CS561 HW4

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1. Problem 1: Unfair dice, convolution.

Prove or disprove if the following statements are true in general:

(a) Consider two independent dice $X \sim p(x)$ and $Y \sim q(y)$ where

$$p(x) \begin{cases} p_i & x = i, i = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

and

$$q(y) \begin{cases} q_i & y = i, i = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

then we can compute the PMF or $Z = X + Y$ as such

- $\mathbb{P}(Z=2) = \mathbb{P}(X=1, Y=1) = p_1q_1$
- $\mathbb{P}(Z=3) = \mathbb{P}(X=1, Y=2) + \mathbb{P}(X=2, Y=1) = p_1q_2 + p_2q_1$
- $\mathbb{P}(Z=4) = \mathbb{P}(X=1, Y=3) + \mathbb{P}(X=2, Y=2) + \mathbb{P}(X=3, Y=1) = p_1q_3 + p_2q_2 + p_3q_1$
- $\mathbb{P}(Z=5) = \mathbb{P}(X=1, Y=4) + \mathbb{P}(X=2, Y=3) + \mathbb{P}(X=3, Y=2) + \mathbb{P}(X=4, Y=1) = p_1q_4 + p_2q_3 + p_3q_2 + p_4q_1$
- $\mathbb{P}(Z=6) = \mathbb{P}(X=1, Y=5) + \mathbb{P}(X=2, Y=4) + \mathbb{P}(X=3, Y=3) + \mathbb{P}(X=4, Y=2) + \mathbb{P}(X=5, Y=1) = p_1q_5 + p_2q_4 + p_3q_3 + p_4q_2 + p_5q_1$
- $\mathbb{P}(Z=7) = \mathbb{P}(X=1, Y=6) + \mathbb{P}(X=2, Y=5) + \mathbb{P}(X=3, Y=4) + \mathbb{P}(X=4, Y=3) + \mathbb{P}(X=5, Y=2) + \mathbb{P}(X=6, Y=1) = p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1$
- $\mathbb{P}(Z=8) = \mathbb{P}(X=1, Y=7) + \mathbb{P}(X=2, Y=6) + \mathbb{P}(X=3, Y=5) + \mathbb{P}(X=4, Y=4) + \mathbb{P}(X=5, Y=3) + \mathbb{P}(X=6, Y=2) + \mathbb{P}(X=7, Y=1) = p_1q_7 + p_2q_6 + p_3q_5 + p_4q_4 + p_5q_3 + p_6q_2 + p_7q_1$
- $\mathbb{P}(Z=9) = \mathbb{P}(X=1, Y=8) + \mathbb{P}(X=2, Y=7) + \mathbb{P}(X=3, Y=6) + \mathbb{P}(X=4, Y=5) + \mathbb{P}(X=5, Y=4) + \mathbb{P}(X=6, Y=3) + \mathbb{P}(X=7, Y=2) + \mathbb{P}(X=8, Y=1) = p_1q_8 + p_2q_7 + p_3q_6 + p_4q_5 + p_5q_4 + p_6q_3 + p_7q_2 + p_8q_1$
- $\mathbb{P}(Z=10) = \mathbb{P}(X=1, Y=9) + \mathbb{P}(X=2, Y=8) + \mathbb{P}(X=3, Y=7) + \mathbb{P}(X=4, Y=6) + \mathbb{P}(X=5, Y=5) + \mathbb{P}(X=6, Y=4) + \mathbb{P}(X=7, Y=3) + \mathbb{P}(X=8, Y=2) + \mathbb{P}(X=9, Y=1) = p_1q_9 + p_2q_8 + p_3q_7 + p_4q_6 + p_5q_5 + p_6q_4 + p_7q_3 + p_8q_2 + p_9q_1$
- $\mathbb{P}(Z=11) = \mathbb{P}(X=1, Y=10) + \mathbb{P}(X=2, Y=9) + \mathbb{P}(X=3, Y=8) + \mathbb{P}(X=4, Y=7) + \mathbb{P}(X=5, Y=6) + \mathbb{P}(X=6, Y=5) + \mathbb{P}(X=7, Y=4) + \mathbb{P}(X=8, Y=3) + \mathbb{P}(X=9, Y=2) + \mathbb{P}(X=10, Y=1) = p_1q_10 + p_2q_9 + p_3q_8 + p_4q_7 + p_5q_6 + p_6q_5 + p_7q_4 + p_8q_3 + p_9q_2 + p_10q_1$
- $\mathbb{P}(Z=12) = \mathbb{P}(X=1, Y=11) + \mathbb{P}(X=2, Y=10) + \mathbb{P}(X=3, Y=9) + \mathbb{P}(X=4, Y=8) + \mathbb{P}(X=5, Y=7) + \mathbb{P}(X=6, Y=6) + \mathbb{P}(X=7, Y=5) + \mathbb{P}(X=8, Y=4) + \mathbb{P}(X=9, Y=3) + \mathbb{P}(X=10, Y=2) + \mathbb{P}(X=11, Y=1) = p_1q_11 + p_2q_10 + p_3q_9 + p_4q_8 + p_5q_7 + p_6q_6 + p_7q_5 + p_8q_4 + p_9q_3 + p_10q_2 + p_11q_1$

Or in simpler terms, for $z = 2, 3, \ldots, 12$

$$p_Z(z) = \sum_{i=\max(1,z-6)}^{\min(6,z-1)} p_i q_{z-i}$$

(b) Let X and Y be random variables and $Z = X + Y$. For any value n that Z may take, we can compute (by the partition rule)

$$\begin{aligned}
\mathbb{P}(Z=n) &= \sum_i \mathbb{P}(X=i, Z=n) \\
&= \sum_i \mathbb{P}(X=i, X+Y=n) \\
&= \sum_i \mathbb{P}(X=i, Y=n-i) \\
&= \sum_i \mathbb{P}(X=i)\mathbb{P}(Y=n-i) \\
p_Z(n) &= \sum_i p_X(i) p_Y(n-i)
\end{aligned}$$

2. Problem 2: Minimal Expectation

Let $X_i \sim U[0,1]$ be i.i.d. random variables and define $Y = \min_{i=1,2,\dots,n} X_i$

(a) Find an expression for the pdf of Y. First, we compute the cdf, which is the chance in which $Y \leq k$ for $k \in [0,1]$.

$$\begin{aligned}
P(Y \leq k) &= 1 - P(Y > k) \\
&= 1 - P(\min_{i=1,2,\dots,n} X_n > k) \\
&= 1 - P(X_i > k : i = 1, 2, \dots, n) \\
&= 1 - \prod_{i=1}^{k} P(X_i > k) \\
&= 1 - (1-k)^n
\end{aligned}$$

and therefore, the cdf F follows that

$$F(k) = \begin{cases} 0 & k < 0 \\ 1 - (1-k)^n & k \in [0,1] \\ 1 & k > 1 \end{cases}$$

we can find the pdf by taking the derivative, resulting in

$$f(k) = \frac{d}{dk} F(k) = \begin{cases} n(1-k)^{n-1} & k \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b) We can compute the expectation $\mathbb{E}[Y]$ as follows

$$\begin{aligned}
\mathbb{E}[y] &= \int_{-\infty}^{\infty} y f(y) dy \\
&= \int_0^1 yn(1-y)^{n-1} dy \\
&= \frac{(1-x)^n (nx+1)}{n+1} \Big|_0^1 \\
&= \frac{1}{n+1}
\end{aligned}$$

(c) The plot looks convincing and close to our analytical solution, since we can compute $\log(\mathbb{E}[y]) = \log(1/(n+1)) = -\log(n+1)$ which matches the plot from the first hw.

3. Problem 3: Expectation Basics

Let $\mathbb{P}(X = 1) = 1/2$, $\mathbb{P}(X = 2) = 1/4$, and $\mathbb{P}(X = 3) = 1/4$.

(a) Compute $\mathbb{E}[X]$:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_i i\mathbb{P}(X = i) \\
&= 1\mathbb{P}(X = 1) + 2\mathbb{P}(X = 2) + 3\mathbb{P}(X = 3) \\
&= 1(1/2) + 2(1/4) + 3(1/4) \\
&= 7/4
\end{aligned}$$

(b) Compute $\mathbb{E}[g(X)]$ when $g(x) = x^2$:

$$\begin{aligned}
\mathbb{E}[g(X)] &= \sum_i g(i)\mathbb{P}(X = i) \\
&= 1^2\mathbb{P}(X = 1) + 2^2\mathbb{P}(X = 2) + 3^2\mathbb{P}(X = 3) \\
&= 1(1/2) + 4(1/4) + 9(1/4) \\
&= 15/4
\end{aligned}$$

(c) Compute $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[(X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2)] \\
&= \mathbb{E}[X^2] - \mathbb{E}[2X\mathbb{E}[X]] + \mathbb{E}[\mathbb{E}[X]^2] \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[2X] + \mathbb{E}[X]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= 15/4 - (7/4)^2 \\
&= 11/16
\end{aligned}$$

(d) Express $\mathbb{E}[-\log(g(X))]$:

$$\mathbb{E}[-\log(g(X))] = \frac{-\log(g(1))}{2} + \frac{-\log(g(2))}{4} + \frac{-\log(g(3))}{4}$$

(e) Compute the entropy $H := \mathbb{E}[-\log_s(p(X))]$:

$$H = \mathbb{E}[-\log_2(p(X))] = \frac{-\log_2(1/2)}{2} + \frac{-\log_2(1/4)}{4} + \frac{-\log_2(1/4)}{4} = 3/2$$

4. Problem 4: Bivariate Random Variables

The joint pmf $p(x, y)$ is defined as follows:

- $\mathbb{P}(X = 1, Y = 1) = 1/8$
- $\mathbb{P}(X = 1, Y = 2) = 1/8$
- $\mathbb{P}(X = 2, Y = 1) = 1/2$
- $\mathbb{P}(X = 2, Y = 2) = 1/4$

(a) Are X and Y dependent? We can compute the marginal probabilities:

- $\mathbb{P}(X = 1) = \mathbb{P}(X = 1, Y = 1) + \mathbb{P}(X = 1, Y = 2) = 1/4$
- $\mathbb{P}(X = 2) = \mathbb{P}(X = 2, Y = 1) + \mathbb{P}(X = 2, Y = 2) = 3/4$
- $\mathbb{P}(Y = 1) = \mathbb{P}(X = 1, Y = 1) + \mathbb{P}(X = 2, Y = 1) = 5/8$
- $\mathbb{P}(Y = 1) = \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 2, Y = 2) = 3/8$

In which we can see that $\mathbb{P}(X = 1)\mathbb{P}(Y = 1) = 5/32 \neq 1/8 = \mathbb{P}(X = 1, Y = 1)$. Therefore, X and Y are not independent.

(b) We can compute the expectations

- $\mathbb{E}[X] = 1\mathbb{P}(X = 1) + 2\mathbb{P}(X = 2) = 1/4 + 6/4 = 7/4$
- $\mathbb{E}[Y] = 1\mathbb{P}(Y = 1) + 2\mathbb{P}(Y = 2) = 5/8 + 6/8 = 11/8$
- $\mathbb{E}[XY] = 1\mathbb{P}(XY = 2) + 2\mathbb{P}(XY = 3) + 4\mathbb{P}(XY = 4) = 1(1/8) + 2(5/8) + 4(1/4) = 19/8$

We can compute the covariance as follows:

$$
\begin{aligned}
\text{Cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\
&= \mathbb{E}[(XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y])] \\
&= \mathbb{E}[XY] - \mathbb{E}[X\mathbb{E}[Y]] - \mathbb{E}[Y\mathbb{E}[X]] + \mathbb{E}[\mathbb{E}[X]\mathbb{E}[Y]] \\
&= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= 19/8 - 77/32 \\
&= -1/32
\end{aligned}
$$

(c) Define a new random variable $Z = X + Y$, then

- $\mathbb{P}(Z = 2) = \mathbb{P}(X = 1, Y = 1) = 1/8$
- $\mathbb{P}(Z = 3) = \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 2, Y = 1) = 1/2 + 1/8 = 5/8$
- $\mathbb{P}(Z = 4) = \mathbb{P}(X = 2, Y = 2) = 1/4$
- Z only takes values $2, 3, 4$ since X and Y only takes value 1 and 2.

(d) A function that generates X, Y given the above pmf

```python
import random
def sample():
    return random.choices(
        population=[[1,1],[1,2],[2,1],[2,2]],
        weights=[1/8,1/8,1/2,1/4],
    )[0]
# Randomly Sample X and Y
X, Y = sample()
```

5. Problem 5: Minimal Expectation

Let X and Y be discrete random variables. Conditional expectation is defined as

$$\mathbb{E}[X|Y] = \sum_x x\mathbb{P}(X = x|Y)$$

for discrete random variables. Then, we can show that

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X|Y]] &= \mathbb{E}[\sum_x x\mathbb{P}(X = x|Y)] \\
&= \sum_y \left(\sum_x x\mathbb{P}(X = x|Y = y) \right) \mathbb{P}(Y = y) \\
&= \sum_y \sum_x x\mathbb{P}(Y = y)\mathbb{P}(X = x|Y = y) \\
&= \sum_x x \sum_y \mathbb{P}(Y = y)\mathbb{P}(X = x|Y = y) \\
&= \sum_x x \sum_y \mathbb{P}(Y = y)\mathbb{P}(X = x, Y = y)/\mathbb{P}(Y = y) \\
&= \sum_x x \sum_y \mathbb{P}(X = x, Y = y) \\
&= \sum_x x\mathbb{P}(X = x) \\
&= \mathbb{E}[X]
\end{aligned}$$

as desired.