Complexity / Složitost (SLOa) – 2021/2022
Homework assignment 3

1. Consider the following modification of the $(GEOGRAPHY)$ $GAME$:

   > A *parity game* is a quadruple (V, E, s, α) where (V, E) is a directed graph (with vertices V and edges $E \subseteq V \times V$), $s \in V$ is an initial vertex and $\alpha\colon V \to \mathbb{N}$ is the *weight* function. The game is played by two players (Player 1, who starts, and Player 2), who alternate in moving a token along the vertices of the graph according to the graph edges. Each vertex can be visited at most once. The game ends when it is not possible to make a move. The winner is Player 1 iff the smallest weight encountered along the game is odd and Player 2 if it is even (note that it does not matter whose turn it was when the game ended).

   $PARITY\ GAME$ asks whether, given a parity game g, Player 1 has a winning strategy for g.

   Prove that $PARITY\ GAME$ is PSPACE-complete.

   Hint: modify the proof that $GEOGRAPHY\ GAME$ is PSPACE-complete from the book of Papadimitriou (an excerpt from the book with the proof can be found here: `http://www.fit.vutbr.cz/~lengal/slo-2022/geography-pspace-complete.pdf`).

   4 points

2. Let $G = (V, E)$ be a finite undirected graph, i.e., $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$. A set of vertices $S \subseteq V$ is a k-*clique* of G iff $|S| = k$ and $\{\{i, j\} \mid i, j \in S, i \neq j\} \subseteq V$. The problem $\oplus CLIQUE$ is defined as follows:

   $$\oplus CLIQUE = \{((V, E), k) : |\{S \subseteq V : S \text{ is a } k\text{-clique of } G\}| \bmod 2 = 1\},$$

   i.e., it is the set of graphs that have an *odd* number of k-cliques. Show that $\oplus CLIQUE$ and $co\oplus CLIQUE$ are PTIME-interreducible (i.e., that there is a polynomial reduction in both directions).

   2.5 points

3. Let $\Sigma = \{0, 1\}$ and $f\colon \Sigma^* \to \Sigma^*$ be a one-way function such that $\forall x \in \Sigma^*\colon |f(x)| = |x|$. Consider the following language L_f:

   $$L_f = \{(x, y, c) \in \Sigma^* \times \Sigma^* \times \mathbb{N} : |x| = |y| \wedge \exists z \in \Sigma^{|y|}\colon f(z) = y \wedge popcount(x \oplus z) \leq c\},$$

   where $x \oplus y$ denotes the XOR of binary strings x and y and $popcount(x)$ denotes the number of occurrences of the symbol "1" in x.

   Prove that $L_f \in \text{UP} \setminus \text{P}$.

   3.5 points