{sk}$ contains more than one element, then for every such element c_i there exists a tree T^{k+1} of G such that $T^{k+1} = c_i \cup C_{sk} \cup T_{sk'}$ where $T_s \subseteq T_{sk'} \subset T_{sk} \subset T$.

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From the given B_f matrix one can construct the circ matrix of order $2^\mu - 1 \times e$ in which each row corresponds to the mod 2 addition of some k rows of B_f. Every circuit of G corresponds to some row of the circ matrix, but not conversely. After setting up the circ matrix, the procedure due to Maxwell and Reed [1] envisages comparison of each row with every other row to determine a disjoint union of circuits. In general, the circ matrix is a large matrix and the amount of testing is thus considerable. We present a criterion, based on the properties stated earlier, by which we determine without any reference to other circs whether or not a circ is a circuit. When a circ is generated it is immediately tested. If it is a circuit, it is listed, thus avoiding the need for its storage in the computer memory.

Let the characteristic submatrix B_{f12} of B_f be called the tree-path matrix and designated F [7]. The rows of F correspond to the chords $c_1, \cdots, c_\mu$ and the columns to the branches $b_1, \cdots, b_r$. The entry f_{ij} is 1 if the path in the tree T between the vertices of c_i includes b_j and is zero otherwise.

Let b_k be a branch present in the path corresponding to c_i. We can obtain a tree T^1 by adding c_i to and deleting b_k from T (property 5). Let F^1 be the tree-path matrix of G based on T^1. Now consider the path corresponding to $c_j (j \neq i)$ in T. If it does not include b_k, the path corresponding to c_j in T^1 is the same as in T. On the other hand, let b_k be present in the path. The ring union of the two paths in T corresponding to c_i and c_j forms a circuit together with c_i and c_j (property 4). Therefore, in this case, the path prescribed by c_j in T^1 consists of c_i and the branches of T corresponding to the nonzero entries in the mod 2 addition of the rows i and j of F.

Thus the construction of F^1 from F is simple and direct. Every row of F (except the ith row) containing a nonzero entry in the kth column is replaced by its mod 2 sum with the ith row, and the resulting zero in the kth column is replaced by 1. Finally, the labels c_i and b_k of the ith row and the kth column are interchanged.

The procedure for enumerating all circuits of a graph involves the following steps. Each circ is represented by a distinct integer index N, where $1 \leq N \leq 2^\mu - 1$. If in the binary representation of N, nonzero entries are present in the digital positions $i, j, k, \cdots$, then the circ S_N is taken as the ring union of the fundamental circuits defined by $c_i, c_j, c_k, \cdots$. Let the addition (mod 2) of the rows of F corresponding to these chords yield nonzero entries in the columns corresponding to $b_\alpha, b_\beta, b_\gamma, \cdots$. Thus, the circ S_N comprises edges $c_i, c_i, c_k, \cdots, b_\alpha, b_\beta, b_\gamma, \cdots$, say m in number. Now there exists a tree in which any $m-1$ elements of S_N are branches if and only if S_N is a circuit. Thus, if $m > r+1$, the circ is not a circuit. If $m = r+1$ and S_N is a circuit, then it is a Hamilton circuit.

To test whether or not S is a circuit when $m \leq r+1$, we examine the existence of a tree having all the elements of S_N except one, say c_i, as its branches. To this end, as a first step, consider a tree T^1 obtained from T by deleting a branch b_i other than $b_\alpha, b_\beta, b_\gamma, \cdots$, and adding a chord c_j. If S_N is a circuit, such a tree always exists (property 6a). Therefore, the row corresponding to c_j in F has a nonzero entry in at least one column other than $b_\alpha, b_\beta, b_\gamma, \cdots$. The path matrix F^1 corresponding to T^1 is then obtained by the procedure given earlier. Repeating the procedure, we obtain T^2 from T^1 by converting c_k as a branch and a branch of T^1 other than $c_j, b_\alpha, b_\beta, b_\gamma, \cdots$, into a chord. If this procedure can be continued till a tree T^x containing the $m-1$ edges $c_j, c_k, \cdots, b_\alpha, b_\beta, b_\gamma, \cdots$, is found, then we conclude that S_N is a circuit. On the other hand, if at any stage it is impossible to obtain a new tree T^{n+1} by converting a chord c_m of a tree T^n into a branch, without converting a branch of T^n which is contained in S_N into a chord, we conclude that S_N is a disjoint union of circuits. The occurrence of the latter event is revealed by the fact that all the columns of F^n having nonzero entries in the row corresponding to c_m relate to the elements of S_N. The order of conversion of $c_j, c_k, \cdots$, into branches is immaterial in the process of identification of S_N as a circuit (property 6b).

Example

Consider the matrix F given below. We illustrate the procedure by testing for the circuit character of S_7 and S_{19}.

$$S_7 = [c_1, c_2, c_3, b_1, b_2]$$

Converting c_2 into a branch and b_4 into a chord, we obtain T^1 having the tree-path matrix F^1. Then it is found that the tree T^2 can be obtained from T^1 by replacing either b_3 or b_5 by c_3. Hence S_7 is a circuit.

$$S_{19} = [c_1, c_2, c_5, b_1, b_4, b_5]$$

Here we notice that c_2 cannot be made a branch without converting either b_4 or b_5 into a chord. Thus, S_{19} is a disjoint union of circuits.

$$F = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad F^1 = \begin{array}{c} \\ c_1 \\ b_4 \\ c_3 \\ c_4 \\ c_5 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & c_2 & b_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The procedure lends itself to a simple extension for obtaining all the paths required for the determination of the switching function of a two-terminal single-contact network. We introduce an edge designated c_1 connecting the two terminals of the network. The matrix F is constructed on a tree for which c_1 is a chord. In this way, only those circs with odd-numbered indices need be considered.

Similarly, we can test whether or not the subgraph formed by a given set of $v-1$ edges contains a circuit and thus list all the trees of the graph. However, in comparison with known methods of listing trees, the computation involved in this procedure is excessive.

A procedure for listing all cutsets of a graph may also be given along lines similar to those for circuit enumeration. The tree-path matrix F is the transpose of the characteristic submatrix Q_{f11} of the fundamental cutset matrix Q_f [5]. We generate a seg by taking the union of some k rows. We then proceed to test whether the seg is a cutset by examining whether there exists a tree for which all elements of the seg but one are chords.

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On Some Extensions of Nyquist's Telegraph Transmission Theory

Abstract—Gibby and Smith derived general conditions for the overall system characteristic which lead to the elimination of intersymbol amplitude and pulsewidth distortion. It is shown that their derivation can be simplified by a more rigorous use of the delta function.

In 1965 Gibby and Smith published a more general interpretation of Nyquist's telegraph transmission theory.[1] They found general conditions on the real and the imaginary parts of the overall system characteristic which lead to the elimination of intersymbol amplitude and pulsewidth distortion without the need of bandwidth restriction.

It will be shown here that by reformulating the constraints on the system response $f(t)$ in terms of the constraints on the *sampled* system response, the derivation of the resulting conditions for the overall system characteristic $F(\omega)$ becomes simple and straightforward.

Undistorted Amplitude

The constraints on the system response $f(t)$ are

$$f(kT) = \begin{cases} 1 \text{ for } k = 0 \\ 0 \text{ for } k \neq 0, \quad k \text{ integer} \end{cases}$$

where T is the time interval between two successive pulses. This is equivalent to the condition

$$f(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT) = \delta(t).$$

Translating this requirement into the frequency domain, one obtains[2]

$$F(\omega) * \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T} \delta\left(\omega - \frac{2\pi k}{T}\right) = 1$$

with $*$ denoting convolution.

Hence, the constraints on $F(\omega)$ for no intersymbol amplitude distortion are given by

$$\sum_{k=-\infty}^{+\infty} F\left(\omega - \frac{2\pi k}{T}\right) = \frac{T}{2\pi}.$$

Undistorted Pulsewidth

In this case the constraints on the system response are

$$f\left(\frac{2k-1}{2} T\right) = \begin{cases} \frac{1}{2} \text{ for } k = 0, k = 1 \\ 0 \text{ for } k \neq 0, k \neq 1, k \text{ integer} \end{cases}$$

or

$$f(t) \cdot \sum_{k=-\infty}^{+\infty} \delta\left(t - \frac{2k-1}{2} T\right) = \tfrac{1}{2} \cdot \delta(t - \tfrac{1}{2}T) + \tfrac{1}{2} \cdot \delta(t + \tfrac{1}{2}T).$$

Translated into the frequency domain,

$$F(\omega) * \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T} \delta\left(\omega - \frac{2\pi k}{T}\right) \exp\left(-j\omega \frac{T}{2}\right) = \tfrac{1}{2}\left\{\exp\left(j\omega \frac{T}{2}\right) + \exp\left(-j\omega \frac{T}{2}\right)\right\}$$

or

$$\sum_{k=-\infty}^{+\infty} (-1)^k F\left(\omega - \frac{2\pi k}{T}\right) = \frac{T}{2\pi} \cos \frac{\omega T}{2}.$$

Combination of Undistorted Amplitude and Undistorted Pulsewidth

Both constraints are satisfied if multiplication of the system response $f(t)$ with the series of equally spaced delta functions

$$\sum_{k=-\infty}^{+\infty} \delta(t - \tfrac{1}{2}kT)$$

results in the function

$$\delta(t) + \tfrac{1}{2} \cdot \delta(t - \tfrac{1}{2}T) + \tfrac{1}{2} \cdot \delta(t + \tfrac{1}{2}T)$$

or, in the frequency domain,

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