Densities and Expectation

Submit a PDF of your answers to Canvas

1. ***Unfair dice, convolution.***

a) Consider two independent dice $X \sim p(x)$ and $Y \sim q(y)$ where

$$p(x) = \begin{cases} p_1 & \text{if } x = 1 \\ \vdots & \\ p_6 & \text{if } x = 6 \\ 0 & \text{otherwise} \end{cases}$$

and

$$q(y) = \begin{cases} q_1 & \text{if } y = 1 \\ \vdots & \\ q_6 & \text{if } y = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Find an expression for the pmf of $X+Y$. It may be helpful to specify the ways in which $X+Y = i$ for $i = 2, 3, \ldots, 12$.

b) Next consider the more general case. Let X and Y be integer valued independent random variables with pmfs given by p_X and p_Y, and define the random variable $Z = X+Y$. Show that

$$p_Z(z) = \sum_i p_X(x_i) p_Y(z - x_i).$$

2. In homework 1, you wrote a few lines of code to find the minimum of n i.i.d. samples of a uniform random variable. Here, you will address the same problem analytically. More precisely, let

$$X_i \overset{i.i.d.}{\sim} U[0,1],$$

and define

$$Y = \min_{i=1,\ldots,n} X_i.$$

a) Find an expression for the pdf of Y. Your answer should depend on n.

b) Find an expression for $\mathbb{E}[Y]$.

c) Does this agree with the plot you made previously?

3. ***Expectation Basics.*** Let $\mathbb{P}(X=1)=1/2$, $\mathbb{P}(X=2)=1/4$, and $\mathbb{P}(X=3)=1/4$.

a) Compute $\mathbb{E}[X]$.

b) Compute $\mathbb{E}[g(X)]$ if $g(x)=x^2$.

c) The variance of an random variable is defined as $\mathsf{var}(X)=\mathbb{E}[(X-\mathbb{E}[X])^2]$. Compute $\mathsf{var}(X)$.

d) Write an expression for $\mathbb{E}[-\log(g(X))]$ in terms of an arbitrary function $g(x)$.

e) Now consider the function $p(x)=\mathbb{P}(X=x)$, where the probabilities were given above. Write an expression for $\mathbb{E}[-\log_2(p(X))]$. This quantity is the *entropy* of X.

4. ***Bivariate Random Variables.*** The joint pmf $p(x,y)$ is defined as follows:

$$\begin{aligned}\mathbb{P}(X=1,Y=1)&=1/8\\ \mathbb{P}(X=1,Y=2)&=1/8\\ \mathbb{P}(X=2,Y=1)&=1/2\\ \mathbb{P}(X=2,Y=2)&=1/4\end{aligned}$$

a) Are X and Y independent? Why or why not?

b) For any two random variables X and Y, the *covariance* is defined as $\mathsf{cov}(X,Y)=\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$. Compute $\mathsf{cov}(X,Y)$.

c) Define a new random variable $Z=X+Y$. Specify the pmf of $p(z)$.

d) Write a function (in code) that generates X and Y at random as specified by the joint PMF. Your function should take no input arguments, and should return $X,Y\in\{1,2\}^2$ with probability specified above.

5. ***Total Probability.*** Let X and Y be discrete random variables. Conditional expectation is defined as $\mathbb{E}[X|Y]=\sum_x x\,\mathbb{P}(X=x|Y)$ for discrete random variables. Use this definition to show that

$$\mathbb{E}[X]=\mathbb{E}\big[\mathbb{E}[X|Y]\big].$$

6. Optional. ***MAP classification and 1-d discriminant analysis.*** Let $X \in \mathbb{R}$ represent a feature, and $Y = 0$ or $Y = 1$ the class label. The distribution of X depends on the label:

$$X|Y = 0 \sim \mathcal{N}(0,1)$$

$$X|Y = 1 \sim \mathcal{N}(4,1)$$

where $\mathcal{N}(\mu, \sigma^2)$ is the Gaussian density: $p(x) = e^{-[(x-\mu)/\sigma]^2/2}/\sqrt{2\pi\sigma^2}$. Let the prior probability of the classes be $\mathbb{P}(Y = 0) = 3/4$ and $\mathbb{P}(Y = 1) = 1/4$.

a) Use a computer to create a plot with both pdfs $p(x|y = 0)$ and $p(x|y = 1)$ on the same axis.

b) Use Bayes and total probability to find an expression for the posterior $p(y|x)$.

c) Use a computer to evaluate $p(y = 0|x = 2)$ using your expression above. What is $p(y = 0|x = 2)$?

d) Recall that maximum a posteriori (MAP) classification rule predicts the label y as follows:

$$\widehat{y} = \arg\max_y p(y|x).$$

Use *maximum a posteriori* to design a classification rule that will predict if $Y = 0$ or $Y = 1$ given $X = x$.

e) What is the *true risk* of your MAP classifier? Use a computer to find a numerical answer.