trim

TEXTS AND READINGS IN MATHEMATICS 23

Introduction to Game Theory

Stef Tijs

HINDUSTAN BOOK AGENCY

TEXTS AND READINGS IN MATHEMATICS **23**

Introduction to Game Theory

Introduction to Game Theory

Stef Tijs
Tilburg University
Tilburg

HINDUSTAN
BOOK AGENCY

Published by
Hindustan Book Agency (India)
P 19 Green Park Extension
New Delhi 110 016
India

email: info@hindbook.com
www.hindbook.com

ISBN 978-81-85931-37-1 ISBN 978-93-86279-17-0 (eBook)
DOI 10.1007/978-93-86279-17-0

Digitally reprinted paper cover edition 2011

ISBN 978-93-80250-24-3

Contents

Preface

This book on game theory evolved from introductory courses in game theory given in the Netherlands in Nijmegen and Tilburg for students in mathematics and students in econometrics and Operations Research, respectively.
After a general introduction in chapter 1, the chapters 2 to 9 deal with non-cooperative game theory. Central are here Nash equilibria, their existence and calculation. Described is the interrelation between non-cooperative game theory on one hand and chapters in Operations Research as linear programming and linear complementary theory on the other hand. Applications in economics are indicated: auctions, oligopoly, mechanisms for cost sharing of a public good. Also potential games arising from congestion situations are treated.
Chapters 10 to 19 deal with cooperative game theory and treat TU-games, bargaining games and NTU-games.
Various solution concepts as the core, the Shapley value, the nucleolus, and the τ-value are introduced. Also some axiomatic characterizations are given. Special attention is devoted to classes of games arising from economic and OR situations such as linear production games, flow games and permutation games.
In each chapter one can find exercices and for most of these exercises one can find solutions or outcomes at the end of the book.
Included is also a collection of extra exercises (without solutions), which are often connected to more than one chapter and which were used as exercises in written exams at Tilburg University.
The book ends with a bibliography which invites to further reading in game theoretical texts.

I am grateful to Peter Borm, Eric van Damme, Vincent Feltkamp and Jeroen Suijs for their valuable comments on earlier versions. Thanks are also due to Annemiek Dankers and Rodica Brânzei for transforming the manuscript into this final version.

August 2002

Chapter 1

Introduction

Game theory is a mathematical theory dealing with models of conflict and cooperation. Non-surprisingly it finds many applications in economics and in other social sciences but also in evolutionary biology. In the fundamental work of John von Neumann and Oskar Morgenstern (1944) "Theory of Games and Economic Behavior" three classes of games are considered:

- games in extensive form (or tree games),
- games in strategic form (or games in normal form) and
- games in coalitional form.

The first two classes belong to non-cooperative game theory, the third class to cooperative game theory. The following examples will help to clarify the introduced notions.
In figure 1 is depicted a tree game, where two players are involved: player 1 and player 2. A play proceeds as follows. First player 1 decides to go to the left (L) or to the right (R), then player 2

decides to go left or right. If player 1 has chosen L and player 2 followed with l_1 then player 1 obtains \$ 5 and player 2 gets \$ 1, etc.
How should you play in this game if you were player 1? And if you were player 2?

For player 2 it seems obvious. He will choose the strategy "go to the right if player 1 has chosen L, go to the left if player 1 chose R", which strategy we will denote by r_1l_2. Player 1 can in anticipation of this behavior of player 2 choose R. The strategy pair (R, r_1l_2) leads to the payoffs 4 and 8 for the players 1 and 2, respectively. In fact, player 1 has two strategies, L and R, and player 2 has four strategies l_1l_2, l_1r_2, r_1l_2 and r_1r_2, leading to eight strategy pairs. In the following bimatrix the corresponding payoff pairs are given

$$\begin{array}{cc} & \begin{array}{cccc} l_1l_2 & l_1r_2 & r_1l_2 & r_1r_2 \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{bmatrix} (5,1) & (5,1) & (3,2) & (3,2) \\ (4,8) & (6,3) & (4,8)^* & (6,3) \end{bmatrix} \end{array}$$

which can be split up into two matrices

$$A = \begin{bmatrix} 5 & 5 & 3 & 3 \\ 4 & 6 & 4 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 8 & 3 & 8 & 3 \end{bmatrix},$$

the *payoff matrices* of player 1 and player 2, respectively. Note, that *unilateral deviation* from (R, r_1l_2) neither pays for player 1 (from payoff 4 to payoff 3 if deviation from R to L) nor for player 2. Such strategy pairs are called *Nash equilibria*.
Question: Is also (R, l_1l_2) a Nash equilibrium?
It is easy to see that this *bimatrix game* has a unique Nash equilibrium. Such a bimatrix game corresponds to a game in strategic

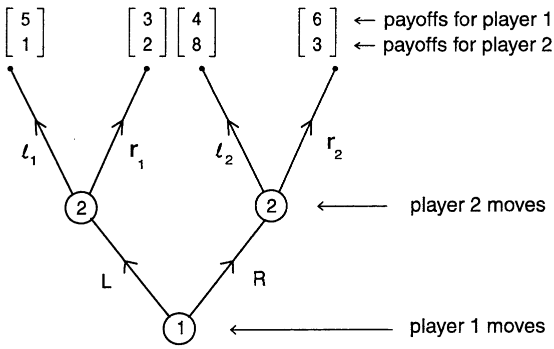

Figure 1. A tree game

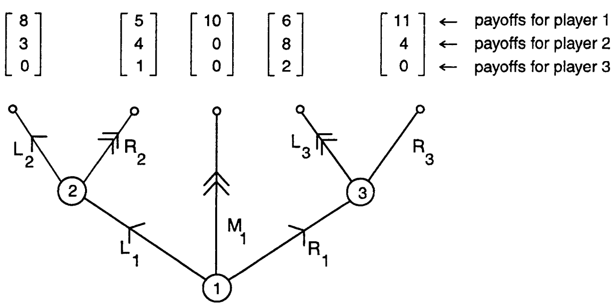

Figure 2. A 3-person game

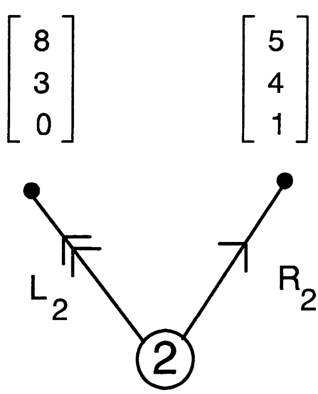

Figure 3. A subgame with one player

form in a natural way. Here, a two-person game in strategic form is an ordered 4–tuple $\langle X_1, X_2, K_1, K_2\rangle$, where X_i is a non-empty set, *the strategy space* of player i ($i = 1, 2$), and $K_i : X_1 \times X_2 \to \mathbb{R}$ is the *payoff function* for player i.
For the above bimatrix game $X_1 = \{L, R\}$, $X_2 = \{l_1l_2, l_1r_2, r_1l_2, r_1r_2\}$ and $K_1(L, l_1r_2) = 5$, $K_2(L, r_1l_2) = 2$, etc.

Consider again the tree game in figure 1. Which payoff can player 1 (player 2) guarantee himself independently of the choice of the other player? And what can the coalition $\{1, 2\}$ get in cooperation? The answers are 4 (by choosing R), $\min\{2, 8\}$ by choosing r_1l_2 and 12 (by choosing the path R, l_2). This corresponds to a *coalitional game* $\langle N, v\rangle$ where $N = \{1, 2\}$ and the *characteristic function* $v : 2^N \to \mathbb{R}$ is given by $v(\emptyset) = 0$, $v(\{1\}) = 4$, $v(\{2\}) = 2$, $v(\{1, 2\}) = 12$.

Again we go from a tree game to a game in strategic form and a game in coalitional form, now for the tree game sketched in figure 2.

- The first move is for player 1. He chooses L_1, M_1 or R_1.
- If player 1 has chosen alternative M_1, then he gets \$ 10 and the other two players get nothing.
- If player 1 chose L_1, then player 2 has to move with available alternatives L_2 and R_2 etc.

How to play this game?
If there is no cooperation: if player 2 is at move, then his choice will be R_2 (because $4 > 3$), denoted by a double arrow; if player

3 has to move, then the choice will be L_3. By anticipation, player 1 will choose alternative M_1 because $10 = \max\{5, 10, 6\}$. The triple (M_1, R_2, L_3) is a Nash equilibrium: unilateral deviations do not pay. The strategy triple (M_1, L_2, R_3) is not a Nash equilibrium, because player 1 can by unilateral deviation (alternative R_1) improve his payoff (from \$ 10 to \$ 11). The strategy triple (M_1, L_2, L_3) is also a Nash equilibrium. This Nash equilibrium is not *subgame perfect* (Selten, 1975) contrary to the Nash equilibrium found earlier (by backwards induction). This follows by looking at the subgame in figure 3. The choice L_2, in case player 2 has to move, is not very credible. Better is R_2 because $4 > 3$. The tree game from figure 2 corresponds to a 3-person game in strategic form, which can be described with the aid of 3 matrices (matrix L_1, M_1 or R_1), player 2 chooses a row and player 3 a column.

$$\begin{array}{cc} & \begin{array}{cc} L_3 & \quad R_3 \end{array} \\ \begin{array}{c} L_2 \\ R_2 \end{array} & \begin{bmatrix} (8,3,0) & (8,3,0) \\ (5,4,1) & (5,4,1) \end{bmatrix} \end{array}$$
matrix L_1

$$\begin{array}{cc} & \begin{array}{cc} L_3 & \quad R_3 \end{array} \\ \begin{array}{c} L_2 \\ R_2 \end{array} & \begin{bmatrix} (10,0,0)^* & (10,0,0) \\ (10,0,0)^* & (10,0,0) \end{bmatrix} \end{array}$$
matrix M_1

$$\begin{array}{cc} & \begin{array}{cc} L_3 & \quad R_3 \end{array} \\ \begin{array}{c} L_2 \\ R_2 \end{array} & \begin{bmatrix} (6,8,2) & (11,4,0) \\ (6,8,2) & (11,4,0) \end{bmatrix} \end{array}$$
matrix R_1

For the corresponding game in strategic form $\langle X_1, X_2, X_3, K_1, K_2, K_3 \rangle$ we have $X_1 = \{L_1, M_1, R_1\}$, $X_2 = \{L_2, R_2\}$, $X_3 = \{L_3, R_3\}$,

$K_1(L_1, L_2, L_3) = 8$, $K_1(L_1, R_2, L_3) = 5$, $K_3(L_1, R_2, L_3) = 1$ etc. The Nash equilibria (M_1, L_2, L_3) and (M_1, R_2, L_3) are denoted with a star * in the matrices.
The corresponding game in coalitional form $\langle N, v\rangle$ is given by $N = \{1, 2, 3\}$, $v(\{1\}) = 10$, $v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 14$ (player 1 chooses R_1, then the payoff to players 1 and 2 together: 6+8 or 11+4), $v(\{1, 3\}) = 11$ (player 1 chooses R_1, player 3 chooses R_3), $v(\{2, 3\}) = 0$ and $v(\{1, 2, 3\}) = 16$ (player 1 chooses R_1, player 3 chooses L_3).

Finally, we remark that non-cooperative game theory deals with conflict situations, where no binding agreements between the players can be made, but pre-play communication is not (always) forbidden.
In cooperative game theory all kind of agreements and often also sidepayments are possible. Consider the bimatrix of exercise 1.4. In a cooperative situation it is possible that the players make the binding agreement to play T and L, respectively, resulting in a payoff 5 for each of the players. In a non-cooperative situation the players can agree in pre-play communication also to play T and L, but when it comes to real action choice both players have the incentive to deviate (cheat) because B (respectively. R) is better than T (respectively L) for player 1 (player 2), whatever player 2 (player 1) chooses. The outcome (T, L) is a Pareto optimal outcome, which is unstable in a noncooperative situation. (B, R) is stable (a Nash equilibrium) but not Pareto optimal.

Exercise 1.1 Consider the 2-person tree game in figure 4.

(i) Find the subgame perfect Nash equilibria of this game.

(ii) Fill in the payoff vectors in the corresponding bimatrix game:

$$\begin{array}{c} \\ L_1L_4 \\ L_1R_4 \\ R_1L_4 \\ R_1R_4 \end{array} \begin{array}{c} \begin{array}{cccc} L_2L_3 & L_2R_3 & R_2L_3 & R_2R_3 \end{array} \\ \begin{bmatrix} (7,5) & (\cdot,\cdot) & (\cdot,\cdot) & (\cdot,\cdot) \\ (\cdot,\cdot) & (\cdot,\cdot) & (\cdot,\cdot) & (\cdot,\cdot) \\ (\cdot,\cdot) & (\cdot,\cdot) & (\cdot,\cdot) & (\cdot,\cdot) \\ (6,5) & (\cdot,\cdot) & (\cdot,\cdot) & (\cdot,\cdot) \end{bmatrix} \end{array}$$

(iii) Why is the third row equal to the fourth row?

(iv) Find all (pure) Nash-equilibria of this 4×4–bimatrix game. Which of these equilibria do not correspond to subgame perfect-equilibria of the corresponding tree game?

(v) Calculate the corresponding game in coalitional form.

Exercise 1.2

(i) Calculate the subgame perfect Nash equilibria of the tree game in figure 5.

(ii) Give also another Nash equilibrium.

(iii) Give the corresponding strategic form game and calculate the (pure) Nash equilibria of this game.

(iv) Calculate the corresponding game in coalitional form.

Exercise 1.3
Let p be a natural number larger than 1. How should you define a p-person strategic game?

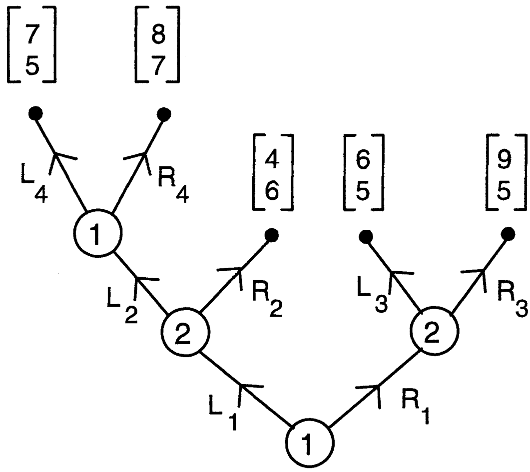

Figure 4.

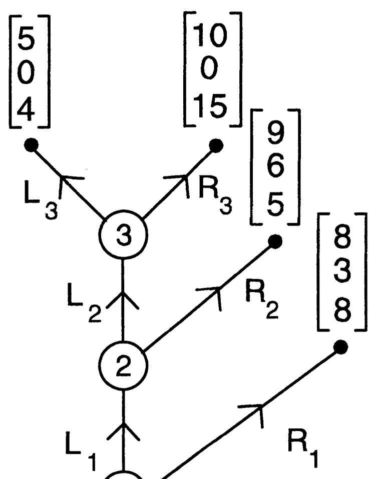

Figure 5.

Exercise 1.4 Find the Nash equilibria of the 2×2–bimatrix game

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} (5,5) & (-4,6) \\ (6,-4) & (-3,-3) \end{bmatrix} \end{array} \quad \text{(prisoner's dilemma).}$$

What is the corresponding coalitional game?

Exercise 1.5 Construct a 2×2-bimatrix game with two (pure) Nash equilibria.

Chapter 2

Games in strategic form

A *p-person game in strategic form* with *player* (decision maker) *set* $P = \{1, 2, \ldots, p\}$ can be described by an ordered $2p$-tuple

$$\langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p \rangle$$

where X_i is the non-empty *strategy set* of player $i \in P$ and $K_i : \prod_{i=1}^{p} X_i \rightarrow \mathbb{R}$ is the *payoff function* of player i, which assigns to each p-tuple $x = (x_1, x_2, \ldots, x_p)$ of strategies a real number $K_i(x)$. A play of such a game proceeds as follows. Eventually after pre-play communication, where no binding agreements can be made, each player i chooses independently of the other players one of his possible strategies, say x_i. Then player 1 obtains a payoff $K_1(x_1, x_2, \ldots, x_p), \ldots$, and player p a payoff $K_p(x_1, x_2, \ldots, x_p)$. The p-tuple $(x_1, x_2, \ldots, x_p)$ is called the *outcome* of the play.
Borel (1921) and von Neumann (1928) noted that not only many parlor games but also many economic situations can be transformed into a game in strategic form, even if the conflict has many stages. We will not go into details [cf. Kuhn (1953)] but note that we have seen (in section 1) some examples where a tree game is

transformed into a bimatrix game. Another example of a different nature is given now.

Example 2.1 We consider a game in which chance plays a role. At the beginning of a play player 1 draws a card from a deck $\{A, K\}$ consisting of an ace and a king; with probability $\frac{1}{2}$ the ace appears. Then player 1 looks at the drawn card and says 'pass' (P) or 'bid' (B). If player 1 passes then he has to pay \$ 1 to player 2. If player 1 bids then the move is for player 2. He can say 'pass' (P) or 'see' (S). If player 2 passes he has to pay \$ 1 to player 1; if player 2 sees then he pays (receives) \$ 2 to (from) player 1 if the drawn card is an ace (king). This game is represented in figure 6.

In this 2-person tree game player 2 is not completely informed about the course of the play till then. He knows what player 1 said but not what chance 'did'. In the figure this is denoted by $- - --$. In the corresponding strategic game, player 1 has four strategies (P, P), (P, B), (B, P) and (B, B) where for example (B, P) means 'bid when an ace is drawn and pass when a king is drawn'. Player 2 has only 2 strategies P and S. Suppose that player 1 uses (B, P) and that player 2 uses S. Then dependent on chance, we have one of the following plays (figure 7) both with probability $\frac{1}{2}$.

The expected payoff from player 2 to player 1 is, given the strategies (B, P) and S, equal to $\frac{1}{2} = \frac{1}{2}(-1) + \frac{1}{2}(2)$. Analogously the expected payoffs in the other seven cases can be calculated result-

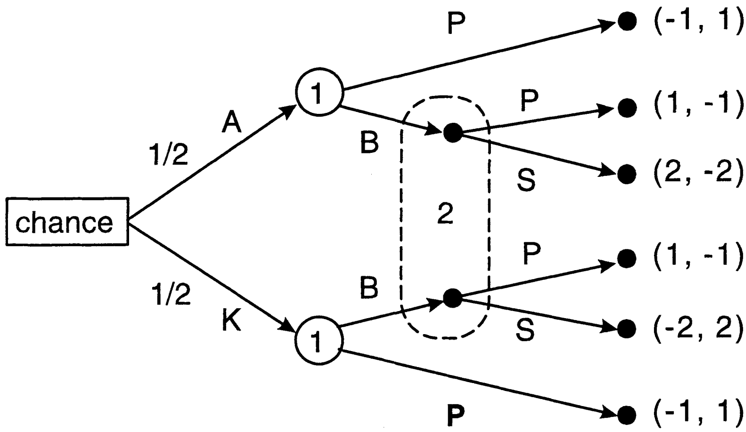

Figure 6. Simple poker

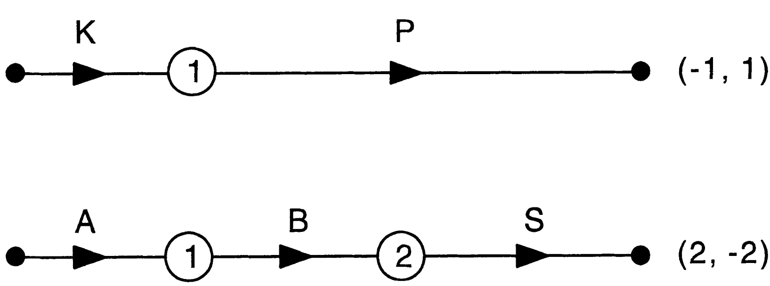

Figure 7.

ing in the bimatrix

$$\begin{array}{c} \\ (P,P) \\ (P,B) \\ (B,P) \\ (B,B) \end{array} \begin{array}{c} \begin{array}{cc} \quad P \quad & \quad S \quad \end{array} \\ \begin{bmatrix} (-1,1) & (-1,1) \\ (0,0) & (-1\frac{1}{2},1\frac{1}{2}) \\ (0,0) & (\frac{1}{2},-\frac{1}{2}) \\ (1,-1) & (0,0) \end{bmatrix} \end{array}$$

So $K_1((P,B),S) = -1\frac{1}{2}$, $K_2((P,B),S) = 1\frac{1}{2}$, etc.

Now we consider three economic situations which reduce almost immediately to a game in strategic form.

Example 2.2 (A. Cournot, 1838; *oligopoly model*). Suppose there are p producers of mineral water. If producer i brings an amount $x_i \in [0,\infty)$ on the market ($i \in \{1,\ldots,p\}$), then his costs are $c(x_i)$. The price of mineral water depends on the total amount $\sum_{i=1}^p x_i$ brought on the market and is denoted by $P(\sum_{i=1}^p x_i)$. This oligopoly situation can be modelled as a game in strategic form $\langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p\rangle$ where for each $i \in \{1,\ldots,p\}$:

$$X_i = [0,\infty) \text{ and } K_i(x_1,x_2,\ldots,x_p) = x_i P\left(\sum_{i=1}^{p} x_i\right) - c(x_i)$$

for all outcomes $x \in [0,\infty)^p$.

Example 2.3 *Auction procedures.*
Let $0 \leq r \leq w_p \leq w_{p-1} \leq \cdots \leq w_1$. A painting is auctioned and the seller wants at least the price r, while the worth of the painting for potential buyer $i \in P$ equals w_i. We consider two types of auctions.

(i) *sealed bid first price auction*: each bidder, independently of the others, proposes a price $x_i \in [r, \infty)$. Then the highest bidder obtains the painting for the highest price. If more players have the same highest bid then the highest bidder with the lowest index obtains the painting. This situation can be described as a p-person game in normal form where $X_i = [r, \infty)$ for each $i \in P$ and

$$K_i(x_1, x_2, \ldots, x_p) = \begin{cases} 0 & \text{if } i \neq i^*(x) \\ w_{i^*(x)} - x_{i^*(x)} & \text{if } i = i^*(x) \end{cases}$$

with $i^*(x) = \min_{i \in H(x)} i$, where $H(x) = \{i \in P | x_i = \max_{k \in P} x_k\}$ is the set of players with the highest bids for outcome x.

(ii) *sealed bid second price auction*: Here the highest bidder with the lowest index obtains the painting for the highest price mentioned by the other bidders. This is again a p-person game with $X_i = [r, \infty)$ but now

$$K_i(x_1, x_2, \ldots, x_p) = \begin{cases} 0 & \text{if } i \neq i^*(x) \\ w_{i^*(x)} - \max_{k \in P \setminus \{i^*(x)\}} x_k & \text{if } i = i^*(x) \end{cases}$$

Exercise 2.1 Show that in the sealed bid second price auction the strategy $x_i = w_i$ is a dominant strategy for player i. How is that for the sealed bid first price auction? (A strategy is called *dominant* if the strategy is at least as good as any other strategy, for all choices of the other players.)

Example 2.4 (*Clarke-Groves mechanism* for the financing of a public good). Let A be a finite set of alternatives and P the set of participants in the society, who have to decide which one of the alternatives will be chosen. For each player $i \in P$ let $w_i : A \rightarrow \mathbb{R}$

be the (worth) function where $w_i(a)$ is the worth of alternative a for player i.
Without loss of generality we suppose that $A = \{a_1, a_2, \ldots, a_k\}$. Suppose that the decision is made as follows.

(i) Each player has to tell what is the worth for him of the different alternatives but he may lie. So player i has to deliver an $x : A \to \mathbb{R}$.

(ii) Depending on the functions delivered by the players the alternative is chosen and it is determined what the players have to pay.

Clarke (1971) and Groves (1973) proposed the following. If $x = (x_1, x_2, \ldots, x_p)$ is the chosen p-tuple of functions, then

(i) $\alpha(x) \in A$ is the chosen outcome, where $\alpha(x)$ is the alternative with the lowest index having the property

$$\sum_{i \in P} x_i(\alpha(x)) = \max_{a \in A} \sum_{i \in P} x_i(a).$$

(ii) Player $i \in P$ has to pay a tax

$$t_i(x) = \max_{a \in A} \left(\sum_{j \in P \setminus \{i\}} x_j(a) \right) - \sum_{j \in P \setminus \{i\}} x_j(\alpha(x)).$$

This corresponds to the p-person game $\langle X_1, \ldots, X_p, K_1, \ldots, K_p \rangle$ with $X_i = \mathbb{R}^A$ for each $i \in P$ and $K_i(x) = w_i(\alpha(x)) - t_i(x)$.

In this game "telling the truth" is optimal as the following theorem shows.

Theorem 2.5 In the game $\langle \mathbb{R}^A, \ldots, \mathbb{R}^A, K_1, \ldots, K_p \rangle$ the strategy $\hat{x}_i = w_i$ is a dominant strategy for each $i \in P$.

Proof. Take $i \in P$ and let $x_1, x_2, \ldots, x_p$ be elements of $\mathbb{R}^A$. We have to prove that

$$K_i(x_1, \ldots, x_{i-1}, w_i, x_{i+1}, \ldots, x_p) \geq$$

$$K_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) \tag{2.1}$$

Note that

$$K_i(x_1, \ldots, x_p) = w_i(\hat{a}) + \sum_{j \neq i} x_j(\hat{a}) - \max_{a \in A} \sum_{j \neq i} x_j(a) \tag{2.2}$$

where $\hat{a}$ is the alternative with lowest index such that

$$\sum_{j=1}^{n} x_j(\hat{a}) = \max_{a \in A} \sum_{j=1}^{n} x_j(a).$$

Further,

$$\begin{aligned} &K_i(x_1, \ldots, x_{i-1}, w_i, x_{i+1}, \ldots, x_n) = w_i(a^*) \\ &+ \textstyle\sum_{j \neq i} x_j(a^*) - \max_{a \in A} \sum_{j \neq i} x_j(a) \end{aligned} \tag{2.3}$$

where a^* is the alternative with the lowest index for which

$$w_i(a^*) + \sum_{j \neq i} x_j(a^*) = \max_{a \in A}(w_i(a) + \sum_{j \neq i} x_j(a)). \tag{2.4}$$

From (2.2) and (2.3) we conclude that

$$\begin{aligned} &K_i(x_1, \ldots, x_{i-1}, w_i, x_{i+1}, \ldots, x_p) - K_i(x) \\ &= (w_i(a^*) + \sum_{j \neq i} x_j(a^*)) - (w_i(\hat{a}) + \sum_{j \neq i} x_j(\hat{a})). \end{aligned}$$

The right hand side of this equality is by (2.4) non-negative, which implies (2.1). $\square$

Exercise 2.2 Consider the situation where $A = \{a_1, a_2\}$, $P = \{1, 2, 3\}$ and $w_i(a_1) = 50$ for $i \in P$, $w_1(a_2) = 20$, $w_2(a_2) = 70$ and $w_3(a_2) = 75$. Let $\langle X_1, X_2, X_3, K_1, K_2, K_3\rangle$ be the Clarke-Groves game corresponding to this situation.

(i) Calculate $K_i(w_1, w_2, w_3)$ for each $i \in P$.

(ii) Show that $K_1(x, w_2, w_3) < K_1(w_1, w_2, w_3)$ if $x(a_1) = 100$, $x(a_2) = 20$.

We introduce some standard terminology. A game in strategic form $\langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p\rangle$ is called a *finite game* if $X_1, X_2, \ldots, X_p$ are finite sets and the game is called a *zero-sum game* if $\sum_{i \in P} K_i = 0$. A two-person game is often denoted by $\langle X, Y, K, L\rangle$. If $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$, then one can represent such a game as an $m \times n$-*bimatrix game*

$$\begin{bmatrix} (K(1,1), L(1,1)) & \ldots & (K(1,j), L(1,j)) & \ldots & (K(1,n), L(1,n)) \\ & & \vdots & & \\ (K(i,1), L(i,1)) & \ldots & (K(i,j), L(i,j)) & \ldots & (K(i,n), L(i,n)) \\ & & \vdots & & \\ (K(m,1), L(m,1)) & \ldots & (K(m,j), L(m,j)) & \ldots & (K(m,n), L(m,n)) \end{bmatrix}$$

where the rows (columns) correspond to the strategies of player 1 (2).

If moreover $L = -K$ then the game can be represented as a matrix game

$$[K(i,j)]_{i=1,j=1}^{m \quad n}$$

where the payoff matrix has m rows and n columns.

Let $x = (x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} X_i$ and $y \in X_i$. Denote by (x_{-i}, y) the element of $\prod_{i=1}^{p} X_i$, which we obtain from x by replacing the

i-th component x_i by y. With the aid of this notation we define the central concept in non-cooperative theory.

Definition 2.6 $\hat{x} = (\hat{x}_1, \cdots, \hat{x}_p) \in \prod_{i=1}^{p} X_i$ is a *Nash equilibrium* (or equilibrium point) of the game $\langle X_1, X_2, \cdots, X_p, K_1, \cdots, K_p \rangle$ if for all $i \in P$ and all $x_i \in X_i$ we have

$$K_i(\hat{x}_{-i}, x_i) \leq K_i(\hat{x}).$$

Hence, $\hat{x}$ is a Nash equilibrium if no player has an incentive to unilaterally deviate from $\hat{x}$ because his reward will not increase with such an action. If, by pre-play communication, the players have agreed (not binding) to play $\hat{x}$, then this agreement is self-enforcing.

Exercise 2.3 Let in $\langle X_1, X_2, \ldots, X_p, K_1, \ldots, K_p \rangle$ each player have a dominant strategy $\hat{x}_i$. Prove that $(\hat{x}_1, \ldots, \hat{x}_p)$ is a Nash equilibrium. Give an example of a game where no player has a dominant strategy and where the game has at least one Nash equilibrium.

Exercise 2.4 Show that the 4×2-bimatrix game in example 2.1 possesses no (pure) Nash equilibrium.

Exercise 2.5 Calculate a Nash equilibrium for the two-person game $\langle [0,1], [3,4], K, L \rangle$ where $K(x,y) = -L(x,y) = |x-y|$ for all $(x,y) \in [0,1] \times [3,4]$.

Exercise 2.6 Let $\langle X, Y, K_1, K_2 \rangle$ be a 2-person game. Let $L_1 = a_1 K_1 + b_1 \mathcal{J}$, $L_2 := a_2 K_2 + b_2 \mathcal{J}$, where $a_1, a_2 \in (0, \infty)$, $b_1, b_2 \in \mathbb{R}$ and $\mathcal{J} : X \times Y \to \mathbb{R}$ is the constant function on $X \times Y$ with value 1. Prove that $\langle X, Y, K_1, K_2 \rangle$ and $\langle X, Y, L_1, L_2 \rangle$ have the same set of Nash equilibria.

Chapter 3

Two-person zero-sum games

Zero-sum games have attractive properties which non-zero sum games in general do not have. As an example consider the next theorem.

Theorem 3.1 Let $\langle X, Y, K, -K \rangle$ be a zero-sum game and suppose that (x_1, y_1) and (x_2, y_2) are Nash equilibria of this game. Then

(i) (x_1, y_2) and (x_2, y_1) are also Nash equilibria [*Interchangeability property*]

(ii) $K(x_i, y_j) = K(x_1, y_1)$ for all $i, j \in \{1, 2\}$ [*Equal payoff property*].

Proof. Note that

$$\begin{aligned} K(x_1, y_1) &= \max_{x \in X} K(x, y_1) \geq K(x_2, y_1) \geq \\ & \min_{y \in Y} K(x_2, y) = K(x_2, y_2) \end{aligned} \tag{3.1}$$

where the first (last) equality follows from the fact that (x_1, y_1) $((x_2, y_2))$ is a Nash equilibrium.
By interchanging the roles of (x_1, y_1) and (x_2, y_2) we obtain also

$$\begin{aligned} K(x_2, y_2) &= \max\nolimits_{x\in X} K(x, y_2) \geq K(x_1, y_2) \geq \\ & \min\nolimits_{y\in Y} K(x_1, y) = K(x_1, y_1). \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we conclude that $K(x_1, y_1) = K(x_2, y_2)$ and, moreover, all inequality signs in these formulae are in fact equalities. This implies that

$$K(x_2, y_1) = \max_{x\in X} K(x, y_1), \quad K(x_2, y_1) = \min_{y\in Y} K(x_2, y),$$

hence (x_2, y_1) is a Nash equilibrium point and $K(x_2, y_1) = K(x_1, y_1)$. Similarly, (3.2) yields that (x_1, y_2) is a Nash equilibrium point with $K(x_1, y_2) = K(x_1, y_1)$. □

The theorem shows that for two-person zero-sum games we have:

(i) Each Nash equilibrium point gives the same payoffs to the players. For example in the matrix game

$$\begin{bmatrix} 1^* & 2 & 1^* \\ 0 & 1 & 0 \\ 1^* & 1 & 1^* \end{bmatrix}$$

there are four (pure) Nash equilibria corresponding to the *'s, with payoff 1 for player 1, and payoff -1 for player 2.
For the bimatrix game $\begin{bmatrix} (0,0) & (2,1) \\ (1,2) & (0,0) \end{bmatrix}$ we see that the payoffs are different for the two (pure) equilibria. This game is well-known under the name "battle of the sexes".

(ii) There is no communication problem in reaching a Nash equilibrium. Player 1 can play a strategy x_1 corresponding to a Nash equilibrium (x_1, y_1), and player 2 a strategy y_2 corresponding to a Nash equilibrium (x_2, y_2). As a result the Nash equilibrium (x_1, y_2) appears as an outcome. Battle of the sexes also shows that this is no longer true for non-zero games.

The above theorem suggests the following

Definition 3.2 Let $\langle X, Y, K, -K\rangle$ be a determined game that is a game with Nash equilibria. Then the payoff to player 1 in the Nash equilibrium outcomes is called the *value* of the game and is denoted by $v(X, Y, K, -K)$. A strategy $x \in X$ for which there is a $y \in Y$ such that (x, y) is a Nash equilibrium point is called an *optimal strategy for player 1*. Similarly, optimal strategies for player 2 are defined. The set of optimal strategies for player $i \in \{1, 2\}$ is denoted by $O_i(X, Y, K, -K)$.

Exercise 3.1 Let $\langle X, Y, K, -K\rangle$ be a determined zero sum game. Prove

(i) $O_1(X, Y, K, -K) \times O_2(X, Y, K, -K)$ is equal to the set of Nash equilibria of the game.

(ii) $\min_{y \in Y} \sup_{x \in X} K(x, y) = \max_{x \in X} \inf_{y \in Y} K(x, y) =$ $v(X, Y, K, -K)$.

The matrix games

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \quad \text{'matching pennies'}$$

$$\begin{array}{cc} & \begin{array}{ccc} St & P & Sc \end{array} \\ \begin{array}{c} St \\ P \\ Sc \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array} \quad \text{'stone, paper, scissors'}$$

have no equilibrium point if one restricts oneself to pure strategies. For such games the notions of lower and upper value are interesting.

Definition 3.3 Let $\langle X, Y, K, -K \rangle$ be a zero sum game. Then

$$\underline{v}(X, Y, K, -K) := \sup_{x \in X} \inf_{y \in Y} K(x, y)$$

is called the *lower value* of the game and

$$\bar{v}(X, Y, K, -K) := \inf_{y \in Y} \sup_{x \in X} K(x, y)$$

the *upper value.*

Exercise 3.2

(i) Show that for each zero-sum game $\langle X, Y, K, -K \rangle$ it holds that $\underline{v}(X, Y, K, -K) \le \bar{v}(X, Y, K, -K)$.

(ii) Show that for 'matching pennies' as well as for 'stone, paper, scissors' the lower value is equal to -1 and the upper value is equal to 1.

(iii) Suppose player 2 has a spy, guaranteeing him the knowledge of the strategy choice of player 1 before he has to choose. Prove that in this case the (new) game is always weakly determined (i.e. the lower value is equal to the upper value which is then also called the value). What is the value in this case?

The concept of mixed strategy was considered already in Borel (1921) and von Neumann (1928) for matrix games. In playing T with probability $\frac{1}{2}$ in matching pennies player 1 can guarantee himself a payoff 0 in expectation. Also player 2 has a strategy (L with probability $\frac{1}{2}$) to guarantee that he has not to pay more than 0 in expectation.

For a matrix game $A = [a_{ij}]_{i=1,}^{m}{}_{j=1}^{n}$ the set of mixed strategies for player 1 is given by

$$\Delta^m := \{p \in \mathbb{R}^m \mid p \geq 0, \sum_{i=1}^{m} p_i = 1\}.$$

An element $p \in \Delta^m$ corresponds to the strategy "play for each $i \in \{1, \ldots, m\}$ row i with probability p_i".
In a $3 \times n$-matrix game the strategy $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ can for example be realized by

(i) drawing a card from a 52 deck and playing row 1 if the card is black, row 2 if the card is hearts, row 3 is the card is diamond,

(ii) looking at a watch and playing row 1 (2, 3) if the second hand is between 12 and 6 (6 and 9, 9 and 12).

Definition 3.4 Let A be an $m \times n$-matrix game. Then the mixed extension of A is the infinite game $\langle \Delta^m, \Delta^n, K, L \rangle$ with

$$\begin{aligned} &K(p,q) := \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = p^\top A q, \\ &L(p,q) = -K(p,q). \end{aligned}$$

Note that $K(p,q)$ is the expected payoff for player 1, if player 1 and player 2 choose independently the mixed strategies p and q, respectively.

For a finite matrix game $A = [a_{ij}]_{i=1,}^{m}{}_{j=1}^{n}$, let us denote the lower value (upper value) of the mixed extension by $\underline{v}(A)$ $(\bar{v}(A))$.

Lemma 3.5

(i) $\underline{v}(A) = \max_{p\in\Delta^m} \min_{j\in\{1,2,\ldots,n\}} p^\top Ae_j$,

(ii) $\bar{v}(A) = \min_{q\in\Delta^n} \max_{i\in\{1,2,\ldots,m\}} e_i^\top Aq$.

(Here e_j is the j-th standard basis vector in $\mathbb{R}^n$.)

Proof. We only show (i). First we note that

$$\inf_{q\in\Delta^n} p^\top Aq = \min_{j\in\{1,2,\ldots,n\}} p^\top Ae_j \text{ for each } p \in \Delta^m.$$

Obviously, the left term is not larger than the right term. Conversely, for each $p \in \Delta^m$

$$p^\top Aq = \sum_{j=1}^n q_j p^\top Ae_j \geq (\min_{j\in\{1,2,\ldots,n\}} p^\top Ae_j) \sum_{j=1}^n q_j =$$
$$= \min_{j\in\{1,2,\ldots,n\}} p^\top Ae_j.$$

So $\inf_{q\in\Delta^n} p^\top Aq = \min_{j\in\{1,2,\ldots,n\}} p^\top Ae_j$.
Note now that $f : \Delta^m \to \mathbb{R}$, defined by $f(p) = \min_j p^\top Ae_j$, is a continuous function on the closed and bounded set Δ^m. Hence, $\max_{p\in\Delta^m} f(p)$ exists.

Combining, we obtain

$$\underline{v}(A) := \sup_{p\in\Delta^m} \inf_{q\in\Delta^n} p^\top Aq = \sup_{p\in\Delta^m} \min_j p^\top Ae_j$$
$$= \max_{p\in\Delta^m} \min_j p^\top Ae_j.$$

□

The next theorem (known as the minimax theorem of J. von Neumann) tells us that matrix games possess a value and optimal strategies for both players. The given proof is due to G. Owen. Another proof will be given in chapter 4. Also, proofs using the duality theorem of linear programming or the lemma of Farkas are well-known.

Theorem 3.6 For each matrix game $A = [a_{ij}]_{i=1,j=1}^{m\ \ n}$ we have $\underline{v}(A) = \bar{v}(A)$.

Proof. Define the gap of A by $g(A) := \bar{v}(A) - \underline{v}(A)$ and the size of A by $s(A) := m + n$. So $s(A) \geq 2$. By exercise 3.2: $g(A) \geq 0$. We prove the equality $\underline{v}(A) = \bar{v}(A)$ or $g(A) = 0$ by induction to the size of A. For $s(A) = 2$, A is of the form $[a]$, so $g(A) = \bar{v}(A) - \underline{v}(A) = a - a = 0$.
Suppose $g(B) = 0$ for all B with $s(B) < r$. Take $A \in \mathbb{R}^{m\times n}$ with $m + n = r$. Using lemma 3.5 we can find a $\hat{p} \in \Delta^m$ and $\hat{q} \in \Delta^n$ such that

$$\hat{p}^\top Ae_j \geq \underline{v}(A) \text{ for all } j \in \{1, \ldots, n\} \tag{3.3}$$

$$e_i^\top A\hat{q} \leq \bar{v}(A) \text{ for all } i \in \{1, \ldots, m\}. \tag{3.4}$$

We consider 3 cases
Case 1. In (3.3) and (3.4) we have only equalities. Then $\hat{p}^\top A\hat{q} = \underline{v}(A) = \bar{v}(A)$. So $g(A) = 0$.
Case 2. There is a k with $\hat{p}^\top Ae_k > \underline{v}(A)$. Then $n > 1$.
Case 3. There is an ℓ with $e_\ell^\top A\hat{q} < \bar{v}(A)$. Then $m > 1$.
Case 3 is similar to case 2. So we prove only for case 2 that $g(A) = 0$. Let A^{-k} be the matrix which we obtain from A by deleting the k-th column. We know that

$$g(A^{-k}) = 0. \tag{3.5}$$

Further, it is easy to show (exercise 3.3) that

$$\bar{v}(A^{-k}) \geq \bar{v}(A), \underline{v}(A^{-k}) \geq \underline{v}(A) \tag{3.6}$$

It is sufficient to show that $\underline{v}(A^{-k}) = \underline{v}(A)$. Because, then $0 = g(A^{-k}) \geq g(A) \geq 0$, which implies that $g(A) = 0$.
Suppose for a moment that $\underline{v}(A^{-k}) > \underline{v}(A)$.
Then there is $\bar{p} \in \Delta^m$ which is maximin for A^{-k}, so

$$\bar{p}^\top Ae_j > \underline{v}(A) \text{ for all } j \in \{1, \ldots, n\} \backslash \{k\}.$$

For each $\varepsilon \in (0, 1)$ we then have

$$(\varepsilon\bar{p} + (1-\varepsilon)\hat{p})^\top Ae_j > \underline{v}(A) \text{ for all } j \neq k.$$

For ε sufficiently near to 0 we have

$$(\varepsilon\bar{p} + (1-\varepsilon)\hat{p})^\top Ae_k > \underline{v}(A).$$

But then $\underline{v}(A) \geq \min_{j=1,\ldots,n}(\varepsilon\bar{p} + (1-\varepsilon)\hat{p})^\top Ae_j > \underline{v}(A)$ a contradiction. Hence $\underline{v}(A^{-k}) = \underline{v}(A)$. □

Exercise 3.3 Show (3.6) in the proof of (the minimax) theorem 3.6.

It is possible to calculate the value and optimal strategies of a matrix game by solving a dual pair of linear programs. We will not discuss this here (see chapter 6), but we will conclude this chapter by describing a graphical method to solve $2 \times n$-matrix games and $m \times 2$-matrix games.
Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$. Define on $[0, 1]$ the (gain) functions $g_1, g_2, \ldots, g_n$ and the (lower hull) function L_1 by $g_j(x) = (1-x, x)Ae_j$ and $L_1(x) = \min_{j\in\{1,\ldots,n\}} g_j(x)$.
Then one easily shows that

(i) $\max L_1(x) = v(A)$, the value of the (mixed extension of the) matrix game A.

(ii) $x \in \arg\max L_1 \Leftrightarrow (1-x, x) \in O_1(A)$, where $O_1(A)$ is the set of optimal strategies for player 1 in A.

(iii) $(q_1, q_2, \ldots, q_n) \in \Delta^n$ is an element of the optimal strategy set $O_2(A)$ for player 2 iff the graph of $\sum_{j=1}^{n} q_j g_j$ lies in $[0,1] \times (-\infty, v(A)]$.

For solving an $m \times 2$-matrix game $[a_{ij}]_{i=1}^{m}{}_{,j=1}^{n}$ one considers the (loss) functions $l_1, l_2, \ldots, l_m$ and the (upper hull) function U, defined by $l_i(y) = e_i^\top A \begin{bmatrix} 1-y \\ y \end{bmatrix}$ $(i = 1, \ldots, m)$ and $U(y) = \max_{i \in \{1,\ldots,m\}} l_i(y)$ for all $y \in [0,1]$. Then $v(A) = \min U$ and $\arg\min U$ corresponds to optimal strategies of player 2, etc. For illustration we give two examples.

Example 3.7 Let A be the 2×3-matrix $\begin{bmatrix} 5 & 2 & 6 \\ 3 & 6 & 7 \end{bmatrix}$. Then for each $x \in [0,1]$, $g_1(x) = 5(1-x)+3x$, $g_2(x) = 2(1-x)+6x$, $g_3(x) = 6(1-x)+7x$. In figure 8 the graph of L_1 is indicated with $----$. The graph $\frac{2}{3}g_1 + \frac{1}{3}g_2 + 0g_3$, corresponding to the unique optimal strategy $(\frac{2}{3}, \frac{1}{3}, 0) \in O_2(A)$ is indicated with $-\cdot-\cdot-\cdot-$.
Further, we deduce from figure 8 that $v(A) = 4$ and $O_1(A) = \{(\frac{1}{2}, \frac{1}{2})\}$.

Example 3.8 In example 2.1 we considered a simple poker game, which resulted in the 4×2-matrix game $\begin{bmatrix} -1 & -1 \\ 0 & -1\frac{1}{2} \\ 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$. We solve

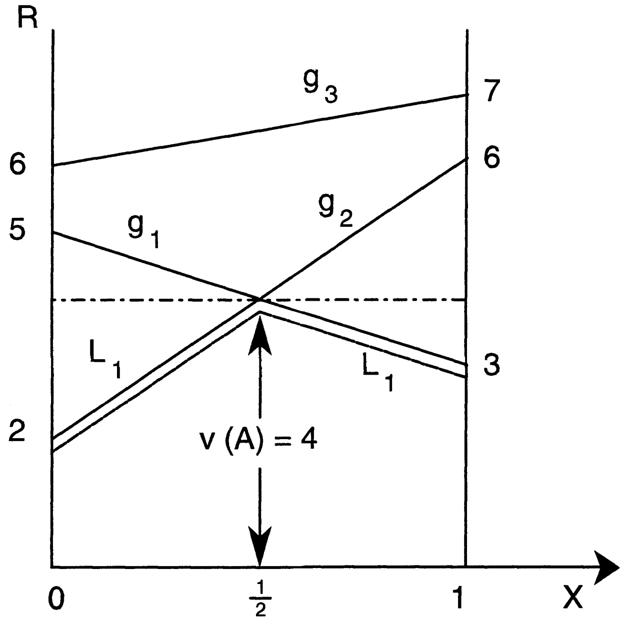

Figure 8. A 2 × 3-game

this game in a graphical way in figure 9. We see that $v(A) = \frac{1}{3}$, $O_1(A) = \{(0, 0, \frac{2}{3}, \frac{1}{3})\}$ and $O_2(A) = \{(\frac{1}{3}, \frac{2}{3})\}$.

So, in the poker game, for player 2 it is optimal to pass with probability $\frac{1}{3}$ and to see with probability $\frac{2}{3}$. For player 1 it is optimal to use the (bluff) strategy (B, B) with probability $\frac{1}{3}$.

Exercise 3.4 Find the value and the optimal strategy spaces for the mixed extension of the matrix game A, where

(i) $A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 7 & 2 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 2 & 2 \end{bmatrix}$

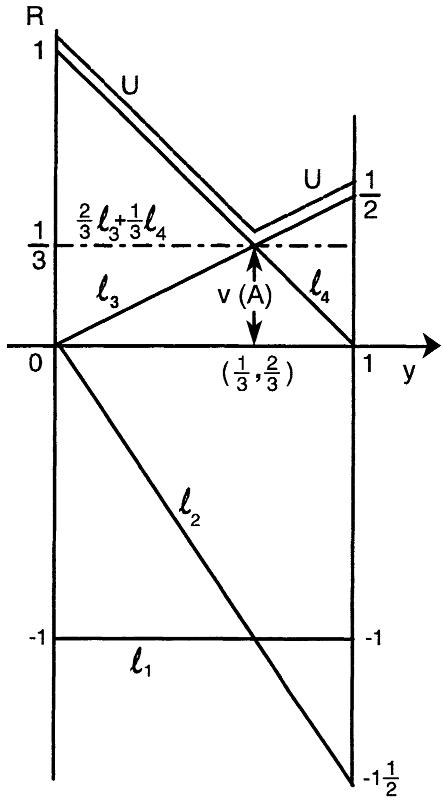

Figure 9. Simple poker

(iii) $A = \begin{bmatrix} -1 & -3 & -2 \\ -3 & -1 & -2 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 7 & 2 \\ -1 & -1 & -1 & -1 \end{bmatrix}$

Exercise 3.5 Let $A = [a_{ij}]_{i=1,j=1}^{2,\ 2}$ be a 2×2-matrix game without any pure Nash equilibrium. Prove that

$$v(A) = t^{-1} \det A, O_1(A) = \{(t^{-1}(a_{22} - a_{21}), t^{-1}(a_{11} - a_{12}))\},$$
$$O_2(A) = \{(t^{-1}(a_{22} - a_{12}), t^{-1}(a_{11} - a_{21}))\},$$

where $t := a_{11} + a_{22} - a_{12} - a_{21} \neq 0$.

Exercise 3.6 Let $A = [a_{ij}]_{i=1,j=1}^{m,\ n}$ be a matrix game with $a_{ij} \in \{-1, 1\}$ for all i, j, and suppose that A possesses a pure Nash equilibrium.
Prove that $O_1(A) = \Delta^m$ or $O_2(A) = \Delta^n$.

Exercise 3.7 Prove that $O_1(A)$ is a convex set for each $m \times n$-matrix A.
[Hint: $O_1(A) = \{p \in \Delta^m | p^\top A e_j \geq v(A)$ for all $j \in \{1, \ldots, n\}\}$.]

Exercise 3.8 (Hide and seek games.) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Calculate $v(A), O_1(A)$ and $O_2(A)$ if

$$A = \begin{bmatrix} a_1 & & & \\ & a_2 & \emptyset & \\ & \emptyset & \ddots & \\ & & & a_n \end{bmatrix}.$$

Chapter 4

Mixed extensions of bimatrix games

The *mixed extension of an $m \times n$-bimatrix game* (A, B) is given by $(\Delta^m, \Delta^n, K, L)$ where

$$K(p,q) = p^\top Aq, L(p,q) = p^\top Bq \text{ for all } p \in \Delta^m, q \in \Delta^n.$$

A fundamental result of J. Nash (1950, 1951) is

Theorem 4.1 (Equilibrium point theorem of J. Nash). The mixed extension of each finite bimatrix game possesses a Nash equilibrium.

A proof of this theorem will be given using

Theorem 4.2 (Fixed point theorem of L. Brouwer).
Let C be a non-empty compact convex set of $\mathbb{R}^k$. Let $f : C \to C$ be a continuous function. Then there is a $\hat{x} \in C$ with $f(\hat{x}) = \hat{x}$.

Proof of theorem 4.1. Let (A, B) be an $m \times n$-bimatrix game. For each $i \in \{1, 2, \ldots, m\}$ let $s_i : \Delta^m \times \Delta^n \to \mathbb{R}$ be the continuous (regret) function defined by

$$s_i(p, q) = \max\{0, e_i^\top Aq - p^\top Aq\}.$$

For each $j \in \{1, 2, \ldots, n\}$ let $t_j : \Delta^m \times \Delta^n \to \mathbb{R}$ be the continuous (regret) function defined by

$$t_j(p, q) = \max\{0, p^\top Be_j - p^\top Bq\}.$$

With the aid of the continuous maps $s = (s_1, s_2, \ldots, s_m) : \Delta^m \times \Delta^n \to \mathbb{R}^m$ and $t = (t_1, t_2, \ldots, t_n) : \Delta^m \times \Delta^n \to \mathbb{R}^n$ we define the continuous function $f : \Delta^m \times \Delta^n \to \Delta^m \times \Delta^n$ by

$$f(p, q) = \left(\frac{p + s(p, q)}{1 + \sum_{i=1}^m s_i(p, q)}, \frac{q + t(p, q)}{1 + \sum_{j=1}^n t_j(p, q)} \right).$$

Since $\Delta^m \times \Delta^n$ is a compact subset of $\mathbb{R}^m \times \mathbb{R}^n$ (which can be identified with $\mathbb{R}^k$, where $k = m + n$), the fixed point theorem of Brouwer implies that there is a $(\hat{p}, \hat{q}) \in \Delta^m \times \Delta^n$ with $f(\hat{p}, \hat{q}) = (\hat{p}, \hat{q})$. We will show that $(\hat{p}, \hat{q})$ is a Nash equilibrium for the mixed extension of (A, B). From $\hat{p} \in \Delta^m$ and $\hat{p}^\top A\hat{q} = \sum_{i=1}^m \hat{p}_i(e_i^\top A\hat{q})$ it follows that there is a $k \in \{1, 2, \ldots, m\}$ with $\hat{p}_k > 0$ and $e_k^\top A\hat{q} \leq \hat{p}^\top A\hat{q}$. But then $s_k(\hat{p}, \hat{q}) = 0$. Now

$$\hat{p}_k > 0, \quad \hat{p}_k = \frac{\hat{p}_k + s_k(\hat{p}, \hat{q})}{1 + \sum_{i=1}^m s_i(\hat{p}, \hat{q})} = \frac{\hat{p}_k}{1 + \sum_{i=1}^m s_i(\hat{p}, \hat{q})}$$

imply: $\sum_{i=1}^m s_i(\hat{p}, \hat{q}) = 0$. So $s_i(\hat{p}, \hat{q}) = 0$ for each $i \in \{1, 2, \ldots, m\}$. Hence, $e_i^\top A\hat{q} \leq \hat{p}^\top A\hat{q}$ for all $i \in \{1, 2, \ldots, m\}$. This implies

(i) $p^\top A\hat{q} \leq \hat{p}^\top A\hat{q}$ for all $p \in \Delta^m$.

Similarly, one can prove

(ii) $\hat{p}^\top Bq \leq \hat{p}^\top B\hat{q}$ for all $q \in \Delta^n$.

From (i) and (ii) we conclude that the fixed point $(\hat{p}, \hat{q})$ of f is a Nash equilibrium of the mixed extension of the bimatrix game. □

With the aid of theorem 4.1 one can also give another proof of the minimax theorem, using exercise 3.1.
In the following we denote the set of Nash equilibria of the mixed extension of (A, B) by $NE(A, B)$.
Let (A, B) be an $m \times n$-bimatrix game, $p \in \Delta^m, q \in \Delta^n$. Then

$$C(p) := \{i \in \{1, \ldots, m\} | p_i > 0\}$$

is called the *carrier* of p and

$$PB_1(q) := \{i \in \{1, \ldots, m\} | e_i^\top Aq = \max_{r \in \{1,\ldots,m\}} e_r^\top Aq\}$$

the set of *pure best answers of player* 1 to q. Further,

$$B_1(q) := \{p \in \Delta^m | p^\top Aq = \max_{r \in \{1,\ldots,m\}} e_r^\top Aq\}$$

is the set of *best answers of player* 1 to q in the game (A, B). Similarly, one defines $C(q), PB_2(p), B_2(p)$. Note that $B_1(q) = \text{conv}(PB_1(q))$ if we identify the pure strategy i with e_i. [Here $\text{conv}(X)$ means the convex hull of X.]
A characterization of Nash equilibria in terms of these sets is given in

Theorem 4.3 Let (A, B) be an $m \times n$-bimatrix game, $\hat{p} \in \Delta^m, \hat{q} \in \Delta^n$. Then $(\hat{p}, \hat{q})$ is a Nash equilibrium of the mixed extension of (A, B) if and only if $C(\hat{p}) \subset PB_1(\hat{q})$ and $C(\hat{q}) \subset PB_2(\hat{p})$.

Proof. $(\hat{p},\hat{q}) \in NE(A,B) \Leftrightarrow \hat{p}^\top A\hat{q} = \max_{p\in\Delta^m} p^\top A\hat{q}, \hat{p}^\top B\hat{q} = \max_{q\in\Delta^n} \hat{p}^\top Bq \Leftrightarrow \hat{p}^\top A\hat{q} = \max_{i\in\{1,\dots,m\}} e_i^\top A\hat{q}, \hat{p}^\top B\hat{q} = \max_{j\in\{1,2,\dots,n\}} \hat{p}^\top Be_j \Leftrightarrow C(\hat{p}) \subset PB_1(\hat{q}), C(\hat{q}) \subset PB_2(\hat{p})$. □

If we write $C(p,q) = C(p) \times C(q), PB(p,q) = PB_1(q) \times PB_2(p)$, then theorem 4.3 summarizes to

$$(p,q) \in NE(A,B) \Leftrightarrow C(p,q) \subset PB(p,q)$$

where $NE(A,B)$ denotes the set of Nash equilibria for the mixed extension of (A,B).

Now we calculate for four games Nash equilibria in order to get an idea of what is possible. We use the following "geometric" observation.

Let $\Gamma = \langle X, Y, K, L\rangle$ be a two-person game and let $NE(\Gamma)$ be the set of Nash equilibria of Γ. Let

$$B_1 : Y \rightarrow\!\!\rightarrow X, B_2 : X \rightarrow\!\!\rightarrow Y$$

be the best response multifunctions for the players 1 and 2 in Γ. Let

$$G_1^* := \{(x,y) \in X \times Y : x \in B_1(y)\}$$
$$G_2 := \{(x,y) \in X \times Y : y \in B_2(x)\}.$$

Note that G_2 is the graph of the multifunction B_2 and G_1^* is the image of the graph $G_1 \subset Y \times X$ of B_1 under the map $Y \times X \to X \times Y$ with $(y,x) \mapsto (x,y)$ for all $(y,x) \in Y \times X$. Then we have:

$(\hat{x},\hat{y}) \in NE(\Gamma) \Leftrightarrow \hat{x} \in B_1(\hat{y}), \hat{y} \in B_2(\hat{x}) \Leftrightarrow (\hat{x},\hat{y}) \in G_1^*, (\hat{x},\hat{y}) \in G_2 \Leftrightarrow (\hat{x},\hat{y}) \in G_1^* \cap G_2$.

Example 4.4 The mixed extension of the bimatrix game

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{cc} L & R \\ \left[\begin{array}{cc} (1,1) & (1,1) \\ (1,1) & (0,0) \end{array}\right] \end{array}$$

can be identified with the game $\langle [0,1], [0,1], K, L \rangle$ where $x \in [0,1]$ denotes the strategy "play with probability x row T and with probability $1-x$ row B" and $y \in [0,1]$ the strategy "play with probability y column L". Further $K(x,y) = xy + (1-x)y + x(1-y) = x + y - xy, L(x,y) = K(x,y)$ for all $x, y \in [0,1]$.
For this game, in the next figure 10, G_1^* and G_2 are indicated with – – – – – –– and – – · – – – · –– respectively. Note that

$$G_1^* \cap G_2 = G_1^* = G_2 = \{(x,y) \in [0,1]^2 | x = 1 \text{ or } y = 1\}.$$

Hence, $NE(A,B) = \{(p,q) \in \Delta_2 \times \Delta_2 | p = e_1 \text{ or } q = e_1\}$.

The payoffs for players 1 and 2 in all equilibria are 1.
The set of Nash equilibria is not convex but consists of two convex line segments.

Example 4.5 For the bimatrix game $\left[\begin{array}{cc} (1,1) & (0,0) \\ (0,0) & (0,0) \end{array}\right]$ the (G_1^*, G_2)-picture is given in figure 11. $G_1^* \cap G_2$ consists of the points (0,0) and (1,1), which implies that $NE(A,B) = \{(e_2, e_2), (e_1, e_1)\}$. The payoffs in the second equilibrium point are for both players more attractive than in the first equilibrium point.

Example 4.6 (Chicken). For the game

$$\begin{array}{c} \\ \text{tough} \\ \text{soft} \end{array} \begin{array}{cc} \text{tough} & \text{soft} \\ \left[\begin{array}{cc} (-900,-900) & (200,0) \\ (0,200) & (100,100) \end{array}\right] \end{array}$$

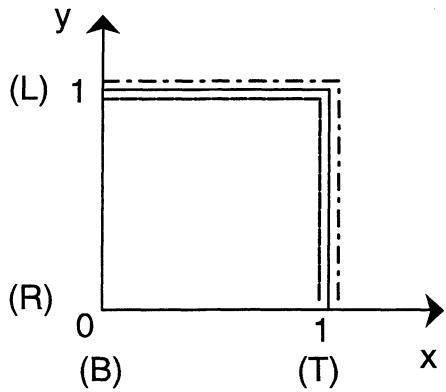

Figure 10.

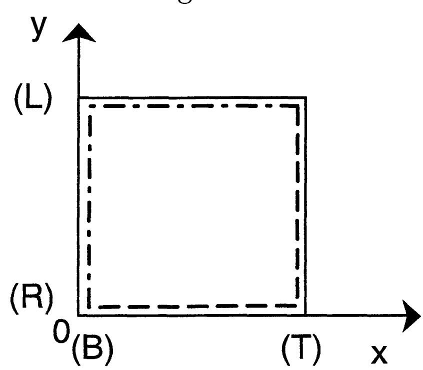

Figure 11. $- - . - - . - - G_2, - - - - - - - G_1^*$

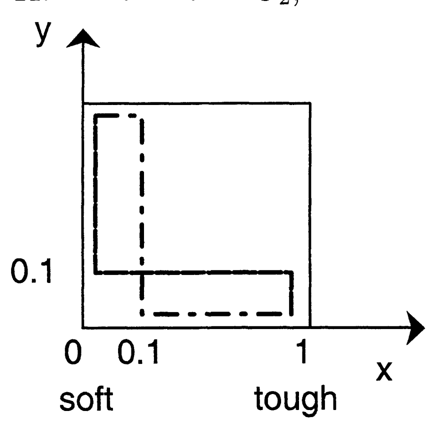

Figure 12. Chicken

the $(G_1^* \cap G_2)$-picture is given in figure 12.

This implies that $(e_1, e_2), (e_2, e_1)$ and $(\hat{p}, \hat{p})$, with $\hat{p} = (0.1, 0.9)^\top$, are the three equilibria. In the symmetric equilibrium point the payoff is 90 for both players.

Example 4.7 For the game (A, B) with $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ we obtain the picture in figure 13.

The prism P is the Cartesian product of the strategy spaces. In P is indicated G_2. G_1^* equals P, so $NE(A, B) = G_2$ consists of three convex pieces:

(i) the line segment with end points $((0, 1), e_1)$ and $((\frac{1}{2}, \frac{1}{2}), e_1)$,

(ii) the triangle with vertices $((\frac{1}{2}, \frac{1}{2}), e_1), ((\frac{1}{2}, \frac{1}{2}), e_2), ((\frac{1}{2}, \frac{1}{2}), e_3)$,

(iii) the quadrangle with vertices $((\frac{1}{2}, \frac{1}{2}), e_2), ((\frac{1}{2}, \frac{1}{2}), e_3), ((1, 0), e_2)$ and $((1, 0), e_3)$.

Exercise 4.1 Let (A, B) be a 2×2-bimatrix game and suppose that $(\hat{p}, \hat{q})$ is the unique Nash equilibrium with $0 < \hat{p}_1 < 1$, $0 < \hat{q}_1 < 1$. Prove that

$$\begin{aligned} \hat{p}_1 &= (b_{11} + b_{22} - b_{21} - b_{12})^{-1}(b_{22} - b_{21}), \\ \hat{q}_1 &= (a_{11} + a_{22} - a_{21} - a_{12})^{-1}(a_{22} - a_{12}) \end{aligned}$$

Exercise 4.2

(i) Calculate the value and optimal strategies for the matrix game $\begin{bmatrix} 1 & 8 \\ 5 & 3 \end{bmatrix}$.

(ii) Calculate the Nash equilibria of (A, B) with

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

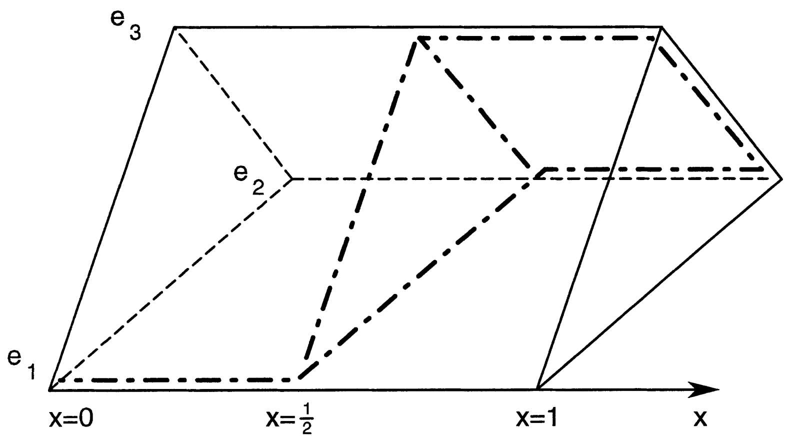

Figure 13. The strategy space of player 1 is identified with [0,1] on the x-axis, the strategy space of player 2 with the triangle, which is the convex hull of e^1, e^2, e^3 in $\mathbb{R}^3$.

Exercise 4.3

(i) Let (A, B) be an $m \times (n+1)$-bimatrix game. Suppose there is a $y \in \text{conv}\{Be_1, Be_2, \dots, Be_n\}$ such that $y > Be_{n+1}$ (i.e. each coordinate of y is larger than the corresponding coordinate of Be_{n+1}). Let (A', B') be the $m \times n$-bimatrix game, obtained from (A, B) by deleting the last column of A and the last column of B.
Prove that $q_{n+1} = 0$ for each $(p, q) \in NE(A, B)$ and that

$$(p, q) \in NE(A', B') \Leftrightarrow (p, (q_1, \dots, q_n, 0)^\top) \in NE(A, B).$$

(ii) Calculate $NE(A, B)$ if

$$(A, B) = \begin{bmatrix} (1,3) & (0,0) & (2,-1) \\ (0,0) & (4,2) & (0,-2) \\ (0,1) & (0,1) & (0,0) \end{bmatrix}$$

[Hint: 'reduce' the game to a 2×2-game.]

Exercise 4.4

Let $a \in (1, \infty)$ and $A_a = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$, $B_a = \begin{bmatrix} 0 & 2-a \\ 1 & 0 \end{bmatrix}$. Calculate $NE(A_a, B_a)$ for each $a \in (1, \infty)$.

Chapter 5

The Nash equilibria of a 3-person game

We consider 3-person games, where each player has a finite number of pure actions: players 1, 2 and 3 have respectively m, n and q pure actions. The payoffs can be described by three 3-dimensional matrices

$$[a_{ijk}]_{i=1}^{m}{}_{,j=1}^{n}{}_{,k=1}^{q}, [b_{ijk}]_{i=1}^{m}{}_{,j=1}^{n}{}_{,k=1}^{q}, [c_{ijk}]_{i=1}^{m}{}_{,j=1}^{n}{}_{,k=1}^{q}.$$

Another way: describe each k-layer separately, giving q trimatrices

$$[a_{ij1}, b_{ij1}, c_{ij1}]_{i=1}^{m}{}_{,j=1}^{n}, [a_{ij2}, b_{ij2}, c_{ij2}]_{i=1}^{m}{}_{,j=1}^{n}, \ldots,$$
$$[a_{ijq}, b_{ijq}, c_{ijq}]_{i=1}^{m}{}_{,j=1}^{n}.$$

This can be described as: player 3 chooses a trimatrix, player 1 a row and player 2 a column. Then the corresponding payoffs follow. As an example consider the 3-person game with two pure actions for each player and payoffs given by the trimatrices

$$\begin{array}{ccc} & j=1 & j=2 \\ \begin{array}{c} i=1 \\ i=2 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} (0,0,0) & (0,2,0) \\ (2,0,0) & (0,0,1) \end{bmatrix}} \\ & \multicolumn{2}{c}{k=1} \end{array}$$

$$\begin{array}{ccc} & j=1 & j=2 \\ \begin{array}{c} i=1 \\ i=2 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} (0,0,2) & (1,0,0) \\ (0,1,0) & (0,0,0) \end{bmatrix}} \\ & \multicolumn{2}{c}{k=2} \end{array}$$

This game corresponds to the situation where three players are involved and each player can take one or two coins in his hand. If all players take the same number of coins, then a payoff zero is the result for each player. If one player has a different number of coins from the other players he obtains a payoff equal to the number of coins in his hand and the other players obtain nothing. We want to calculate the Nash equilibria of the mixed extension of this game. Each strategy space can be identified with [0,1], where $x \in [0,1]$ means "take with probability x one coin and with probability $1-x$ two coins". Again with the aid of graphs of best response multifunctions the Nash equilibrium set can be discovered. The payoff functions $K_i : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ for player i are given by

$$
\begin{aligned}
K_1(x,y,z) &= x(1-y)(1-z)+2(1-x)yz \\
&= (2-(1+y)(1+z))x+2yz \\
K_2(x,y,z) &= y(1-x)(1-z)+2(1-y)xz \\
&= (2-(1+x)(1+z))y+2xz \\
K_3(x,y,z) &= z(1-x)(1-y)+2(1-z)xy \\
&= (2-(1+x)(1+y))z+2xy.
\end{aligned}
$$

From this last formula we see that

$$
B_3(x,y) = \begin{cases} \{1\} & (1+x)(1+y) < 2 \quad \text{(one coin)} \\ [0,1] & (1+x)(1+y) = 2 \quad \text{(indifferent)} \\ \{0\} & (1+x)(1+y) > 2 \quad \text{(two coins)} \end{cases}
$$

The graph of the best response multifunction B_3 for player 3 is the set $\{(x,y,z) \in [0,1]^3 | z \in B_3(x,y)\}$.
This graph consists of three pieces

(i) a piece in the xy-plane bounded by the hyperbola $(1+x)(1+y)=2$ and the lines $y=1, x=1$.

(ii) a piece in the plane $z=1$.

(iii) a cylindric piece.

Similar in structure are the graphs

$\{(x,y,z) \in [0,1]^3 | x \in B_1(y,z)\}$ and $\{(x,y,z) \in [0,1]^3 | y \in B_2(x,z)\}$

of the best response multifunctions B_1 and B_2, respectively.

The intersection of these graphs consists of one interior point $(\sqrt{2}-1, \sqrt{2}-1, \sqrt{2}-1)$ and six line segments $[(0,0,1),(1,0,1)]$, $[(0,1,0),(1,1,0)]$, $[(0,0,1),(0,1,1)]$, $[(1,0,0),(1,1,0)]$, $[(1,0,0),$ $(1,0,1)]$, $[(0,1,0),(0,1,1)]$. This intersection coincides with the set of Nash equilibria. On each of these line segments one of the players is indifferent because he always obtains a payoff zero. The payoff for each player in the symmetric Nash equilibrium equals $6-4\sqrt{2}$.

Exercise 5.1 Let $\Gamma = \langle X, Y, Z, K_1, K_2, K_3 \rangle$ be a 3-person strategic game with $X = Y = Z = \{1,2,3,4\}$

(i) If, for each $i \in \{1,2,3\}$, $K_i(x,y,z) = x+y+z+4i$, then Γ has a unique Nash equilibrium. Prove this.

(ii) Let, for each $i \in \{1,2,3\}$, $K_i(x,y,z) = 10$ if $x=y=z$ and $K_i(x,y,z) = 0$ otherwise. Describe all pure Nash equilibria and show that mixed Nash equilibria lead to smaller payoffs than pure Nash equilibria.

Exercise 5.2

(i) Find two pure Nash equilibria of the $2 \times 2 \times 2$-game given by

$$\begin{array}{cc} & \begin{array}{cc} j=1 & j=2 \end{array} \\ \begin{array}{c} i=1 \\ i=2 \end{array} & \begin{bmatrix} (4,3,2) & (3,4,4) \\ (3,4,3) & (4,3,5) \end{bmatrix} \\ & k=1 \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} j=1 & j=2 \end{array} \\ \begin{array}{c} i=1 \\ i=2 \end{array} & \begin{bmatrix} (2,1,6) & (0,0,8) \\ (0,0,7) & (1,2,9) \end{bmatrix} \\ & k=2 \end{array}$$

[Hint: note that $k = 2$ is a dominant strategy for player 3.]

(ii) Find Nash equilibria for the coordination game $\langle \{0,1\}, \{0,1\}, \{0,1\}, K, K, K \rangle$, where $K(i,i) = 8 + i$ for $i \in \{0,1\}$ and $K(i,j) = 0$ if $i \neq j$.

Chapter 6

Linear programming and matrix games

Let A be an $m \times n$-matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$. Then the corresponding primal and dual program are given by

$$\inf\{x^\top c | x \in P\}, \text{ where } P = \{x \in \mathbb{R}^m | x \geq 0, x^\top A \geq b^\top\},$$

$$\sup\{b^\top y | y \in D\}, \text{ where } D = \{y \in \mathbb{R}^n | y \geq 0, Ay \leq c\}.$$

We recall some of the properties for the dual pair of programs:

P.1. For all $x \in P$ and $y \in D : x^\top c \geq b^\top y$.

P.2. If $\inf\{x^\top c | x \in P\} = -\infty$, then $D = \emptyset$.

P.3. If $\sup\{b^\top y | y \in D\} = +\infty$, then $P = \emptyset$.

P.4. (Duality theorem). If $P \neq \emptyset$ and $D \neq \emptyset$, then

$$\inf\{x^\top c | x \in P\} = \sup\{b^\top y | y \in D\}$$

and there are optimal solutions $\hat{x} \in P$ and $\hat{y} \in D$ for the programs which means that

$$\hat{x}^\top c = \min\{x^\top c | x \in P\}, b^\top \hat{y} = \max\{b^\top y | y \in D\}.$$

Corresponding to A, b and c we can also construct a matrix game, where the antisymmetric $(m+n+1) \times (m+n+1)$-payoff matrix S is given by

$$S = \begin{bmatrix} O & A & -c \\ -A^\top & O & b \\ c^\top & -b^\top & 0 \end{bmatrix}$$

A relation of S with the dual pair of linear programs is given in

Exercise 6.1

(i) Prove that each matrix game A with $A = -A^\top$, has value equal to 0.

(ii) Let $(r, s, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ be an optimal strategy for player 1 (and player 2) in the mixed extension of S and suppose that $t \neq 0$. Prove that in this case $t^{-1}r$ and $t^{-1}s$ are optimal solutions of the primal and the dual program, respectively, corresponding to A, b, c.

The next theorem shows that solving an $m \times n$-matrix game A with $v(A) > 0$ is equivalent to solving a dual pair of linear programs, corresponding to $A, 1_n, 1_m$. [Here $1_m \in \mathbb{R}^m$ is the vector with each coordinate equal to 1.]
So one can use linear programming algorithms (such as the simplex method or interior point methods) to solve matrix games.

Theorem 6.1 Let A be an $m \times n$-matrix with $v(A) > 0$. Then

(i) p is an optimal strategy for player 1 in A iff $u := v(A)^{-1}p$ is an optimal solution of the linear program

$$\inf\{x^\top 1_m | x \geq 0, x^\top A \geq 1_n^\top\}.$$

(ii) q is an optimal strategy for player 2 in A iff $w = v(A)^{-1}q$ is an optimal solution of the linear program

$$\sup\{1_n^\top y | y \geq 0, Ay \leq 1_m\}.$$

Proof. p is optimal for player 1 in $A \Leftrightarrow p \in \mathbb{R}_+^m, p^\top 1_m = 1, p^\top A \geq v(A)1_n^\top \Leftrightarrow u \in \mathbb{R}_+^m$, $u^\top 1_m = v(A)^{-1}$, $u^\top A \geq 1_n^\top$. Similarly, q is optimal for player 2 in $A \Leftrightarrow w \in \mathbb{R}_+^n$, $1_n^\top w = v(A)^{-1}$, $Aw \leq 1_m$. Take strategies p and q optimal in A, which is possible because of the minimax theorem. Then

$$\inf\{x^\top 1_m | x \geq 0, x^\top A \geq 1_n^\top\} \leq u^\top 1_m = v(A)^{-1} = 1_n^\top w \leq \sup\{1_n^\top y | y \geq 0, Ay \leq 1_m\}.$$

In view of P.1, we can conclude that

$$\inf\{x^\top 1_m | x \geq 0, x^\top A \geq 1_n^\top\} = \sup\{1_n^\top y | y \geq 0, Ay \leq 1_m\} = v(A)^{-1}$$

and that u and w are optimal solutions for the primal and the dual program, respectively. Conversely, if $v(A) > 0$, it follows in a straightforward way that for optimal solutions u and w for the primal and dual program corresponding for $A, 1_n, 1_m$ the strategies $p = v(A)u, q = v(A)w$ are optimal in the game A for players 1 and 2, respectively. $\square$

Exercise 6.2 Let A be an $m \times n$-matrix, $d \in \mathbb{R}$ and let J be the $m \times n$-matrix with in each entry a 1.
Prove that $v(A + dJ) = v(A) + d$ and that the optimal strategy spaces for both players coincide in A and $A + dJ$. [From this fact follows that the condition $v(A) > 0$ in theorem 6.1 is essentially no restriction.]

Exercise 6.3 Prove that for each optimal strategy p (and q) for player 1 (2) in the matrix game $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, the strategy $(2 + v(A))^{-1}(p, q, v(A))$ is optimal in B, where B is given by

$$B = \begin{bmatrix} 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 0 & -1 \\ -1 & -3 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

Chapter 7

Linear complementarity and bimatrix games

For a given function $f : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ we mean by the *complementarity problem*, corresponding to f, the problem CP: Find $x \in \mathbb{R}^k$, such that $x \geq 0, f(x) \geq 0, x^\top f(x) = 0$. If f is an affine function, say $f(x) = q + Mx$ for all $x \in \mathbb{R}^n$ for some $n \times n$-matrix M and $q \in \mathbb{R}^n$, then we call the problem a *linear complementarity problem (LCP)*. Let us denote the solution set of CP by $O(f)$ and the solution set of LCP, corresponding to (q, M), by $O(q, M)$. Hence, $O(f) = \{\hat{x} \in \mathbb{R}^k | \hat{x} \geq 0, f(\hat{x}) \geq 0, \hat{x}^\top f(\hat{x}) = 0\}$ etc. Many problems in Operations Research, mathematical economics and game theory can be translated into a complementarity problem. We give some examples in the following exercises.

Exercise 7.1 (Fixed points and complementarity)
Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n_+$ be given.

(i) Prove that $\{x \in \mathbb{R}^n | g(x) = x\} \subset O(f)$, where for each $x \in \mathbb{R}^n$: $f_j(x) := \max\{0, x_j - g_j(x)\}$ for $j \in \{1, 2, \ldots, n\}$.

(ii) Prove that $O(f^*) \subset \{x \in \mathbb{R}^n | g(x) = x\}$ if $f^*(x) = x - g(x)$ for all $x \in \mathbb{R}^n$.

Exercise 7.2 (*LP*'s as *LCP*'s).
Let A be an $m \times n$-matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$. Consider the linear complementarity problem (q, M) with $q = \begin{bmatrix} c \\ -b \end{bmatrix}$ and $M = \begin{bmatrix} O & -A \\ A^\top & O \end{bmatrix}$. Let $z = \begin{bmatrix} x \\ y \end{bmatrix} \in O(q, M)$, where $x \in \mathbb{R}^m, y \in \mathbb{R}^n$. Prove that x and y are optimal solutions for the primal and the dual linear program, respectively, which correspond to A, b and c. Conversely, prove, in case x and y are optimal solutions for the dual pair, that $\begin{bmatrix} x \\ y \end{bmatrix} \in O(q, M)$.

In the next theorem we relate bimatrix game problems to *LCP*-problems. The Lemke algorithm for *LCP*'s can then help us to find a Nash equilibrium for a bimatrix game.

Theorem 7.1 Let (A, B) be an $m \times n$-bimatrix game with $A > 0$ and $B < 0$. Consider the *LCP* problem (r, M) with

$$r = \begin{bmatrix} 1_m \\ -1_n \end{bmatrix} \in \mathbb{R}^{m+n} \text{ and } M = \begin{bmatrix} O & -A \\ -B^\top & O \end{bmatrix}.$$

Then

(i) if $(p, q) \in NE(A, B)$, then $\begin{bmatrix} -\frac{p}{p^\top Bq} \\ \frac{q}{p^\top Aq} \end{bmatrix} \in O(r, M)$,

(ii) if $\begin{bmatrix} x \\ y \end{bmatrix} \in O(r, M)$, then $x \neq 0, y \neq 0$ and $\left(\frac{x}{\Sigma_{i=1}^m x_i}, \frac{y}{\Sigma_{j=1}^n y_j} \right) \in NE(A, B)$.

Proof.

(i) Let $(p,q) \in NE(A,B)$ and let $z = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x = -\frac{p}{p^\top Bq}, y = \frac{q}{p^\top Aq}$. Then

$$\begin{aligned} e_i^\top Ay &\le 1 \text{ for all } i \in \{1,2,\ldots,m\}, \\ x^\top Be_j &\le -1 \text{ for all } j \in \{1,2,\ldots,n\}. \end{aligned}$$

Hence, $r + Mz \ge 0$ and $z \ge 0$. Furthermore,

$$z^\top(r + Mz) = \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j - x^\top Ay - x^\top By = 0.$$

So, $z \in O(r,M)$.

(ii) Suppose $\begin{bmatrix} x \\ y \end{bmatrix} \in O(r,M)$. Then $x \ge 0, y \ge 0$ and

$$e_i^\top Ay \le 1 \text{ for all } i \in \{1,\ldots,m\} \tag{7.1}$$

$$x^\top Be_j \le -1 \text{ for all } j \in \{1,\ldots,n\}. \tag{7.2}$$

Further,

$$\sum_{i=1}^{m} x_i - x^\top Ay - \sum_{j=1}^{n} y_j - x^\top By = 0.$$

Combined with (7.1) and (7.2) this implies that

$$\sum_{i=1}^{m} x_i - x^\top Ay = -\sum_{j=1}^{n} y_j - x^\top By = 0. \tag{7.3}$$

From (7.2) we conclude $x \ne 0$, and then from (7.3) $y \ne 0$. Then $p = \left(\frac{x}{\Sigma_{i=1}^m x_i}, \frac{y}{\Sigma_{j=1}^n y_j}\right) \in NE(A,B)$. □

Chapter 8

Potential games

These games are studied in D. Monderer and L.S. Shapley (1996). See also Voorneveld (1999). Let $\Gamma = \langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p \rangle$ be a p-person game and let $P : \prod_{i=1}^{p} X_i \to \mathbb{R}$ be a real-valued function on the Cartesian product of the strategy spaces of Γ. Then P is called a *potential* of Γ if for each $i \in \{1, 2, \ldots, p\}$, each $x^{-i} \in \prod_{k \neq i} X_k$ and all x_i, $x_i' \in X_i$ we have

$$K_i(x^{-i}, x_i') - K_i(x^{-i}, x_i) = P(x^{-i}, x_i') - P(x^{-i}, x_i).$$

A game for which a potential exists is called a *potential game.*

Example 8.1 Consider the following situation, where two players are involved. Player 1 has to travel from A to C and player 2 from B to D. See figure 14.

Player 1 can travel via B or via D and player 2 via A or via C. The costs of using AB is 2 (5) if one player (both players) uses (use) AB etc. This situation reduces to the bimatrix cost game

$$\begin{array}{cc} & \begin{array}{cc} \text{via } A & \text{via } C \end{array} \\ \begin{array}{c} \text{via } B \\ \text{via } D \end{array} & \begin{bmatrix} (8,9) & (8,7) \\ (11,12) & (7,6)^* \end{bmatrix} \end{array}$$

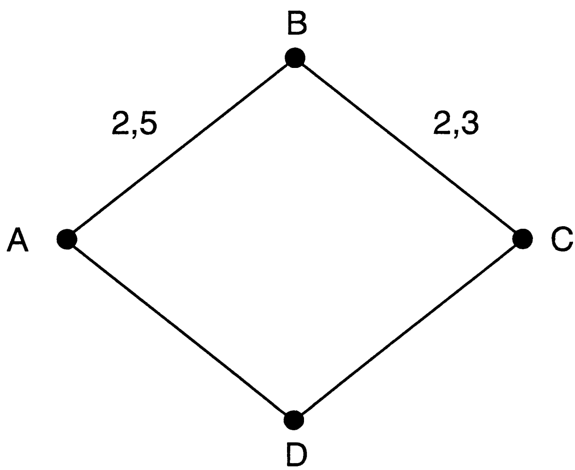

Figure 14.

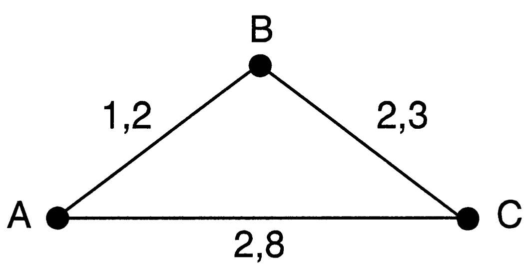

Figure 15.

in which (via D, via C) with costs 7 and 6 for player 1 and 2, is a pure Nash equilibrium. A cost potential for this game is given by $P = \begin{bmatrix} 14 & 12 \\ 17 & 11 \end{bmatrix}$.

Not all bimatrix games possess a potential as we can conclude from

Theorem 8.2 Let Γ be a finite p-person game and suppose that P is a potential of Γ. Then

(i) $NE\langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p\rangle =$ $NE\langle X_1, X_2, \ldots, X_p, P, P, \ldots, P\rangle$,

(ii) Γ has at least one pure Nash equilibrium.

Proof.

(i) follows directly from the definition of a potential.

(ii) follows from (i) and the remark that $\hat{x}$ is a Nash equilibrium for $\langle X_1, X_2, \ldots X_p, P, P, \ldots, P\rangle$ if $P(\hat{x}) = \max\{P(x) \mid x \in \prod_{i=1}^{p} X_i\}$. $\square$

Exercise 8.1 Let Γ be a p-person game with a potential P and suppose that all payoff functions are bounded. Prove that for each $\varepsilon > 0$, Γ has an ε-Nash equilibrium. [$\hat{x}$ is called and ε-Nash equilibrium if for all $i \in \{1, 2, \ldots, p\}$ and all $x_i \in X_i$:

$$K_i(\hat{x}^{-i}, x_i) \leq K_i(\hat{x}) + \varepsilon$$

which means that unilateral deviation from $\hat{x}$ pays at most ε.]

Exercise 8.2 Let $\langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p \rangle$ be a game corresponding to an oligopoly situation with linear price function and cost functions $c_1, c_2, \ldots, c_p$ with continuous derivatives. So $X_i = [0, \infty)$, $K_i(x_1, \ldots, x_n) = x_i(a - b\sum_{j=1}^{p} x_j) - c_i(x_i)$ for each $(x_1, \ldots, x_p) \in \mathbb{R}_+^p$.
Prove that $P : \mathbb{R}_+^p \to \mathbb{R}$, defined by

$$P(x_1, x_2, \ldots, x_p) = a\sum_{j=1}^{p} x_j - b\sum_{j=1}^{p} x_j^2 - b\sum_{1 \le i < j \le p} x_i x_j - \sum_{j=1}^{p} c_j(x_j)$$

is a potential for this game.
[Hint: It is sufficient to show that $\frac{\partial P}{\partial x_i} = \frac{\partial K_i}{\partial x_i}$ for all $i \in \{1, \ldots, p\}$.]

Interesting classes of cost games, arising from congestion models with a potential were introduced by R.W. Rosenthal (1973). A *congestion model* can be described as a 4-tuple $\langle N, M, (X_i)_{i \in N}, (c_j)_{j \in M} \rangle$ where

(i) N is the set of players involved (drivers on roads, producers),

(ii) M is the set of facilities $\{1, 2, \ldots, m\}$ involved (such as road segments, primary production factors),

(iii) X_i is the set of strategies of player i, where strategies of player i consist of suitable non-empty subsets of M,

(iv) $c_j : \{1, 2, \ldots, n\} \to \mathbb{R}$ is for facility j the cost function, where $c_j(k)$ denotes the costs to each user of facility j in case there are precisely k users.

The *congestion game* corresponding to the congestion model is the cost game in strategic form $\langle X_1, X_2, \ldots, X_n, C_1, C_2, \ldots, C_n \rangle$, where the costs $C_i(B_1, B_2, \ldots, B_n)$ for player i is equal to

$$\sum_{j \in B_i} c_j(t_j(B_1, B_2, \ldots, B_n))$$

with $t_j(B_1, B_2, \ldots, B_n) := |\{r \in N \mid j \in B_r\}|$, the number of users of facility j if $B_1, B_2, \ldots, B_n$ are the chosen strategies. (The number of elements in a finite set S is denoted by $|S|$.)

Example 8.3 The situation in example 8.1 can be seen as a congestion situation, where $N = \{1, 2\}, M = \{1, 2, 3, 4\}$ if we identify AB, BC, AD, DC with 1, 2, 3 and 4. Furthermore, $X_1 = \{\{1, 2\}, \{3, 4\}\}$ and $X_2 = \{\{1, 3\}, \{2, 4\}\}$. Moreover $c_1(1) = 2$, $c_1(2) = 5$, $c_2(1) = 3$, $c_2(2) = 6$, $c_3(1) = 4$, $c_3(2) = 10$, $c_4(1) = 1$ and $c_4(2) = 3$.

Example 8.4 There are three machines 1, 2, 3 used by firms 1 and 2. Firm 1 can produce using machines 1 and 2 or 1 and 3. Firm 2 can produce using machines 1 and 2, 1 and 3 or 2 and 3. Costs for using machine 1 are 5 and 6 respectively, corresponding to one and two users respectively. For machine 2 the costs are 3 and 4, and for machine 3 the costs are 2 and 5 respectively. This corresponds to a congestion situation. The corresponding cost game is given by

$$\begin{array}{cc} & \begin{array}{ccc} \{1,2\} & \{1,3\} & \{2,3\} \end{array} \\ \begin{array}{c} \{1,2\} \\ \{1,3\} \end{array} & \left[\begin{array}{ccc} (6+4, 6+4) & (6+3, 6+2) & (5+4, 4+2) \\ (6+2, 6+3) & (6+5, 6+5) & (5+5, 3+5) \end{array} \right] \end{array}$$

$$= \left[\begin{array}{ccc} (10, 10) & (9, 8) & (9, 6)^* \\ (8, 9) & (11, 11) & (10, 8) \end{array} \right].$$

The unique pure Nash equilibrium corresponds to the situation where player 1 produces using machines 1 and 2 and player 2 produces using machines 2 and 3. A (cost) potential for this game is given by

$$P = \begin{bmatrix} 18 & 16 & 14 \\ 16 & 18 & 15 \end{bmatrix}.$$

Exercise 8.3 Let $N, M, (X_i)_{i\in N}, (c_j)_{j\in M}$ be a congestion situation and let $t_j(B_1, B_2, \ldots, B_n)$ be as above. Prove that $P : \prod_{i\in N} X_i \to \mathbb{R}$, defined by

$$P(B_1, B_2, \ldots, B_n) = \sum_{j\in\cup_{i\in N} B_i} \left(\sum_{k=1}^{t_j(B_1,\ldots,B_n)} c_j(k)\right)$$
$$\text{for all } B_i \in X_i \quad (i \in N)$$

is a cost potential for the corresponding cost game.

Exercise 8.4 Find a potential for the (cost) congestion game, corresponding to the situation in figure 15 where player 1 has to go from A to C, player 2 from C to A and where the costs of AB is 1 (2) if one (two) player(s) use AB etc. Note that this game has two pure NE.

Chapter 9

Other topics in non-cooperative game theory

Till now we concentrated on games in strategic form. Special attention was paid to Nash equilibria. Existence of NE was treated in chapters 3 and 4. In chapters 6 and 7 LP and LCP problems were considered in connection with calculation of Nash equilibria for zero-sum games and general games. In chapter 3 we saw that the set of NE is nice for zero sum games and that here all equilibrium payoffs are the same. These beautiful results do not hold for general non-cooperative games as we saw in chapters 4 and 5. This phenomenon has led to (i) theory of refinements of Nash equilibria and to (ii) equilibrium selection theory.
In equilibrium selection theory the objective is to find for each game one 'best' NE. We refer to Harsanyi and Selten (1988) for details. In the refinement theory one drops NE with 'less nice'

properties and keeps the 'good' ones. Examples of refinements are best response strong and quasi-strong equilibria (Harsanyi, 1973), perfect equilibria (Selten, 1975), proper equilibria (Myerson, 1978), persistent equilibria (Kalai and Samet, 1984) and stable equilibria (Kohlberg and Mertens, 1986). For a systematic treatment we refer to E. van Damme (1991).
In strategic games, considered till now, the players each choose once a strategy and the game is over. There exist also dynamic theories such as the theory of stochastic games (Filar and Vrieze, 1997) or the theory of differential games (Isaacs, 1975). Also situations where the payoff functions of the opponent are not (completely) known gave rise to the theory of repeated games with incomplete information (Sorin, 2002). Surprisingly the theory of Nash equilibria plays also a role in biology, i.e. in the theory of evolution of conflict behavior (Maynard Smith, 1982).
The rest of this book will be devoted to cooperative game theory. For further reading we refer to the bibliography.

Chapter 10

Games in coalitional form

Now we start with a systematic study of cooperative game theory. First, cooperative games with side payments (TU-games, transferable utility games) will be treated, and later, games without side payments, such as bargaining games and NTU-games (non transferable utility games). A survey of solution concepts and axiomatic characterizations will be given. Also attention will be paid to interesting subclasses of games. Applications will be indicated. Cooperative game theory is concerned primarily with *coalitions* – groups of players – who coordinate their actions and pool their winnings. For each set S of players, $v(S)$ denotes the amount they can gain if they form a coalition, excluding the other players. One of the problems is how to divide extra earnings (or cost savings) among the members of the formed coalition.
For $N = \{1, 2, \ldots, n\}$ denote the collection of subsets of N by 2^N.

Definition 10.1 A *cooperative n-person game in coalitional form*

is an ordered pair $\langle N, v\rangle$, where $N := \{1, 2, \ldots, n\}$ (the *set of players*) and $v : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. The function v is called the characteristic function of the game, $v(S)$ is called the *worth* (or *value*) of coalition S.
In section 1 we have already introduced examples of coalitional games arising from tree games. Now we give some other examples.

Example 10.2 (*Glove game*). Let $N = \{1, 2, \ldots, n\}$ be divided into two disjoint subsets L and R. Members of L possess a left hand glove, members of R a right hand glove. A single glove is worth nothing, a right-left pair of gloves \$1. This situation can be reduced to an n-person game $\langle N, v\rangle$ where

$$v(S) := \min\{|L \cap S|, |R \cap S|\}$$

for each $S \in 2^N$.

Exercise 10.1 Construct the characteristic function of the variant of the glove game of example 10.2, where each member of L possesses p left hand gloves and each member of R owns q right hand gloves.

Example 10.3 (Three cooperating communities.) Communities 1, 2 and 3 want to be connected with a nearby power source. The possible transmission links and their costs are shown in figure 16. The *cost game* $\langle N, c\rangle$ associated with this situation is given by $N = \{1, 2, 3\}$ and $c(\emptyset) = 0$, $c(\{1\}) = 100$, $c(\{2\}) = 90$, $c(\{3\}) = 80$, $c(\{1, 2\}) = 130$, $c(\{1, 3\}) = 110$, $c(\{2, 3\}) = 110$, $c(N) = 140$.

The game $\langle N, v\rangle$ defined by

$$v(S) := \sum_{i \in S} c(i) - c(S) \text{ for each } S \in 2^N$$

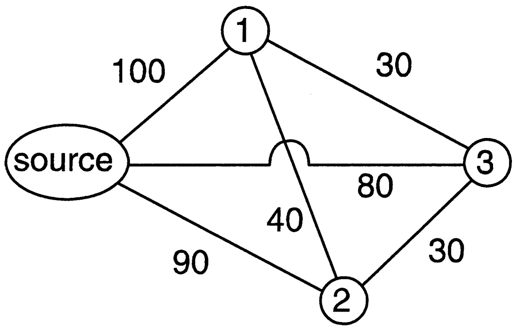

Figure 16.

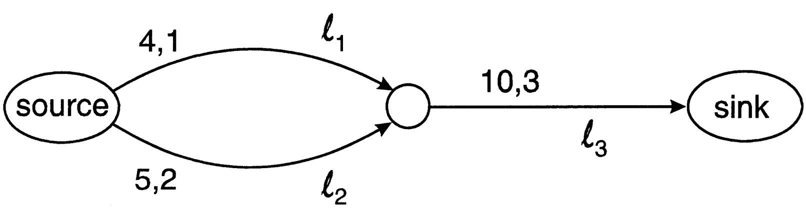

Figure 17.

is called the *cost savings game* corresponding to $< N, c >$. The cost savings $v(S)$ for coalition S is the difference in costs corresponding to the situation where all members of S work alone and the situation where all members of S work together.

Consider the following situation (cf. Tijs, Parthasarathy, Potters and Rajendra Prasad (1984)).

(i) There are n persons $1, 2, \ldots, n$ and each person i, $1 \leqslant i \leqslant n$, possesses a machine M_i and has a job J_i to be processed.

(ii) Any machine M_j can process any job J_i, but no machine is allowed to process more than one job.

(iii) Coalition formation and sidepayments are allowed.

(iv) If a person does not cooperate, his job has to be processed on his own machine.

(v) The cost of processing job J_i on machine M_j equals k_{ij}, where $1 \leqslant i, j \leqslant n$.

This situation can be reduced to the n-person cost game $\langle N, c\rangle$ where for each coalition $S \in 2^N \backslash \{\emptyset\}$:

$$c(S) = \min_{\sigma} \sum_{i \in S} k_{i\sigma(i)} \tag{10.1}$$

with the minimum taken over all S-permutations $\sigma : S \to S$. Such a σ corresponds to a plan where job J_i of player $i \in S$ is processed on machine $M_{\sigma(i)}$ of player $\sigma(i)$. The game given by (10.1) is called the *permutation game*, corresponding to the *cost matrix* $K = [k_{ij}]_{i=1,j=1}^{n \quad n}$.

Example 10.4 Consider the 3-person permutation game $\langle N, c\rangle$ with cost matrix

$$K = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}.$$

Then $\langle N, c\rangle$ and the corresponding cost savings game $\langle N, w\rangle$ are given in the next table. It follows for example that $c(1,2) = \min\{1+6, 3+2\} = 5$, corresponding to the costs if job J_1 (J_2) is processed on machine M_2 (M_1).

S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$c(S)$	0	1	6	12	5	7	17	13
$w(S)$	0	0	0	0	2	6	1	6

Nice examples of coalitional games are games arising from flow situations, where each arc is owned by one of the players. In a flow situation there is one source and one sink. On the arcs there are capacity restrictions. Furthermore, for each arc is given who owns the arc. The value of a coalition S is the maximal flow through the network from source to sink (per time unit), where one uses only arcs which are owned by members of S.

Example 10.5 Consider the network in figure 17 with one source, one sink and one intermediate node, and three arcs ℓ_1, ℓ_2, ℓ_3 with capacities 4, 5 and 10, respectively, and owners 1, 2 and 3, respectively.
The coalition $\{1,3\}$ can only use the arcs ℓ_1 and ℓ_3, so the maximal flow (per time unit) for $\{1,3\}$ is 4. This results in $v(\{1,3\}) = 4$ for the corresponding flow game $\langle N, v\rangle$. This game is given by $N = \{1,2,3\}$, $v(\{i\}) = 0$ if $i \in N$, $v(\{1,2\}) = 0$, $v(\{1,3\}) = 4$, $v(\{2,3\}) = 5$ and $v(N) = 9$.

Most of cooperative (reward) games derived from practical situations have the following *superadditivity property*

$$v(S \cup T) \geqslant v(S) + v(T) \text{ for all } S, T \in 2^N \text{ with } S \cap T = \emptyset.$$

Such games are called *superadditive.* In a superadditive game the value of the union of two disjoint coalitions is at least as large as the sum of the values of the subcoalitions separately. So in a superadditive game breaking up a coalition into parts does not pay.

Exercise 10.2

(i) Prove that the games $\langle N, v\rangle$ of examples 10.2, example 10.3 and 10.5 are superadditive and that $\langle N, c\rangle$ in example 10.3 is subadditive. (A game $\langle N, v\rangle$ is called *subadditive* if $\langle N, -v\rangle$ is superadditive.)

(ii) Prove that $v\left(\cup_{i=1}^{k} S_i\right) \geqslant \sum_{i=1}^{k} v(S_i)$ if $S_1, S_2, \ldots, S_k$ are mutually disjoint coalitions (i.e. $S_i \cap S_j = \emptyset$ if $i \neq j$) and v is superadditive.

A game is called *additive* if

$$v(S \cup T) = v(S) + v(T) \text{ for all } S, T \in 2^N \text{ with } S \cap T = \emptyset.$$

An n-person additive game $\langle N, v\rangle$ is determined by the vector $a = (v(1), v(2), \ldots, v(n))$ in $\mathbb{R}^n$ because

$$v(S) = \sum_{i \in S} a_i \text{ for all } S \in 2^N.$$

How to divide the earnings in an additive game is no problem. Such games are also called *inessential.*
The game $\langle N, w\rangle$ is called *S-equivalent (strategically equivalent)* to the game $\langle N, v\rangle$ if there exists a positive real number k and an additive game $\langle N, a\rangle$ such that $w = kv + a$.

Exercise 10.3

(i) Prove that 'S-equivalence' is an equivalence relation in the set of n-person games (i.e. that S-equivalence has the reflexivity property, the symmetry property and the transitivity property).

(ii) A game $\langle N, w\rangle$ is called *zero normalized* if

$$w(\{i\}) = 0 \text{ for each } i \in N.$$

Prove that each game is S-equivalent to a zero-normalized game.

(iii) A game $\langle N, w\rangle$ is called *zero-one normalized* if $\langle N, w\rangle$ is zero-normalized and $w(N) = 1$. Prove that each game $\langle N, v\rangle$, for which $v(N) > \sum_{i=1}^{n} v(\{i\})$ holds, has a unique S-equivalent zero-one normalized game.

(iv) Find the zero-one normalization of the game $\langle\{1,2\}, v\rangle$ where $v(\{1\}) = -2$, $v(\{2\}) = -3, v(\{1,2\}) = -4$.

Exercise 10.4 Calculate the permutation game corresponding to the cost matrix $\begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$.
Is this game subadditive? Calculate also the corresponding cost savings game.

Chapter 11

The imputation set and the core

Let $\langle N, v\rangle$ be an n-person game. A vector $x \in \mathbb{R}^n$ is called an *imputation* if

(i) x is *individual rational* i.e. $x_i \geq v(\{i\})$ for all $i \in N$,

(ii) x is *efficient (Pareto optimal)* i.e. $\sum_{i=1}^n x_i = v(N)$.

The set of imputations of $\langle N, v\rangle$ is denoted by $I(v)$. An element $x \in I(v)$ can be interpreted as a payoff distribution of the earnings $v(N)$ of the grand coalition N, which gives player i a payoff x_i which is at least as much as he can obtain when he operates alone. For an additive game v, $I(v)$ consists of one point $\{(v(1), v(2), \ldots, v(n))\}$.[1]

Note that

$$I(v) \neq \emptyset \text{ iff } v(N) \geq \sum_{i=1}^n v(\{i\}).$$

[1]Instead of $v(\{i\}), v(\{i,j\})$ etc. we often write $v(i), v(i,j)$, etc.

For an essential game where $v(N) > \sum_{i=1}^{n} v(i), I(v)$ is the convex hull of the points: $f^1, f^2, \ldots, f^n$ where $f_k^i = v(\{k\})$ if $k \neq i$ and $f_i^i = v(N) - \sum_{k \in N \setminus \{i\}} v(\{k\})$. This is a direct consequence of the following theorem from the theory of linear inequalities (by taking for A the $n \times (n+2)$-matrix with columns $e_1, e_2, \ldots, e_n, 1_n, -1_n$ and $b = (v(1), v(2), \ldots, v(n), v(N), -v(N))$, where 1_n is the vector in $\mathbb{R}^n$ with all coordinates equal to 1).

Theorem 11.1 Let A be an $n \times p$-matrix, $b \in \mathbb{R}^p$ and P the polyhedral set of solutions of the set of inequalities $x^\top A \geq b^\top$. For $x \in P$ let $\mathrm{tight}(x)$ be the set of columns $\{Ae_j \mid x^\top Ae_j = b_j\}$ of A where the corresponding inequalities are equalities for x. Then we have: x is an extreme point of P iff $\mathrm{tight}(x)$ is a complete system of vectors in $\mathbb{R}^n$.

Proof.

(i) First we prove the 'if' part. Suppose that x is not an extreme point of P. Then there are $x^1, x^2 \in P, x^1 \neq x^2$ such that $x = \frac{1}{2}(x^1 + x^2)$. For $z := x^1 - x^2$ we then have: $z^\top Ae_j = 0$ for all $Ae_j \in \mathrm{tight}(x)$. So z is orthogonal to the linear subspace generated by $\mathrm{tight}(x)$ and unequal to 0. Hence, $\mathrm{tight}(x)$ is not a complete system in $\mathbb{R}^n$.

(ii) For the 'only if' part, suppose that $\mathrm{tight}(x)$ does not generate $\mathbb{R}^n$. Then we can take $z \in \mathbb{R}^n \setminus \{0\}$ with $z^\top Ae_j = 0$ for all $Ae_j \in \mathrm{tight}(x)$. For $\varepsilon > 0$, sufficiently small, we then have $x^1 := x + \varepsilon z \in P, x^2 := x - \varepsilon z \in P$ and $x = \frac{1}{2}(x^1 + x^2)$. Then x is not an extreme point of P. □

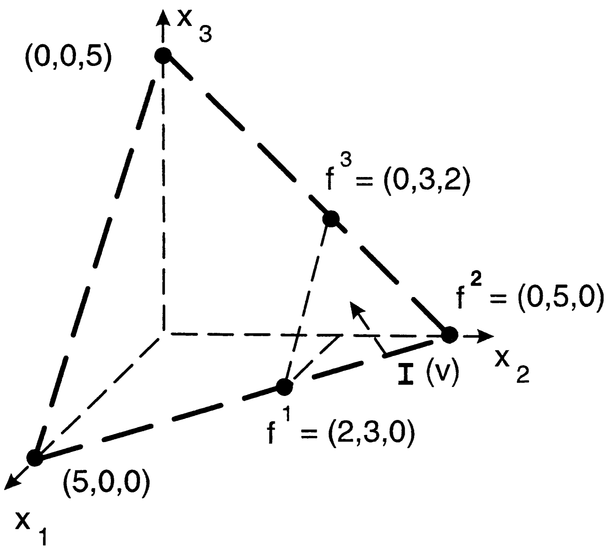

Figure 18.

Example 11.2 Let $\langle N, v\rangle$ be a 3-person game with $v(1) = v(3) = 0$, $v(2) = 3$, $v(1,2,3) = 5$. Then $I(v)$ is the triangle with vertices $f^1 = (2,3,0), f^2 = (0,5,0)$ and $f^3 = (0,3,2)$. (See figure 18.)

Exercise 11.1 For an n-person zero-one game (see exercise 10.3) the imputation set is the convex set Δ^n generated by the standard basis $e^1, e^2, \ldots, e^n$ in $\mathbb{R}^n$. Prove this.

The *core* of a game $\langle N, v\rangle$ is the set

$$C(v) := \{x \in I(v) \mid \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in 2^N \backslash \{\emptyset\}\}.$$

If $x \in C(v)$, then no coalition $S \neq N$ has an incentive to split off if x is the proposed reward allocation in N, because the total amount $x(S) := \sum_{i\in S} x_i$ allocated to S is not smaller than the amount $v(S)$ which they can obtain by forming the subcoalition.

Exercise 11.2 Let $\langle N, w\rangle$ be S-equivalent to $\langle N, v\rangle$, say $w = kv + a$. Show that $C(w) = kC(v) + a$ ($:= \{x \in \mathbb{R}^n \mid x = ky + a$ for some $y \in C(v)\}$).

The equality of the two sets expresses that the core is *relative invariant with respect to S-equivalence.*

Elements of $C(v)$ can easily be obtained because the core is defined with the aid of linear inequalities. The core is a polytope.

Exercise 11.3 Let $\langle N, v\rangle$ be an n-person game. For each $i \in N$ let $M_i(v) := v(N) - v(N\backslash\{i\})$. Prove that for each $x \in C(v)$: $x_i \leq M_i(v)$ for all $i \in N$.

Example 11.3 A core element of the game in example 10.5 is (4,5,0). A core element of the 3-person cost savings game in example 10.4 is (2,0,4). The core is empty for the 2-person game v with $v(\{1,2\}) = v(\{1\}) = v(\{2\}) = 1$.

Bondareva (1963) and Shapley (1967) gave independently a characterization of games with a non-empty core. For this we need the notions of balanced collection and balanced games.

Let $N := \{1, 2, \ldots, n\}$ and 2^N the family of subsets of N. A map $\lambda : 2^N\backslash\{\emptyset\} \to \mathbb{R}_+$ is called *balanced map* if

$$\sum_{S \in 2^N\backslash\{\emptyset\}} \lambda(S)e^S = e^N.$$

Here e^S is the *characteristic vector* for coalition S with

$$e_i^S = 1 \text{ if } i \in S \text{ and } e_i^S = 0 \text{ if } i \in N\backslash S.$$

A collection B of coalitions is called a *balanced collection* if there is a balanced map λ such that

$$B = \{S \in 2^N \backslash \{\emptyset\} \mid \lambda(S) > 0\}.$$

Example 11.4

(i) Let $N_1, N_2, \ldots, N_k$ be a partition of N (i.e. $N = \cup_{r=1}^k N_r$, $N_s \cap N_t = \emptyset$ if $s \neq t$). Then $\{N_1, N_2, \ldots, N_k\}$ is a balanced collection, corresponding to the balanced map λ with $\lambda(S) = 1$ if $S \in \{N_1, N_2, \ldots, N_k\}$ and $\lambda(S) = 0$, otherwise.

(ii) For $N = \{1, 2, 3\}$, the set $B = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is balanced and corresponds to the balanced map λ with

$$\lambda(S) = 0 \text{ if } |S| \in \{1, 3\} \text{ and } \lambda(S) = \frac{1}{2} \text{ if } |S| = 2.$$

Exercise 11.4

(i) Show that for a balanced map λ it holds that $\sum_S \lambda(S) > 1$, unless the corresponding balanced collection equals $\{N\}$.

(ii) If B is a balanced collection unequal to $\{N\}$, then

$$B^c := \{S \in 2^N \backslash \{\emptyset\} \mid N \backslash S \in B\}$$

is also a balanced collection. Give the corresponding balanced map.

(iii) Let $S \in 2^N \backslash \{\emptyset, N\}$. Prove that $\{S, (N \backslash \{i\})_{i \in S}\}$ is a balanced collection.

Definition 11.5 An n-person game $\langle N, v\rangle$ is called a *balanced game* if for each balanced map $\lambda : 2^N\backslash\{\emptyset\} \to \mathbb{R}_+$:

$$\sum_S \lambda(S)v(S) \leq v(N). \tag{11.1}$$

The importance of this notion follows from theorem 11.7 proved by Bondareva (1963) and Shapley (1967). This theorem characterizes games with a non-empty core. In the proof we use the following duality theorem from linear programming theory.

Theorem 11.6 Let A be an $n \times p$ matrix, $b \in \mathbb{R}^p$ and $c \in \mathbb{R}^n$. Then $\min\{x^\top c \mid x^\top A \geq b^\top\} = \max\{b^\top y \mid Ay = c, y \geq 0\}$ if $\{x \in \mathbb{R}^n \mid x^\top A \geq b^\top\} \neq \emptyset$ and $\{y \in \mathbb{R}^p \mid Ay = c, y \geq 0\} \neq \emptyset$.

Theorem 11.7 Let $\langle N, v\rangle$ be an n-person game. Then the following two assertions are equivalent:

(i) $C(v) \neq \emptyset$,

(ii) $\langle N, v\rangle$ is a balanced game.

Proof. First we note that $C(v) \neq \emptyset$ iff

$$v(N) = \min\left\{\sum_{i=1}^n x_i \mid x(S) \geq v(S) \text{ for all } S \in 2^N\backslash\{\emptyset\}\right\} \tag{11.2}$$

By the duality theorem 11.6, equality (11.2) holds iff

$$v(N) = \max\{\Sigma\lambda(S)v(S) \mid \Sigma\lambda(S)e^S = e^N, \lambda \geq 0\} \tag{11.3}$$

(Take for A the matrix with the characteristic vectors e^S as columns.) Now ((11.3) holds iff (11.1) holds. Hence (i) and (ii) are equivalent. □

Exercise 11.5 Find the core of the game $\langle N, v\rangle$ with $N = \{1, 2, 3\}$ and

(i) $v(S) = 0$ for all $S \neq N, v(N) = 1$,

(ii) $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, v(\{1,3\}) = v(\{2,3\}) = 1, v(\{1,2\}) = 0$ and $v(N) = 1$.

Chapter 12

Linear production games

Linear production games are introduced in Owen (1975).

Consider the linear production situation where products $P_1, P_{\mathbb{m}}, \dots, P_j, \dots, P_m$ can be made, using resources $G_1, G_{\mathbb{m}}, \dots, G_k, \dots, G_q$. Suppose further that for the production of α units of $P_j (\alpha \geq 0)$ we need αa_{j1} units of G_1, $\alpha a_{j\mathbb{m}}$ units of $G_{\mathbb{m}}$, etc. Furthermore, the price per unit of product P_j is c_j. Let $A := [a_{jk}]_{j=1,k=1}^{m \ \ q}$ be the corresponding *production matrix.* We suppose that $A \geq 0$ and that each row of A has a positive coordinate. If there is available a bundle $b \in (\mathbb{R}^q_+)^\top$ of resources, then feasible production plans can be described as vectors $x \in \mathbb{R}^m_+$ with $x^\top A \leq b$, where x corresponds to the plan: make for each $j \in \{1, 2, \dots, m\}$ x_j units of product P_j. The value of the products made by such a plan at price c is equal to the inner product $x^\top c$ of x and c.

Let $\operatorname{val}(b) := \max\{x^\top c \mid x \geq 0, x^\top A \leq b\}$ be the value corresponding to an optimal production plan, given the resource bundle b. That $\operatorname{val}(b)$ is well-defined follows from the fact that $\{x \in \mathbb{R}^m \mid x \geq 0, x^\top A \leq b\}$ is a compact set, in view of the conditions put on A.

Now we suppose that a group $N = \{1, 2, \ldots, n\}$ of agents possesses resources and that b^i is the resource bundle owned by player i. Then the amount available for a coalition $S \in 2^N$, if they cooperate, is $\sum_{i \in S} b^i$. The above situation corresponds to a *linear production game* $\langle N, v \rangle$, where

$$v(S) := \operatorname{val}\left(\sum_{i \in S} b^i\right) \text{ for each } S \in 2^N \backslash \{\emptyset\}.$$

Example 12.1 Consider the linear production situation

$$\begin{array}{c c} & \begin{array}{cc} G_1 & G_2 \end{array} \quad c \\ \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} \end{array} \qquad \begin{array}{rcl} b^1 & = & (5, 8) \\ b^2 & = & (5, 2) \\ b^3 & = & (0, 2) \end{array}$$

where two resources G_1 and G_2 are involved and two products P_1 and P_2, with prices per unit 5 and 7. Suppose there are available 10 units of G_1 and 12 units of G_2: players 1, 2 and 3 own, respectively, resource bundles (5,8), (5,2) and (0,2). This situation corresponds to a 3-person linear production game $\langle N, v \rangle$ with

$$\begin{aligned} v(S) \ : &= \max\{5x_1 + 7x_2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + 2x_2 \leq b_1(S), \\ & \quad 2x_1 + x_2 \leq b_2(S)\} \end{aligned} \tag{12.1}$$

where $b_k(S) = \left(\sum_{i \in S} b^i\right)_k$ for $k \in \{1, 2\}$. So $b_k(S)$ is the total amount of G_k owned by coalition S.

If one wishes to calculate v, it is easier not to use (12.1) but the next formula (12.2), which we obtain from the duality theorem of linear programming theory. The reason is that for all dual programs corresponding to the $2^n - 1 = 7$ coalitions the feasible region is the same. We have

$$\begin{aligned} v(S) &= \min\{b_1(S)y_1 + b_2(S)y_2 \mid y_1 \geq 0, y_2 \geq 0, y_1 + 2y_2 \geq 5, \\ & \quad 2y_1 + y_2 \geq 7\} \end{aligned} \tag{12.2}$$

The feasible region of the dual problem has three extreme points: $\hat{y} = (3, 1)$, $\hat{y}' = (5, 0)$, $\hat{y}'' = (0, 7)$. For each of the seven problems in (12.2) the minimum is attained in one of these three extreme points. The next table describes where the minimum is attained and gives also the characteristic function v.

S	$b(S)$	minimum	$v(S)$
(1)	(5,8)	$\hat{y}$	23
(2)	(5,2)	$\hat{y}''$	14
(3)	(0,2)	$\hat{y}'$	0
(1,2)	(10,10)	$\hat{y}$	40
(1,3)	(5,10)	$\hat{y}, \hat{y}'$	25
(2,3)	(5,4)	$\hat{y}$	19
(1,2,3)	(10,12)	$\hat{y}$	42

For the grand coalition N the value is equal to 42 and is attained in $\hat{y} = (3, 1)$. Interpret 3 and 1 as (shadow) prices for G_1 and G_2. According to these prices the bundle $b^1 = (5, 8)$, owned by player 1, has value $5 \cdot 3 + 8 \cdot 1 = 23$. For players 2 and 3 the values of their bundles are 17 and 2 respectively. These values correspond to the imputation (23,17,2) of $\langle N, v\rangle$. Note that (23,17,2) is even a core element of $\langle N, v\rangle$ and that to find this vector we only need to solve the dual linear program in (12.2) for $S = N$.

Linear production games turn out to be *totally balanced games,* which are games where each subgame is balanced. This follows from theorem 12.2.

Theorem 12.2 Let $\langle N, v\rangle$ be the linear production game corresponding to the linear production situation with production matrix A, price c and commodity bundles $b^1, b^2, \ldots, b^n$ owned by players $1, 2, \ldots, n$, respectively. Then $C(v) \neq \emptyset$.

Proof. Let $\hat{y}$ be an optimal solution of the dual problem

$$\min\left\{\left(\sum_{i=1}^{n} b^i\right) y \mid y \geq 0, Ay \geq c\right\} \tag{12.3}$$

for the grand coalition. We claim that $(x_1, x_2, \ldots, x_n)$ with $x_i := b^i\hat{y}$ is a core element. By the duality theorem the minimum is equal to $v(N)$. So $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} b^i\hat{y} = v(N)$. Now take $S \subset N$, $S \neq \emptyset$. First note that $\hat{y}$ is feasible for the program $\min\left\{\sum_{i\in S} b^i y \mid y \geq 0, Ay \geq c\right\}$, and this minimum is by the duality theorem equal to $v(S) = \max\left\{x^\top c \mid x \geq 0, x^\top A \leq \sum_{i\in S} b^i\right\}$. Hence, $\sum_{i\in S} x_i = \sum_{i\in S} b^i\hat{y} \geq v(S)$. This proves that x is a core element. $\square$

Note that the core element x in theorem 12.2 is obtained as follows.

(i) First calculate shadow prices $\hat{y}_1, \hat{y}_2, \ldots$ for the commodities by solving the linear program (12.3).

(ii) With respect to the shadow price vector $\hat{y}$ calculate the values of the owned bundles and give this value to the players, resulting in a vector x in the core of v.

Exercise 12.1 Consider the linear production situation, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, c = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, b^1 = (6,0), b^2 = (0,5) \text{ and } b^3 = (0,1).$$

Calculate the linear production game and find a core element for this game.

Exercise 12.2 Find a core element for the linear production game corresponding to the production situation, given by

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, c = \begin{bmatrix} 30 \\ 40 \end{bmatrix}, b^1 = (3,4), b^2 = (2,4) \text{ and } b^3 = (5,4).$$

One can prove that for each non-negative totally balanced game $\langle N, v\rangle$ there is a linear production situation $A, c, b^1, b^2, \ldots, b^n$, such that the corresponding linear production game coincides with $\langle N, v\rangle$. Take A the $(2^n - 1) \times n$-matrix, where each column corresponds with a player and each row with a non-empty coalition. For column i and row S we have $a_{Si} = \begin{cases} 1 \text{ if } i \in S \\ 0 \text{ if } i \notin S \end{cases}$. So the i-th row corresponds to the characteristic vector e^S of coalition S. Further $c_S = v(S)$ and $b^i = e^i$ for each $S \in 2^N \backslash \{\emptyset\}$ and $i \in N$. A special case is treated in

Exercise 12.3 Let $\langle N, v\rangle$ be the 2-person game with $v(1) = 1, v(2) = 3$ and $v(1,2) = 5$. Prove that this game coincides with the linear production game corresponding to

$$A = \begin{array}{@{}c@{\quad}c@{}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} & \begin{array}{c} \{1\} \\ \{2\} \\ \{1,2\} \end{array} \end{array} \qquad c = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad \begin{array}{l} b^1 = (1,0) \\ b^2 = (0,1) \end{array}.$$

Chapter 13

Dominance, the D-core and stable sets

Let $\langle N, v\rangle$ be an n-person game. Let $y, z \in I(v), S \in 2^N \backslash \{\emptyset\}$. We say that y *dominates* z *via coalition* S, denoted by y dom_S z, if

(i) $y_i > z_i$ for all $i \in S$,

(ii) $\sum_{i \in S} y_i \leq v(S)$.

Interpretation: The payoff distribution y is better than z for all members $i \in S$ if (i) holds and the payoffs $(y_i)_{i \in S}$ are attainable for the members of S by cooperation if (ii) holds. Against each z in $D(S) := \{z \in I(v) \mid \exists_{y \in I(v)} y \ \mathrm{dom}_S \ z\}$ the players of S can protest successfully. $D(S)$ consists of the imputations which are dominated via S.

Example 13.1 Let $\langle N, v\rangle$ be the 3-person game with $v(\{1,2\}) = 2, v(N) = 1$ and $v(S) = 0$ if $S \neq \{1,2\}, N$.

Then $D(S) = \emptyset$ if $S \neq \{1,2\}$ and $D(\{1,2\}) = \{x \in I(v) \mid x_3 > 0\}$. The elements x in $I(v)$ which are undominated satisfy $x_3 = 0$.

Exercise 13.1

(i) Prove that for each game $\langle N, v\rangle, D(S) = \emptyset$ if $|S| \in \{1, n\}$

(ii) Determine for each S the set $D(S)$ for the cost savings game in example 10.3. Determine also the undominated elements.

For $x, y \in I(v)$, we say that x *dominates* y (notation: x dom y) if there is an $S \in 2^N \backslash \{\emptyset\}$ such that x dom_S y. In the sequel, we write $x(S)$ for $\sum_{i \in S} x_i$.

Exercise 13.2

(i) Prove that dom and dom_S are irreflexive relations and that dom_S is transitive and antisymmetric.

(ii) Construct a game $\langle N, v\rangle$ and find x and y in $I(v)$ such that x dom y and y dom x.

(iii) Construct a game $\langle N, v\rangle$ and find $x, y, z \in I(v)$ with x dom y and y dom z and not x dom z.

Exercise 13.3

(i) Prove that in the unique constant sum zero-one normalized 3-person game v each element of the imputation set is dominated by another element

(ii) Prove that in this game each $x \in I(v) \backslash A$ is dominated by an element of $A := \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$.

(iii) If $c \in [0, \frac{1}{2})$ and $B := \{x \in I(v) \mid x_3 = c\}$, then each element of $I(v) \backslash B$ is dominated by an element of B. Show this.

The *D-core* of a game $\langle N, v \rangle$ is the set

$$DC(v) := I(v) \backslash \cup_{S \in 2^N \backslash \{\emptyset\}} D(S),$$

i.e. the set of all undominated elements in $I(v)$.
For the game in example 13.1 the D-core is non-empty and the core is empty.
In general we have

Theorem 13.2 The core is a subset of the D-core for each game.

Proof. Let $\langle N, v \rangle$ be a game and $x \notin DC(v)$. Then there is a $y \in I(v)$ and a coalition S such that $y \text{ dom}_S\, x$. Then $v(S) \geq y(S) > x(S)$, which implies that $x \notin C(v)$. □

For superadditive games the core and the D-core coincide as the following theorem shows.

Theorem 13.3 Let $\langle N, v \rangle$ be a superadditive game. Then $DC(v) = C(v)$.

Proof.

(i) Firstly, we show that for an $x \in I(v)$ with $x(S) < v(S)$ for some S, there is an $y \in I(v)$ such that $y \text{ dom}_S\, x$. Define y as follows. If $i \in S$, then $y_i := x_i + |S|^{-1}(v(S) - x(S))$. If $i \notin S$, then $y_i := v(\{i\}) + (v(N) - v(S) - \sum_{i \in N \backslash S} v(\{i\}))|N \backslash S|^{-1}$. Then $y \in I(v)$, where for the proof of $y_i \geq v(\{i\})$ for $i \in N \backslash S$ we use the superadditivity of the game. Furthermore, $y \text{ dom}_S\, x$.

(ii) To prove $DC(v) = C(v)$ we have, in view of theorem 13.2, only to show that $DC(v) \subset C(v)$. Suppose $x \in DC(v)$. Then there is no $y \in I(v)$ with y dom x. In view of (i) we then have $x(S) \geq v(S)$ for all $S \in 2^N \backslash \{\emptyset\}$. Hence, $x \in C(v)$.

□

Note that in the proof of theorem 13.3 we only used that

$$v(N) \geq v(S) + \sum_{i \in N \backslash S} v(\{i\}) \text{ for all } S \in 2^N \backslash \{\emptyset\},$$

which is weaker than superadditivity.
For $A \subset I(v)$ we denote by $\text{dom}(A)$ the set $\{x \in I(v) \mid a \text{ dom } x$ for some $a \in A\}$. The set $\text{dom}(A)$ consists of all imputations which are dominated by some element of A. Note that $DC(v) = I(v) \backslash \text{dom}(I(v))$.

Definition 13.4 For a game $\langle N, v \rangle$, a subset M of $I(v)$ is called a *stable set* (or a *von Neumann-Morgenstern solution*) iff the following two properties hold:
(NM.1) *Internal stability*: $M \cap \text{dom}(M) = \emptyset$
(NM.2) *External stability*: $I(v) \backslash M \subset \text{dom}(M)$.

Note that M is a stable set iff M and $\text{dom}(M)$ form a partition of $I(v)$. In a stable set M no point of M is dominated by another point of M and each imputation outside M is dominated by an imputation in M. J. von Neumann and O. Morgenstern interpreted a stable set as a set corresponding to a "standard of behavior", which, if generally accepted, is self-enforcing. Outcomes in such a set are not dominated by points of the stable set.
A game can have many stable sets. For a long time it was an open problem, whether a game has always at least one stable set.

W. Lucas (1968) constructed a 10-person game with no stable set and a non-empty core. Lucas and Rabie (1982) constructed a 14-person game without stable sets and with an empty core.

Example 13.5 Let $\langle N, v\rangle$ be a 3-person glove game (see example 10.2) where $L = \{1,2\}$ and $R = \{3\}$. Then $v(i) = 0$ for all $i \in N$, $v(1,2) = 0$, $v(1,3) = v(2,3) = v(1,2,3) = 1$. Then $DC(v) = C(v) = \{(0,0,1)\}$. Furthermore, for each $\varepsilon \in (0,1)$, the line segment with end points (0,0,1) and $(\varepsilon, 1-\varepsilon, 0)$ is a stable set. See figure 19.

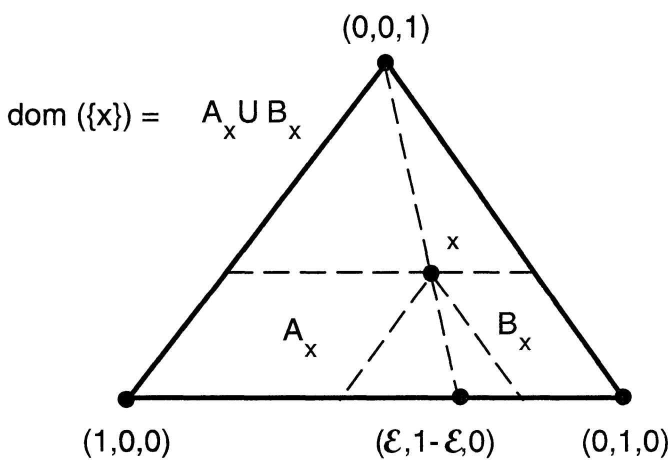

Figure 19.

Exercise 13.4 Show that for the game of exercise 13.3 the following sets are stable sets.

$$\begin{aligned}
A &:= \{(\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{2},0,\tfrac{1}{2}),(0,\tfrac{1}{2},\tfrac{1}{2})\} \text{ the \textit{symmetric solution}}\\
D_1(c) &:= \{x \in I(v) \mid x_1 = c\},\\
D_2(c) &:= \{x \in I(v) \mid x_2 = c\},\\
D_3(c) &:= \{x \in I(v) \mid x_3 = c\}, \text{ where } c \in [0,\tfrac{1}{2}).
\end{aligned}$$

(The $D_i(c)$-type solutions are called *discriminatory solutions.*)
(One can show that there are no other stable sets.)
The relation between the D-core and stable sets is described in the following theorem.

Theorem 13.6

(i) Let M be a stable set of $\langle N, v\rangle$. Then $DC(v) \subset M$.

(ii) If $DC(v)$ is a stable set, then there are no other stable sets.

Exercise 13.5 Prove theorem 13.6.

Exercise 13.6 Show that the core of the 3-person game v is a stable set, where

$$v(\{i\}) = 0, v(\{i,j\}) = 2, v(\{1,2,3\}) = 6 \text{ for all } i, j \in N.$$

Exercise 13.7 For $A \subset I(v)$ let $U(A) := I(v) \backslash \mathrm{dom}(A)$. Show that

(i) $U(A) = A \Leftrightarrow A$ is a stable set,

(ii) $DC(v) = U(I(v))$.

Chapter 14

The Shapley value

The Shapley value, introduced in the fifties (Cf. Shapley (1953)), is one of the most interesting solution concepts in cooperative game theory which has drawn much attention. See for example A. Roth (1988). The Shapley value associates to each n-person game one (payoff) vector in $\mathbb{R}^n$. We will discuss a few formulae, properties and an axiomatization for this value.
Before introducing the Shapley value, we define for games $\langle N, v\rangle$ and permutations $\sigma : N \to N$ marginal vectors $m^\sigma(v)$. The payoff vector $m^\sigma(v)$ corresponds to a situation, where the players enter a room one by one in the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$ and where each player is given the marginal contribution he creates by entering. Hence, $m^\sigma(v)$ is the vector in $\mathbb{R}^n$ with

$$\begin{aligned} m^\sigma_{\sigma(1)}(v) \quad &: \quad = v(\{\sigma(1)\}), m^\sigma_{\sigma(2)}(v) := v(\{\sigma(1), \sigma(2)\}) - \\ & \qquad v(\{\sigma(1)\}), \ldots, \\ m^\sigma_{\sigma(k)}(v) \quad &: \quad = v(\{\sigma(1), \sigma(2), \ldots, \sigma(k)\}) - v(\{\sigma(1), \sigma(2), \ldots, \\ & \qquad \sigma(k-1)\}) \end{aligned}$$

for each $k \in N$. If we introduce the *set of predecessors of i in σ* by

$P_\sigma(i) := \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$, then $m^\sigma_{\sigma(k)}(v) = v(P_\sigma(\sigma(k)) \cup \{\sigma(k)\}) - v(P_\sigma(\sigma(k)))$ or

$$m_i^\sigma(v) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)) \tag{14.1}$$

Example 14.1 Let $\langle N, v \rangle$ be the 3-person game with $v(1) = v(2) = v(3) = 0$, $v(1,2) = 4$, $v(1,3) = 7$, $v(2,3) = 15$, $v(1,2,3) = 20$. Then the marginal vectors are given in table 14.1, where $\sigma : N \to N$ is identified with $(\sigma(1), \sigma(2), \sigma(3))$.

σ	$m_1^\sigma(v)$	$m_2^\sigma(v)$	$m_3^\sigma(v)$
(1,2,3)	0	4	16
(1,3,2)	0	13	7
(2,1,3)	4	0	16
(2,3,1)	5	0	15
(3,1,2)	7	13	0
(3,2,1)	5	15	0

Table 14.1

The average of the six marginal vectors is $\frac{1}{6}(21, 45, 54)$, which is by definition the Shapley value of $\langle N, v \rangle$.

Definition 14.2 The Shapley value $\Phi(v)$ of a game $\langle N, v \rangle$ is the average of the marginal vectors of the game, i.e.

$$\Phi(v) := \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^\sigma(v). \tag{14.2}$$

(Here $\pi(N)$ denotes the set of permutations of N.)

Example 14.3

(i) For a 2-person game $\langle N, v\rangle$ we have
$\Phi(v) = \left(v(1) + \frac{v(N)-v(1)-v(2)}{2}, v(2) + \frac{v(N)-v(1)-v(2)}{2}\right).$

(ii) The Shapley value $\Phi(v)$ for an additive game is equal to $(v(1), v(2), \ldots, v(n))$.

(iii) Let u_S be the *unanimity game for* S, defined by $u_S(T) = 1$ if $S \subset T, u_S(T) = 0$ otherwise. Then $\Phi(u_S) = \frac{1}{|S|}e^S$.

One can give with the aid of (14.2) a probabilistic interpretation of the Shapley value as follows. Suppose we draw from an urn, containing the elements of $\pi(N)$, a permutation σ (probability $(n!)^{-1}$). Then we let the players enter a room one by one in the order σ and give each player the marginal contribution created by him. Then the i-th coordinate $\Phi_i(v)$ of $\Phi(v)$ is the expected payoff to player i according to this random procedure.
Using (14.1) we can rewrite (14.2) obtaining

$$\Phi_i(v) = \frac{1}{n!} \sum_{\sigma\in\pi(N)} v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)). \tag{14.3}$$

The terms behind this summation sign are of the form $v(S\cup\{i\}) - v(S)$, where S is a subset of N not containing i. For how many orderings do we have $P_\sigma(i) = S$? The answer is $|S|!(n-1-|S|)!$, where the first factor $|S|!$ corresponds to the number of orderings of S and the second factor $(n-1-|S|)!$ to the number of orderings of $N\backslash(S \cup \{i\})$. Using this, we can rewrite (14.3) and obtain

$$\Phi_i(v) = \sum_{S:i\notin S} \frac{|S|!(n-1-|S|)!}{n!}(v(S \cup \{i\}) - v(S)). \tag{14.4}$$

Note that $\frac{|S|!(n-1-|S|)!}{n!} = \frac{1}{n}\begin{pmatrix} n-1 \\ |S| \end{pmatrix}^{-1}$. This gives rise to a second probabilistic interpretation of the Shapley value. Create as follows a subset S to which i does not belong. First, draw at random a number out of the urn consisting of possible sizes $0, 1, 2, \ldots, n-1$, where each number has probability n^{-1} to be drawn. If size s is chosen, draw a set out of the urn consisting of subsets of $N\backslash\{i\}$ of size s, where each set has the same probability $\begin{pmatrix} n-1 \\ s \end{pmatrix}^{-1}$ to be drawn. If S is drawn, then pay player i the amount $v(S \cup \{i\}) - v(S)$.

Then, obviously in view of (14.4), the expected payoff for player i in this random procedure is the Shapley value for player i of the game $\langle N, v\rangle$.

Let us denote with G^n the collection of all characteristic functions v, corresponding to an n-person coalitional game $\langle N, v\rangle$, and A^n the collection of all additive characteristic functions a corresponding to a game $\langle N, a\rangle$. Then G^n is a linear (vector) space, because it is obvious what we mean with $v+w$, αv for $v, w \in G^n$ and $\alpha \in \mathbb{R}$. The dimension of the linear space G^n is equal to $2^n - 1$. One basis for this linear space is $\{\delta^S \mid S \in 2^N\backslash\{\emptyset\}\}$, where $\delta^S : 2^N \to \mathbb{R}$ is defined by $\delta^S(T) = 1$ if $T = S$ and $\delta^S(T) = 0$, otherwise.

A more interesting basis is $\{u^S \mid S \in 2^N\backslash\{\emptyset\}\}$ formed by the unanimity games (see example 14.3). It is not difficult to show that $v = \sum_{S \in 2^N\backslash\{\emptyset\}} c_S u_S$ with

$$c_S = \sum_{T:T\subset S} (-1)^{|S|-|T|} v(T).$$

Let us call a map $f : G^n \to \mathbb{R}^n$ a *(one-point) solution for games.* The Shapley value is such a solution and later some others will be introduced. We mention some desirable properties for solutions.

(i) $f : G^n \to \mathbb{R}^n$ is called *individual rational* (IR) if

$$f_i(v) \geq v(\{i\}) \text{ for all } v \in G^n \text{ and } i \in N.$$

(ii) $f : G^n \to \mathbb{R}^n$ is called *efficient* (EFF) if

$$\sum_{i=1}^{n} f_i(v) = v(N) \text{ for all } v \in G^n.$$

(iii) $f : G^n \to \mathbb{R}^n$ is called *relative invariant w.r.t. S-equivalence* (SEQ) if for all v and w in G^n with $w = kv + a(k \in (0, \infty)$, $a \in A^n) : f(kv + a) = kf(v) + a$.

(iv) $f : G^n \to \mathbb{R}^n$ has the *dummy player property* (DUM) if $f_i(v) = v(i)$ for all $v \in G^n$ and all dummy players i in v, that is players $i \in N$ such that

$$v(S \cup \{i\}) = v(S) + v(i) \text{ for all } S \in 2^{N \setminus \{i\}}.$$

(v) $f : G^n \to \mathbb{R}^n$ has the *anonymity property* (AN) if $f(v^\sigma) = \sigma^*(f(v))$ for all $\sigma \in \pi(N)$. Here v^σ is the game with

$$\begin{aligned} v^\sigma(\sigma(U)) \quad &: \quad = v(U) \text{ for all } U \in 2^N \text{ or} \\ v^\sigma(S) \quad &= \quad v(\sigma^{-1}(S)) \text{ for all } S \in 2^N \end{aligned}$$

and $\sigma^* : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $(\sigma^*(x))_{\sigma(k)} := x_k$ for all $x \in \mathbb{R}^n$ and $k \in N$.

(vi) $f : G^n \to \mathbb{R}^n$ is called an *additive solution* (ADD) if $f(v + w) = f(v) + f(w)$ for all $v, w \in G^n$.

Exercise 14.1

(i) Show that the Shapley value is not individual rational.

(ii) Show that $\Phi_k(v) \geq v(k)$ for each $k \in N$ if v is zero-monotonic (that is $v(S) - \sum_{i \in S} v(i) \leq v(T) - \sum_{i \in T} v(i)$ for all S, T with $S \subset T$).

(iii) Show that Φ satisfies ADD, DUM and EFF.

Proposition 14.4. Φ is an anonymous solution.

Proof.

(i) First we show that

$$\rho^*(m^\sigma(v)) = m^{\rho\sigma}(v^\rho) \text{ for all } v \in G^n \text{ and } \rho, \sigma \in \pi(N).$$

This follows because for all $i \in N$: $(m^{\rho\sigma}(v^\rho))_{\rho\sigma(i)} =$
$v^\rho(\{\rho\sigma(1), \ldots, \rho\sigma(i)\}) - v^\rho(\{\rho\sigma(1), \ldots, \rho\sigma(i-1)\}) =$
$v(\{\sigma(1), \ldots, \sigma(i)\}) - v(\{\sigma(1), \ldots, \sigma(i-1)\}) =$
$(m^\sigma(v))_{\sigma(i)} = (\rho^*(m^\sigma(v)))_{\rho\sigma(i)}$.

(ii) Take $v \in G^n$ and $\rho \in \pi(N)$. Then, using (i), the fact that $\rho \to \rho\sigma$ is a surjection on $\pi(N)$ and the linearity of ρ^*:

$$\begin{aligned} \Phi(v^\rho) &= \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^\sigma(v^\rho) = \frac{1}{n!} \sum_\sigma m^{\rho\sigma}(v^\rho) = \\ &= \frac{1}{n!} \sum_\sigma \rho^*(m^\sigma(v)) = \rho^* \left(\frac{1}{n!} \sum_\sigma m^\sigma \right) = \rho^*(\Phi(v)). \end{aligned}$$

This proves the anonymity of Φ. □

One of the well-known axiomatizations (that is a characterizing list of properties) of the Shapley value is given in the next theorem.

Theorem 14.5 (cf. Shapley (1953)). There is a unique solution $f : G^n \to \mathbb{R}^n$ satisfying EFF, AN, DUM and ADD. This solution is the Shapley value.

Proof.

(i) It follows from proposition 14.4 and exercise 14.1 that Φ satisfies the four properties.

(ii) Conversely, suppose f satisfies the four properties. We have to show that $f = \Phi$. Take $v \in G^n$. Then $v = \sum_T c_T u_T$ with $c_T = \sum_{S:S \subset T} (-1)^{|T|-|S|} v(S)$. By the additivity property we then have $f(v) = \sum_T f(c_T u_T), \Phi(v) = \sum_T \Phi(c_T u_T)$.

So we have only to show that for all T and $c \in \mathbb{R}$:

$$f(cu_T) = \Phi(cu_T) \tag{14.5}$$

Take T and c. Note first that for all $i \in N \backslash T$:

$$cu_T(S \cup \{i\}) - cu_T(S) = 0 = cu_T(\{i\}) \text{ for all } S.$$

So, by the dummy property, we have

$$f_i(cu_T) = \Phi_i(cu_T) = 0 \text{ for all } i \in N \backslash T. \tag{14.6}$$

Now suppose that $i, j \in T, i \neq j$. Then there is a $\sigma \in \pi(N)$ with $\sigma(i) = j$, $\sigma(j) = i$, $\sigma(k) = k$ for $k \in N \backslash \{i, j\}$. It easily follows that $cu_T = \sigma(cu_T)$. Then the anonymity implies that $\Phi(cu_T) = \Phi(\sigma(cu_T)) = \sigma^* \Phi(cu_T)$, $\Phi_{\sigma(i)}(cu_T) = \Phi_i(cu_T)$. So

$$\Phi_i(cu_T) = \Phi_j(cu_T) \text{ for all } i, j \in T \tag{14.7}$$

and similarly $f_i(cu_T) = f_j(cu_T)$ for all $i, j \in T$.
Then efficiency, (14.6) and (14.7) imply that

$$f_i(cu_T) = \Phi_i(cu_T) = |T|^{-1}c \text{ for all } i \in T \tag{14.8}$$

Now (14.6) and (14.8) imply (14.5). So $f(v) = \Phi(v)$ for all $v \in G^n$.
□

Let $\langle N, v\rangle$ be an n-person game. For each non-empty coalition S we define a real number d_S, the *dividend* (cf. Harsanyi (1959)) in a recursive manner as follows.

$$\begin{aligned} d_T(v) &:= v(T) \text{ for all } T \text{ with } |T| = 1 \\ d_T(v) &:= |T|^{-1}(v(T) - \sum_{S \subset T, S \neq T} |S| d_S(v)) \text{ if } |T| > 1. \end{aligned}$$

The relation between dividends and the Shapley value is described in the next theorem. The Shapley value of a player in a game turns out to be the sum of all dividends of coalitions to which the player belongs.

Theorem 14.6 Let $v = \sum c_T u_T$. Then

(i) $|T| d_T(v) = c_T$ for all T.

(ii) The Shapley value $\Phi_i(v)$ for player i is equal to the sum of the dividends of the coalitions to which player i belongs, that is $\Phi_i(v) = \sum_{T: i \in T} d_T(v)$.

Proof. We have seen in the proof of theorem 14.5 that $\Phi(c_T u_T) = |T|^{-1} c_T e^T$ for each T, so by ADD, $\Phi(v) = \sum_T c_T |T|^{-1} e^T$. Hence, $\Phi_i(v) = \sum_{T: i \in T} c_T |T|^{-1}$. The only thing to show is that

$$c_T |T|^{-1} = d_T(v) \text{ for all } T. \tag{14.9}$$

We prove this by induction. If $|T| = 1$, then $c_T = v(T) = d_T(v)$. Suppose (14.9) holds for all $S \subset T$, $S \neq T$. Then $|T|d_T(v) = v(T) - \sum_{S\subset T, S\neq T} |S|d_S(v) = v(T) - \sum_{S\subset T, S\neq T} c_S = c_T$ because $v(T) = \sum_{S\subset T} c_S$.

□

Now we turn to the description of the Shapley value of Owen (1972, 1982) by means of the multilinear extension of a game. Let $\langle N, v\rangle$ be an n-person game. Consider the function $f : [0,1]^n \to \mathbb{R}$ on the hypercube $[0,1]^n$, defined by

$$f(x_1, x_2, \ldots, x_n) = \sum_{S\in 2^N} \left(\prod_{i\in S} x_i \prod_{i\in N\setminus S} (1 - x_i) \right) v(S) \quad (14.10)$$

In view of theorem 11.1, the set of extreme points of $[0,1]^n$ is equal to $\{e^S \mid S \in 2^N\}$. By exercise 14.2(i),

$$f(e^S) = v(S) \text{ for each } S \in 2^N. \quad (14.11)$$

So f can be seen as an extension of $\tilde{v} : \mathrm{ext}([0,1]^n) \to \mathbb{R}$ with $\tilde{v}(e^S) := v(S)$. In view of exercise 14.2(ii), f is called *the multilinear extension* of ($\tilde{v}$ or) v.

Exercise 14.2

(i) Prove (14.11).

(ii) Show that f is a multilinear function. [A function $g : \mathbb{R}^n \to \mathbb{R}$ is called multilinear iff g is of the form

$g(x) = \sum_{T\subset N} c_T (\prod_{i\in T} x_i)$.]

(iii) Prove that f is the unique multilinear extension of $\tilde{v}$ to $[0,1]^n$.

One can give a probabilistic interpretation of $f(x)$. Suppose that each of the players $i \in N$, independently, decides whether to co-operate (probability x_i) or not (probability $1 - x_i$). Then with probability $\prod_{i\in S} x_i \prod_{i\in N\setminus S}(1 - x_i)$ the coalition S forms, which has worth $v(S)$. Then $f(x)$ as given in (14.10) can be seen as the expectation of the worth of the formed coalition.

We denote by $D_k f(x)$ the derivative of f with respect to the k-th coordinate in x. Then we have the following result of Owen (1972), describing the Shapley value $\Phi_k(v)$ as the integral along the main diagonal of $[0,1]^n$ of $D_k f$.

Theorem 14.7 $\Phi_k(v) = \int_0^1 (D_k f)(t, t, \ldots, t)dt$ for each $k \in N$.

Proof.

$$D_k f(x) = \sum_{T:k\in T} \left(\prod_{i\in T\setminus\{k\}} x_i \prod_{i\in N\setminus T}(1 - x_i)\right) v(T)$$

$$- \sum_{S:k\notin S} \left(\prod_{i\in S} x_i \prod_{i\in N\setminus(S\cup\{k\})}(1 - x_i)\right) v(S)$$

$$= \sum_{S:k\notin S} \left(\prod_{i\in S} x_i \prod_{i\in N\setminus(S\cup\{k\})}(1 - x_i)\right) (v(S\cup\{k\}) - v(S)).$$

Hence, $\int_0^1 (D_k f)(t, t, \ldots, t)dt = \sum_{S:k\notin S} \left(\int_0^1 t^{|S|}(1 - t)^{n-|S|-1}dt\right)$ $(v(S \cup \{k\}) - v(S))$.

Using the well-known (beta)-integral formula

$$\int_0^1 t^{|S|}(1 - t)^{n-|S|-1}dt = \frac{|S|!(n - |S| - 1)!}{n!}$$

we obtain $\int_0^1 (D_k f)(t, \ldots, t)dt = \sum_{S:k\notin S} \frac{|S|!(n-|S|-1)!}{n!}(v(S \cup \{k\}) - v(S)) = \Phi_k(v)$ by (14.4). □

Example 14.8 Let $\langle N, v\rangle$ be the 3-person game with $v(1) = v(2) = v(3) = v(1, 2) = 0$, $v(1, 3) = 1$, $v(2, 3) = 2$, $v(1, 2, 3) = 4$.

Then $f(x_1, x_2, x_3) = x_1(1 - x_2)x_3 + 2(1 - x_1)x_2x_3 + 4x_1x_2x_3 = x_1x_3 + 2x_2x_3 + x_1x_2x_3$ for all $x \in [0,1]^3$. So $D_1 f(x) = x_3 + x_2x_3$. By theorem 14.7 we obtain:

$$\Phi_1(v) = \int_0^1 D_1 f(t,t,t)dt = \int_0^1 t + t^2 dt = \frac{5}{6},$$

Exercise 14.3 Calculate $\Phi_2(v)$ and $\Phi_3(v)$ using Owen's integral formula for the game v of example 14.8.

Exercise 14.4 Find a 3-person balanced game, where the Shapley value is not an element of the core.

Chapter 15

The τ-value

The τ-value is introduced in Tijs (1981) for quasi-balanced games and is based on two vectors, $M(v)$ and $m(v)$, for a game $\langle N, v\rangle$. The vector $M(v)$, called the *N-marginal vector,* has as i-th coordinate $M_i(v) := v(N) - v(N\setminus\{i\})$, which is the marginal contribution of player i to the grand coalition. $M_i(v)$ is also called the *utopia payoff* for player i in the grand coalition. If he wants more, then it is advantageous for the other players in N to throw player i out. The i-th coordinate of the *minimum right vector* $m(v)$ is defined by $m_i(v) := \max_{S:S\ni i}\left(v(S) - \sum_{j\in S\setminus\{i\}} M_j(v)\right)$. Note that $m_i(v) \geq R_v(S,i) := v(S) - \sum_{j\in S\setminus\{i\}} M_j(v)$ for each coalition S to which i belongs. $R_v(S,i)$ is called the *remainder* of i in the coalition S given v: it is the amount which remains for player i if coalition S forms and all the other players in S obtain their utopia payoff. A player i has a reason to ask at least $m_i(v)$ in the grand coalition N, by arguing that he can obtain that amount also by drumming up a coalition $\hat{S}$ with $m_i(v) = R_v(\hat{S}, i)$ and making all other players in $\hat{S}$ happy with the utopia payoff. The N-marginal vector and the minimum right vector are upper and lower bounds, respectively, for the core of the game as is seen in

Theorem 15.1 Let $\langle N, v\rangle$ be an n-person game and let $x \in C(v)$. Then $m(v) \le x \le M(v)$ i.e. $m_i(v) \le x_i \le M_i(v)$ for all $i \in N$.

Proof.

(i) $x_i = x(N) - x(N\backslash\{i\}) = v(N) - x(N\backslash\{i\}) \le v(N) - v(N\backslash\{i\})$ $= M_i(v)$ for each $i \in N$.

(ii) For each S with $i \in S$ we have in view of (i):
$x_i = x(S) - x(S\backslash\{i\}) \ge v(S) - \sum_{j\in S\backslash\{i\}} M_j(v) = R_v(S,i)$.
So $x_i \ge \max_{S:i\in S} R_v(S,i) = m_i(v)$ for each $i \in N$. □

An n-person game $\langle N, v\rangle$ is called *quasi-balanced* if

$(Q1)$ $m(v) \le M(v)$,

$(Q2)$ $\sum_{i=1}^{n} m_i(v) \le v(N) \le \sum_{i=1}^{n} M_i(v)$.

The set of n-person quasi-balanced games is denoted by Q^n.

Exercise 15.1 Prove that each balanced game is quasi-balanced.

For a game $v \in Q^n$, the τ-*value* $\tau(v)$ is the unique efficient payoff vector on the closed interval $[m(v), M(v)]$ in $\mathbb{R}^n$. See figure 20. Equivalently, $\tau(v)$ is the unique convex combination of $m(v)$ and $M(v)$, lying in the hyperplane E of efficient payoff vectors.

Example 15.2 Consider the market with three agents 1, 2, 3, where agent 1 possesses a house, with worth a dollars for him, which he wants to sell. The agents 2 and 3 are potential buyers,

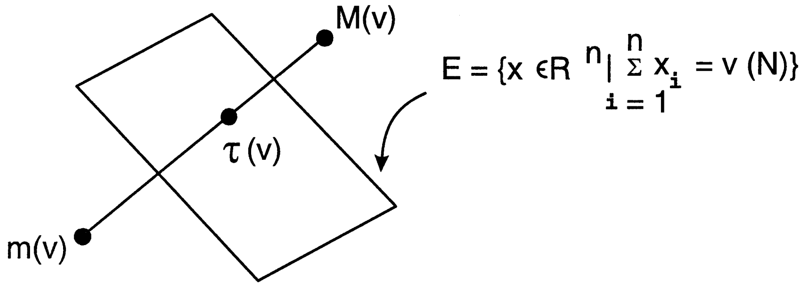

Figure 20.

where for agents 2 and 3 the house of 1 has worth b and c dollars, respectively. We suppose that $a < b < c$. This market situation reduced to the 3-person game $\langle N, w\rangle$ with $w(i) = 0$ for $i \in \{1,2,3\}$, $w(2,3) = 0$, $w(1,2) = b-a$, $w(1,3) = w(1,2,3) = c-a$. The zero-one normalization $\langle N, v\rangle$ of this game $\langle N, v\rangle$ is given by: $v(i) = 0$ for all $i \in N$, $v(1,3) = v(1,2,3) = 1$, $v(2,3) = 0$, $v(1,2) = q := (c-a)^{-1}(b-a)$.
Then $M(v) = (1,0,1-q)$, $m_1(v) = \max\{v(1), v(1,2)-M_2(v), v(1,3) -M_3(v), v(N) - M_2(v) - M_3(v)\} = \max\{0,q,q,q\} = q$, $m_2(v) = m_3(v) = 0$.
So $\tau(v) = m(v) + \alpha(M(v) - m(v)) = (q,0,0) + \alpha(1-q,0,1-q)$, when α is such that $\sum_{i=1}^{3} \tau_i(v) = 1$. Hence, $\alpha = \frac{1}{2}$, $\tau(v) = \left(\frac{1}{2} + \frac{1}{2}q, 0, \frac{1}{2} - \frac{1}{2}q\right)$.
Note that the core of v is the line segment with endpoints $(1,0,0)$ and $(q,0,1-q)$ and that $\tau(v)$ is in the barycenter of that line segment. Note further that $\Phi(v) \notin C(v)$. The payoff vector $\tau(v)$ corresponds to the situation in the original game $\langle N, v\rangle$, where agent 1 sells his house to agent 3 for a price $\frac{1}{2}(b+c)$; $\tau(w) = \left(\frac{1}{2}(b+c) - a, 0, c - \frac{1}{2}(b+c)\right)$.

Exercise 15.2 Prove that, for $v \in Q^2, \tau(v)$ coincides with the Shapley value for 2-person games and that $\tau(v)$ is the middle of the core $C(v)$. Show also that $C(v) = I(v)$.

We collect some properties of the τ-value in the next

Theorem 15.3 For $\tau : Q^n \to \mathbb{R}^n$ we have:

(i) Efficiency: $\sum_{i=1}^n \tau_i(v) = v(N)$ for all $v \in Q^n$,

(ii) Individual rationality: $\tau_i(v) \geq v(i)$ for all $v \in Q^n$, $i \in N$,

(iii) Dummy player property: $\tau_i(v) = v(i)$ if i is a dummy player in $v \in Q^n$,

(iv) Anonymity,

(v) Invariance w.r.t. strategic equivalence,

(vi) Weak proportionality: $\tau(v)$ is proportional to $M(v)$ if $v \in Q_0^n := \{v \in Q^n \mid m(v) = 0\}$.

Proof. We leave the proof of (i), (iv) and (v) to the reader.
For the proof of (ii) note that $v \in Q^n$ implies $m(v) \leq \tau(v) \leq M(v)$. Hence, $\tau_i(v) \geq m_i(v) \geq R_v(\{i\}, i) = v(i)$.
For the proof of (iii) note that if i is a dummy player in v, then $v(i) = R_v(\{i\}, i) \leq m_i(v) \leq \tau_i(v) \leq M_i(v) = v(i)$. So $\tau_i(v) = v(i)$.
For the proof of (vi), note that $\tau(v) = m(v) + \alpha(M(v) - m(v)) = \alpha M(v)$ if $v \in Q_0^n$. $\square$

In the next theorem a characterization of the τ-value is given. Here a role is played by the minimal right property. We say that a rule $f : Q^n \to \mathbb{R}^n$ has the *minimal right property* iff $f(v) = m(v) + f(v - m(v))$ for all $v \in Q^n$.

The minimal right property says that the payoffs w.r.t. f in v are the same as for the case that first minimal rights are paid to the players and then the amounts according to f in the adapted game $v - m(v)$, where $m(v)(S) = \sum_{i \in S} m_i(v)$ for each $S \in 2^N$.

Note that SEQ implies the minimal right property. So τ has this property.

Theorem 15.4 (Tijs (1987)). There is a unique allocation rule $f : Q^n \to \mathbb{R}^n$ with the efficiency property, the weak proportionality property and the minimal right property and that is the τ-value.

Proof. We have already seen that τ has the three mentioned properties. Conversely, suppose that f has the three properties. Take $v \in Q^n$. We have to prove that $f(v) = \tau(v)$. We prove first that for a game $v \in Q^n$, the game $w := v - m(v) \in Q_0^n$. This follows by noting that for all $i \in N$ and all S with $i \in S$:

$$\begin{aligned} R_w(S,i) &= w(S) - \sum_{j \in S \setminus \{i\}} M_j(w) \\ &= v(S) - \sum_{j \in S} m_j(v) - \sum_{j \in S \setminus \{i\}} (M_j(v) - m_j(v)) \\ &= R_v(S,i) - m_i(v), \text{ so} \end{aligned}$$

$$m_i(w) = \max_{S: i \in S} R_w(S,i) = \max_{S: i \in S} R_v(S,i) - m_i(v) = 0.$$

Then by the minimum right property and the weak proportionality property: $f(v) = m(v) + f(v - m(v)) = m(v) + \alpha M(v - m(v)) = m(v) + \alpha M(v) - \alpha m(v) = (1 - \alpha) m(v) + \alpha M(v)$ for a suitable α. By efficiency, $\sum_{i=1}^n f_i(v) = v(N)$. So $f(v)$ is the unique efficient point on the line segment $[m(v), M(v)]$ or $f(v) = \tau(v)$. $\square$

Exercise 15.3

(i) Let $\langle N, v\rangle$ be the 3-person game with $v(1,2,3) = 5$, $v(i) = 0$ for all $i \in N$, $v(1,2) = v(1,3) = 2$ and $v(2,3) = 3$.
Show that $\tau(v)$ is a multiple of $M(v)$.

(ii) Let $\langle N, v\rangle$ be the 99-person game with $v(N) = 1$, $v(S) = \frac{1}{2}$ if $\{1,2\} \subset S \neq N$, $v(2,3,4,\ldots,99) = v(1,3,4,\ldots,99) = \frac{1}{4}$ and $v(S) = 0$ otherwise. Prove that

$$\tau(v) = \frac{1}{200}(3,3,2,2,\ldots,2) \quad \text{and that } \tau(v) \notin C(v).$$

Exercise 15.4 Consider the glove game of example 10.2.

(i) Prove that in case $1 \leq |L| < |R| : C(v) = \{M(v)\} = \{\tau(v)\}$ and that $\Phi(v) \notin C(v)$.

(ii) Prove that in case $|L| = |R| : \tau(v) = \Phi(v) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

Chapter 16

The nucleolus

The nucleolus, introduced by D. Schmeidler (1969), is a one-point solution concept, defined for the family of those games, which have a non-empty imputation set. In the definition of the nucleolus the lexicographic order $\leq_L$ on $\mathbb{R}^p$ plays a role.
For $x, y \in \mathbb{R}^p$, we say that x is *lexicographically smaller* than y (notation $x \leq_L y$) if $x = y$ or if there exists an $s \in \{1, 2, \ldots, p\}$ such that $x_i = y_i$ for all $i < s$ and $x_s < y_s$. For example, for $p = 3$ we have $(0, 100, 100) \leq_L (1, -10, -10)$ and
$(10, 4, 100) \leq_L (10, 5, 6)$.
One can prove that a compact subset C of $\mathbb{R}^p$ always has a unique lexicographic minimum, that is a unique d such that $d \in C, d \leq_L c$ for all $c \in C$.
For example, the lexicographic minimum of

$$D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, x_1 \geq 0\} \text{ is } (0, -1).$$

For a game $\langle N, v \rangle$ with $I(v) \neq \emptyset$ one looks for each $x \in I(v)$ at the *excess vector* (or *complaint vector*) $\theta(x) \in \mathbb{R}^p$ with $p = 2^n - 1$ which has as its coordinates the excesses $e(S, x) := v(S) - \sum_{i \in S} x_i$

for each $S \in 2^N \backslash \{\emptyset\}$, and these excesses are written down in a decreasing order. For example, for the game $\langle \{1,2\}, v \rangle$ with $v(1) = 2$, $v(2) = 4$, $v(1,2) = 10$ for $\hat{x} = (4,6)$ we have $\theta(\hat{x}) = (e(\{1,2\},\hat{x}), e(\{1\},\hat{x}), e(\{2\},\hat{x})) = (0,-2,-2)$. It is easy to show that $\theta(\hat{x})$ is the lexicographic minimum of $\{\theta(x) \mid x \in I(v)\}$.
For all games $\langle N, v \rangle$ with $I(v) \neq \emptyset$ it turns out that $\{\theta(x) \mid x \in I(v)\}$ is compact and that there is a unique $\hat{x} \in I(v)$ such that $\theta(\hat{x})$ is the lexicographic minimum of $\{\theta(x) \mid x \in I(v)\}$.

Definition 16.1 Let $\langle N, v \rangle$ be a game with $I(v) \neq \emptyset$. The nucleolus $Nu(v)$ of the game is the unique imputation such that $\theta(Nu(v))$ is the lexicographic minimum of the set $\{\theta(x) \in \mathbb{R}^p \mid x \in I(v)\}$.

Roughly speaking, the nucleolus minimizes the maximal complaint.

Exercise 16.1 Let $\langle \{1,2\}, v \rangle$ be a 2-person game with $r := v(1,2) - v(1) - v(2) > 0$. Then, given $f^1 = (v(1,2) - v(2), v(2))$ and $f^2 = (v(1), v(1,2) - v(1))$, each element of $I(v)$ is of the form $x^\alpha := (1-\alpha)f^1 + \alpha f^2$ with $\alpha \in [0,1]$.
Calculate $\theta(x^\alpha)$ for each $\alpha \in [0, \frac{1}{2}]$ and $\alpha \in [\frac{1}{2}, 1]$ and prove that the nucleolus corresponds to $\alpha = \frac{1}{2}$. Prove also that the nucleolus coincides here with the Shapley value and the τ-value.

An interesting property of the nucleolus is given in

Theorem 16.2 Let $\langle N, v \rangle$ be a game with a non-empty core. Then $Nu(v) \in C(v)$.

Proof. Note that for $x \in I(v)$ we have

$$x \in C(v) \Leftrightarrow e(S,x) \leq 0 \text{ for all } S \in 2^N \backslash \{\emptyset\} \Leftrightarrow \theta(x) \leq 0. \qquad (16.1)$$

Take $z \in C(v)$. Then, by definition of the nucleolus, $\theta(Nu(v)) \leq_L \theta(z)$. So $\theta_1(Nu(v)) \leq \theta_1(z) \leq 0$, from which it follows that all coordinates of $\theta(Nu(v))$ are non-positive. Then from (16.1) we conclude that $Nu(v) \in C(v)$. $\square$

Example 16.3 Consider a 3-person market, where player 1 is a seller, player 2 is a (weak) buyer and player 3 a (strong) buyer. Player 1 has a chair with value 0 for him and with value 100 (200) for player 2 (3). This corresponds to the game $\langle N, v\rangle$ with $N = \{1,2,3\}$, $v(1,2) = 100$, $v(1,3) = v(1,2,3) = 200$ and $v(S) = 0$, otherwise. The core of this game consists of the vectors x in $\mathbb{R}^3_+$ with $x_2 = 0, x_1 + x_3 = 200$ and $x_1 \geq 100$. Using theorem 16.2, we know that the nucleolus is of the form $x_\alpha := (\alpha, 0, 200 - \alpha)$ with $\alpha \in [100, 200]$.
In figure 21 the graphs of the excess functions $\alpha \mapsto e(S, x_\alpha)$ on $[100, 200]$ are drawn.
Note that $e(S, x_\alpha) = -\alpha$ if $S = \{1\}$, $e(S, x_\alpha) = \alpha - 200$ if $S = \{3\}$ or $S = \{2,3\}$, $e(S, x_\alpha) = 100 - \alpha$ if $S = \{1,2\}$, and $e(S, x_\alpha) = 0$ otherwise. From this figure it follows that x_α with $\alpha = 150$ is the nucleolus of the game. Note that there is some similarity with solving $2 \times n$-matrix games (see Potters and Tijs (1992)).

Exercise 16.2 Prove that the τ-value for the game of example 16.3 is equal to the nucleolus. Calculate also the Shapley value of this game.

Exercise 16.3 Calculate the nucleolus of $\langle N, v\rangle$ where

(i) $\langle N, v\rangle$ is the game of exercise 11.5(ii),

(ii) $\langle N, v\rangle$ is the 4-person game with

$$v(S) = |S|^2 \text{ for all } S \in 2^N.$$

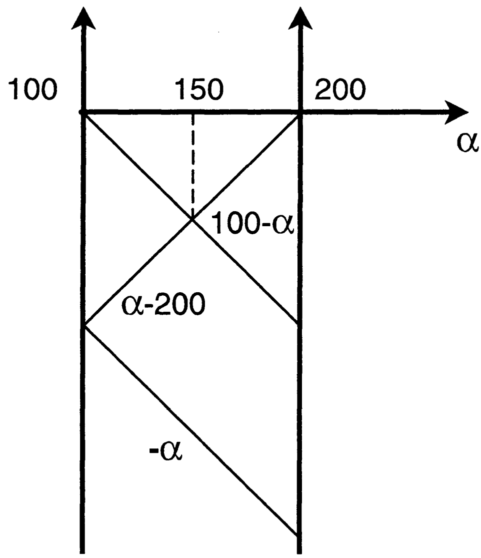

Figure 21.

Chapter 17

Bargaining games

A two-person *bargaining game* is an ordered pair (F, d), where F is a subset of $\mathbb{R}^2$ and where d is a point in $\mathbb{R}^2$. The elements of F are called *feasible outcomes* (utility pairs), which the players can reach if they cooperate. In case of no cooperation the *disagreement outcome* d results, with utility d_i for player $i \in \{1, 2\}$. The problem is on which outcome to agree. Many solutions are proposed and much work has been done in this field (Cf. A. Roth (1979), A. Roth (1985), H. Peters (1992)). The basis was laid by J. Nash (1950). Let $\mathcal{B}$ be the family of bargaining games (F, d) with the following properties:

(B.1) F is non-empty, convex, closed and comprehensive,[1]

(B.2) $\{x \in F \mid x \geq d\}$ is bounded,

(B.3) there exists an $x^0 \in F$ with $x^0 > d$.

[1] F is called *comprehensive* if $x \in F, y \leq x$ implies that $y \in F$.

In the following we need the notion of *comprehensive hull* of a set A, denoted by compr(A), where for $A \subset \mathbb{R}^S$

$$\text{compr}(A) := \{x \in \mathbb{R}^S \mid x \leq a \text{ for some } a \in A\}.$$

A map $f : \mathcal{B} \to \mathbb{R}^2$, with $f(F, d) \in F$ for each $(F, d) \in \mathcal{B}$, is called a *bargaining solution.* Examples are given below.

(i) The *dictatorial solution* D_i is defined as follows. $D_i(F, d)$ is the Pareto optimal point p of F with $p \geq d$ where the i-th coordinate is maximal.

(Recall that a point $p \in F$ is a *Pareto optimal point* if $F \cap \{x \in \mathbb{R}^2 \mid x \geq p\} = \{p\}$.) So $D_1(G, 0) = (2, 1)$ and $D_2(G, 0) = (1, 2)$ if G =compr(conv($\{(1, 2), (2, 1)\}$)). The Pareto set of G, denoted by Par(G) is the line segment with end points (1,2) and (2,1). (See figure 22.)

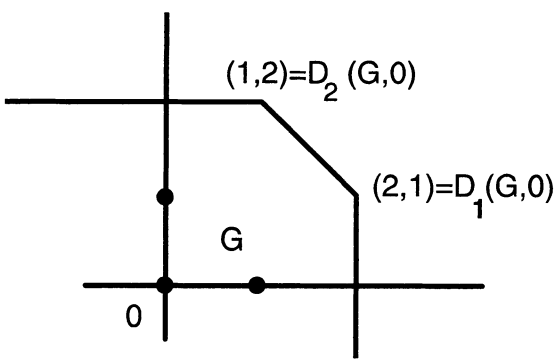

Figure 22.

(ii) The *egalitarian solution* E. For $(F, d) \in \mathcal{B}$, $E(F, d)$ is the point (d_1+m, d_2+m), where $m := \sup\{t | (d_1+t, d_2+t) \in F\}$.

Exercise 17.1 Prove that $E(F,d)$ is an element of the *weak Pareto boundary* $W\mathrm{Par}(F)$ of F where

$$W\mathrm{Par}(F) := \{x \in F \mid y \notin F \text{ for all } y > x\}.$$

Give an example (F,d) for which $E(F,d) \notin \mathrm{Par}(F)$.

(iii) The *Nash bargaining solution* (1950) $N : \mathcal{B} \longrightarrow \mathbb{R}^2$.
$N(F,d)$ is the unique point in $F_d := \{x \in F \mid x \geq d\}$ where

$$(x_1, x_2) \mapsto (x_1 - d_1)(x_2 - d_2)$$

is maximal. So $\{N(F,d)\} = \arg\max_{x \in F_d}((x_1,x_2) \mapsto (x_1 - d_1)(x_2 - d_2))$.

(iv) The *non-symmetric Nash solution with parameter* $t \in (0,1)$ is denoted by N^t. Here $N^t(F,d)$ is the unique point in F_d where $(x_1,x_2) \mapsto (x_1 - d_1)^t (x_2 - d_2)^{1-t}$ takes its maximum. Note that $N^{\frac{1}{2}}$ is the original Nash bargaining solution.

(v) The *Raiffa-Kalai-Smorodinsky solution* (1953, 1975) $K : \mathcal{B} \longrightarrow \mathbb{R}^2$. Here $K(F,d)$ is the unique Pareto optimal point of F lying on the line through d and the utopia point $u(F,d) \in \mathbb{R}^2$, where for $i \in \{1,2\}$:

$$u_i(F,d) := \max\{x_i \mid x \in F_d\}.$$

Exercise 17.2 Prove that $N(F,d) \in \mathrm{Par}(F)$. Give a bargaining game where the Nash solution and the Raiffa-Kalai-Smorodinsky solution differ.

Now we introduce some interesting properties which bargaining solutions may have or may not have.

(i) IR (*Individual Rationality*). A bargaining solution $f : \mathcal{B} \to \mathbb{R}^2$ is called individual rational if for all $(F, d) \in \mathcal{B}$ we have $f(F, d) \geq d$.

(ii) EFF (*efficiency*). A bargaining solution f is called efficient if $f(F, d) \in \mathrm{Par}(F)$ for all $(F, d) \in \mathcal{B}$.

(iii) MON (*Monotonicity*). A bargaining solution f is called monotonic if for all $(F, d), (G, d) \in \mathcal{B}$ with $F \subset G$ we have $f(F, d) \leq f(G, d)$.

(iv) RMON (*restricted monotonicity*). A bargaining solution is said to have RMON-property if for all $(F, d), (G, d) \in \mathcal{B}$ with $F \subset G$ and the utopia point $u(F, d)$ equal to $u(G, d)$, we have $f(F, d) \leq f(G, d)$.

(v) IIA (*independence of irrelevant alternatives*). $f : \mathcal{B} \to \mathbb{R}^2$ has the IIA-property if for all $(F, d), (G, d) \in \mathcal{B}$ with $F \subset G$ and $f(G, d) \in F$ we have $f(F, d) = f(G, d)$.

(vi) COV (*covariance with affine transformations*). f has the COV-property if for all $(F, d) \in \mathcal{B}$ and all transformations of the form

$$(x_1, x_2) \mapsto (a_1 x_1 + b_1, a_2 x_2 + b_2) \text{ where } a_1, a_2 > 0$$

we have $af(F, d) + b = f(aF + b, ad + b)$. (Here $aF = \{ax = (a_1 x_1, a_2 x_2) \in \mathbb{R}^2 \mid (x_1, x_2) \in F\}$.)

(vii) SYM (*symmetry*). A solution $f : \mathcal{B} \to \mathbb{R}^2$ is called symmetric if for all (F, d) we have $f(V(F), V(d)) = V(f(F, d))$, where $V : \mathbb{R}^2 \to \mathbb{R}^2$ is the map with $V(x_1, x_2) = (x_2, x_1)$.

Note that the egalitarian solution does not have the property COV, but it has the properties MON, SYM and also WEFF (*weak efficiency*) which means that the egalitarian solution assigns to each bargaining game a point of the weak Pareto set of the outcome space. The next theorem shows that three properties together with a weak form of COV namely COTR characterize the egalitarian solution. We say that $f : \mathcal{B} \to \mathbb{R}^2$ has the COTR-property (*covariance with translation*) if $f(F+a, d+a) = f(F,d) + a$ for all $a \in \mathbb{R}^2$.

Theorem 17.1 (*Axiomatic characterization of the egalitarian solution*). $E : \mathcal{B} \to \mathbb{R}^2$ is the unique solution with the properties MON, SYM, WEFF and COTR.

Proof. Let $f : \mathcal{B} \to \mathbb{R}^2$ be a bargaining solution with the properties MON, SYM, COTR and WEFF. We have to show that $f = E$. Take $(F,d) \in \mathcal{B}$. Let $e := E(F,d)$ and let $G := \{x \in \mathbb{R}^2 \mid x \leq e\}$. We have to prove that $f(F,d) = e$. Because $V(G-d) = G-d$ and $V(0) = 0$, we conclude from SYM and WEFF: $f(G-d, 0) = e-d$. Then by COTR: $f(G,d) = e$. Since $G \subset F$, MON implies

$$f(F,d) \geq f(G,d) = e. \tag{17.1}$$

If $e \in \mathrm{Par}(F)$, then (17.1) implies $f(F,d) = e$. Suppose $e \notin \mathrm{Par}(F)$. Then without loss of generality we suppose that $e_2 = u_2(F,d)$. For each $n \in \mathbb{N}$ let

$$F_n := \mathrm{compr}(cl(\mathrm{conv}(F \cup \{(d_1, e_2 + \frac{1}{n})\}))).$$

Since $E(F_n, d) \in \mathrm{Par}(F_n)$, we know from the foregoing that $E(F_n, d) = f(F_n, d)$. Further $\lim_{n\to\infty} E(F_n, d) = e = \lim_{n\to\infty} f(F_n, d)$.

Since $F \subset F_n$ for all n, by MON we have $f(F,d) \leq f(F_n,d)$. Taking the limit gives $f(F,d) \leq e$. Together with (17.1) we find $f(F,d) = e$. □

It is natural to look for monotonic and efficient solutions. It is impossible to find one as the next proposition shows

Proposition 17.2 Monotonic and efficient solutions do not exist.

Proof. Suppose $f : \mathcal{B} \longrightarrow \mathbb{R}^2$ is monotonic and efficient. Let $G :=$compr(conv($\{(1,2),(2,1)\}$)), $H := \{x \in \mathbb{R}^2 \mid x \leq (1,2)\}$ and $K = \{x \in \mathbb{R}^2 \mid x \leq (2,1)\}$. The efficiency of f then implies $f(H,0) = (1,2)$ and $f(K,0) = (2,1)$. Since $H \subset G$ and $K \subset G$, the monotonicity of f implies $f(G,0) \geq f(H,0) = (1,2)$, $f(G,0) \geq f(K,0) = (2,1)$. So $f(G,0) \geq (2,2)$. But $(2,2) \notin G$ and then $f(G,0) \notin G$, a contradiction. □

Now we want to characterize all efficient solutions which are also restricted monotonic and have the COV-property. Each continuous map $k : [0,1] \longrightarrow [0,1] \times [0,1]$ with

(i) $k(0) \in \text{conv}\{(0,1),(1,0)\}$ and $k(1) = (1,1)$,

(ii) $k(x) \geq k(y)$ if $x \geq y$,

gives rise to a solution $f^k : \mathcal{B} \longrightarrow \mathbb{R}^2$ with the properties EFF, COV and RMON. First we define $f^k(F,d)$ for problems (F,d) with $d = 0$ and utopia point $u(F,d) = (1,1)$ as the unique Pareto optimal point of F lying on the curve $\{k(x) | 0 \leq x \leq 1\}$. For general (G,d) we define

$$f^k(G,d) := d + u(G-d,0)f^k(u(G-d,0)^{-1}(G-d,0)).$$

We call f^k the *solution corresponding to the monotonic curve k*.

Theorem 17.3 (Peters-Tijs (1985)). Let $f : \mathcal{B} \to \mathbb{R}^2$ be a solution. Then f has the properties EFF, COV and RMON iff there is a monotonic curve $k : [0,1] \to [0,1] \times [0,1]$ such that $f = f^k$.

Proof. The 'if' part is trivial. For the 'only if' part, suppose there is given an $f : \mathcal{B} \to \mathbb{R}^2$ with the properties EFF, COV and RMON. We construct k as follows. Take $k(t) := f(G^t, 0)$ where

$$G^t := \text{compr}(\text{conv}(\{t,1),(1,t)\})).$$

By EFF we know that $k(0) \in \text{Par}(G^0)$ $=\text{conv}\{(0,1),(1,0)\}$ and $k(1) \in \text{Par}(G^1) = \{(1,1)\}$. RMON implies that $k(x) \geq k(y)$ if $x \geq y$. Also k is continuous because for $0 \leq s \leq t \leq 1$ we have in view of EFF and RMON: $k_1(s) \leq k_1(t), k_2(s) \leq k_2(t)$, $k_1(s) + k_2(s) = s+1, k_1(t) + k_2(t) = t+1$, which implies: $k_i(t) - k_i(s) \in [0, t-s]$ for $i \in \{1,2\}$. To prove that $f = f^k$, we have in view of COV only to show that $f(F,0) = f^k(F,0)$ if $u(F,0) = (1,1)$. Let $t := f_1^k(F,0) + f_2^k(F,0)$. Now $f(G^t,0) = f^k(G^t,0)$ by definition of k. Let $D := G^t \cap F$. Then by RMON: $f^k(D,0) \leq f^k(G^t,0)$, $f^k(D,0) \leq f^k(F,0)$, $f(D,0) \leq f(G^t,0)$, $f(D,0) \leq f(F,0)$ and by EFF all these inequalities are equalities. This implies $f(F,0) = f(D,0) = f(G^t,0) = f^k(G^t,0) = f^k(D,0) = f^k(F,0)$. □

As an easy corollary we obtain from this theorem

Theorem 17.4 (*Axiomatic characterization of the Raiffa-Kalai-Smorodinsky solution*).
The Raiffa-Kalai-Smorodinsky solution is the unique solution with the properties RMON, EFF, COV and SYM.

Exercise 17.3 Show that there is a monotonic curve k_i such that $D_i = f^{k_i}$ for $i \in \{1,2\}$.

Now we want to concentrate on IIA-solutions. If we don't ask for other conditions we obtain many IIA-solutions, such as the egalitarian solution, the non-symmetric Nash solutions, the dictatorial solutions. Furthermore, each function $f : \mathbb{R}^2 \to \mathbb{R}$ which has the property that $\arg\max_{x \in S} f$ consists of one point for each $(S, d) \in \mathcal{B}$ gives rise to an IIA-solution: assign to (S, d) the unique point where f restricted to S is maximal.

In the next theorem (due to R. de Koster, et al. (1983)) we describe the solutions satisfying IIA, IR, COV and EFF. We will make use of the following

Exercise 17.4 Let $t \in (0, 1)$, let $(S, d) \in \mathcal{B}$

(i) Prove that $\arg\max_{x \in S_d}\{(x_1, x_2) \mapsto (x_1 - d_1)^t (x_2 - d_2)^{1-t}\}$ consists of one point. Call this point (a_1, a_2).

(ii) Prove that there is a unique straight line through (a_1, a_2), which separates (weakly) the convex sets S and $\{x \in \mathbb{R}^2 \mid x \geq d, (x_1 - d_1)^t (x_2 - d_2)^{1-t} \geq (a_1 - d_1)^t (a_2 - d_2)^{1-t}\}$.

(iii) Show that the slope of the line L is equal to $\frac{1}{t-1}$ times the slope of the line through (d_1, d_2) and (a_1, a_2).

(iv) Show that $p := \left(d - 1 + \frac{a_1 - d_1}{t}, d_2\right)$ and $q := \left(d_1, d_2 + \frac{a_2 - d_2}{1-t}\right)$ are on L and that $a = tp + (1-t)q$. (See figure 23.)

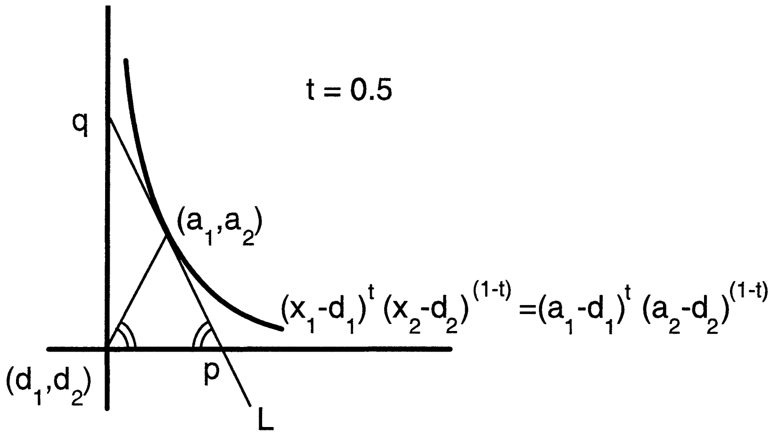

Figure 23.

Theorem 17.5 Let $f : \mathcal{B} \to \mathbb{R}^2$ be a bargaining solution with the properties IIA, COV, IR and EFF. Then $f \in \{D_1, D_2\} \cup \{N^t \mid 0 < t < 1\}$.

Proof. Let H be the half space $\{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 1\}$. By EFF and IR, $f(H, 0)$ is of the form $(t, 1-t)$, where $0 \leq t \leq 1$. We consider three cases: $0 < t < 1$, $t = 0$ and $t = 1$

Case 1. We show that $f = N^t$ if $t \in (0,1)$. Take $(S, d) \in \mathcal{B}$. Let $(a_1, a_2) := N^t(S, d)$ and let K be the closed half space with $(d_1, d_2) \in K$ and with boundary the straight line L of exercise 17.4. Let A be the affine map with

$$A(x_1, x_2) := \left(d_1 + \frac{a_1 - d_1}{t} x_1, d_2 + \frac{a_2 - d_2}{1-t} x_2\right)$$

Then $A(H) = K, A(0) = d$. So by COV: $f(K, d) = A(t, 1-t) = (a_1, a_2)$.
By IIA we obtain $f(S, d) = f(K, d) = (a_1, a_2)$ since $S \subset K$.

Since $(a_1, a_2) \in S$ and $\{(x_1, x_2) \in \mathbb{R}^2_+ \mid (x_1 - d_1)^t(x_2 - d_2)^{1-t} \leq (a_1 - d_1)^t(a_2 - d_2)^{1-t}\} \supset S_d$ we obtain $N^t(S, d) = (a_1, a_2)$. So $N^t = f$.

Case 2. Suppose $f(H, 0) = (0, 1)$. We show that $f = D_2$. Take $(S, d) \in \mathcal{B}$. Let $p = (p_1, p_2)$ be the unique point on the Pareto boundary of S_d with $p_2 = u_2(S, d)$ and let $s = (d_1, p_2)$. We want to show that $f(S, d) = p$.
Suppose $f(S, d) = q \neq p$. (See figure 24). By EFF then $(T, d) \in \mathcal{B}$ where T is the closed half space below the line through s and q. Let $D := T \cap S$. By IIA we obtain $f(D, d) = q$. On the other hand, by COV it follows from $f(H, 0) = (0, 1)$ that $f(T, d) = s$. By IIA, we obtain $f(D, d) = f(T, d) = s$, a contradiction because $s \neq q$. So $f(S, d) = p$.

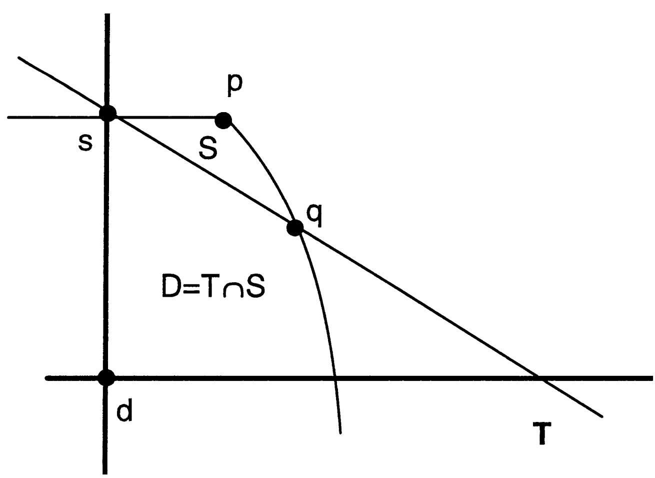

Figure 24.

Case 3. This case is similar to case 2 and we obtain then $f = D_1$. □

From theorem 17.5 we immediately obtain

Theorem 17.6 (*Axiomatic characterization of the Nash solution*). The Nash solution is the unique solution with the properties IR, IIA, COV, EFF and SYM.

Exercise 17.5 Prove theorem 17.6.

Exercise 17.6 Prove that the dictatorial solution D_i $(i = 1, 2,)$ is not monotonic and not symmetric. Prove that $V \circ D_1 = D_2 \circ V$. Prove that the egalitarian solution is not covariant with affine transformations and that E is symmetric.

In their 1975-paper Kalai-Smorodinsky characterize the RKS-solution with the aid of individual monotonicity (IMON). In the following theorem we describe the relation between IMON and RMON. First we give

Definition 17.7 A solution $f : \mathcal{B} \to \mathbb{R}^2$ is called *individual monotonic* if for all $(S, d), (T, d) \in \mathcal{B}$ with $S \subset T$ and $u_i(S, d) = u_i(T, d)$ $(i \in \{1, 2\})$ we have $f_{3-i}(S, d) \leq f_{3-i}(T, d)$.

Theorem 17.8 For a solution f with properties COV and EFF the properties RMON and IMON imply each other.

Proof. Obviously, IMON implies RMON. Conversely, suppose f has the properties COV, EFF and RMON. In view of COV we consider only bargaining problems with 0 as disagreement point and we restrict to individual monotonicity w.r.t player 2 $(i = 1)$.

So take $(S,0),(T,0) \in \mathcal{B}$ with $S \subset T$ and $u_1(S,0) = u_1(T,0)$. We have to show

$$f_2(S,0) \leq f_2(T,0) \tag{17.2}$$

In view of the fact that EFF and RMON imply $f(\mathrm{compr}(F_d),d) = f(F,d)$ for all $(F,d) \in \mathcal{B}$ we suppose w.l.o.g that $S = \mathrm{compr}(S_0), T = \mathrm{compr}(T_0)$. Let $U := T \cap \{x \in \mathbb{R}^2 \mid x_2 \leq u_2(S,0)\}$ and $V := (1,\alpha)U$ (see figure 25) where $\alpha := (u_2(S,0))^{-1}u_2(T,0)$. Then $(U,0)$ and $(V,0) \in \mathcal{B}$. From $S \subset U$ and $T \subset V$ by RMON we obtain

$$f(S,0) \leq f(U,0) \tag{17.3}$$

$$f(T,0) \leq f(V,0). \tag{17.4}$$

By COV we obtain

$$f_1(V,0) = f_1(U,0). \tag{17.5}$$

Then (17.4) and (17.5) imply

$$f_1(T,0) \leq f_1(V,0) = f_1(U,0). \tag{17.6}$$

Since $f(U,0), f(T,0) \in \mathrm{Par}(T)$, inequality (17.6) implies

$$f_2(T,0) \geq f_2(U,0). \tag{17.7}$$

Now (17.7) and (17.3) imply (17.2). □

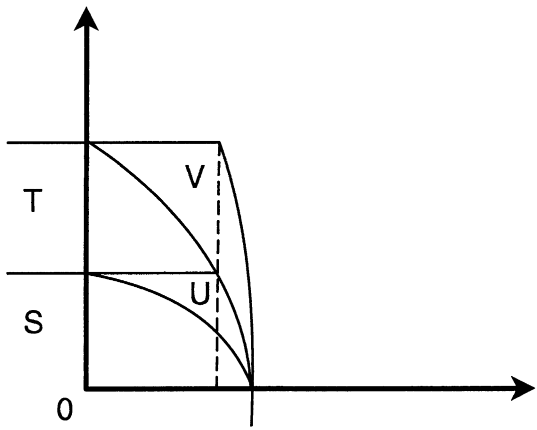

Figure 25.

Exercise 17.7 Show that N has not the IMON property.

Chapter 18

NTU-games

An NTU-*game* is a pair $\langle N, V \rangle$ where $N = \{1, 2, \ldots, n\}$ and where V is a map assigning to each $S \in 2^N \backslash \{\emptyset\}$ a subset $V(S)$ of $\mathbb{R}^S$ such that the following properties hold for each S:
(P.1) $V(S)$ is a non-empty closed subset of $\mathbb{R}^S$.
(P.2) $V(S)$ is comprehensive i.e. if $x \in V(S)$ and $y \in \mathbb{R}^S$ such that $y \leq x$, then $y \in V(S)$.
(P.3) $\{x \in \mathbb{R}^n \mid x_i \geq v(i), x \in V(N)\}$ is bounded, where $V(\{i\}) = (-\infty, v(i)]$.
(NTU is abbreviation of 'Non Transferable Utility').

Interpretation. The elements of N are players, who can cooperate. If coalition S forms, then each of the payoff vectors $x \in V(S)$ is attainable, giving reward (utility) x_i to player $i \in S$.

Example 18.1 Each n-person TU-game $\langle N, v \rangle$ gives rise to an n-person NTU-game $\langle N, V \rangle$, where

$$V(S) = \left\{ x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq v(S) \right\}.$$

Example 18.2 A 2-person *bargaining game* (F, d) can be seen as a 2-person NTU-game $\langle\{1,2\}, V\rangle$ where $V(\{1\}) = (-\infty, d_1]$, $V(\{2\}) = (-\infty, d_2]$, $V(\{1,2\}) = F$.

Example 18.3 An exchange market $\langle N, \mathbb{R}^m_+, f^1, f^2, \ldots, f^n, u_1, u_2, \ldots, u_n\rangle$ gives rise to an NTU-game. Here N is the set of agents, $\mathbb{R}^m_+$ is the commodity space, $f^i \in \mathbb{R}^m_+$ is the initial commodity bundle of agent $i \in N$, and $u_i : \mathbb{R}^m_+ \to \mathbb{R}$ the (smooth) utility function of agent i. An admissible reallocation of coalition S is a collection of commodity bundles $(x^i)_{i\in S}$ such that $x^i \in \mathbb{R}^m_+$ for each $i \in S$ and $\sum_{i\in S} x^i = \sum_{i\in S} f^i$. $A(S)$ is the set of admissible reallocations of S.
The corresponding NTU-game is defined by

$$V(S) = \{t \in \mathbb{R}^S \mid \exists_{(x^i)_{i\in S}\in A(S)}[u_i(x^i) \geq t_i \text{ for all } i \in S]\}.$$

Example 18.4 Three voters 1, 2, 3 have to decide about one of two alternatives a_1, a_2. The majority decides. The utilities of the voters are given in the next table.

	a_1	a_2
1	5	1
2	2	3
3	4	3

This situation corresponds to a 3-person NTU-game $\langle N, V\rangle$ where $N = \{1,2,3\}$, $V(1) = (-\infty, 1]$, $V(2) = (-\infty, 2]$, $V(3) = (-\infty, 3]$, $V(1,2) = \text{compr}\{(5,2),(1,3)\}$, $V(1,3) = \text{compr}\{(5,4),(1,3)\}$, $V(2,3) = \text{compr}\{(2,4),(3,3)\}$, $V(1,2,3) = \text{compr}\{(5,2,4),(1,3,3)\}$.

Also games in strategic form give rise to NTU-games.

Let $\Gamma = \langle X_1, X_2, \ldots, X_n, K_1, K_2, \ldots, K_n\rangle$ be a game in strategic (or normal) form. A corresponding NTU-game $\langle N, V_\Gamma\rangle$ is defined by $N = \{1, 2, \ldots, n\}$ and for $S \in 2^N \backslash \{\emptyset, N\}$:
$V_\Gamma(S) = \{t \in \mathbb{R}^S \mid \exists_{x_S \in X_S} \forall_{x_{N\backslash S} \in X_{N\backslash S}} \forall_{i \in S}[K_i(x_S, x_{N\backslash S}) \geq t_i]\}$,
$V_\Gamma(N) = \text{compr}\{K(x_N) \mid x_N \in X_N\}$.
Here $X_S := \prod_{i\in S} X_i$ is the set of *coordinated actions* of coalition S.

Exercise 18.1 An NTU-game is called *superadditive* iff for all $S, T \in 2^N \backslash \{\emptyset\}$ with $S \cap T = \emptyset$ we have $V(S) \times V(T) \subset V(S \cup T)$. Prove that V_Γ is superadditive.

Conversely, given an NTU-game $\langle N, V\rangle$ we can construct (cf. Borm and Tijs (1992)) a game in strategic form, the *claim game* $\Gamma_V = \langle X_1, X_2, \ldots, X_n, K_1, K_2, \ldots, K_n\rangle$ where for each $i \in N$: $X_i = P_i \times \mathbb{R}$ with $P_i := \{S \in 2^N \mid i \in S\}$, $K_i((S_1, t_1), (S_2, t_2), \ldots,$
$(S_n, t_n)) = t_i$ iff $S_i = S_j$ for all $j \in S_i$ and $(t_j)_{j\in S_i} \in V(S_i)$,
$K_i((S_1, t_1), (S_2, t_2), \cdots (S_n, t_n) = \min\{t_i, v(i)\}$ otherwise where
$v(i) = \max\{t \mid t \in V(\{i\})\}$.
A strategy $(S_i, t_i) \in P_i \times \mathbb{R}$ can be interpretated as "I want that coalition S_i forms, and I want the payoff t_i". If the wishes with respect to the coalition formation are consistent for all $j \in S_i$ and $(t_j)_{j\in S_i}$ is attainable for S_i, then S_i forms and player i obtains the payoff t_i. Otherwise, the 'individual rational payoff' $v(i)$ results or t_i if $t_i < v(i)$.

Theorem 18.5 Let $\langle N, V\rangle$ be an NTU-game with corresponding claim game Γ. Then

(i) $V \subset V_\Gamma$

(ii) If $\langle N, V\rangle$ is superadditive, then $V_\Gamma = V$.

Proof.

(i) Take $S \in 2^N \backslash \{\emptyset\}$ and take $t \in V(S)$. Let $\hat{x}_i = (S, t_i)$ for each $i \in S$. Then for each $x_{N\backslash S} \in X_{N\backslash S}$ and for each $i \in S$: $K_i(\hat{x}_S, x_{N\backslash S}) = t_i$. Hence $t \in V_\Gamma(S)$.

(ii) Take $S \in 2^N \backslash \{\emptyset\}$ and $t \in V_\Gamma(S)$. Then there are strategies $\hat{x}_i = (\hat{S}_i, \hat{u}_i) \in X_i$ for each $i \in S$ such that $K_i(\hat{x}_S, x_{N\backslash S}) \geq t_i$ for all $x_{N\backslash S} \in X_{N\backslash S}$. Especially $K_i(\hat{x}_S, (\{j\}, v(j))_{j \in N\backslash S}) \geq t_i$ for all $i \in S$.

The strategies $\hat{x}_i (i \in S)$ and the strategies $(\{j\}, v(j))$ $(j \in N\backslash S)$ determine a partition $T_1, T_2, \ldots, T_r$ of S. Here a part T_l is either a group of players k with $T_l = S_k$ for all $k \in T_l$ and $(\hat{u}_i)_{i \in T_l} \in V(T_l)$ or one player: $T_l = \{k\}$ and then $K_k(\hat{x}_s, (\{k\}, v(j))_{j \in N\backslash S}) \leq v(i)$. By comprehensiveness

$(t_i)_{i \in T_k} \in V(T_k)$ for $k \in \{1, \ldots, r\}$.

Then $(t_i)_{i \in S} \in \prod_{k=1}^r V(T_k) \subset V(S)$, where the last inclusion follows from the superadditivity of V. Hence $V_\Gamma \subset V$. □

Exercise 18.2 A TU-game is called superadditive iff $v(S \cup T) \geq v(S) + v(T)$ for all disjoint $S, T \in 2^N$. Prove: $\langle N, v \rangle$ is superadditive iff the corresponding NTU-game $\langle N, V \rangle$ is superadditive.

Exercise 18.3 Let $\Gamma = \langle X_1, X_2, \ldots, X_n, K_1, K_2, \ldots, K_n \rangle$ be a strategic form game. Define the NTU-game $\langle N, V_* \rangle$ by $V_*(S) = \{t \in \mathbb{R}^S \mid \forall_{x_{N\backslash S} \in X_{N\backslash S}} \exists_{x_S \in X_S} \forall_{i \in S} [K_i(x_S, x_{N\backslash S}) \geq t_i]\}$ if $S \neq N$, and $V_*(N) = \{t \in \mathbb{R}^N \mid \exists_{x_N \in X_N} K(x_N) \geq t\}$.

(i) Prove that $V_\Gamma \subset V_*$

(ii) Construct a game Γ such that V_* is not superadditive.

Exercise 18.4 Let $\Gamma = \langle X_1, \ldots, X_n, K_1, K_2, \ldots, K_n \rangle$ be a finite game. For $S \subset N$, let $\tilde{X}_S$ be the set of *correlated strategies* of coalition S, that is the set of probability measures on X_S. Define games V_Γ^α and V_Γ^β by
$V_\Gamma^\alpha(S) = \{t \in \mathbb{R}^S \mid \exists_{\hat{x}^S \in \tilde{X}_S} \forall_{y^{N\setminus S} \in \tilde{X}_{N\setminus S}} \forall_{i \in S}[K_i(\hat{x}^S, y^{N\setminus S}) \geq t_i]\}$
$V_\Gamma^\beta(S) = \{t \in \mathbb{R}^S \mid \forall_{x^{N\setminus S} \in \tilde{X}_{N\setminus S}} \exists_{x^S \in \tilde{X}_S} \forall_{i \in S}[K_i(x^S, x^{N\setminus S}) \geq t_i]\}$.
Prove that V_Γ^α is superadditive and that $V_\Gamma^\alpha \subset V_\Gamma^\beta$.

Now we introduce cores and strong cores for NTU-games, similar to D-cores for TU-games.
Let $\langle N, V \rangle$ be an NTU-game. For each $S \in 2^N \setminus \{\emptyset\}$, let

$$\begin{aligned} \mathrm{dom}(S) \ &: \ = \{z \in \mathbb{R}^S \mid \exists_{y \in V(S)} \forall_{i \in S}[y_i > z_i]\} \\ w\mathrm{dom}(S) \ &: \ = \{z \in \mathbb{R}^S \mid \exists_{y \in V(S)}[y \neq z, \forall_{i \in S}[y_i \geq z_i]]\}. \end{aligned}$$

The elements of $\mathrm{dom}(S)$ and $w\mathrm{dom}(S)$ are elements dominated and weakly dominated respectively by coalition S. Then the *core* $C(V)$ and the *strong core* $SC(V)$ of V are defined by

$$\begin{aligned} C(V) \quad &:= \quad \{x \in V(N) \mid \neg\exists_S[x_S \in \mathrm{dom}(S)]\} \\ SC(V) \quad &:= \quad \{x \in V(N) \mid \neg\exists_S[x_S \in w\mathrm{dom}(S)]\}. \end{aligned}$$

Since $\mathrm{dom}(S) \subset w\mathrm{dom}(S)$ for all S we have: $SC(V) \subset C(V)$.

Exercise 18.5

(i) Prove that for a superadditive game $\langle N, V \rangle$ corresponding to a TU-game $\langle N, v \rangle$, the D-core and the strong core coincide.

(ii) Prove that both $C(V) \neq \emptyset$ and $SC(V) \neq \emptyset$ for a superadditive 2-person game $\langle \{1, 2\}, V \rangle$.

Theorem 18.6 Let Γ be the claim game corresponding to the NTU-game $\langle N, V \rangle$. Then

(i) for each $x \in V(N) \backslash \mathrm{int}\ V(N)$ with $x_i \geq v(i)$ for each $i \in N$, the strategy N-tuple $(N, x_i)_{i \in N}$ is a Nash equilibrium of Γ.

(ii) each $x \in SC(V)$ corresponds to a strong Nash equilibrium of the claim game Γ.

Exercise 18.6 Prove theorem 18.6.
(An n-tupel $x = (x_1, x_2, \ldots, x_n) \in \Pi_{i \in N} X_i$ for a strategic game $\langle X_1, X_2, \ldots, X_n, K_1, \ldots, K_n \rangle$ is called a *strong Nash equilibrium* if there is no $S \subset N, S \neq \emptyset$ with an $y_S \in \Pi_{i \in S} X_i$ such that $K_i(y_S, x_{N \backslash S}) > K_i(x)$ for all $i \in S$.)

Chapter 19

The NTU-value

Let $\langle N, V\rangle$ be an NTU-game. A (*comparison weight*) vector $\lambda \in \Delta := \{\mu \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \mu_i = 1\}$ is called *V-feasible* if for all $S \in 2^N \backslash \{\emptyset\}$:

$$v_\lambda(S) := \sup \left\{ \sum_{i \in S} \lambda_i x_i \mid x \in V(S) \right\} < \infty.$$

So, a V-feasible λ generates a TU-game $\langle N, v_\lambda\rangle$. Let us denote the Shapley value of v_λ by $\Phi(v_\lambda)$. For $\langle N, V\rangle$, the *Shapley NTU-value* is defined by

$$SH(V) := \{x \in V(N) \mid \exists_{\lambda \in \Delta}[\lambda \text{ is } V\text{-feasible}, \Phi(v_\lambda) = \lambda x]\}.$$

Here λx is the vector in $\mathbb{R}^n$ with $\lambda_i x_i$ as the i-th coordinate. A vector $x \in V(N)$ with $\Phi(v_\lambda) = \lambda x$ is called a Shapley value of V corresponding to λ. A game can have more than one Shapley value. The first two theorems show that the NTU-value coincides for TU-games with the TU-value of Shapley and for bargaining games with the Nash bargaining solution.

Theorem 19.1 (Coincidence of SH and Φ for TU-games.)
Let $\langle N, v\rangle$ be a TU-game and let $\langle N, V\rangle$ be the corresponding NTU-game. Then $\{\Phi(v)\} = SH(V)$.

Proof. Take $\lambda \in \Delta$. If $\lambda^T \neq \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, then λ is not V-feasible. Let λ^p be the projection of λ on the hyperplane $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$. Then $\lambda^p \neq 0$, $L := \{v(N)e^1 + \alpha\lambda^p \mid \alpha \in \mathbb{R}\}$ is a subset of $V(N)$ and $\sup\{\sum_{i=1}^n \lambda_i x_i \mid x \in L\} = \infty$].
For $\hat{\lambda}^T := \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, $v_{\hat{\lambda}}(S) = \sup\left\{\frac{1}{n}\sum_{i\in S} x_i \mid x \in V(S)\right\} = \frac{1}{n}v(S)$ for each S. So $\Phi(v_{\hat{\lambda}}) = \frac{1}{n}\Phi(v) = \hat{\lambda}\Phi(v)$. Further, $\Phi(v) \in V(N)$. So $SH(V) = \{\Phi(v)\}$. □

Theorem 19.2 (Coincidence of SH and N for bargaining games.)
Let $(F, d) \in \mathcal{B}$ and let $\langle N, V\rangle$ be the corresponding NTU-game. Then $\{N(F, d)\} = SH(V)$.

Proof.

(i) Let $\hat{x} := N(F, d)$. We prove that $\hat{x} \in SH(V)$. Take $\hat{\lambda} := \alpha(\hat{x}_2 - d_2, \hat{x}_1 - d_1)$ where $\alpha := (\hat{x}_1 + \hat{x}_2 - d_1 - d_2)^{-1}$. Then $\hat{\lambda} \in \Delta$. The line L through $\hat{x}$ with

$$L := \{(x_1, x_2) \in \mathbb{R}^2 \mid \hat{\lambda}_1 x_1 + \hat{\lambda}_2 x_2 = \hat{\lambda}_1\hat{x}_1 + \hat{\lambda}_2\hat{x}_2\}$$

separates the convex sets F and

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - d_1)(x_2 - d_2) \geq (\hat{x}_1 - d_1)(\hat{x}_2 - d_2)\}.$$

So $\hat{\lambda}$ is V-feasible and

$$\begin{aligned} v_{\hat{\lambda}}(\{1\}) &= \hat{\lambda}_1 d_1, \quad v_{\hat{\lambda}}(\{2\}) = \hat{\lambda}_2 d_2, \\ v_{\hat{\lambda}}(\{1,2\}) &= \hat{\lambda}_1\hat{x}_1 + \hat{\lambda}_2\hat{x}_2. \end{aligned}$$

Using the fact that $\hat{\lambda}_2(\hat{x}_2 - d_2) = \hat{\lambda}_1(\hat{x}_1 - d_1)$ we obtain

$$\begin{aligned}
\Phi_1(v_{\hat{\lambda}}) &= \frac{1}{2}(v_{\hat{\lambda}}(1,2) + v_{\hat{\lambda}}(1) - v_{\hat{\lambda}}(2)) \\
&= \frac{1}{2}(\hat{\lambda}_1\hat{x}_1 + \hat{\lambda}_2\hat{x}_2 + \hat{\lambda}_1 d_1 - \hat{\lambda}_2 d_2) \\
&= \frac{1}{2}\hat{\lambda}_1(\hat{x}_1 + d_1) + \frac{1}{2}\hat{\lambda}_2(\hat{x}_2 - d_2) \\
&= \frac{1}{2}\hat{\lambda}_1(\hat{x}_1 + d_1) + \frac{1}{2}\hat{\lambda}_1(\hat{x}_1 - d_1) \\
&= \hat{\lambda}_1\hat{x}_1
\end{aligned}$$

and, similarly $\Phi_2(v_{\hat{\lambda}}) = \hat{\lambda}_2\hat{x}_2$. Hence $\Phi(v_{\hat{\lambda}}) = \hat{\lambda}\hat{x} = \hat{\lambda}N(F,d)$. So $N(F,d) \in SH(V)$.

(ii) Now we prove: if $y \in SH(V)$, then $y = N(F,d)$. Take $y \in SH(V)$. Then there is a V-feasible $\lambda \in \Delta$ such that $v_\lambda(1) = \lambda_1 d_1$, $v_\lambda(2) = \lambda_2 d_2$, $v_\lambda(1,2) = \lambda_1 y_1 + \lambda_2 y_2$. Note that $v_\lambda(1,2) > \lambda_1 d_1 + \lambda_2 d_2 = v_\lambda(1) + v_\lambda(2)$ in view of (B.3) of section 17 and the definition of v_λ. Now $\Phi(v_\lambda) = \lambda y$ is equivalent to

$$\begin{aligned}
\lambda_1 y_1 &= \frac{1}{2}(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_1 d_1 - \lambda_2 d_2) \\
\lambda_2 y_2 &= \frac{1}{2}(\lambda_1 y_1 + \lambda_2 y_2 - \lambda_1 d_1 + \lambda_2 d_2).
\end{aligned}$$

This implies

$$\lambda_1(y_1 - d_1) = \lambda_2(y_2 - d_2). \tag{19.1}$$

If $\lambda_1 = 0$, (19.1) implies $\lambda_2 = 1, y_2 = d_2$. But then

$$v_\lambda(1,2) = \lambda_2 y_2 = \lambda_2 d_2 = v_\lambda(2) = v_\lambda(1) + v_\lambda(2)$$

and that is impossible. So $\lambda_1 > 0$. Similarly $\lambda_2 > 0$. Then (19.1) implies that $\lambda = \beta(y_2 - d_2, y_1 - d_1)$ with $\beta = (y_1 + y_2 - d_1 - d_2)^{-1}$. Note that

$$\begin{aligned}(\lambda_1, \lambda_2) \cdot (x_1, x_2) &\leq (\lambda_1, \lambda_2) \cdot (y_1, y_2) \text{ for all } x \in F \\ (\lambda_1, \lambda_2) \cdot (x_1, x_2) &\geq (\lambda_1, \lambda_2) \cdot (y_1, y_2) \text{ for all } x \in H^+\end{aligned}$$

with $H^+ = \{(x_1, x_2) \mid (x_1 - d_1)(x_2 - d_2) \geq (y_1 - d_1)(y_2 - d_2)\}$. From this follows: $y = N(F, d)$. We have proved $N(F, d) = SH(V)$. □

Exercise 19.1

Let $d = (-1, 0)$ and $F = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 \leq 1\}$.

(i) Calculate for the bargaining problem (F, d) the Nash solution and the Raiffa-Kalai-Smorodinsky solution.

(ii) Calculate $SH(V)$ for the 2-person NTU-game $< N, V >$ corresponding to (F, d).

Is $SH(V)$ a subset of the core of $< N, V >$?

(iii) Give a bargaining solution which satisfies IIA as well as RMON.

Exercise 19.2.

Let $< N, V >$ be the NTU-game with $N = \{1, 2, 3\}$,

$$\begin{aligned}V(\{i\}) &= (-\infty, 0] \text{ for each } i \in N, \\ V(\{1,2\}) &= \{x \in \mathbb{R}^{\{1,2\}} | x_1 \leq 3, x_2 \leq 3\} \\ V(\{1,3\}) &= \{x \in \mathbb{R}^{\{1,3\}} | x_1 \leq 3, x_3 \leq 6\} \\ V(\{2,3\}) &= \{x \in \mathbb{R}^{\{2,3\}} | x_2 \leq 3, x_3 \leq 6\} \\ V(N) &= \mathit{compr}(\mathit{conv}(\{10,0,0), (0,10,0), (0,0,20)\})).\end{aligned}$$

$$\text{Let } \lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \text{ and } \mu = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5}).$$

(i) Calculate $\Phi(v_\lambda)$ and show that it does not lead to an element of $SH(V)$.

(ii) Show that $\Phi(v_\mu)$ leads to an element of $SH(V)$.

Appendix A

Solutions of Exercises

1.1

(i) (R_1R_4, L_2R_3) and (L_1R_4, L_2L_3).

(ii)

$$\begin{array}{c} \\ L_1L_4 \\ L_1R_4 \\ R_1L_4 \\ R_1R_4 \end{array}
\begin{array}{c} \begin{array}{cccc} L_2L_3 & L_2R_3 & R_2L_3 & R_2R_3 \end{array} \\ \begin{bmatrix} (7,5) & (7,5) & (4,6) & (4,6) \\ (8,7)^* & (8,7) & (4,6) & (4,6) \\ (6,5) & (9,5)^* & (6,5)^* & (9,5)^* \\ (6,5) & (9,5)^* & (6,5)^* & (9,5)^* \end{bmatrix} \end{array}$$

(iii) If in the beginning of the play, player 1 goes to the right (R_1), then the left part of the tree is no longer relevant for the payoffs, and so also not the choice of player 1 (L_4 or R_4) in his second decision point. This explains why the third row is equal to the fourth row in the bimatrix game.

(iv) There are seven pure Nash equilibria, corresponding to the cells of the matrix marked with *a* *. Five of them do not correspond to a subgame perfect equilibrium.

(v) $\langle N, v\rangle$ with $N = \{1,2\}$, $v(\emptyset) = 0$, $v(\{1\}) = 6$, $v(\{2\}) = 5$, $v(\{1,2\}) = 15$. The coalition $\{1,2\}$ can obtain a total payoff 15 with the aid of the strategy pair (L_1R_4, L_2L_3).

1.2

(i) (L_1, R_2, R_3) is the unique subgame perfect Nash equilibrium.

(ii) (L_1, R_2, L_3) or (R_1, L_2, L_3).

(iii) $\langle X_1, X_2, X_3, K_1, K_2, K_3\rangle$ with $X_i = \{L_i,\ R_i\}$ for each $i \in \{1,2,3\}$. $K_1(L_1, L_2, L_3) = 5$, $K_2(L_1, L_2, L_3) = 0$,

$K_3(L_1, L_2, L_3) = 4$, $K_1(L_1, L_2, R_3) = 10$, etc.

Pure Nash equilibria: $(L_1, R_2, R_3), (L_1, R_2, L_3), (R_1, L_2, L_3)$.

(iv) $N = \{1,2,3\}, v(\{1\}) = 8, v(\{2\}) = 3, v(\{3\}) = 5, v(\{1,2\}) = 15$, $v(\{1,3\}) = 16$, $v(\{2,3\}) = 11$, $v(\{1,2,3\}) = 25$.

1.3 See section 2.

1.4 (B, R) is the unique Nash equilibrium. $N = \{1,2\}$, $v(\emptyset) = 0$, $v(\{1\}) = -3$, $v(\{2\}) = -3$, $v(\{1,2\}) = 10$.

1.5 The bimatrix game
$$\begin{array}{c} \\ T \\ B \end{array}\begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} (1,2) & (0,0) \\ (0,0) & (2,1) \end{bmatrix} \end{array}$$
has two pure Nash equilibria: (T, L) and (B, R).

2.1

(i) First note that $K_i(w_i, x_{-i})$ is non-negative for all x_{-i}, because this payoff is 0 if player i, bidding w_i, does not get the painting and this payoff is non-negative otherwise, since then $\max_{k \in P\setminus\{i\}} x_k \leq w_i$. So, $K_i(w_i, x_{-i}) \geq K_i(x_i, x_{-i}) = 0$

if $i \neq i^*(x)$. If $i = i^*(x)$ and $\max_{k \in P\setminus\{i\}} x_k > w_i$, then $K_i(w_i, x_{-i}) \geq 0 > K_i(x_i, x_{-i})$. If $i = i^*(x)$ and $\max_{k \in P\setminus\{i\}} x_k \leq w_i$, then $K_i(w_i, x_{-i}) = K_i(x_i, x_{-i}) = w_i - \max_{k \in P\setminus\{i\}} x_k$. So we have proved that w_i is a dominant strategy of i in the sealed bid second price auction.

(ii) For the sealed bid first price auction there does not exist a dominant strategy.

2.2

(i) Note that $\alpha(w_1, w_2, w_3) = a_2$ because $\sum_{i \in P} w_i(a_1) < \sum_{i \in P} w_i(a_2)$, and that

$$t_1(w_1, w_2, w_3) = \max\{100, 145\} - w_2(a_2) - w_3(a_2) = 0,$$

$$t_2(w_1, w_2, w_3) = \max\{100, 95\} - w_1(a_2) - w_3(a_2) = 5,$$

$$t_3(w_1, w_2, w_3) = \max\{100, 90\} - w_1(a_2) - w_2(a_2) = 10.$$

So, $K_1(w_1, w_2, w_3) = w_1(a_2) - t_1(w_1, w_2, w_3) = 20 - 0 = 20,$

$K_2(w_1, w_2, w_3) = w_2(a_2) - t_2(w_1, w_2, w_3) = 70 - 5 = 65,$

$K_3(w_1, w_2, w_3) = w_3(a_2) - t_3(w_1, w_2, w_3) = 75 - 10 = 65.$

(ii) Note that $\alpha(x_1, w_2, w_3) = a_1$ and $t_1(x_1, w_2, w_3) = 45$.

So, $K_1(x_1, w_2, w_3) = w_1(a_1) - t_1(x_1, w_2, w_3) = 50 - 45 = 5 < K_1(w_1, w_2, w_3) = 20$.

2.3 If $\hat{x}_i$ is a dominant strategy for each $i \in P$, then $K_i(x_{-i}, x_i) \leq K_i(x_{-i}, \hat{x}_i)$ for each $i \in P$, $x_{-i} \in \prod_{r \in P\setminus\{i\}} X_r$ and $x_i \in X_i$. So, also $K_i(\hat{x}_{-i}, x_i) \leq K_i(\hat{x})$ for each $i \in P$ and each $x_i \in X_i$. Hence $\hat{x}$ is a Nash equilibrium. In the bimatrix game $\begin{bmatrix} (1,1) & (0,0) \\ (0,0) & (1,1) \end{bmatrix}$ no player has a dominant strategy and the game has two pure Nash equilibria.

2.5 (0,3) is a Nash equilibrium of the two-person game because $|0-3| \geq |x-3|$ for each $x \in [0,1]$ and $-|0-3| \geq -|0-y|$ for each $y \in [3,4]$. Further $(0,3)$ is the unique Nash equilibrium of this game and it is a Nash equilibrium in dominant strategies.

2.6 $(\hat{x},\hat{y}) \in NE(X_1,X_2,K_1,K_2) \Leftrightarrow K_1(\hat{x},\hat{y}) = \max_{x\in X} K_1(x,\hat{y})$, $K_2(\hat{x},\hat{y}) = \max_{y\in Y} K_2(\hat{x},y) \Leftrightarrow L_1(\hat{x},\hat{y}) = \max_{x\in X} L_1(x,\hat{y})$, $L_2(\hat{x},\hat{y}) = \max_{y\in Y} L_2(\hat{x},y) \Leftrightarrow (\hat{x},\hat{y}) \in NE(X_1,X_2,L_1,L_2)$.

3.1

(i) Let $\Gamma = (X,Y,K,-K)$. Take $(x,y) \in O_1(\Gamma) \times O_2(\Gamma)$. Then there is a $y_1 \in Y$ and an $x_1 \in X$ such that $(x,y_1) \in NE(\Gamma)$ and $(x_1,y) \in NE(\Gamma)$. Then by theorem 3.1 also $(x,y) \in NE(\Gamma)$. So, $O_1(\Gamma) \times O_2(\Gamma) \subset NE(\Gamma)$. Conversely, take $(x,y) \in NE(\Gamma)$. Then $x \in O_1(\Gamma)$ and $y \in O_2(\Gamma)$, so $(x,y) \in O_1(\Gamma) \times O_2(\Gamma)$. Hence, $O_1(\Gamma) \times O_2(\Gamma) = NE(\Gamma)$.

(ii) Let $(\hat{x},\hat{y}) \in NE(\Gamma)$. Then

(1) $\inf_{y\in Y} \sup_{x\in X} K(x,y) \leq \sup_{x\in X} K(x,\hat{y}) = K(\hat{x},\hat{y}) =$
$\inf_{y\in Y} K(\hat{x},y) \leq \sup_{x\in X} \inf_{y\in Y} K(x,y)$

where the equalities follows from the fact that $(\hat{x},\hat{y}) \in NE(\Gamma)$. So, we have proved that the upper value is not larger than the lower value. The converse is also true (see exercise 3.2). So the game Γ has a value and from (1) it follows then that $K(\hat{x},\hat{y}) = \sup_{x\in X} \inf_{y\in Y} K(x,y) = \inf_{y\in Y} \sup_{x\in X} K(x,y)$, and that in the second term $\sup_{x\in X}$ can be replaced by $\max_{x\in X}$ and in the third term $\inf_{y\in Y}$ can be replaced by $\min_{y\in Y}$.

3.2

(i) $\inf_{z\in Y} K(x,z) \leq K(x,y)$ for each $x \in X$, $y \in Y$ implies $\sup_{x\in X}\inf_{z\in Y} K(x,z) \leq \sup_{x\in X} K(x,y)$ for all $y \in Y$ and then $\sup_{x\in X}\inf_{z\in Y} K(x,z) \leq \inf_{y\in Y}\sup_{x\in X} K(x,y)$ or $\underline{v}(X,Y,K,-K) \leq \bar{v}(X,Y,K,-K)$.

(iii) In the new game player 2 can choose an action dependent on the chosen action by player 1. So, a strategy of player 2 is a map $f : X \to Y$. The set of strategies of player 2 is Y^X. The new game is $\langle X, Y^X, K', -K'\rangle$, where $K'(x,f) = K(x,f(x))$ for each $x \in X$, $f \in Y^X$. We prove that in case K is lower bounded $v(X,Y^X,K',-K') = \underline{v}(X,Y,K,-K)$ by showing

(a) $\underline{v}(X,Y^X,K',-K') = \underline{v}(X,Y,K,-K)$, and

(b) $\bar{v}(X,Y^X,K',-K') \leq \underline{v}(X,Y,K,-K)+\varepsilon$ for each $\varepsilon > 0$.

To prove (a) note that $\underline{v}(X,Y^X,K',-K') = \sup_{x\in X}\inf_{f\in Y^X} K'(x,f) = \sup_x \inf_f K(x,f(x)) = \sup_x \inf_y K(x,y) = \underline{v}(X,Y,K,-K)$.

To prove (b) take $\varepsilon > 0$. Construct $f^\varepsilon \in Y^X$ such that

(1) $K(x,f^\varepsilon(x)) \leq \inf_{y\in Y} K(x,y) + \varepsilon$.

Then $\bar{v}(X,Y^X,K',-K') = \inf_{f\in Y^X}\sup_{x\in X} K'(x,f) \leq \sup_{x\in X} K'(x,f^\varepsilon) = \sup_{x\in X} K(x,f^\varepsilon(x)) \leq \sup_{x\in X}(\inf_{y\in Y} K(x,y)+\varepsilon) = \underline{v}(X,Y,K,-K)+\varepsilon$, where the second inequality follows from (1). [We leave it to the reader to consider the case that K is unbounded.]

3.3 The inequality (4) is a special case of the following proposition: let $\langle X, Y, K, -K\rangle$ and $\langle X, Z, K', -K'\rangle$ be zero-sum games, where $Z \subset Y$ and $K' : X \times Z \to \mathbb{R}$ is the restriction of $K : X \times Y \to \mathbb{R}$ to $X \times Z$. Then $\bar{v}(X, Z, K', -K') \geq \bar{v}(X, Y, K, -K)$ and $\underline{v}(X, Z, K', -K') \geq \underline{v}(X, Y, K, -K)$.
We leave the proof of this proposition to the reader.

3.4

(i) $v(A) = \frac{2}{3}$, $O_1(A) = \{(\frac{1}{3}, \frac{2}{3})\}$, $O_2(A) = \{(\frac{1}{3}, \frac{2}{3}, 0, 0)\}$.

(ii) $v(A) = 2, O_2(A) = \{(\frac{1}{2}, \frac{1}{2})\}, O_1(A) = \text{conv}\{(\frac{1}{2}, \frac{1}{2}, 0), (0, 0, 1)\}$.

(iii) $v(A) = -2, O_1(A) = \{(\frac{1}{2}, \frac{1}{2})\}, O_2(A) = \text{conv}\{(\frac{1}{2}, \frac{1}{2}, 0), (0, 0, 1)\}$.

[Compare the outcomes of (ii) and (iii); see also exercise 6.1.]

(iv) $v(A) = \frac{2}{3}$, $O_1(A) = \{(\frac{1}{3}, \frac{2}{3}, 0)\}$, $0_2(A) = \{(\frac{1}{3}, \frac{2}{3}, 0, 0)\}$.

[Compare the outcomes of (i) and (iv) and note that $e_1 A \gg e_3 A$.]

3.6 Let (k, l) be a pure Nash equilibrium of A. Then $a_{il} \leq a_{kl} \leq a_{kj}$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. We consider two cases: $a_{kl} = -1$ and $a_{kl} = 1$. In the first case $a_{il} = -1$ for all $i \in \{1, 2, \ldots, m\}$. So, $v(A) = -1$, $e_l \in O_2(A)$ and $O_1(A) = \Delta^m$.
In the second case $a_{kj} = 1$ for each $j \in \{1, 2, \ldots, n\}$ and then $v(A) = 1$, $e_k \in O_1(A)$ and $O_2(A) = \Delta^n$.

3.7 $O_1(A)$ is convex since it is the intersection of closed convex half spaces.

3.8 $v(A) = \left(\sum\limits_{i=1}^{n} a_i^{-1}\right)^{-1}$, $O_1(A) = O_2(A) =$
$\left\{\sum\limits_{i=1}^{n} a_i^{-1}(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})\right\}$.

4.1 We only prove the formula for $\hat{p}_1$. First we note that $b_{22} \neq b_{21}$, because $b_{22} = b_{21}$ should imply that for player 2 e_1 is a dominant strategy (if $b_{11} \geq b_{12}$) or e_2 is a dominant strategy (if $b_{12} \geq b_{11}$), and then it is easy to show that the game has a pure Nash equilibrium, which is not the case.
By theorem 4.3 we have for (A, B) that $\{1,2\} = C(\hat{q}) \subset PB_2(\hat{p}) \subset \{1,2\}$. So, $\hat{p}Be_1 = \hat{p}Be_2$ or $\hat{p}_1 b_{11} + \hat{p}_2 b_{21} - \hat{p}_1 b_{12} - \hat{p}_2 b_{22} = 0$. But then $\hat{p}_1(b_{11} - b_{21} - b_{12} + b_{22}) = b_{22} - b_{21}$ because $\hat{p}_2 = 1 - \hat{p}_1$.

4.2

(i) $v(A) = 4\frac{1}{9}$, $O_1(A) = \{(\frac{2}{9}, \frac{7}{9})\}$, $O_2(A) = \{(\frac{5}{9}, \frac{4}{9})\}$.

(ii) $(e_1, e_1), (e_2, e_2)$, $((\frac{2}{3}, \frac{1}{3}), (\frac{3}{5}, \frac{2}{5}))$.

4.3

(ii) Note that $Be_3 < Be_1$. So by (i):

$$\{((\tfrac{2}{5}, \tfrac{3}{5}, 0), (\tfrac{4}{5}, \tfrac{1}{5}, 0)), (e_1, e_1), (e_2, e_2)\} = NE(A, B).$$

4.4 $NE(A_\alpha, B_\alpha) = \{e_1, e_2\}$ if $1 < a < 2$, $NE(A_\alpha, B_\alpha) = \{(e_1, e_1)\}$ if $\alpha > 2$, and $NE(A_2, B_2) = \{(e_1, q) | q \in \Delta_2\}$.

5.1

(i) Clearly, $(4,4,4)$ is the unique Nash equilibrium.

(ii) Pure Nash equilibria are $(1,1,1), (2,2,2)$ and $(3,3,3)$, where for each of the players there is a payoff 10. A mixed strategy tuple leads to payoffs smaller than 10, because the nondiagonal payoffs are 0.

5.2

(i) (1,1,2) and (2,2,2) are pure Nash equilibria.

(ii) (0,0,0) and (1,1,1) are pure Nash equilibria.

6.1

(i) Let A be an $m \times m$-matrix with $A = -A^T$. Then $v(A) =$

$\max_{p \in \Delta^m} \min_{q \in \Delta^n} p^T A q = \max_{p \in \Delta^m} \min_{q \in \Delta^n} p^T(-A^T)q =$

$\max_{p \in \Delta^m} \min_{q \in \Delta^n} -(q^T A p)^T = -\min_{p \in \Delta^m} \max_{q \in \Delta^m} q^T A p =$

$-v(A)$. So, $v(A) = 0$.

(ii) $(r, s, t) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times (\mathbb{R}_+ \backslash \{0\})$ optimal in S implies

$(r^T, s^T, t)S \geq (v(S)1^T_m, v(S)1^T_n, v(S)) = (0, 0, 0)$, which is equivalent to

$$-(As)^T + tc^T \geq 0,\ r^T A - tb^T \geq 0,\ -r^T c + s^T b \geq 0 \tag{1}$$

and to

$$A(t^{-1}s) \leq c,\ (t^{-1}r)^T A \geq b^T,\ b^T(t^{-1}s) \geq (t^{-1}r)^T c \tag{2}$$

By (P.1) the last inequality in (2) is an equality, since (2) implies that $t^{-1}s$ is feasible for the dual program and $t^{-1}r$ feasible for the primal program. But then $t^{-1}s$ and $t^{-1}r$ are optimal for the dual and the primal program, respectively.

6.2 By exercise 3.1 we have $NE(C, -C) = O_1(C) \times O_2(C)$ for each $m \times n$-matrix C. Then: $(\hat{p}, \hat{q}) \in O_1(A) \times O_2(A) = NE(A, -A) \Leftrightarrow p^T A \hat{q} \leq p^T A q = v(A) \leq \hat{p}^T A q$ for all $(p, q) \in \Delta^m \times \Delta^n \Leftrightarrow p^T(A + dJ)\hat{q} \leq \hat{p}^T(A + dJ)\hat{q} = v(A) + d \leq \hat{p}^T(A + dJ)q$ for all $(p, q) \in \Delta^m \times \Delta^n \Leftrightarrow (\hat{p}, \hat{q}) \in NE(A + dJ, -A - dJ) = O_1(A + dJ) \times O_2(A + dJ)$. So $O_i(A) = O_i(A + dJ)$ for each $i \in \{1, 2\}$ and $v(A + dJ) = v(A) + d$.

6.3 $p \in O_1(A)$, $q \in O_2(A)$ implies
(1) $p^T A \geq (v(A), v(A))$, $Aq \leq \binom{v(A)}{v(A)}$, $p \in \Delta^2$, $q \in \Delta^2$
and (1), exercise 6.1 (i) and $B^T = -B$ imply

(2) $z = (2 + v(A))^{-1}(p, q, v(A)) \in \Delta^5$, $zB \geq 0 = v(B)1_5$.
So, $z \in O_1(B)$.
Note that $v(A) = 1\frac{1}{2}$, $O_1(A) = \{(\frac{3}{4}, \frac{1}{4})\}$, $O_2(A) = \{(\frac{1}{2}, \frac{1}{2})\}$.
So, $z = \frac{2}{7}(\frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) \in O_1(B)$.

7.1

(i) Let $g(\hat{x}) = \hat{x}$. Then $\hat{x} = g(\hat{x}) \geq 0$, $f(\hat{x}) = 0 \geq 0$, $\hat{x}^T f(\hat{x}) = 0$. So, $\hat{x} \in O(f)$.

(ii) Let $\hat{x} \in O(f^*)$. Then $\hat{x} \geq 0$, $f^*(\hat{x}) \geq 0$, $\hat{x}^T f^*(\hat{x}) = 0$. This implies that $f_i^*(\hat{x}) = 0$ if $\hat{x}_i > 0$ and that $0 \leq f_i^*(\hat{x}) = \hat{x}_i - g_i(\hat{x}) = 0 - g_i(\hat{x}) \leq 0$ if $\hat{x}_i = 0$. So, $f^*(\hat{x}) = 0$, $\hat{x} = g(\hat{x})$.

7.2 Equivalent are (1), (2), (3), (4) and (5) with
(1) x is a solution of the primal problem and y is a solution of the dual problem,
(2) $x^T A \geq b^T$, $x \geq 0$, $Ay \leq c$, $y \geq 0$, $x^T c = b^T y$,
(3) $-Ay + c \geq 0$, $A^T x - b \geq 0$, $x \geq 0$, $y \geq 0$, $(x^T, y^T)\begin{bmatrix} c \\ -b \end{bmatrix} = 0$,
(4) $\begin{bmatrix} 0 & -A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ -b \end{bmatrix} \geq 0$, $\begin{bmatrix} x \\ y \end{bmatrix} \geq 0$,

$$(x^T, y^T)\left(\begin{bmatrix} 0 & -A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ -b \end{bmatrix}\right) = 0,$$

(5) $\begin{bmatrix} x \\ y \end{bmatrix} \in O(q, M)$.

8.1 Take $\varepsilon > 0$. It follows from the boundedness of all payoff functions that also P is a bounded function. This implies that we can take $\hat{x} \in X$ such that $P(x) \leq P(\hat{x}) + \varepsilon$ for each $x \in X$. Then for each $i \in \{1,2,\ldots,p\}$ and each $x_i \in X_i : K_i(\hat{x}^{-i}, x_i) - K_i(\hat{x}) = P(\hat{x}^{-i}, x_i) - P(\hat{x}) \leq \varepsilon$. So, $\hat{x}$ is an ε-Nash equilibrium of Γ.

8.2 By the main theorem of calculus $P(x^{-i}, x_i') - P(x^{-i}, x_i) = \int_{x_i}^{x_i'} \frac{\partial P}{\partial x_i}(x^{-i}, t)dt$ and for each $i \in \{1,2,\ldots,p\} : K_i(x^{-i}, x_i') - K_i(x^{-i}, x_i) = \int_{x_i}^{x_i'} \frac{\partial K_i}{\partial x_i}(x^{-i}, t)dt$. So, to prove that the given P is a potential for the oligopoly game, we have only to show that $\frac{\partial P}{\partial x_i} = \frac{\partial K_i}{\partial x_i}$ for $i \in \{1,2,\ldots,p\}$. Now, $\frac{\partial K_i}{\partial x_i}(x) = (a - b\sum_{j=1}^{p} x_j) - x_i b - c'(x_i) = a - 2bx_i - b \sum_{j \in P\setminus\{i\}} x_j - c'(x_i) = \frac{\partial P}{\partial x_i}(x)$.

8.3 For mathematical convenience we extend c_j to $\{0,1,\ldots,n\}$ with $c_j(0) = 0$. Also $B_i \subset M$ with $B_i = \emptyset$ we allow. Then

(i) $P(B_1,\ldots,B_i,\ldots,B_m) = \sum_{j\in M} \left(\sum_{k=0}^{t_j(B_1,\ldots,B_n)} c_j(k) \right)$. If we put $u_i(j) = 1$ if $j \in B_i$ and $u_i(j) = 0$ otherwise, then

(ii) $C_i(B_1,\ldots,B_i,\ldots,B_n) = \sum_{j\in M} c_j(t_j(B_1,\ldots,B_i,\ldots,B_n))u_i(j)$.

From (i) and (ii) follows

(iii) $P(B_1,\ldots,B_i,\ldots,B_m) - C_i(B_1,\ldots,B_i,\ldots,B_n) = \sum_{j\in M} \left(\sum_{k=0}^{t_j(B_1,\ldots,B_n)-u_i(j)} c_j(k) \right) = P(B_1,\ldots,B_{i-1},\emptyset, B_{i+1},\ldots,B_n)$.

In a similar way follows

(iv) $P(B_1,\ldots,B_i',\ldots,B_m) - C_i(B_1,\ldots,B_i',\ldots,B_n) = P(B_1,\ldots,B_{i-1},\emptyset,B_{i+1},\ldots,B_n)$.

From (iii) and (iv) follows that P is a potential for the congestion game.

8.4 The congestion game corresponding to the given congestion situation is given by the bimatrix
$$\begin{array}{cc} & \begin{array}{cc} D_2 & I_2 \end{array} \\ \begin{array}{c} D_1 \\ I_1 \end{array} & \begin{bmatrix} (8,8) & (2,3) \\ (3,2) & (5,5) \end{bmatrix} \end{array},$$
where D_1 (D_2) stands for the direct road AC (CA), and I_1 (I_2) for the indirect road ABC (CBA). The strategy pairs (I_1, D_2) and (D_1, I_2) are Nash equilibria for the (cost) bimatrix game. A potential is given by $\begin{bmatrix} 0 & -5 \\ -5 & -2 \end{bmatrix}$. Another potential is $\begin{bmatrix} 10 & 5 \\ 5 & 8 \end{bmatrix}$, which we obtain by using the formula in exercise 8.3.

10.1 $v(S) = \min\{p|L \cap S|, q|R \cap S|\}$.

10.2

(i) We concentrate first on the games $\langle N, v\rangle$ and $\langle N, c\rangle$ in example 10.3.

To prove that $< N, v >$ is superadditive we show that

$v(S \cup T) \geq v(S) + v(T)$ for all non-empty S and T with $S \cap T = \emptyset$.

$$\begin{array}{lllllllllll}
60 & = & v(1,2) & \geq & v(1) & + & v(2) & = & 0 & + & 0, \\
70 & = & v(1,3) & \geq & v(1) & + & v(3) & = & 0 & + & 0, \\
60 & = & v(2,3) & \geq & v(2) & + & v(3) & = & 0 & + & 0, \\
130 & = & v(1,2,3) & \geq & v(1) & + & v(2,3) & = & 0 & + & 60, \\
130 & = & v(1,2,3) & \geq & v(2) & + & v(1,3) & = & 0 & + & 70, \\
130 & = & v(1,2,3) & \geq & v(3) & + & v(1,2) & = & 0 & + & 60.
\end{array}$$

To prove that $< N, c >$ is subadditive we show that

$c(S \cup T) \leq c(S) + c(T)$ for all $S, T \in 2^N \backslash \{\emptyset\}$ with $S \cap T = \emptyset$.

$$\begin{array}{lclclclclcl}
130 & = & c(1,2) & \leq & c(1) & + & c(2) & = & 100 & + & 90, \\
110 & = & c(1,3) & \leq & c(1) & + & c(3) & = & 100 & + & 80, \\
110 & = & c(2,3) & \leq & c(2) & + & c(3) & = & 90 & + & 80, \\
140 & = & c(1,2,3) & \leq & c(1) & + & c(2,3) & = & 100 & + & 110, \\
140 & = & c(1,2,3) & \leq & c(2) & + & c(1,3) & = & 90 & + & 110, \\
140 & = & c(1,2,3) & \leq & c(3) & + & c(1,2) & = & 80 & + & 130.
\end{array}$$

Now we prove that the glove game in example 10.2 is superadditive. Suppose $S \cap T = \emptyset$. Then

$v(S) + v(T) = \min\{|L \cap S|, |R \cap S|\} + \min\{|L \cap T|, |R \cap T|\}\}$

$= \min\{|L \cap S| + |L \cap T|, |L \cap S| + |R \cap T|,$

$|R \cap S| + |L \cap T|, |R \cap S| + |R \cap T|\}$

$\leq \min\{|L \cap S| + |L \cap T|, |R \cap S| + |R \cap T|\}$

$= \min\{|L \cap (S \cup T)|, |R \cap (S \cup T)|\} = v(S \cup T).$

For the third equality we use the fact $S \cap T = \emptyset$.

(ii) We prove by induction to k that for all k and $S_1, S_2 \ldots, S_k$ mutually disjoint we have

$v(\bigcup_{i=1}^{k} S_i) \geq \sum_{i=1}^{k} v(S_i)$ if v is a superadditive game.

For $k = 1 : v(S_1) \geq v(S_1)$. Suppose that the formula holds for $k = m$. Let $S_1, S_2, \ldots, S_m, S_{m+1}$ be mutually disjoint sets. Then $v(\bigcup_{i=1}^{m+1} S_i) = v((\bigcup_{i=1}^{m} S_i) \cup S_{m+1}) \geq v(\bigcup_{i=1}^{m} S_i) + v(S_{m+1}) \geq \sum_{i=1}^{m} v(S_i) + v(S_{m+1}) = \sum_{i=1}^{m+1} v(S_i)$, where the first inequality follows from the superadditivity and the second from the induction hypothesis.

10.3

(i) (a) $v \sim v$ (*reflexivity*): take $k = 1, a = 0 \in \mathbb{R}^n$.

(b) $v \sim w \Longrightarrow w \sim v$ (*symmetry*): if $w(S) = kv(S) + a(S)$ for each $S \in 2^N$, then $v(S) = mw(S)+b(S)$ with $m = \frac{1}{k}$ and $b_i = k^{-1}(-a_i)$ for each $S \in 2^N$.

(c) $v \sim w, w \sim u \Longrightarrow v \sim u$ (*transivity*): $w(S) = kv(S) + a(S)$ and $u(S) = lw(S) + b(S)$ imply: $u(S) = klv(S) + la(S) + b(S)$. Note that $kl > 0$ and $la(S) + b(S) = \sum_{i\in S}(la_i + b_i)$.

(ii) $v \sim w$ with $w(S) = v(S) - \sum_{i\in S} v(i)$. Here $k = 1, a = (-v(1), -v(2), \ldots, -v(n))$ and w is 0-normalized.

(iii) Take $w(S) = \frac{v(S)-\sum_{i\in S} v(i)}{v(N)-\sum_{i\in N} v(i)}$. Then $w \sim v$ and w is $(0,1)$-normalized.

(iv) $w(1) = 0, w(2) = 0, w(1,2) = 1$ (w is the $(0,1)$-normalization of each essential 2-person game.)

10.4 $c(1) = k_{11} = 5, c(2) = k_{22} = 4$ and $c(1,2) = \min_{\sigma\in\{(1,2),(2,1)\}} \sum_{i=1}^{2} k_{i\sigma(i)} = \min\{k_{11}+k_{22}, k_{12}+k_{21}\} = \min\{9,3\} = 3$. Since $3 = c(1,2) \leq c(1) + c(2) = 5 + 4$ the game is subadditive. The corresponding cost savings game is given by

$$\begin{array}{lclcl} v(1) & = & c(1) & - & c(1) & = & 0, \\ v(2) & = & c(2) & - & c(2) & = & 0, \\ v(1,2) & = & c(1)+c(2) & - & c(1,2) & = & 5+4-3=6 \end{array}$$

11.1 For an n-person zero-one normalized game $< N, v >$ we have $I(v) = \{x \in \mathbb{R}^n | x_i \geq v(i) = 0$, for each $i \in N, \sum_{i=1}^{n} x_i = 1\} = conv\{e^1, e^2, \ldots, e^n\}$. [Note that $f^i = e^i$ for each $i \in N$.]

11.2 Suppose $w = kv + a$. Then $C(w) =$ "$kC(v) + a$"$(= \{x \in \mathbb{R}^n | \exists_{y\in C(v)}[x = ky + a]\})$ follows from $x \in C(w) \Longrightarrow y := \frac{x-a}{k} \in C(v)$, $z \in C(v) \Longrightarrow x := kz + a \in C(w)$

11.3 If $x \in C(v)$, then for each $i \in N : x_i = x(N) - x(N\backslash\{i\}) = v(N) - x(N\backslash\{i\}) \leq v(N) - v(N\backslash\{i\}) = M_i(v)$.

11.4

(i) There is an $S \in B, S \neq N$. Take $i \in N\backslash S$. Then

$\sum_{T\in B} \lambda(T)e^T = e^N$ implies : $\sum_{T\in B, i\in T} \lambda(T) = 1$. Then $\sum_{T\in B} \lambda(T) \geq \sum_{T\in B, i\in T} \lambda(T) + \lambda(S) \geq 1 + \lambda(S) > 1$.

(ii) $e^N = \sum_{S\in B} \lambda(S)e^S = \sum_{S\in B} \lambda(S)(e^N - e^{N\backslash S})$. So

$\sum_{S\in B} \lambda(S)e^{N\backslash S} = (\sum_{S\in B} \lambda(S) - 1)e^N$ and $\sum_{S\in B} \lambda(S) - 1 > 0$ by (i), which implies that B^c is a balanced collection.

(iii) This follows from (ii) with $B = \{N\backslash S, \{i\}_{i\in S}\}$, a partition.

11.5

(i) $C(v) = \{x \in \mathbb{R}^3 | \sum_{i\in S} x_i \geq v(S) = 0$ for all $S \neq N, \sum_{i=1}^{3} x_i = 1\} = \{x \in \mathbb{R}^3_+ | \sum_{i=1}^{3} x_i = 1\} = I(v) = conv(\{e_1, e_2, e_3\})$.

(ii) $C(v) = \{x \in \mathbb{R}^3_+ | x_1 + x_2 + x_3 = 1, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1\} = \{(0, 0, 1)\}$.

12.1 Main problem: $\max\{6x_1 + 8x_2 | x \geq 0,\ 2x_1 + x_2 \leq b_1^S,\ x_1 + 4x_2 \leq b_2^S\}$.

Dual problem: $\min\{\ b_1^S y_1 + b_2^S y_2 | y \geq 0, 2y_1 + y_2 \geq 6, y_1 + 4y_2 \geq 8\}$.

(i) The feasible region of the dual problem does not depend on S and $\hat{y} = (2\frac{2}{7}, 1\frac{3}{7})$ is optimal for N. (Make a picture of this region.)

$$\begin{array}{lllllll}
b^N & = & (6,6) & , & v(N) & = & 22\frac{2}{7}, \\
x_1 & = & b^1\hat{y} & = & 6 * 2\frac{2}{7} & = & 13\frac{5}{7} \\
x_2 & = & b^2 y & = & 5 * 1\frac{3}{7} & = & 7\frac{1}{7} \\
x_3 & = & b^3 y & = & 1 * 1\frac{3}{7} & = & 1\frac{3}{7}.
\end{array}$$

So $x \in C(v)$.

(ii) $\hat{y}^1 = (8,0), \hat{y}^2 = (0,6)$ and $\hat{y}^3 = (2\frac{2}{7}, 1\frac{3}{7})$ are the extreme points of the dual region.

The minimum is in (at least) one of these points.

To find the production game we make the following scheme.

S	$=$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$b(S)$	$=$	$(6,0)$	$(0,5)$	$(0,1)$	$(6,5)$	$(6,1)$	$(0,6)$	$(6,6)$
$(\hat{y}^1)^T b(S)$	$=$	48	0^*	0^*	48	48	0^*	48
$(\hat{y}^2)^T b(S)$	$=$	0^*	30	6	30	6^*	36	36
$(\hat{y}^3)^T b(S)$	$=$	$13\frac{5}{7}$	$7\frac{1}{7}$	$1\frac{3}{7}$	$20\frac{6^*}{7}$	$15\frac{2}{7}$	$8\frac{4}{7}$	$22\frac{2^*}{7}$
$v(S)$	$=$	0	0	0	$20\frac{6}{7}$	6	0	$22\frac{2}{7}$

12.2 $\hat{y} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \in argmin \; \{(\sum_{i=1}^{3} b^i)y | y \geq 0, Ay \geq c\} =$ $\arg\min\{10y_1 + 12y_2 | y \geq 0, y_1 + 3y_2 \geq 30, 4y_1 + 2y_2 \geq 40\}$. Then $\hat{x} := (b^1\hat{y}, b^2\hat{y}, b^3\hat{y}) = (50, 44, 62) \in C(v)$.

12.3 Given A, b^1, b^2, c let $< N, w >$ be the corresponding production game. We have to prove that $v = w$.
$w(N) = \min\{y_1 + y_2 | y_1 \geq 1, y_2 \geq 3,\ y_1 + y_2 \geq 5\} = 5 = v(N)$,
$w(\{2\}) = \min\{y_2 | y_1 \geq 1, y_2 \geq 3, y_1 + y_2 \geq 5\} = 3 = v(\{2\})$,
$w(\{1\}) = \min\{y_1 | y_1 \geq 1, y_2 \geq 3, y_1 + y_2 \geq 5\} = 1 = v(\{1\})$.

13.1

(i) Take $x, y \in I(v)$. Then $y \; dom_{\{i\}} \; x$ implies $y_i > x_i, y_i \leq v(i)$, so $x_i < y_i \leq v(i)$ which is impossible because $x \in I(v)$. So $D(\{i\}) = \emptyset$.

Suppose $z \in D(N)$. Then there is a $y \in I(v)$ such that $y \text{ dom}_N\ z$ or $y_i > z_i \forall_{i \in N}, \sum_{i \in N} y_i \leq v(N)$.

This implies $\sum_{i \in N} z_i < \sum_{i \in N} y_i \leq v(N)$. Then $z \notin I(v)$, a contradiction. Hence $D(N) = \emptyset$.

(ii) For the cost savings game in example 10.3 we have:

$D(\{1,2\}) = \{x \in I(v) | x_3 > 70\}$. Such an $x \in I(v)$ with $x_3 > 70$ is dominated via $\{1,2\}$ by $y = (x_1 + \frac{1}{2}(x_3 - 70), x_2 + \frac{1}{2}(x_3 - 70), 70)$. Further $D(\{2,3\}) = \{x \in I(v) | x_1 > 70\}, D(\{1,3\}) = \{x \in I(v) | x_2 > 60\}$, and $D(S) = \emptyset$ for the other S. The set of undominated elements is equal to $C(v) = \{x \in \mathbb{R}^3_+ | x(N) = 130, x_1 + x_2 \leq 60, x_1 + x_3 \leq 70, x_2 + x_3 \leq 60\}$.

13.2

(i) That dom and dom_S are irreflexive relations follows from: $\neg \exists_{S \in 2^N \setminus \{\emptyset\}} \forall_{i \in S} [x_i > x_i]$ for $x \in I(v)$. To prove that dom_S is transitive take $x, y, z \in I(v)$ such that $x \text{ dom}_S\ y$, $y \text{ dom}_S\ z$. Then $x_i > y_i > z_i$ for all $i \in S$. So $x \text{ dom}_S\ z$. To prove that dom_s is antisymmetric, suppose $x \text{ dom}_S\ y$. Then $\forall_{i \in S}[x_i > y_i] \Longrightarrow \neg \forall_{i \in S}[y_i > x_i] \Longrightarrow \neg y \text{ dom}_S\ x$.

(ii) Let $N = \{1,2,3,4\}, v(N) = v(1,2) = v(3,4) = 20\ v(S) = 0$ otherwise. Then for $x = (10,10,0,0), y = (0,0,10,10)$ in $I(v) : x \text{ dom}_{\{1,2\}}\ y$, $y \text{ dom}_{\{3,4\}}\ x$.

(iii) Take $< N, v >$ as in (ii) and also x, y. Let $z = (0,12,4,4) \in I(v)$. Then x dom y via $\{1,2\}$; y dom z via $\{3,4\}$ and $\neg x$ dom z. Only for coalition $\{1\}$ x is better than z but not $x_1 \leq v(1) = 0$.

13.3

(i) Take $x \in I(v)$. Then $x \geq 0, x_1+x_2+x_3 = 1$, so (a) $x_1+x_2 < 1$ or (b) $x_1 + x_3 < 1$ or (c) $x_2 + x_3 < 1$.

In case (a): $(x_1 + \varepsilon, x_2 + \varepsilon, 0)\ \mathrm{dom}_{\{1,2\}}\ x$ where $\varepsilon := \frac{1}{2}(1 - x_1 - x_2)$.

In case (b):$(x_1 + \varepsilon', 0, x_3 + \varepsilon')\ \mathrm{dom}_{\{1,3\}}\ x$ with $\varepsilon' := \frac{1}{2}(1 - x_1 - x_3)$.

In case (c): $(0, x_2 + \delta,\ x_3 + \delta)\ \mathrm{dom}_{\{2,3\}}\ x$ with $\delta := \frac{1}{2}(1 - x_2 - x_3)$.

(iii) Let $y \in I(v)\backslash B$. We consider the following three cases:

(a) $y_3 > c$, (b) $y_3 < c, y_1 < 1 - c$, (c) $y_3 < c, y_2 < 1 - c$.

In case (a): $(y_1 + \frac{1}{2}(y_3 - c), y_2 + \frac{1}{2}(y_3 - c), c)\ \mathrm{dom}_{\{1,2\}}\ y$.

In case (b): $(1 - c, 0, c)\ \mathrm{dom}_{\{1,3\}}\ y$.

In case (c): $(0, 1 - c, c)\ \mathrm{dom}_{\{2,3\}}\ y$.

13.4

(i) Let $a = (0, \frac{1}{2}, \frac{1}{2})$, $b = (\frac{1}{2}, 0, \frac{1}{2})$, $c = (\frac{1}{2}, \frac{1}{2}, 0)$.From exercise 13.3 we know that $\{a, b, c\}$ is externally stable. To prove internal stability we have to show $\neg a\,\mathrm{dom}\,b$, $\neg b\,\mathrm{dom}\,a$, $\neg a\,\mathrm{dom}\,c$, $\neg c\,\mathrm{dom}\,a$, $\neg b$ dom c and $\neg c$ dom b. We prove

$\neg a\,\mathrm{dom}\,b$ only. $a_1 > b_1$ and $a_2 > b_2$ does not hold. So $\neg a\,\mathrm{dom}_{\{1,2\}}\,b$. Similarly $\neg a\,\mathrm{dom}_{\{1,3\}}\,b$, $\neg a\,\mathrm{dom}_{\{2,3\}}\,b$. With exercise 13.1(i) we conclude: $\neg a\,\mathrm{dom}\,b$. So $A = \{a, b, c\}$ is a stable set.

(ii) Now we prove that $D_3(c) := \{x \in I(v) | x_3 = c\}$ is a stable set for $c \in [0, \frac{1}{2})$.

We showed in exercise 13.3 already the external stability. To prove internal stability, we have to show for each $(x_1, x_2, c) \in D_3(c), (y_1, y_2, c) \in D_3(c) : \neg(x_1, x_2, c)\text{dom}(y_1, y_2, c)$ or

(a) $\neg(x_1, x_2, c)\text{dom}_{\{1,2\}}(y_1, y_2, c)$

(b) $\neg(x_1, x_2, c)\text{dom}_{\{1,3\}}(y_1, y_2, c)$

(c) $\neg(x_1, x_2, c)\text{dom}_{\{2,3\}}(y_1, y_2, c)$.

(a) follows from $x_1 + x_2 = y_1 + y_2 = 1 - c$, so $x_1 \leq y_1$ or $x_2 \leq y_2$.

(b) and (c) follow from $\neg(c > c)$.

13.5

(i) To prove $DC(v) \subset M$ it is sufficient to show that $I(v) \backslash M \subset I(v) \backslash DC(v)$.

Take $x \in I(v) \backslash M$. By the external stability of M there is a $y \in M$ with y dom x.The elements in $DC(v)$ are not dominated. So $x \notin DC(v), x \in I(v) \backslash DC(v)$.

(ii) Suppose that $DC(v)$ is a stable set. Let M also be stable.

By (i) $DC(v) \subset M$. To prove that $M = DC(v)$, we have to show that $M \backslash DC(v) = \emptyset$.

Suppose there is an $x \in M \backslash DC(v)$. By the external stability of $DC(v)$ there is a $y \in DC(v)(\subset M)$ such that y dom x. This is in contradiction with the internal stability of M. So $M \backslash DC(v) = \emptyset$ holds.

13.6 We have $DC(v) = C(v)$ because v is superadditive. So $C(v)$ is internally stable. For the external stability we consider for $x \in I(v) \backslash C(v)$ three possibilities: (i) $x_2 + x_3 < 2$, (ii) $x_1 + x_3 < 2$, (iii) $x_1 + x_2 < 2$.

In case (iii) take $z_3 = (x_1 + \varepsilon_3, x_2 + \varepsilon_3, 4) \in C(v)$ with $\varepsilon_3 = \frac{1}{2}(2-x_1-x_2)$. Then z_3 dom x. In case (ii) take $z_2 = (x_1+\varepsilon_2, 4, x_3+\varepsilon_2) \in C(v)$. Then z_2 $\mathrm{dom}_{\{1,3\}}$ x. In case (i): z_1 $\mathrm{dom}_{\{2,3\}}$ x, where $z_1 = (4, x_2+\varepsilon_1, x_3+\varepsilon_1) \in C(v)$. So $C(v)$ is also externally stable.

13.7

(i) $U(A) = A \Longleftrightarrow (I(v)\setminus \mathrm{dom}(A)) = A \Longleftrightarrow$

$A \cap \mathrm{dom}(A) = \emptyset, I(v)\setminus A = \mathrm{dom}(A) \Longleftrightarrow A$ is a stable set.

(ii) $U(I(v)) = I(v)\setminus \mathrm{dom}(I(v))$, so $U(I(v))$ is equal to the set of undominated elements that is to $DC(v)$.

14.1

(i) Let $N = \{1,2,3\}, v(1,2) = -2, v(S) = 0$ if $S \neq \{1,2\}$.

Then $\Phi(v) = \frac{1}{3!}((0,-2,2) + (0,0,0) + (-2,0,2) + (0,0,0) + (0,0,0) + (0,0,0)) = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$. So $\Phi_1(v) = -\frac{1}{3} < v(\{1\})$. Note that v is not superadditive: $v(1,2) < v(1) + v(2)$.

(ii) v is zero-monotonic implies that $v(S\cup\{i\})-v(S) \geq v(\{i\})$ for each S with $i \notin S$. Then $\Phi_i(v) = \sum_{S:i\notin S} \frac{|S|!(n-1-|S|)!}{n!}(v(S \cup \{i\}) - v(S)) \geq \sum_{S:i\notin S} \frac{|S|!(n-1-|S|)!}{n!}\, v(i) = v(i)$.

(iii) ADD follows from $m^\sigma(v+w) = m^\sigma(v)+m^\sigma(w)$. DUM follows from (14.4) by noting that $\sum_{S:i\in S} \frac{|S|(n-1-|S|)!}{n!} = 1$. EFF: Φ is a convex combination of m^σ's and $m^\sigma(N) = v(N)$ for each σ.

14.2

(i) Note that $\Pi_{i\in S}(e^T)_i \Pi_{i\in N\setminus S}(1-(e^T)_i) = 1$ if $S = T$ and the product is equal to 0 otherwise. Then by (14.10):

$f(e^T) = \sum_{S\in 2^N}(\Pi_{i\in S}(e^T)_i \Pi_{i\in N\setminus S}(1-(e^T)i))v(S) = v(T)$.

(ii) Working out a product $\Pi_{i\in S}x_i\Pi_{i\in N\backslash S}(1-x_i)v(S)$ yields the sum of expressions of the form

$v(S)\Pi_{i\in S}x_i(-1)^{|U|}\Pi_{i\in U}x_i\Pi_{i\in N\backslash(S\cup U)}1 = v(S)(-1)^U\Pi_{i\in S\cup U}x_i$ where $U \subset N\backslash S$.

So $f(x_1,\ldots,x_n) = \sum_S(\Pi_{i\in S}x_i\Pi_{i\in N\backslash S}(1-x_i))v(S)$ is sum of terms of the form $c_T\Pi_{i\in T}x_i$.

(iii) If $f(x) = \sum c_T\Pi_{i\in T}x_i$ and $f(e^S) = v(S)$ for all S, then the c_T's are uniquely determined.

The linear system of equalities in $c_T : f(e^S) = \sum_{T\subset S} c_T = v(S)(S \in 2^N)$ has a unique solution.

(Check it. Hint: start with $\{i\},\{i,j\},\ldots$)

14.3 $f(x_1,x_2,x_3) = x_1x_3 + 2x_2x_3 + x_1x_2x_3$. So
$\Phi_2(v) = \int_0^1 D_2f(t,t,t)dt = \int_0^1(2t+t^2)dt = \frac{4}{3}$.
$\Phi_3(v) = \int_0^1 D_3f(t,t,t)dt = \int_0^1(t+2t+t^2)dt = 1\frac{5}{6}$.
Check: $\Phi_1(v) + \Phi_2(v) + \Phi_3(v) = \frac{5}{6} + \frac{4}{3} + 1\frac{5}{6} = 4 = v(N)$.

14.4 $\Phi(v) \notin C(v)$ for the balanced game $< N,v >$ in example 16.3.

15.1 Let $< N,v >$ be a balanced game. Take $x \in C(v)$. By theorem 15.1: $m(v) \le x \le M(v)$.
From this follow $Q.1 : m(v) \le M(v)$ and $Q.2 : \sum_{i=1}^n m_i(v) \le (\sum_{i=1}^n x_i =)v(N) \le \sum_{i=1}^n M_i(v)$. So $< N,v >$ is quasi-balanced.

15.2 $v \in Q^2 \Longrightarrow m_1(v) \le M_1(v)$. $m_1(v) = \max\{v(1), v(1,2) - M_2(v)\} = \max\{v(1), v(1,2) - (v(1,2) - v(1))\} = v(1), M_1(v) = v(1,2)-v(2)$. So $v(1) \le v(1,2)-v\,(2)$, v is superadditive, $I(v) \neq \emptyset$. Then $C(v) = \{x \in \mathbb{R}^2 | x_1+x_2 = v(1,2), x\,(S) \ge v(S)$ for each $S\} = I(v)$. Further $\Phi(v) = (\frac{1}{2}v(1)+\frac{1}{2}(v(1,2)-v(2)), \frac{1}{2}v(2)+\frac{1}{2}(v(1,2)-$

$v(1))$) and $\tau(v) = \frac{1}{2}(M(v) + m(v)) = \frac{1}{2}((v(12) - v(2), v(1,2) - v(1)) + (v(1), v(2)) = \Phi(v)$. So $\Phi(v) = \tau(v) = \frac{1}{2}(f^1 + f^2)$, which is the middle of the core of v.

15.3

(i) $M(v) = (v(1,2,3) - v(2,3), \ldots) = (2,3,3), m_1(v) = \max\{v(1),$
$v(1,2) - M_2, v(1,3) - M_3, v(1,2,3) - M_3 - M_2\} =$
$\max\{0,-1,-1,-1\} = 0, m_2(v) = \max\{0,0,0,0\} = 0, m_3(v) =$
$\max\{0,0,0,0\} = 0$.

So $m(v) = 0$ and $v \in Q^n$. Then $\tau(v) = \alpha m(v) + (1-\alpha)M(v) = (1-\alpha)M(v)$, a multiple of $M(v) : \tau(v) = \frac{5}{8}(2,3,3)$.

(ii) $M(v) = (\frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \ldots, \frac{1}{2}), m(v) = \ldots = (0, 0, \ldots, 0)$.

So $\tau(v) = (1-\alpha) M(v)$ with $(1-\alpha)\frac{200}{4} = 1, 1-\alpha = \frac{4}{200}$. Hence $\tau(v) = \frac{4}{200}(\frac{3}{4}, \frac{3}{4}, \frac{2}{4}, \ldots, \frac{2}{4}) = \frac{1}{200}(3,3,2,\ldots,2)$.

Note that $\tau_1(v) + \tau_2(v) = \frac{6}{200} < v(1,2) = \frac{1}{2}$. So $\tau(v) \notin C(v)$.

15.4

(i) $1 \le |L| < |R| \Rightarrow M_i(v) = \begin{cases} 0 \text{ if } i \in R \\ 1 \text{ if } i \in L. \end{cases}$

Note that $M(v) \in C(v)$. So $v \in Q^n$ and $\tau(v) = 0\ m(v) + 1M(v) = M(v)$. Further, if $i \in R$, $x_i > 0$, then $x(N \backslash i) < v(N) = v(N \backslash i)$, so $x \notin C(v)$. Hence $C(v) = \{M(v)\}$. For the Shapley value $\Phi_i(v)$ we have $\Phi_i(v) > 0$ if $i \in R$. [Look at $\sigma = (LR \cdots)$.] So $\Phi(v) \notin C(v)$.

(ii) $|L| = |R| \Rightarrow M(v) = (1, 1, \cdots, 1)$, $m(v) = (0, 0, \cdots, 0)$.

So $\tau(v) = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ and also $\Phi(v) = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ by symmetry considerations.

16.2 For the game of example 16.3 we have: $v(1,2) = 100, v(1,3) = v(1,2,3) = 200, v(S) = 0$ otherwise.
Then $M_1(v) = v(1,2,3) - v(2,3) = 200 - 0 = 200$, $M_2(v) = v(1,2,3) - v(1,3) = 200 - 200 = 0$, $M_3(v) = v(1,2,3) - v(1,2) = 200 - 100 = 100$, so $M(v) = (200, 0, 100)$. Further $m_1(v) = \max\{v(1), v(1,2) - M_2(v), v(1,3) - M_3(v), v(N) - M_2(v) - M_3(v)\} = \max\{0, 100 - 0, 200 - 100, 200 - 0 - 100\} = 100$, $m_2(v) = \max\{v(2), v(1,2) - M_1(v), v(2,3) - M_3(v), v(N) - M_1(v) - M_3(v)\} = \max\{0, 100 - 200, 0 - 100, 200 - 200 - 100\} = 0$, $m_3(v) = \cdots = 0$. So $m(v) = (100, 0, 0)$. $C(v)$ is line segment $[(200,0,0),(100,0,100)]$. $\tau(v)$ is the center of this equal to the line segment: $(150, 0, 50)$, so $\tau(v) = Nu(v)$. Further $\Phi(v) = (116\frac{2}{3}, 16\frac{2}{3}, 66\frac{2}{3}) \notin C(v)$.

16.3

(i) $C(v) = \{(0,0,1)\}, Nu(v) \in C(v)$ (by theorem 16.2). So $Nu(v) = (0,0,1)$.

(ii) $< N, v >$ is a symmetric 4-person game with $v(N) = 16$. Then $Nu(v) = (4,4,4,4)$. This follows from the claim that for $x \in I(v)$ with $x_i \neq x_j$ for some $i, j, x \neq Nu(v)$.

This can be seen from $\theta(y) <_L \theta(x)$ for $y \in I(v)$ with $y_k = x_k$ for $k \in N \backslash \{i,j\}, y_i = y_j = \frac{x_i + x_j}{2}$.

[Let $K \subset N$ and suppose $i, j \notin K$. Take $S = \{i\} \cup K, T = \{j\} \cup K$. Suppose w.l.o.g. that $x_i > x_j$. Then on one hand $e(S,x) = v(S) - x_i - x(K) < v(S) - x_j - x(K) = e(T,x)$ and on the other hand $e(S,y) = e(T,y) = v(S) - \frac{x_i + x_j}{2} - x(K) = \frac{e(S,x) + e(T,x)}{2} < e(T,x)$. So $\theta(y) <_L \theta(x)$.]

17.1 Suppose $y > E(F,d)$ and $y \in F$. Then by comprehensiveness also $z := (E_1(F,d) + \varepsilon, E_2(F,d) + \varepsilon) \in F$ where $\varepsilon := \min\{y_1 -$

$E_1(F,d), y_2 - E_2(F,d)\} > 0$ but this contradicts the definition of $E(F,d)$. So $E(F,d) \in W\text{Par }(F)$.

17.2

(i) Suppose $x \gneqq N(F,d)$. Then $(x_1-d_1)(x_2-d_2) > (N_1(F,d)-d_1)(N_2(F,d)-d_2)$ and then $x \notin F$. From this follows: $N(F,d) \in$ Par (F).

(ii) Let $d = (0,0), F = \{x \in \mathbb{R}^2_+ | x_1 + x_2 \leq 2, x_2 \leq 1, x_1 \leq 2\}$. Then $d = (0,0), u = (2,1)$ and $\alpha d + (1-\alpha)u \in \text{Par}(F)$ for $\alpha = \frac{1}{3}$. So $K(F,d) = (\frac{4}{3}, \frac{2}{3}), N(F,d) = (1,1)\} \in \arg\max\{x_1 x_2 | x_1 \geq 0, 0 \leq x_2 \leq 1, x_1 + x_2 = 2\}$.

17.6

(i) Let $d = (0,0), F = \mathit{compr}(\mathit{conv}\{(1,2),(2,1)\}), G =$

$\mathit{compr}(\mathit{conv}(\{0,3),(3,0)\})$. Then $F \subset G$ and $D_1(F,d) = (2,1) \not\leq (3,0) = D_1(G,d)$, $D_2(F,d) = (1,2) \not\leq (0,3) = D_2(G,d)$. So D_1 and D_2 are not monotonic. Further D_1 and D_2 are not symmetric: $D_1(V(F),V(d)) = D_1(F,d) = (2,1) \neq V(D_1(F,d)) =$

$V(2,1) = (1,2), D_2(V(F),V(d)) = D_2(F,d) = (1,2) \neq (2,1) = V(D_2(F,d))$.

(iii) Let $d = (0,0), H =$ compr conv $\{(1,0),(0,1)\}$, $K = A(H)$ with $A(x_1,x_2) = (2x_1,x_2)$. Then $E(K,0) = (\frac{2}{3},\frac{2}{3}) \neq A(\frac{1}{2},\frac{1}{2}) = A(E(H,0))$, so E is not covariant with affine transformations.

$V(d_1+m, d_2+m) = (d_2+m, d_1+m)$ where $m = \sup\{t | d_1+t, d_2+t) \in F\} = \sup\{t | d_2+t, d_1+t) \in V(F)\}$. So $E(V(F),V(d)) = V(E(F,d))$, so E is symmetric.

18.1 Take $(t^S, t^T) \in V_\Gamma(S) \times V_\Gamma(T)$. Then there are $\hat{x}^S$ and $\hat{x}^T$ such that

(i) $K_S(\hat{x}^S, z_{N\setminus S}) \geq t^S$ for all $z_{N\setminus S} \in X^{N\setminus S}(= \prod_{i\in N\setminus S} x_i)$

(ii) $K_T(\hat{x}^T, z_{N\setminus T}) \geq t^T$ for all $z_{N\setminus T} \in X^{N\setminus T}$.

From (i) and (ii) follows $K_{S\cup T}(\hat{x}^S, \hat{x}^T, z) \geq (t^S, t^T)$ for all $z \in X^{N\setminus(S\cup T)}$. Hence $(t^S, t^T) \in V_\Gamma(S \cup T)$.

18.2

(i) Take S, T with $S \cap T = \emptyset$. Let $v(S\cup T) \geq v(S) + v(T)$. Take $(x_i)_{i\in S} \in V(S), (x_i)_{i\in T} \in V(T)$. Then $\sum_{i\in S} x_i \leq v(S)$, $\sum_{i\in T} x_i \leq v(T)$ and then $\sum_{i\in S\cup T} x_i \leq v(S) + v(T) \leq v(S\cup T)$ or $(x_i)_{i\in S\cup T} \in V(S\cup T)$. So $V(S) \times V(T) \subset V(S\cup T)$. So, if $\langle N, v\rangle$ is superadditive, then $\langle N, V\rangle$ is superadditive.

(ii) Take S, T with $S\cap T = \emptyset$ and let $V(S) \times V(T) \subset V(S\cup T)$. Take $x \in V(S)$ with $\sum_{i\in S} x_i = v(S)$ and $y \in V(T)$ with $\sum_{i\in T} x_i = v(T)$. Then $(x, y) \in V(S\cup T)$. So $v(S) + v(T) = \sum_{i\in S} x_i + \sum_{i\in T} y_i \leq v(S\cup T)$. So if $\langle N, V\rangle$ is superadditive, then $\langle N, v\rangle$ is superadditive.

18.5

(ii) Take $\hat{x} \in \arg\max_{x\in V(1,2), x_1\geq V(1), x_2\geq V(2)}(x_1 + x_2)$. Then by superadditivity such an $\hat{x}$ exists and is an element of $SC(V) \subset C(V)$.

19.1 We prove that $N(F, d) = K(F, d) = (0, 1)$ using COV, SYMM, EFF. Let A be the linear map $(x_1, x_2) \overset{A}{\longmapsto} (x_1 + 1, x_2)$. Then $(A(F), A(d))$ is a symmetric problem (i.e. $V(A(F), A(d)) = (A(F), A(d))$). Then $N(A(F), A(d)) = K(A(F), A(d)) = (\alpha, \alpha)$ with $\alpha + \alpha = 2$ (because of EFF), so $\alpha = 1$. $N(F, d) = A^{-1}(1, 1) = (0, 1) = K(F, d)$, by covariance.

(ii) $SH(V) = \{N(F,d)\} = \{(0,1)\} \subset$ Core $(N,V) = \{x \in \mathbb{R}^2 | x_1 \geq -1,\ x_2 \geq 0,\ x_1 + x_2 = 1\}$, where the first equality follows from theorem 19.1.

(iii) D_1 satisfies IIA and RMON (check it!)

19.2

(i) $v_\lambda(i) = 0$ for $i \in \{1,2,3\}$, $v_\lambda(1,2) = 2$, $v_\lambda(1,3) = v_\lambda(2,3) = 3$, $v_\lambda(1,2,3) = 6\frac{2}{3}$. Then $\Phi(v_\lambda) = \frac{1}{6}(12\frac{1}{3}, 12\frac{1}{3}, 15\frac{1}{3}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) *$ $(6\frac{1}{6}, 6\frac{1}{6}, 7\frac{2}{3})$. Since $(6\frac{1}{6}, 6\frac{1}{6}, 7\frac{2}{3}) \notin V(N)$, λ does not lead to an element in $SH(V)$.

(ii) $v_\mu(i) = 0$ for $i \in \{1,2,3\}$, $v_\mu(1,2) = v_\mu(1,3) = v_\mu(2,3) = 2\frac{2}{5}$, $v_\mu(1,2,3) = 4$. Then $\Phi(v_\mu) = (1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3}) = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5}) *$ $(3\frac{1}{3}, 3\frac{1}{3}, 6\frac{2}{3})$. Now $(3\frac{1}{3}, 3\frac{1}{3}, 6\frac{2}{3}) = \frac{1}{3}(10,0,0) + \frac{1}{3}(0,10,0) + \frac{1}{3}(0,0,20) \in V(N)$, so $(3\frac{1}{3}, 3\frac{1}{3}, 6\frac{2}{3}) \in SH(V)$.

Appendix B

Extra exercises

E.1 Let $\langle N, v\rangle$ be the TU-game with $N = \{1,2,3\}$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{3\}) = 10$, $v(S) = 20$ for $|S| = 2$ and $v(N) = 50$.

(i) Give the extreme points of $I(v)$.

(i) Calculate the Shapley value $\Phi(v)$.

(iii) Calculate the 0-normalization $\langle N, w\rangle$ of $\langle N, v\rangle$.

(iv) Calculate $\tau(w)$ and check if $\tau(w) \in C(w)$.

E.2

(i) Give an axiomatic characterization of the Nash bargaining solution $N : \mathcal{B} \to \mathbb{R}^2$.

(ii) Is the D-core of a linear production game non-empty?

(iii) Show that $I(u_N)$ is the unique stable set for the unanimity game $\langle N, u_N\rangle$.

[Hint: prove that $I(u_N) = C(u_N)$.]

E.3

Let $\langle N, V\rangle$ be the NTU-game with $N = \{1,2\}$, $V(\{1\}) = V(\{2\}) = (-\infty, 1]$ and $V(\{1,2\}) = \{x \in \mathbb{R}^2 | x_1 \leq 4,\ x_2 \leq 5\}$.

(i) Is $\langle N, V\rangle$ a superadditive game?

(ii) Calculate $C(V)$ and $SC(V)$.

(iii) Calculate $SH(V)$.

[Hint: does $\langle N, V\rangle$ correspond to a bargaining game?]

E.4

a) (Re)prove the fact that every strictly determined zero-sum game has at least one Nash equilibrium.

b) Give an example of a 2×3 matrix game with value 7 and where the second row is a dominant strategy for player 1.

c) Let A be an $m \times n$ matrix game with $v(A) = 7$. Determine $v(-A^T)$.

E.5 Determine all Nash equilibria in mixed strategies of the 2×4-bimatrix game (A, B) given by

$$(A,B) = \begin{bmatrix} (0,3) & (0,0) & (1,1) & (2,2) \\ (0,0) & (2,2) & (0,2) & (0,2) \end{bmatrix}.$$

E.6 Let $\langle N, v\rangle$ be the TU-game with $N = \{1,2,3\}$ and $v = 2u_{\{1,2\}} + 6u_{\{1,3\}}$.

(i) Prove that $C(v) = \{x \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 8, 0 \leq x_2 \leq M_2(v), 0 \leq x_3 \leq M_3(v)\}$.

(ii) Show that (0,2,6) is an extreme point of the core $C(v)$.

(iii) Calculate $\Phi(v)$ and show that $\Phi(v) \in C(v)$.

(iv) Is $\langle N, V\rangle$ a quasi-balanced game?

E.7

(i) Find a core element for the 3-person linear production game $\langle N, v\rangle$ corresponding to the production situation, given by $A = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b^1 = (0, 19)$, $b^2 = (4, 0)$ and $b^3 = (12, 0)$.

(ii) Does this core element lie in each stable set?

(iii) Does the inequality $2v(\{1,2\})+v(\{1,3\})+v(\{2,3\})+v(\{3\}) \leq 3v(\{1,2,3\})$ hold?

E.8 Let (F, d) be the bargaining game with $d = (0,0)$ and $F = \{x \in \mathbb{R}^2 | x_1 \leq 2,\ x_2 \leq 4,\ 2x_1 + x_2 \leq 4\}$. Let $\langle N, V\rangle$ be the 2-person NTU-game $\langle\{1,2\}, V\rangle$ corresponding to (F, d).

(i) Prove that $N(F, d) = K(F, d)$.

(ii) Calculate $SH(V)$.

(iii) Prove that $C(V) = \text{Par}(F)$.

E.9 Determine all Nash equilibria in mixed strategies of

$$(A, B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccc} \quad f_1 \quad & \quad f_2 \quad & \quad f_3 \quad & \quad f_4 \quad \end{array} \\ \begin{bmatrix} (-3,5) & (2,0) & (1,3) & (0,0) \\ (-3,0) & (1,4) & (0,3) & (3,4) \end{bmatrix} \end{array}.$$

E.10

(i) Let $< X, Y, K, -K >$ be the zero-sum game with $X = Y = [0, 10]$ and $K(x, y) = (x-5)(y-6)+5$ for each $(x, y) \in X \times Y$. Calculate the best answer multifunctions $B_1 : Y \twoheadrightarrow X$, $B_2 : X \twoheadrightarrow Y$ and $NE(X, Y, K, -K)$.

(ii) Let (A, B) be a 2×2-bimatrix game with $A > 0$ and $B < 0$. Find 2×2-matrices P and Q such that Nash equilibria of (A, B) are related to solutions of the LCP problem (r, M) with $r = (1, 1, -1, -1)^\top$ and $M = \left[\begin{array}{cc|cc} 0 & 0 & \multicolumn{2}{c}{P} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{Q} & 0 & 0 \\ & & 0 & 0 \end{array}\right]$.

What is the relation between Nash equilibria of (A, B) and solutions of (r, M)?

E.11 Consider the 2×2-bimatrix game (A, B) given by:

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} (1,2) & (1,2) \\ (1,2) & (4,5) \end{bmatrix} \end{array}$$

(i) Is (A, B) a potential game?

(ii) Give a strong Nash equilibrium and a Nash equilibrium in dominant strategies for the game (A, B).

(iii) Find all Nash equilibria of the mixed extension of (A, B).

(iv) Find a tree game where player 1 has the first move and also a tree game where player 2 has the first move, such that for both tree games a corresponding strategic game is equal to (A, B). Calculate for one of these tree games a subgame perfect equilibrium.

E.12 Let (F, d) be a bargaining problem and let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map with $A(x_1, x_2) = (2x_1 + 3, x_2 + 5)$ for all $(x_1, x_2) \in \mathbb{R}^2$. Suppose that $N(F, d) = (10, 11)$.

(i) Calculate $N(A(F), A(d))$.

(ii) Suppose that $K(F, d) = N(F, d)$. Calculate $K(A(F), A(d))$.

(iii) Calculate an element of $DC(v)$ of the linear production game corresponding to the linear production situation with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $b^1 = (2, 0)$, $b^2 = (2, 0)$, $b^3 = (0, 4)$.

E.13

Let $< N, v >$ be the 3-person game with $N = \{1, 2, 3\}$, $v(\{i\}) = 0$ for $i \in N$, $v(1, 2) = v(1, 3) = 1$, $v(2, 3) = 6$, $v(1, 2, 3) = 6$.

(i) Prove that $C(v) \subset \{x \in \mathbb{R}^3_+ | x_1 = 0\}$.

(ii) Prove that $\Phi(v) \notin C(v)$, $\tau(v) \in C(v)$.

(iii) Calculate $\mathrm{Nu}(v)$.

(iv) Calculate $SH(V)$ for the NTU-game corresponding to $< N, v >$.

E.14 Let $A = \begin{bmatrix} 12 & 0 & 5 \\ 0 & 12 & 5 \end{bmatrix}$ and $J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(i) Calculate $v(A), O_1(A)$ and $O_2(A)$.

(ii) Calculate $v(7A + 3J)$ and $O_1(7A + 3J)$.

(iii) Suppose for $C \in \mathbb{R}^{2\times 3}$ we have $A - 3J \leq C \leq A + 3J$ (entry wise). Prove that $|v(C) - v(A)| \leq 3$.

(iv) Calculate $\min\{x_1 + x_2 | x_1 \geq 0, x_2 \geq 0, (x_1, x_2)(7A + 3J) \geq (1, 1, 1)\}$ and find also an optimal vector for this LP-problem using the result in (ii).

E.15 Consider the 2×3-bimatrix game

$$(A, B) = \begin{bmatrix} (5,16) & (5,16) & (5,16) \\ (5,8) & (6,9) & (9,9) \end{bmatrix}.$$

(i) Find all pure Nash equilibria of (A, B) in dominant strategies.

(ii) Find all Nash equilibria of the mixed extension of (A, B).

(iii) Is (A, B) a potential game?

(iv) Find a tree game for which a corresponding strategic game is equal to (A, B), and find for this tree game the subgame perfect equilibria.

(v) Give a strong Nash equilibrium of (A, B).

E.16 Let $\langle N, v\rangle$ be the 3-person TU-game with $N = \{1, 2, 3\}$, $v(1, 2) = 4$, $v(1, 3) = 5$, $v(1, 2, 3) = 9$ and $v(S) = 0$, otherwise.

(i) Write v as a linear combination of unanimity games.

(ii) Show that $\phi(v) = \tau(v)$.

(iii) Prove that $DC(v)$ is the unique stable set of $\langle N, v\rangle$.

(iv) Construct a linear production situation A, b^1, b^2, b^3, c with $A = [1 \;\; 1], c = (1)$ such that the corresponding linear production game is equal to $\langle N, v\rangle$.

E.17

(i) Let $\langle N, V\rangle$ be the 3-person NTU-game with $N = \{1,2,3\}$

$V(S) = \{x \in \mathbb{R}^S | x \leq 0\}$ for all $S \in 2^N \backslash \{\emptyset, N\}$ and $V(N) = \{x \in \mathbb{R}^N | x_1 \leq 5, x_2 \leq 10, x_3 \leq 20\}$. Show that the weight vector $\lambda = \left(\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right)$ leads to an element of $SH(V)$.

(ii) Calculate for the game of (i) the strong core $SC(V)$. Does the equality $SC(V) = C(V)$ hold?

(iii) Let $\varphi : \mathcal{B} \to \mathbb{R}^2$ be the bargaining solution that assigns to a bargaining problem (F, d) the unique element in $\operatorname{argmax}\{(x_1 - d_1)^2(x_2 - d_2) | x \in F, x \geq d\}$. Show that φ satisfies IIA and COV.

(iv) Let $\langle N, v\rangle$ be an n-person game with $N = \{1, 2, \ldots, n\}$ and $I(v) \neq \emptyset$. Let $k > 0$. Prove that $I(kv) \neq \emptyset$ and $Nu(kv) = kNu(v)$.

[Hint: compare $e(S, kx)$ in $\langle N, kv\rangle$ with $e(S, x)$ in $\langle N, v\rangle$, where $x \in I(v)$.]

E.18 Let A be the matrix game $\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix}$.

(i) Find $v(A), O_1(A)$ and $O_2(A)$.

(ii) Find with the aid of (i) an optimal vector for the linear program $\min\{x_1 + x_2 | x^T A \geq (1,1,1), x_1 \geq 0, x_2 \geq 0\}$.

(iii) Let $B \in \mathbb{R}^{2\times 3}$ and $B \geq A$ (i.e. $b_{ij} \geq a_{ij}$ for all $i \in \{1,2\}, j \in \{1,2,3\}$). Is $v(B) \geq v(A)$?

(iv) Does there exist a tree game with corresponding strategic game $(A, -A)$?

E.19 For each $a \in \mathbb{R}$, let $A(a) = \begin{bmatrix} 2+a & 4 \\ 4 & 6 \end{bmatrix}$,

$B(a) = \begin{bmatrix} 3+a & 5 \\ 5 & 7 \end{bmatrix}$

(i) For which $a \in \mathbb{R}$ has the bimatrix game $(A(a), B(a))$ a Nash equilibrium in dominant strategies?

(ii) For which $a \in \mathbb{R}$ has $(A(a), B(a))$ two pure Nash equilibria?

(iii) Calculate for $(A(3), B(3))$ all Nash equilibria (pure and mixed).

(iv) Give for each $a \in \mathbb{R}$ a potential for $(A(a), B(a))$.

E.20 Let $a \in \mathbb{R}^n$. Let $\langle N, v\rangle$ be the n-person game with $N = \{1, 2, \ldots, n\}$ and $v(S) = \sum\limits_{i\in S} a_i$ for all S with $|S| \geq n-1$, $v(S) \leq \sum\limits_{i\in S} a_i$, otherwise.

(i) Prove that $C(v) = \{a\}$.

(ii) Show that v is quasi-balanced and that $\tau(v) = M(v)$.

(iii) Prove that $\tau(v) = Nu(v)$.

(iv) Prove that for each $S \in 2^N \backslash \{\emptyset\}$ it holds that

$$v(S) + \textstyle\sum_{i \in S} v(N\backslash\{i\}) \leq |S| v(N).$$

(v) Let M be a stable set of $\langle N, v\rangle$. Give an element of M.

E.21

(i) Find a bargaining solution $f : \mathcal{B} \to \mathbb{R}^2$ such that f satisfies EFF, RMON and IIA.

(ii) Show that the Raiffa-Kalai-Smorodinsky solution does not satisfy IIA.

(iii) Construct a 2-person NTU-game which is not superadditive.

(iv) Calculate the sum of the dividends $\sum\limits_{T:i\in T} d_T(v)$ for player i in a game $\langle N, v\rangle$ where player i is a dummy player and $\langle N, v\rangle$ is a zero normalized game.

E.22

(i) For which bimatrix game can you find an equilibrium if you know that $(1, 0, \frac{1}{2}, 0)^T$*is* a solution to the linear complementarity problem (r, M) where $r^T = (1, 1, -1, -1)$ and

$$M = \begin{pmatrix} 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & -2 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}?$$

(ii) Which sealed bid price auction corresponds to a strategic game where each player has a dominant strategy?

E.23 For each $a \in \mathbb{R}$ let $A(a)$ be the 2×2-matrix game $\begin{pmatrix} 2a & a \\ 4 & 1 \end{pmatrix}$.

(i) Give all $a \in \mathbb{R}$ for which player 1 has a dominant pure strategy.

(ii) Show that for each $a \in \mathbb{R}$ the matrix game $A(a)$ has a pure Nash equilibrium.

(iii) Calculate the value $v(A(a))$ for each $a \in \mathbb{R}_+$ and show that the function: $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(a) := v(A(a))$ for each $a \in \mathbb{R}$ is in one point not differentiable.

(iv) For which $a \in \mathbb{R}$ is $(A(a), -A(a))$ a potential game? Give also a potential in that case.

E.24 Let $\langle N, v\rangle$ be a 4-person game with $N = \{1, 2, 3, 4\}$. Let $(2, 2, 2, 2) \in C(v)$ and let M be a stable set of $\langle N, v\rangle$.

(i) Is $DC(v)$ a non-empty set?

(ii) Is $\langle N, v\rangle$ a quasi-balanced game?

(iv) Prove that $v(\{1, 2, 3\}) + v(\{2, 3, 4\}) + v(\{1, 4\}) \leq 16$.

(iv) Prove that the nucleolus $Nu(v)$ is an element of M.

E.25 Let $\langle N, V\rangle$ be the NTU-game with $N = \{1, 2\}$, $V(\{1\}) = (-\infty, 3]$, $V(\{2\}) = (-\infty, 4]$ and $V(\{1, 2\}) = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 11\}$.

(i) Is $\langle N, V\rangle$ superadditive?

(ii) Calculate the core $C(V)$.

(iii) Find a bargaining problem (F, d) such that the corresponding NTU-game is $\langle N, V\rangle$. Calculate $N(F, d)$ and $SH(N, V)$.

(iv) Find a 2-person TU-game $\langle N, v\rangle$ such that the corresponding NTU-game is $\langle N, V\rangle$. Show that the τ-value $\tau(v)$ is the unique element in $SH(N, V)$.

Bibliography

[1] Aumann, R.J. (1985). An axiomatization of the non-transferable utility value. Econometrica, 53, 667-678.

[2] Aumann, R.J. and S. Hart (Eds.). Handbook of Game Theory, Volume 1, 2, 3 (1992, 1994, 2002). Amsterdam: North Holland.

[3] Aumann, R.J. and L.S. Shapley (1974). Values of non-atomic games. Princeton: Princeton University Press.

[4] Binmore, K. (1992). Fun and games. A text on game theory. Lexington: D.C. Heath and Company.

[5] Bondareva, O.N. (1963). Some applications of linear programming methods to the theory of cooperative games (in Russian). Problemy Kibernet. 10, 119-139.

[6] Borel, E. (1921). La théorie du jeu et les equations intégrales à noyau symétrique. Comptes Rendues de l'Académie des Sciences 173, 1304-1308.

[7] Borm, P., H. Keiding, R. McLean, S. Oortwijn and S.H. Tijs (1992). The compromise value for NTU-games. International Journal of Game Theory, 21, 175-189.

[8] Borm, P. and H. Peters (Eds.). Chapters in Game Theory. In honor of Stef Tijs (2002). Boston: Kluwer.

[9] Borm, P. and S. Tijs (1992). Strategic claim games corresponding to an NTU-game. Games and Economic Behavior 4, 58-71.

[10] Clarke, E.H. (1971). Multipart pricing of public goods. Public Choice, 11, 17-33.

[11] Cournot, A. (1838). Recherches sur les principes mathématiques de la théorie des richesses. Paris: Hachette.

[12] Curiel, I. (1997). Cooperative Game Theory and Applications. Cooperative Games arising from Combinatorial Optimization Problems. Boston: Kluwer.

[13] Damme, E. van (1991). Stability and Perfection of Nash Equilibrium. Berlin: Springer Verlag.

[14] Driessen, T. (1988). Cooperative games, solutions and applications. Dordrecht: Kluwer.

[15] Filar, J.A. and K. Vrieze (1997). Competitive Markov decision processes. New York: Springer Verlag.

[16] Fudenberg, D. and J. Tirole (1991). Game Theory. Cambridge: M.I.T. Press.

[17] Greenberg, J. (1990). The theory of social situations: an alternative game-theoretic approach. Cambridge: Cambridge University Press.

[18] Groves, T. (1973). Incentives in teams. Econometrica, 47, 627-636.

[19] Harsanyi, J.C. (1959). A bargaining model for the cooperative n-person game. Annals of Mathematics Study 40, 325-355.

[20] Harsanyi, J.C. (1973). Games with randomly disturbed payoffs: a new rationale for mixed strategy equilibrium points. International Journal of Game Theory 2, 1-23.

[21] Harsanyi, J.C. and R. Selten (1988). A general theory of equilibrium selection in games. Cambridge: M.I.T. Press.

[22] Ichiishi, T. (1983). Game theory for economic analysis. New York: Academic Press.

[23] Isaacs, R. (1975). Differential games. Huntington, New York: Robert E. Krieger Publ. Comp.

[24] Kalai, E. and D. Samet (1984). Persistent equilibria in strategic games. International Journal of Game Theory 13, 129-144.

[25] Kalai, E. and D. Samet (1985). Monotonic solutions to general cooperative games. Econometrica 53, 307-328.

[26] Kalai, E. and M. Smorodinsky (1975). Other solutions to Nash's bargaining problem. Econometrica 43, 513-518.

[27] Kalai, E. and E. Zemel (1982). Totally balanced games and games of flow. Mathematics of Operations Research 7, 476-478.

[28] Kohlberg, E. and J.F. Mertens (1986). On the strategic stability of equilibrium. Econometrica 54, 1003-1038.

[29] Kreps, D. (1990). Game theory and economic modelling. Oxford: Oxford University Press.

[30] Kreps, D. and R. Wilson (1982). Sequential equilibrium. Econometrica 50, 863-894.

[31] Kuhn, H. (1953). Extensive games and the problem of information, in: Contributions to the theory of games, Eds. H. Kuhn and A. Tucker. Princeton: Princeton University Press, pp. 193-216.

[32] Lemke, C.E. and J.T. Howson (1964). Equilibrium points of bimatrix games. SIAM 12, 413-423.

[33] Lucas, W.F. (1968). A game with no solution. Bulletin of the American Mathematical Society, 74, 237-239.

[34] Lucas, W.F. and M. Rabie (1982). Games with no solutions and empty core. Mathematics of Operations Research 7, 491-500.

[35] Maynard Smith, J. (1982). Evolution and the theory of games. Cambridge: Cambridge University Press.

[36] Milgram, P. and R. Weber (1982). A theory of auctions and competitive bidding. Econometrica 50, 1089-1122.

[37] Monderer, D. and L.S. Shapley (1996). Potential games. Games and Economic Behavior 14, 124-143.

[38] Moulin, H. (1986). Game theory for the social sciences. New York: New York University Press.

[39] Moulin, H. (1988). Axioms of cooperative decision making. Cambridge: Cambridge University Press.

[40] Myerson, R. (1978). Refinements of the Nash equilibrium concept. International Journal of Game Theory 7, 73-80.

[41] Myerson, R. (1991). Game theory: analysis of conflict. Harvard: Harvard University Press.

[42] Neumann, J. von. (1928). Zur Theorie der Gesellschaftsspiele. Mathematische Annalen 100, 295-320.

[43] Neumann, J. von, and O. Morgenstern (1944). Theory of games and economic behavior, Princeton: Princeton University Press.

[44] Nash, J.F. (1950). The bargaining problem. Econometrica 28, 155-162.

[45] Nash, J.F. (1951). Non-cooperative games. Annals of Mathematics 54, 286-295.

[46] Osborne, M. and A. Rubinstein (1990). Bargaining and markets, New York: Academic Press.

[47] Owen, G. (1972). Multilinear extensions of games. Management Science 18, 64-79.

[48] Owen, G. (1975). On the core of linear production games. Mathematical Programming 9, 358-370.

[49] Owen, G. (1982). Game Theory, 2nd ed. New York: Academic Press.

[50] Perea, A. (2001). Rationality in Extensive Form Games. Boston: Kluwer.

[51] Peters, H. (1992). Axiomatic bargaining theory. Dordrecht: Kluwer.

[52] Peters, H. and S. Tijs (1985). Characterization of all individually monotonic bargaining solutions. International Journal of Game Theory 14, 219-228.

[53] Potters, J. and S. Tijs (1992). The nucleolus of a matrix game and other nucleoli. Mathematics of Operations Research 17, 164-174.

[54] Maschler, M., J. Potters, and S. Tijs (1992). The general nucleolus and the reduced game property. International Journal of Game Theory 21, 85-106.

[55] Raiffa, H. (1953). Arbitration schemes for generalized two-person games. Annals of Mathematics Studies 28, 361-378.

[56] Rasmusen, E. (1989). Games and information. Oxford: Blackwell.

[57] Rosenthal, R.W. (1973). A class of games possessing pure-strategy Nash Equilibria. International Journal of Game Theory 2, 65-67.

[58] Roth, A.E. (1979). Axiomatic models of bargaining. Berlin and New York: Springer-Verlag.

[59] Roth, A.E. (1985). Game-theoretic models of bargaining. Cambridge: Cambridge University Press.

[60] Roth, A.E. (1988). The Shapley value: Essays in honor of Lloyd S. Shapley. Cambridge: Cambridge University Press.

[61] Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. Econometrica 50, 97-109.

[62] Schmeidler, D. (1969). The nucleolus of a characteristic function game. SIAM J on Applied Mathematics, 17, 1163-1170.

[63] Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. International Journal of Game Theory 4, 25-55.

[64] Shapley, L.S. (1953). A value for N-person games, in: Contributions to the theory of games II, H.W. Kuhn and A.W. Tucker eds. Princeton: Princeton University Press, pp. 307-317.

[65] Shapley, L.S. (1967). On balanced sets and cores. Naval Research Logistics Quarterly 14, 453-460.

[66] Shapley, L.S. (1969). Utility comparison and the theory of games, in: La Decision: Aggregation et Dynamique des Ordres de Preference. Editions du Centre National de la Recherche Scientifique, Paris pp. 251-263.

[67] Slikker, M. and A. van de Nouweland (2000): Social and Economic Networks in Cooperative Game Theory. Boston: Kluwer.

[68] Sorin, S. (2002). A first course on zero-sum repeated games. New York: Springer Verlag.

[69] Suijs, J. (1999). Cooperative Decision-Marking under Risk. Boston: Kluwer.

[70] Tijs, S.H. (1981). Bounds for the core of a game and the τ-value, in: Game Theory and Mathematical Economics, eds. O. Moeschlin and D. Pallaschke. Amsterdam: North Holland, pp. 123-132.

[71] Tijs, S.H. (1987). An axiomatization of the τ-value. Mathematical Social Sciences 13, 177-181.

[72] Tijs, S.H. and G.J. Otten (1993). Compromise values in cooperative game theory. Top 1, 1-51.

[73] Tijs, S.H., T. Parthasarathy, J. Potters and V. Rajendra Prasad (1984). Permutation games: another class of totally balanced games. Operations Research Spektrum 6, 119-123.

[74] Tirole, J. (1988). The theory of industrial organization. Cambridge: M.I.T. Press.

[75] Voorneveld, M. (1999). Potential games and interactive decisions with multiple criteria. CentER Dissertation Series, No. 61, Tilburg University, Tilburg, The Netherlands.

[76] Vorobev, N.N. (1977). Game Theory. New York: Springer Verlag.

Index

Texts and Readings in Mathematics

1. R. B. Bapat: Linear Algebra and Linear Models (Second Edition)
2. Rajendra Bhatia: Fourier Series (Second Edition)
3. C. Musili: Representations of Finite Groups
4. H. Helson: Linear Algebra (Second Edition)
5. D. Sarason: Complex Function Theory (Second Edition)
6. M. G. Nadkarni: Basic Ergodic Theory (Second Edition)
7. H. Helson: Harmonic Analysis (Second Edition)
8. K. Chandrasekharan: A Course on Integration Theory
9. K. Chandrasekharan: A Course on Topological Groups
10. R. Bhatia (ed.): Analysis, Geometry and Probability
11. K. R. Davidson: C* – Algebras by Example
12. M. Bhattacharjee et al.: Notes on Infinite Permutation Groups
13. V. S. Sunder: Functional Analysis — Spectral Theory
14. V. S. Varadarajan: Algebra in Ancient and Modern Times
15. M. G. Nadkarni: Spectral Theory of Dynamical Systems
16. A. Borel: Semisimple Groups and Riemannian Symmetric Spaces
17. M. Marcolli: Seiberg – Witten Gauge Theory
18. A. Bottcher and S. M. Grudsky: Toeplitz Matrices, Asymptotic Linear Algebra and Functional Analysis
19. A. R. Rao and P. Bhimasankaram: Linear Algebra (Second Edition)
20. C. Musili: Algebraic Geometry for Beginners
21. A. R. Rajwade: Convex Polyhedra with Regularity Conditions and Hilbert's Third Problem
22. S. Kumaresan: A Course in Differential Geometry and Lie Groups
23. Stef Tijs: Introduction to Game Theory
24. B. Sury: The Congruence Subgroup Problem
25. R. Bhatia (ed.): Connected at Infinity
26. K. Mukherjea: Differential Calculus in Normed Linear Spaces (Second Edition)
27. Satya Deo: Algebraic Topology: A Primer (Corrected Reprint)
28. S. Kesavan: Nonlinear Functional Analysis: A First Course
29. S. Szabó: Topics in Factorization of Abelian Groups
30. S. Kumaresan and G. Santhanam: An Expedition to Geometry
31. D. Mumford: Lectures on Curves on an Algebraic Surface (Reprint)
32. J. W. Milnor and J. D. Stasheff: Characteristic Classes (Reprint)
33. K. R. Parthasarathy: Introduction to Probability and Measure (Corrected Reprint)
34. A. Mukherjee: Topics in Differential Topology
35. K. R. Parthasarathy: Mathematical Foundations of Quantum Mechanics
36. K. B. Athreya and S. N. Lahiri: Measure Theory
37. Terence Tao: Analysis I (Second Edition)
38. Terence Tao: Analysis II (Second Edition)

39. W. Decker and C. Lossen: Computing in Algebraic Geometry
40. A. Goswami and B. V. Rao: A Course in Applied Stochastic Processes
41. K. B. Athreya and S. N. Lahiri: Probability Theory
42. A. R. Rajwade and A. K. Bhandari: Surprises and Counterexamples in Real Function Theory
43. G. H. Golub and C. F. Van Loan: Matrix Computations (Reprint of the Third Edition)
44. Rajendra Bhatia: Positive Definite Matrices
45. K. R. Parthasarathy: Coding Theorems of Classical and Quantum Information Theory
46. C. S. Seshadri: Introduction to the Theory of Standard Monomials
47. Alain Connes and Matilde Marcolli: Noncommutative Geometry, Quantum Fields and Motives
48. Vivek S. Borkar: Stochastic Approximation: A Dynamical Systems Viewpoint
49. B. J. Venkatachala: Inequalities: An Approach Through Problems
50. Rajendra Bhatia: Notes on Functional Analysis
51. A. Clebsch (ed.): Jacobi's Lectures on Dynamics (Second Revised Edition)
52. S. Kesavan: Functional Analysis
53. V. Lakshmibai and Justin Brown: Flag Varieties: An Interplay of Geometry, Combinatorics, and Representation Theory
54. S. Ramasubramanian: Lectures on Insurance Models
55. Sebastian M. Cioaba and M. Ram Murty: A First Course in Graph Theory and Combinatorics
56. Bamdad R. Yahaghi: Iranian Mathematics Competitions, 1973-2007
57. Aloke Dey: Incomplete Block Designs
58. R.B.Bapat: Graphs and Matrices