Mixture Distributions and the EM Algorithm

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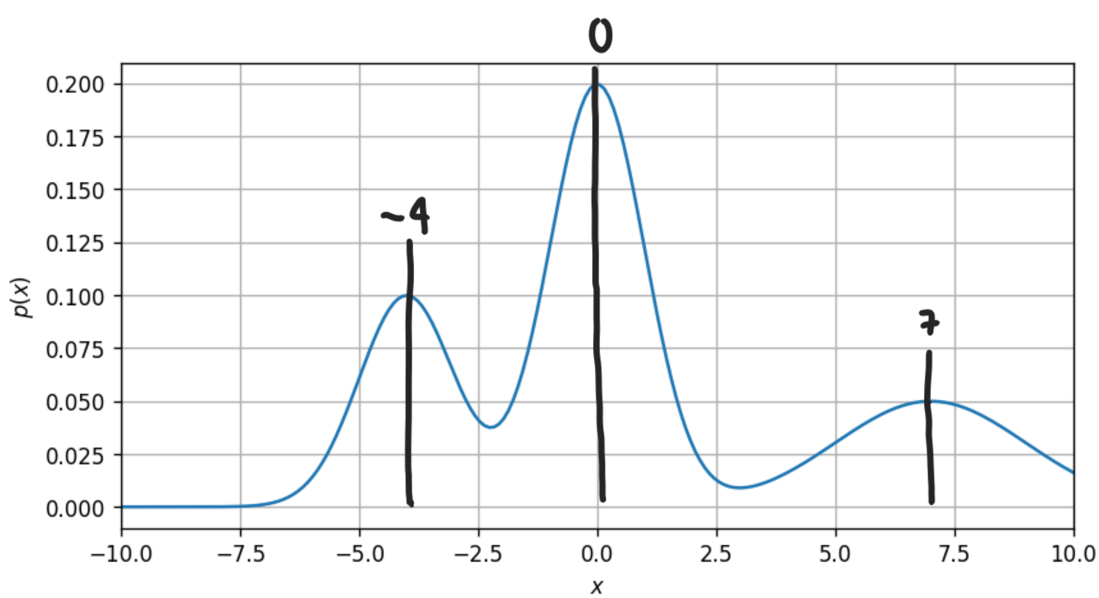
1. A mixture distribution is a convex combination of several base distributions:

$$p(\boldsymbol{x}) = \sum_k \pi_k p_k(\boldsymbol{x}),$$

where π_k is the mixing proportion or mixing prior, which represents the probability $\boldsymbol{x}$ is drawn from base distribution $p_k(\boldsymbol{x})$. Show that if $\pi_1, \ldots, \pi_k \geq 0$ and $\sum_k \pi_k = 1$ then $p(\boldsymbol{x})$ is a valid distribution; i.e, $p(\boldsymbol{x}) \geq 0$ and $\int p(\boldsymbol{x})\, dx = 1$.

$$\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{\infty} \sum_k \pi_k P_k(x)dx = \sum_k \int_{-\infty}^{\infty} \pi_k P_k(x)dx = \sum_k \pi_k(1) = 1$$

2. Consider the density shown in the figure below. You decide to approximate the distribution with a Gaussian mixture. Two of the normals have unit variance, while one has variance equal to 2. Two of the mixing priors are equal to 0.25.



Find an expression for $p(x)$ — Your answer does not have to be exact; just look at the graph to estimate parameters.

$0.25 \times N(-4,1) + 0.5 \times N(0,1) + 0.25 \times N(7,2)$

3. In this problem, you will run the EM algorithm for a Gaussian mixture model by hand in 1-dimension. You collect 6 data points:

$$\mathscr{D} = \{-9, -8, -7, 5, 7, 9\}.$$

a) Implement the first few rounds of the EM algorithm with $K = 2$. Use the following initial starting conditions: $\pi_1 = \pi_2 = 1/2$, $\mu_1 = -2$, $\mu_2 = 2$, $\boldsymbol{\Sigma}_1 = [1]$ and $\boldsymbol{\Sigma}_2 = [1]$. To simplify your calculations, you can make a hard assignment. In other words, you can approximate the responsibility r_{ik} as 1 for a single value of k, and zero for other values k.

b) What is the resulting expression for $p(x)$? $\frac{1}{2}\mathcal{N}(-8, \frac{2}{3}) + \frac{1}{2}\mathcal{N}(7, \frac{8}{3})$

c) How many iterations did it take for the algorithm to converge? 2

$-9 \quad -8 \quad -7 \qquad | \qquad 5 \quad 7 \quad 9$

Itr 1 $\quad (-2, 1) \;|\; (2, 1)$

2 $\quad (-8, \frac{2}{3}) \;|\; (7, \frac{8}{3})$

$\downarrow \qquad \downarrow$

$\mu_1 \;\; \sigma_1^2 \qquad \mu_2 \;\; \sigma_2^2$