Gaussian Random Vectors, Gaussian Discriminant Analysis

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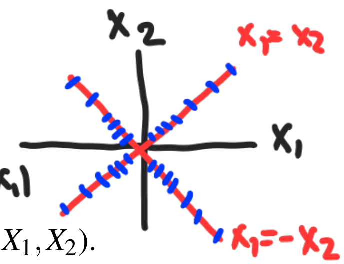
1. If a random vector $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a Gaussian random vector, then the marginals are Gaussian, i.e, $p(x_j) \sim \mathcal{N}(\mu_j, \sigma_j)$. In this problem, we consider the converse. Let $X_1 \sim \mathcal{N}(0,1)$ and define the random variable

$$p\ X_2 \sim N(0,1) \quad X_2 = \begin{cases} X_1 & \text{with probability } 1/2 \\ -X_1 & \text{with probability } 1/2. \end{cases}$$



a) What is $p(x_2)$? $p(X_2) = \sum_{x_1} p(x_2|x_1)p(x_1) = \frac{p(-x_1)}{2} + \frac{p(x_1)}{2} = p(x_1)$

b) Sketch $p(x_1, x_2)$ in the x_1, x_2 plane (or describe the joint distribution of X_1, X_2).

c) Is $[X_1\ X_2]^T$ a Gaussian random vector? **NO**

2. For binary classification the MAP classification rule can be expressed using the log-likelihood ratio: if

$$\log\left(\frac{p(\boldsymbol{x}|y=0)}{p(\boldsymbol{x}|y=1)}\right) > 0 \tag{1}$$

then $\widehat{y} = 0$, else $\widehat{y} = 1$.

In *Gaussian discriminant analysis*, we assume that the class conditional distributions are Gaussian:

$$p(\boldsymbol{x}|y=0) = \frac{1}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}_0|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)\right)$$

and

$$p(\boldsymbol{x}|y=1) = \frac{1}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}_1|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T \boldsymbol{\Sigma}_1^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)\right).$$

a) In linear discriminant analysis, $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_1$, and (1) is equivalent to the following linear classification rule: if

$$\boldsymbol{w}^T \boldsymbol{x} > c$$

then $\widehat{y} = 0$, else $\widehat{y} = 1$. Find $\boldsymbol{w}$ and c in terms of $\boldsymbol{\Sigma}_0$, $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$.

b) In quadratic discriminant analysis, $\boldsymbol{\Sigma}_0 \neq \boldsymbol{\Sigma}_1$, and (1) is equivalent to the following quadratic rule: if

$$\boldsymbol{x}^T \boldsymbol{B} \boldsymbol{x} + \boldsymbol{w}^T \boldsymbol{x} > c$$

then $\widehat{y} = 0$, else $\widehat{y} = 1$. Find an expression for $\boldsymbol{B}$, $\boldsymbol{w}$ and c in terms of $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Sigma}_1$, μ_0 and μ_1.

a) assume $\Sigma_0 = \Sigma_1$ then $|\Sigma_0|^{1/2} = |\Sigma_1|^{1/2}$ denote $A = \Sigma_0^{-1}$

$$\log\left(\frac{p(x|y=0)}{p(x|y=1)}\right) > 0 \rightarrow (x-\mu_0)^T\Sigma_0^{-1}(x-\mu_0) < (x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)$$

$$x^TAx - \mu_0^TAx + \mu_0^TA\mu_0 - x^TA\mu_0$$

$$-\mu_0^TAx + \mu_0^TA\mu_0 - x^TA\mu_0 < -\mu_1^TAx + \mu_1^TA\mu_1 - x^TA\mu_1$$

$$\mu_0^TA^Tx$$

$$\underbrace{\mu_0^TA\mu_0 - \mu_1^TA\mu_1}_{C} < \underbrace{(\mu_0-\mu_1)^T(A+A^T)}_{W^T}x$$

b)

$$-\frac{1}{2}\log(\Sigma_0) - (x-\mu_0)^T\Sigma_0^{-1}(x-\mu_0) > -\frac{1}{2}\log(\Sigma_1) - (x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)$$

$$x^T\underbrace{(\Sigma_1^{-1} - \Sigma_0^{-1})}_{B}x + \underbrace{\left[\mu_0^T(\Sigma_0^{-1}+(\Sigma_0^{-1})^T) - \mu_1^T(\Sigma_1^{-1}+(\Sigma_1^{-1})^T)\right]}_{W^T}x > \frac{1}{2}\log(\Sigma_0) - \frac{1}{2}\log(\Sigma_1) + \underbrace{\mu_0^T\Sigma_0^{-1}\mu_0 - \mu_1^T\Sigma_1^{-1}\mu_1}_{C}$$