Covariance, Random Vectors

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1. ***Variance of the sample mean.*** Let X be a random variable with $E[X] = \mu$ and $\text{var}(X) = \sigma^2$, and let $X_1, X_2, \ldots$ be i.i.d. samples of X. Define $Y = \sum_{i=1}^{n} X_i$.

   a) Last time you showed that $E[Y] = n\mu$ using the properties of expectation. What is $\text{var}(Y)$?

   b) The sample mean is defined as $\frac{1}{n}\sum_{i=1}^{n} X_i$. What is the variance of the sample mean, i.e, $\text{var}(\frac{1}{n}\sum_{i=1}^{n} X_i)$?

$\Sigma_x = E[xx^T] - \mu_x \mu_x^T$

2. Consider a random vector $\boldsymbol{x} \in \mathbb{R}^{n\times 1}$ with mean $\mu_{\boldsymbol{x}} \in \mathbb{R}^{n\times 1}$ and covariance $\Sigma_{\boldsymbol{x}}$, and a deterministic vector $\boldsymbol{a} \in \mathbb{R}^{n\times 1}$.

   a) Consider the random variable $Y = \boldsymbol{a}^T\boldsymbol{x}$. Find an expression for $E[Y]$.

   b) Find an expression for $\text{var}(Y)$ in terms of $\boldsymbol{a}$, $\Sigma_{\boldsymbol{x}}$ and $\mu_{\boldsymbol{x}}$.

   c) Does this agree with your answer in the problem above? How/why? *Hint* Find $\boldsymbol{a}$ and $\boldsymbol{x}$ so that $\boldsymbol{a}^T\boldsymbol{x} = \frac{1}{n}\sum_i X_i$.

1) a) $\text{Var}(Y) = E[Y^2] - E[Y]^2$

$= E[(\Sigma x_i)^2] - n^2 E[X]^2$

$= E[\Sigma x_i^2] + (n^2-n) E[x_i x_j] - n^2 E[x]^2$

$= n E[x_i^2] - n E[x_i]^2 = n\sigma^2$

b) $\text{Var}(\frac{Y}{n}) = \frac{\text{var}(Y)}{n^2}$

$= \frac{n\sigma^2}{n^2}$

$= \frac{1}{n}\sigma^2$

2) a) $E[Y] = E[a^T x] = a^T E[x] = a^T \mu_x$

b) $\text{Var}\, Y = E[(a^T(X - EX))(a^T(X-EX))^T]$

$= a^T E[(X-EX)(X-EX)^T] a$

$= a^T \Sigma_x a$

c) $a = \begin{pmatrix} 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{pmatrix}$ does the trick