CPSC 578: Assignment #4

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Problem 1

Find the tangent and normal for a cubic Bezier segment defined by the points (2,5), (3, 7), (4,5), (5,0) and the point where the spline segment has an x value that is the average of the x values at the endpoints of the segment.

A: The Bezier spline defined by the above four points can be represented as follows (The following parameter t should be in the range $[0, 1]$):

$$P = \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} * \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

If we replace P_i with the given four above points, we can get the formula of the curve:

$$P(t) = (2, 5) + (3, 6)t + (0, -12)t^2 + (0, 1)t^3$$

$\therefore$ the tangent of the curve is:

$$P'(t) = (3, 6) + (0, -24)t + (0, 3)t^2 = (3, 6 - 24t + 3t^2)$$

The normal $n(t)$ of the curve should satisfy the following condition:

$$n(t) \cdot P'(t) = (0, 0)$$

$\therefore$ the normal of the curve is:

$$n(t) = (2 - 8t + t^2, 1)$$

And so, we have calculated the tangent and normal vector of the given Bezier curve.
Let us assume in point $P(t)$, the x value is the average of the x values at the endpoints, then we can get:

$$2 + 3t = \frac{2+3+4+5}{4} = 3.5$$

Thus, when $t = 0.5$, the point has an x value of the average of the x values at the endpoints.

Problem 2

The de Casteljau algorithm can be expressed as producing a new sequence of 8 control points $P'_0, P'_1, ...P'_7$ from an original set of 4 points P_0, P_1, P_2, P_3 by multiplying by S.
i. Show that all 8 new points are within the convex hull defined by original 4 points.
ii. Show that the two new segments have C2 continuity independent of the values of the original 4 points.

A:
i. Apparently, each of the new eight point can be represented as the weighted sum of the original four points, while the weights are the rows in the transform matrix S.

$\therefore$ if $w >= 0 \&\& \sum w = 1$, then the new points are in the convex hull defined by original four points

$\because$ the sum of each row is 1

$\therefore$ the 8 new points are within the convex hull defined by original 4 points.

ii. To test whether the two new segments have C2 continuity, we compute the second-derivative of the ending point of the first segment $\ddot{P}_1(1)$ and that of the starting point of the second segment $\ddot{P}_2(0)$, according to the definition of C2 continuity, iif $\ddot{P}_1(1) == \ddot{P}_2(0)$ then the two segments have C2 continuity.

$\because \ddot{P}_1(1) = 2(-3P'_1 - 6P'_2 + 3P'_3) + 6(-P'_1 + 3P'_2 - 3P'_3 + P'_4)$, while $\ddot{P}_2(0) = 2(3P'_5 - 6P'_6 + 3P'_7)$

If we replace P' with the original P using the transformation matrix, we get:

$\ddot{P}_1(1) = \frac{3}{4}(P_1 - P_2 - P_3 + P_4)$ and $\ddot{P}_2(0) = \frac{3}{4}(P_1 - P_2 - P_3 + P_4)$

$\therefore \ddot{P}_1(1) = \ddot{(P)}_2(0)$

$\therefore$ the two new segments have C2 continuity.

Problem 3

Choose four 2D points that a define Catmull-Rom spline segment. Give a value of u and the corresponding point P on the spline segment that is not in the convex hull of the original 4 points. Explain how you know that the point is not in the convex hull.

A: Given any 4 points P_0, P_1, P_2, P_3, the point in the corresponding spline can be represented as:

$$P(t) = \frac{1}{2} * \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} * \begin{pmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{pmatrix} * \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

That is:

$$P(t) = \begin{pmatrix} -\frac{1}{2}t + t^2 - \frac{1}{2}t^3 & 1 - \frac{5}{2}t^2 + \frac{3}{2}t^3 & \frac{1}{2}t + 2t^2 - \frac{3}{2}t^3 & -\frac{1}{2}t^2 + \frac{1}{2}t^3 \end{pmatrix} * \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

If we calculate the derivatives of the basis matrix, we can calculate the max and min of each row.
Notice that, when $t = \frac{1}{3}$, the first row: $-\frac{1}{2}t + t^2 - \frac{1}{2}t^3 = -\frac{2}{27} < 0$, doesn't satisfy the condition of convex hull.
$\therefore$ point $P(\frac{1}{3})$ is not in the convex hull of the original 4 points.