Transformations of random variables, VAEs

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1. This problem will explore a question central to variational auto encoders: How can a deterministic function take a simple random variable as input, and create an arbitrarily complicated, higher dimensional distribution?

Variational autoencoders (VAEs) operate by passing a random variable with a simple distribution (like a 2-dimenional normal distribution) through a function to create a more complicated distribution. This allows one to sample randomly from the complicated distribution by sampling from the simple distribution and then passing the sample through the function. As an example, consider the decoder function of a VAE trained on MNIST. It takes as input a random vector $\boldsymbol{z} \in \mathbb{R}^2$, with $\boldsymbol{z} \sim \mathcal{N}(0, \boldsymbol{I})$, and outputs a new random vector $\widehat{\boldsymbol{x}} \in \mathbb{R}^{784}$ with a distribution that approximates the unknown distribution over MNIST images. Transformations of random variables are central to the operation of a VAE.

In this problem we will consider a transformations of random variables.

a) Let $Z \sim U[0,1]$. Find a function $g(\cdot)$ so that $X = g(Z)$ follows an exponential distribution, i.e, so that

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

b) Find a function that $f(\cdot)$ that generates a uniform random variable from a exponentially distributed random variable.

c) Let $Z \sim U[0,1]$ (and note that Z is a univariate random variable). Devise a deterministic function that maps Z to a two dimensional random vector $\boldsymbol{x}$, where $\boldsymbol{x} = [X_1 \;\; X_2]^T$ is uniform over $[0,1]^2$. You may find it helpful to specify the function as a procedure.

d) Extend your procedure to generate a random variable $\boldsymbol{x} \in \mathbb{R}^n$ that is uniform over $[0,1]^n$.

$0\,[$——$]\,1$ $\qquad Z \sim U[0,1]$ $\quad U(X) = F_X(X)$ is uniform on $[0,1]$

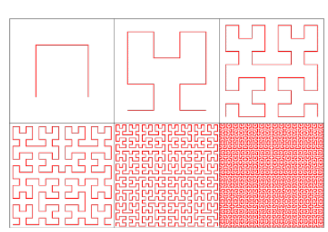
$\uparrow \qquad \downarrow$

$0\,[$————$)\,\infty$ $\quad X \sim \mathrm{Exp}(\lambda) \quad X \quad F_X^{-1}(Z)$ is $\exp(\lambda)$

a) inverse of b: $1-e^{-\lambda x}$ is $X = \frac{-1}{\lambda}(\ln(1-Z))$

b) CDF of exp is $Z = 1-e^{-\lambda x}$

c) any space filling curve i.e. ⟶



d) all cantor sets are homeomorphic