CS561 HW11

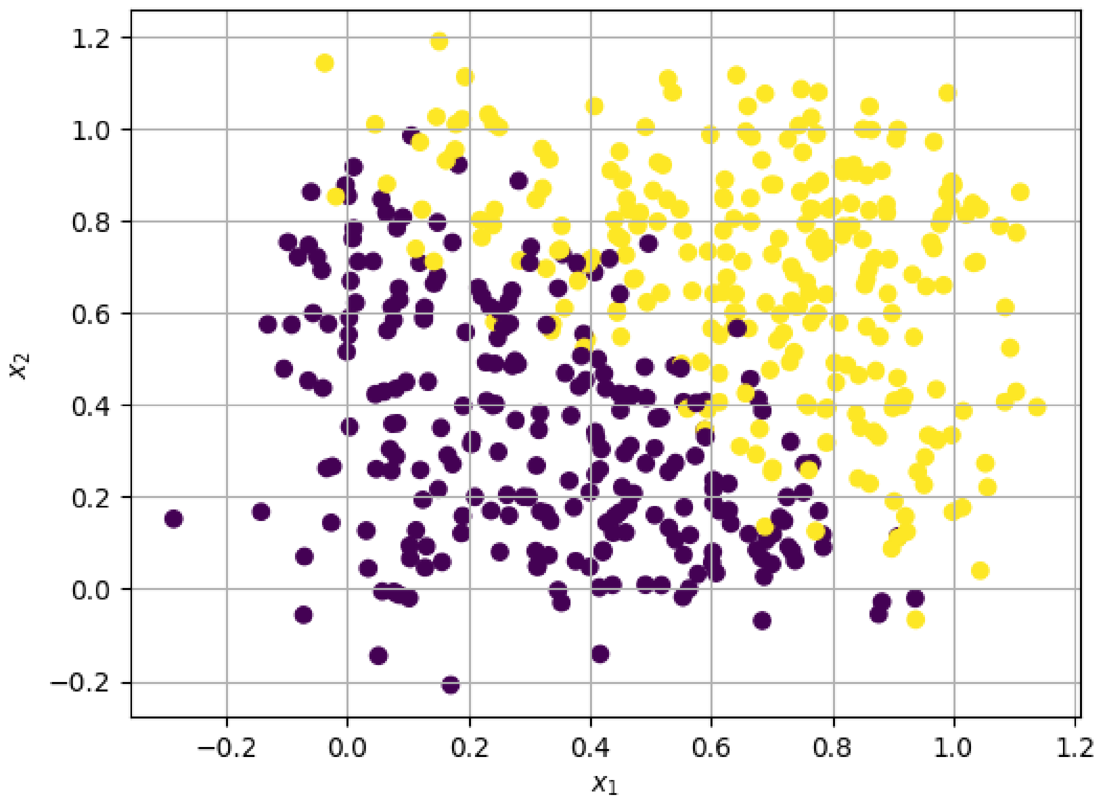
November 18, 2023

1 Problem 1

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
import pickle

# Load the dataset
with open('two_class_dataset.pkl', 'rb') as f:
    X, y = pickle.load(f)

# Create a scatter plot the datapoints and assign a color based on the class
# note X has three rows (the first is a row of 1)
plt.scatter(X[1,:], X[2,:] , c=y)
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.grid()
plt.show()
```



[2]:
```
#1a and 1b

def nll(_w):
    """
    compute the logistic loss (which is equal to the negative log likelihood,␣
 ↪or nnl)
    given a numpy array X with each column a datapoint, a list of labels,
    and a weight vector w.  Returns a scalar.
    """
    assert np.shape(X)[1]== len(y)
    assert np.shape(X)[0]==np.shape(_w)[0]
    _nnl = 0
    for i in range(len(y)):
        _nnl+= np.log(1+np.exp(-y[i]*(X[:,i]@_w)))
    return _nnl

def grad(_w):
    """
    compute the gradient of the logistic loss given a numpy array X with each
    column a datapoint, a list of labels, and a wieght vector w. Returns a␣
 ↪column vector numpy array.
```

```
    """
    assert np.shape(X)[1]== len(y)
    assert np.shape(X)[0]==np.shape(_w)[0]
    _grad_nnl = 0
    for i in range(len(y)):
        _grad_nnl+= ((-y[i])/(1+np.exp(y[i]*(X[:,i]@_w))) * X[:,i]).reshape(_w.
 ↪shape)
    return _grad_nnl
```

[3]:
```
#1c

max_its = 300
tau =.03  # convergence is sensitive to step size

w = np.zeros((3,1))  ## pick a random starting point

## run gradient descent
w_new  = w - tau*grad(w)
it = 1

while it < max_its : ### specify stopping criteria here #### and it < max_its :
    w = w_new
    w_new  = w - tau*grad(w)
    it += 1
    # print stats on every tenth iteration
    if it%10 == 0:
        print('iteration:', it, ',  objective value:', nll(w))

print(w)
print(grad(w))
```

```
iteration: 10 ,  objective value: [348.99531389]
iteration: 20 ,  objective value: [189.44304556]
iteration: 30 ,  objective value: [106.60976708]
iteration: 40 ,  objective value: [102.24921622]
iteration: 50 ,  objective value: [101.58130585]
iteration: 60 ,  objective value: [101.07965026]
iteration: 70 ,  objective value: [100.67011276]
iteration: 80 ,  objective value: [100.33231786]
iteration: 90 ,  objective value: [100.05133661]
iteration: 100 ,  objective value: [99.81590688]
iteration: 110 ,  objective value: [99.61739109]
iteration: 120 ,  objective value: [99.44906916]
iteration: 130 ,  objective value: [99.30564684]
iteration: 140 ,  objective value: [99.1829069]
iteration: 150 ,  objective value: [99.07745693]
iteration: 160 ,  objective value: [98.98654406]
```

```
iteration: 170 ,  objective value: [98.90791678]
iteration: 180 ,  objective value: [98.83972059]
iteration: 190 ,  objective value: [98.78041821]
iteration: 200 ,  objective value: [98.72872787]
iteration: 210 ,  objective value: [98.68357523]
iteration: 220 ,  objective value: [98.6440554]
iteration: 230 ,  objective value: [98.60940289]
iteration: 240 ,  objective value: [98.57896746]
iteration: 250 ,  objective value: [98.55219473]
iteration: 260 ,  objective value: [98.52861041]
iteration: 270 ,  objective value: [98.50780743]
iteration: 280 ,  objective value: [98.48943539]
iteration: 290 ,  objective value: [98.47319184]
iteration: 300 ,  objective value: [98.45881508]
[[-12.52406529]
 [ 12.7059725 ]
 [ 12.43331915]]
[[ 0.12206812]
 [-0.12448892]
 [-0.12057335]]
```

[4]:

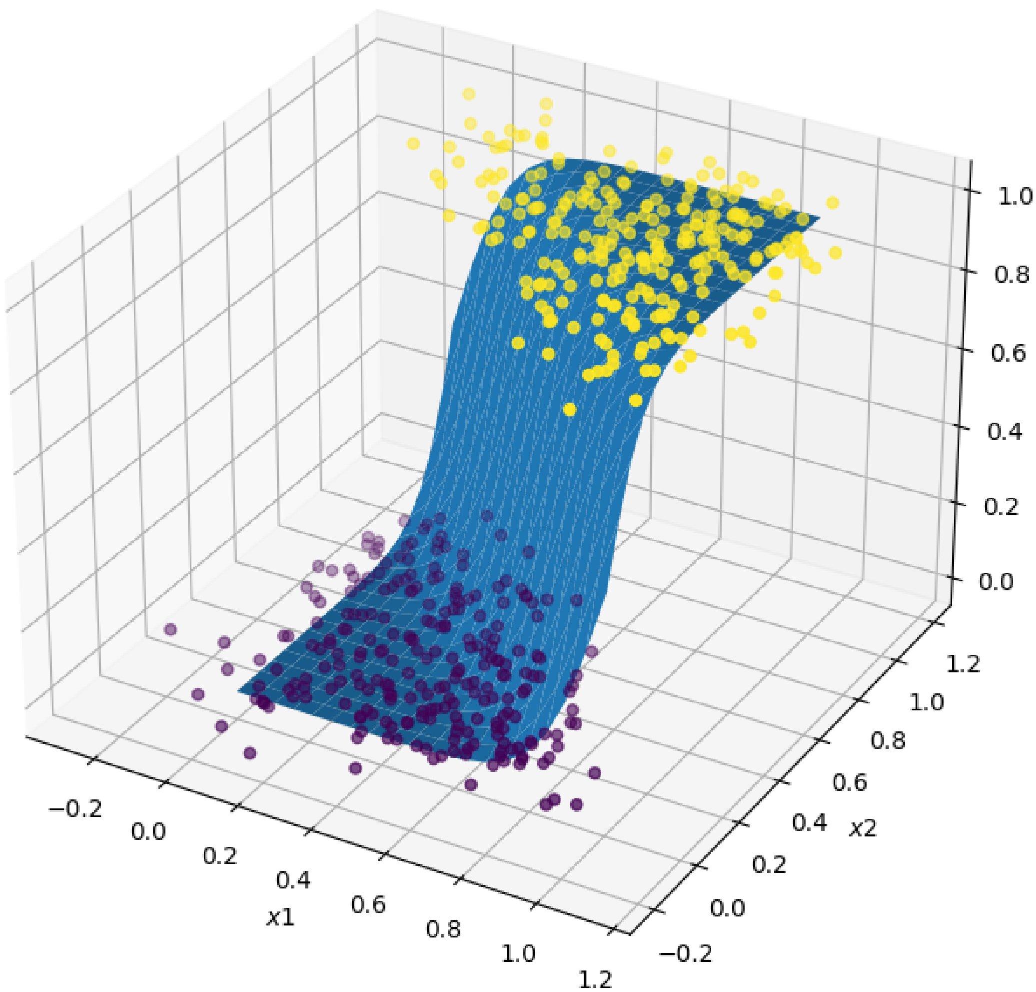
```
%matplotlib inline
from mpl_toolkits.mplot3d import axes3d

# Plot the logistic surface and the data points
xx = np.linspace(-0,1,20)
yy = np.linspace(-0,1,20)
XX,YY = np.meshgrid(xx,yy)
ZZ = 1/(1+np.exp(-(w[0]+XX*w[1]+YY*w[2])))

fig = plt.figure(figsize=(8,8))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(XX, YY, ZZ)
ax.scatter(X[1,:], X[2,:], (np.array(y)+1)/2, c=y)
plt.xlabel('$x1$')
plt.ylabel('$x2$')
```

[4]:

```
Text(0.5, 0.5, '$x2$')
```



[5]:

```
#1d

prediction = 1/(1+np.exp(-X.T@w))
prediction = list((prediction[:,0]>0.5).astype(int)*2-1)
error = np.array(prediction)!=np.array(y)
error_rate = (np.sum(error)/len(y))

print(f"Error rate is {error_rate}")
```

```
Error rate is 0.098
```

2 Problem 2

2.a) Normally, the K means involves finding the closest cluster center for each each data point as:

$$z_i = \arg\min_k ||x_i - \mu_k||_2^2$$

In Gaussian mixture models, the responsibility of each cluster k for a data point x_i is computed as

$$r_{i,k} = \frac{\pi_k \mathcal{N}(x_i|\mu_k, \Sigma_k)}{\sum_{k'} \pi_{k'} \mathcal{N}(x_i|\mu_{k'}, \Sigma_{k'})}$$

Where we can (hard) assign each point its cluster as follows

$$z_i' = \arg\max_k r_{i,k}$$

Assuming $\pi_k = \frac{1}{K}$ (equal weights per class) and $\Sigma_k = I$, we argue z_i and z_i' are equivalent as follows: first we simplify $r_{i,k}$ with the information

$$\begin{aligned}
r_{i,k} &= \frac{\pi_k \mathcal{N}(x_i|\mu_k, \Sigma_k)}{\sum_{k'} \pi_{k'} \mathcal{N}(x_i|\mu_{k'}, \Sigma_{k'})} \\
&= \frac{\mathcal{N}(x_i|\mu_k, I)}{\sum_{k'} \mathcal{N}(x_i|\mu_{k'}, I)} \\
\log(r_{i,k}) &= -(x_i - \mu_k)^T(x_i - \mu_k) + \sum_{k'} (x_i - \mu_{k'})^T(x_i - \mu_{k'}) \\
-\log(r_{i,k}) &= (x_i - \mu_k)^T(x_i - \mu_k) - \sum_{k'} (x_i - \mu_{k'})^T(x_i - \mu_{k'})
\end{aligned}$$

This means

$$\begin{aligned}
z_i' &= \arg\max_k r_{i,k} \\
&= \arg\min_k - \log(r_{i,k}) \\
&= \arg\min_k \left((x_i - \mu_k)^T(x_i - \mu_k) - \sum_{k'} (x_i - \mu_{k'})^T(x_i - \mu_{k'}) \right) \\
&= \arg\min_k \left((x_i - \mu_k)^T(x_i - \mu_k) \right) \\
&= \arg\min_k ||x_i - \mu_k||_2^2 \\
&= z_i
\end{aligned}$$

Thus, z_i' and z_i are equivalent, as desired.

2.b) The soft and hard assignments are nearly equivalent in cases where each clusters are extremely far apart in terms of L_2 distance, which means $\pi_k \mathcal{N}(x_i|\mu_k, \Sigma_k) >> \pi_{k'} \mathcal{N}(x_i|\mu_{k'}, \Sigma_{k'})$ for $k' \neq k$, and thus the fraction

$$r_{i,k} = \frac{\pi_k \mathcal{N}(x_i|\mu_k, \Sigma_k)}{\sum_{k'} \pi_{k'} \mathcal{N}(x_i|\mu_{k'}, \Sigma_{k'})}$$

can be approximated to 1. As each $r_{i,k}$ must sum up to 1, the responsibility from other clusters k' can be approximated to 0.

3 Problem 3

Recall that the cumulative distribution function of a random variable with pdf $f(x)$ is given by

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

3.a) The expectation is defined as $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(t)f(t)dt$

$$\begin{aligned}
\mathbb{E}[\mathbb{1}\{X \leq x\}] &= \int_{-\infty}^{\infty} \mathbb{1}\{X \leq x\} f(t)dt \\
&= \int_{-\infty}^{x} \mathbb{1}\{X \leq x\} f(t)dt + \int_{x}^{\infty} \mathbb{1}\{X \leq x\} f(t)dt \\
&= \int_{-\infty}^{x} 1 f(t)dt + \int_{x}^{\infty} 0 f(t)dt \\
&= \int_{-\infty}^{x} f(t)dt \\
&= F(x)
\end{aligned}$$

3.b) Let $X_1, X_2, \ldots, X_n$ denote i.i.d. samples from $f(x)$. We define

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i \leq x\}$$

Using the results from above, we can compute

$$\begin{aligned}
\mathbb{E}[\hat{F}_n(x)] &= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i \leq x\}\right] \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{1}\{X_i \leq x\}\right] \\
&= \frac{1}{n} \sum_{i=1}^{n} F(x) \\
&= F(x)
\end{aligned}$$

3.c) As $n\hat{F}_n(x)$ is a sum of i.i.d bernoulli variables, we can see $n\hat{F}_n(x) = \sum_{i=1}^{n} \mathbb{1}\{X_i \leq x\} \sim \text{Bin}(n, \mathbb{P}(X \leq x))$. Recall that $\mathbb{P}(X \leq x) = F(x)$. The variance of this binomial distribution is $\text{Var}(n\hat{F}_n(x)) = n(F(x))(1 - F(x))$. Therefore,

$$\text{Var}(\hat{F}_n(x)) = \frac{1}{n^2} n(F(x))(1 - F(x)) = \frac{1}{n}(F(x))(1 - F(x))$$

3.d) From previous problems, $\mathbb{E}[(\hat{F}_n(x) - F_n(x))^2] = \text{Var}(\hat{F}_n(x)) = \frac{1}{n}(F(x))(1 - F(x))$. Denote by $p = F(x) \in [0, 1]$, which means $p - \frac{1}{2} \in [-0.5, 0.5]$. Consider $p(1 - p) = p - p^2 = \frac{1}{4} - (p - \frac{1}{2})^2 \leq \frac{1}{4}$. Therefore, $\mathbb{E}[(\hat{F}_n(x) - F_n(x))^2] \leq \frac{1}{4n}$.