“By Example” introduction into the Mathematica PairwiseComparisons` Package

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To install the package on the computer start Mathematica and choose File/Install..., then select PairwiseComparisons.m as the package source and press OK.

To install the package at the Raspberry PI just copy them into /opt/Wolfram/WolframEngine/10.0/AddOns/ExtraPackages and restart the application

Introduction

Loading the package

```
<< PairwiseComparisons`;
```

Printing function usage

```
? KoczkodajIdx
```

KoczkodajIdx[M] returns the value of the Koczkodaj inconsistency computed for the matrix M

Printing full (with implementation) function usage

```
?? KoczkodajIdx
```

KoczkodajIdx[M] returns the value of the Koczkodaj inconsistency computed for the matrix M

```
KoczkodajIdx[PairwiseComparisons`Private`matrix_] :=
 Last[KoczkodajTheWorstTriad[PairwiseComparisons`Private`matrix]]
```

Print all the public functions in the package

```
? PairwiseComparisons`*
```

▼ PairwiseComparisons`

AIJadd	EigenvalueRank-Sym	HREGeomFullR-ank	KoczkodajImpro-veMatrixStep	RankToVector
AIJgeom	ErrorMatrix	HREGeomInter-mediateRank	KoczkodajTheW-orstTriad	RecreatePCMat-rix
COP1Check	GeometricRank	HREGeomMatrix	KoczkodajTheW-orstTriads	SaatyIdx
COP1ViolationL-ist	GeometricResc-aledRank	HREGeomParti-alRank	KoczkodajTriadI-dx	SaatyIdxSym
COP2Check	GetMatrixEntry	HREGeomResc-aledRank	LocalDiscrepan-cyMatrix	SetDiagonal
COP2ViolationL-ist	GlobalDiscrepa-ncy	HREMatrix	PrincipalEigenV-alue	VersionPC
DeleteColumns	HarkerMatrix	HREPartialRank	PrincipalEigenV-alueSym	
DeleteRows	HREConstantTe-rmVector	HRERescaledR-ank	PrincipalEigenV-ector	
DeleteRowsAnd-Columns	HREFullRank	KoczkodajConsi-stentTriad	PrincipalEigenV-ectorSym	
EigenvalueRank	HREGeomCons-tantTermVect-or	KoczkodajIdx	RankOrder	

Assign the matrix to the variable M

$$M = \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 1 & 4 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{pmatrix}$$

$$\left\{\{1, 2, 3\}, \left\{\frac{1}{2}, 1, 4\right\}, \left\{\frac{1}{3}, \frac{1}{4}, 1\right\}\right\}$$

Eigenvalue based method

Calculate the principal eigenvalue of the matrix M

```
PrincipalEigenValue[M]
```

```
3.10785
```

Calculate the principal eigenvector of the matrix M

```
PrincipalEigenVector[M]
```

```
{4.16017, 2.8845, 1.}
```

Calculate the value of the Saaty (eigenvalue based) inconsistency index of M

```
SaatyIdx[M]
```

```
0.0539237
```

Calculate the eigenvalue based ranking based on M

```
EigenvalueRank[M]
```

```
{0.517134, 0.35856, 0.124306}
```

Calculate the geometric mean based ranking based on M

Geometric Mean method (Logarithmic Least Square Method)

```
N[GeometricRank[M]]
```

```
{1.81712, 1.25992, 0.43679}
```

Calculate the geometric mean based ranking based on M rescaled so that the sum of entries is 1

```
N[GeometricRescaledRank[M]]
```

```
{0.517134, 0.35856, 0.124306}
```

Please note that if you want to have numeric (not symbolic) output the input data also should be “numeric”.
Thus, it is not a bad idea to add a numeric conversion N@ to the input matrix. For example:

```
N[GeometricRank[N@M]]
```

```
{1.81712, 1.25992, 0.43679}
```

The above applies to all the methods in the package, except eigenvalue based methods such as PrincipalEigenValue,
PrincipalEigenVector, SaatyIdx and EigenvalueRank, which are “by design” numeric.

Heuristic Rating Estimation Method (additive)

Calculate the HRE Matrix for the given matrix M where the unknown concepts are $\{c_1, c_2, c_3\}$ and the known concepts are $\{c_4 = 5, c_5 = 9\}$. It is assumed that the unknown concepts has value 0 whilst the known concepts have the values greater than 0.

Further references could be found in papers :
* Konrad Kułakowski,
Heuristic Rating Estimation Approach to The Pairwise Comparisons Method
http : // arxiv.org/abs/1309.0386
* Konrad Kułakowski, A heuristic rating estimation algorithm for the pairwise comparisons method
http : // dx.doi.org/10.1007/s10100 – 013 – 0311 – x

$$\text{HREMatrix}\left[\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5} \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 9 \end{pmatrix}\right]$$

$$\begin{pmatrix} 1 & -\frac{m_{1,2}}{4} & -\frac{m_{1,3}}{4} \\ -\frac{m_{2,1}}{4} & 1 & -\frac{m_{2,3}}{4} \\ -\frac{m_{3,1}}{4} & -\frac{m_{3,2}}{4} & 1 \end{pmatrix}$$

Calculate the HRE constant term vector for M
where C_U equals $\{c_1, c_2, c_3\}$ and C_K equals $\{c_4 = 5, c_5 = 9\}$

$$\text{HREConstantTermVector}\left[\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5} \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 9 \end{pmatrix}\right]$$

$$\begin{pmatrix} \frac{5 m_{1,4}}{4} + \frac{9 m_{1,5}}{4} \\ \frac{5 m_{2,4}}{4} + \frac{9 m_{2,5}}{4} \\ \frac{5 m_{3,4}}{4} + \frac{9 m_{3,5}}{4} \end{pmatrix}$$

Auxiliary function that transform an upper triangle matrix into a full and reciprocal matrix

$$\text{M} = \text{RecreatePCMatrix}\left[\begin{pmatrix} 1 & \frac{3}{5} & \frac{4}{7} & \frac{5}{8} & \frac{5}{10} \\ 0 & 1 & \frac{5}{7} & \frac{5}{2} & \frac{10}{3} \\ 0 & 0 & 1 & \frac{7}{2} & 4 \\ 0 & 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$\left\{\left\{1, \frac{3}{5}, \frac{4}{7}, \frac{5}{8}, \frac{1}{2}\right\}, \left\{\frac{5}{3}, 1, \frac{5}{7}, \frac{5}{2}, \frac{10}{3}\right\}, \left\{\frac{7}{4}, \frac{7}{5}, 1, \frac{7}{2}, 4\right\}, \left\{\frac{8}{5}, \frac{2}{5}, \frac{2}{7}, 1, \frac{4}{3}\right\}, \left\{2, \frac{3}{10}, \frac{1}{4}, \frac{3}{4}, 1\right\}\right\}$$

Defining known and unknown alternatives. It is assumed
that c_2 and c_3 are known and equal 5 and 7 correspondingly

$$\text{mk} = \begin{pmatrix} 0 \\ 5 \\ 7 \\ 0 \\ 0 \end{pmatrix}$$

```
{{0}, {5}, {7}, {0}, {0}}
```

Calculate the HRE ranking vector for unknown alternatives i.e. for c_1, c_4 and c_5 only

```
mu = N[HREPartialRank[M, mk]]
```

$$\begin{pmatrix} 2.52765 \\ 2.88338 \\ 2.61696 \end{pmatrix}$$

Calculate the full HRE ranking for the given input matrix M and the vector mk

```
mu = N[HREFullRank[M, mk]]
{2.52765, 5., 7., 2.88338, 2.61696}
```

Calculate the full HRE ranking rescaled so that all its entries sum up to 1

```
mu = N[HRERescaledRank[M, mk]]
{0.126206, 0.249651, 0.349511, 0.143967, 0.130665}
```

Heuristic Rating Estimation Method (multiplicative/geometric)

Further references could be found in papers:
 * Konrad Kułakowski, Grobler-Dębska Katarzyna, Wąs Jarosław, Heuristic rating estimation - geometric approach, http://arxiv.org/abs/1404.6981

$$\textbf{HREGeomMatrix}\left[\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5} \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ q_4 \\ q_5 \end{pmatrix}\right]$$

$$\begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

Calculate the HRE constant term vector

$$\textbf{HREGeomConstantTermVector}\left[\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5} \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ q_4 \\ q_5 \end{pmatrix}\right]$$

$$\begin{pmatrix} \frac{\log(q_4\, q_5\, m_{1,2}\, m_{1,3}\, m_{1,4}\, m_{1,5})}{\log(10)} \\ \frac{\log(q_4\, q_5\, m_{2,1}\, m_{2,3}\, m_{2,4}\, m_{2,5})}{\log(10)} \\ \frac{\log(q_4\, q_5\, m_{3,1}\, m_{3,2}\, m_{3,4}\, m_{3,5})}{\log(10)} \end{pmatrix}$$

Calculate the HRE geometric ranking vector for unknown alternatives i.e. for c_1, c_4 and c_5 only

```
mu = N[HREGeomPartialRank[M, mk]]
```

$$\begin{pmatrix} 2.11273 \\ 2.49035 \\ 2.13344 \end{pmatrix}$$

Calculate the full HRE geometric ranking for the given input matrix M and the vector mk

```
mu = N[HREGeomFullRank[M, mk]]
{2.11273, 5., 7., 2.49035, 2.13344}
```

Calculate the full HRE geometric ranking, rescaled so that all its entries sum up to 1

```
mu = N[HREGeomRescaledRank[M, mk]]
{0.11276, 0.266858, 0.373602, 0.132914, 0.113866}
```

Show intermediate (before raisng up to the power) HRE geometric partial rank vector

```
N[HREGeomIntermediateRank[M, mk]]
```

$$\begin{pmatrix} 0.324843 \\ 0.396261 \\ 0.329081 \end{pmatrix}$$

Bana e Costa and Vansnick's Condition of Order Preservation test

Let calculate the eigenvalue ranking

```
rank = EigenvalueRank[M]
{0.119092, 0.27476, 0.356526, 0.130954, 0.118669}
```

.... and check whether the first Bana e Costa and Vansnick condition “condition of order preservation - COP” is satisfied

```
COP1Check[M, mu]
```

```
True
```

then check whether the second Bana e Costa and Vansnick condition (preserving intensity of preferences postulate) is satisfied

```
COP2Check[M, mu]
```

```
False
```

Prints the list of pairs for which the 1st COP is not satisfied

```
COP1ViolationList[M, mu]
```

```
{}
```

Prints the list of pairs of pairs for which the 2nd COP is not satisfied

```
COP2ViolationList[M, mu]
```

```
{{False, {{1, 2}, {1, 5}}}, {False, {{1, 2}, {4, 2}}}, {False, {{1, 2}, {5, 2}}},
 {False, {{1, 3}, {1, 5}}}, {False, {{1, 3}, {4, 2}}}, {False, {{1, 3}, {4, 3}}},
 {False, {{1, 3}, {5, 2}}}, {False, {{1, 3}, {5, 3}}}, {False, {{1, 4}, {1, 5}}},
 {False, {{1, 4}, {2, 3}}}, {False, {{1, 5}, {1, 2}}}, {False, {{1, 5}, {1, 3}}},
 {False, {{1, 5}, {1, 4}}}, {False, {{1, 5}, {2, 3}}}, {False, {{1, 5}, {5, 4}}},
 {False, {{2, 1}, {2, 4}}}, {False, {{2, 1}, {2, 5}}}, {False, {{2, 1}, {5, 1}}},
 {False, {{2, 3}, {1, 4}}}, {False, {{2, 3}, {1, 5}}}, {False, {{2, 4}, {2, 1}}},
 {False, {{2, 4}, {3, 1}}}, {False, {{2, 5}, {2, 1}}}, {False, {{2, 5}, {3, 1}}},
 {False, {{3, 1}, {2, 4}}}, {False, {{3, 1}, {2, 5}}}, {False, {{3, 1}, {3, 4}}},
 {False, {{3, 1}, {3, 5}}}, {False, {{3, 1}, {5, 1}}}, {False, {{3, 2}, {4, 1}}},
 {False, {{3, 2}, {5, 1}}}, {False, {{3, 4}, {3, 1}}}, {False, {{3, 5}, {3, 1}}},
 {False, {{4, 1}, {3, 2}}}, {False, {{4, 1}, {5, 1}}}, {False, {{4, 2}, {1, 2}}},
 {False, {{4, 2}, {1, 3}}}, {False, {{4, 3}, {1, 3}}}, {False, {{4, 5}, {5, 1}}},
 {False, {{5, 1}, {2, 1}}}, {False, {{5, 1}, {3, 1}}}, {False, {{5, 1}, {3, 2}}},
 {False, {{5, 1}, {4, 1}}}, {False, {{5, 1}, {4, 5}}}, {False, {{5, 2}, {1, 2}}},
 {False, {{5, 2}, {1, 3}}}, {False, {{5, 3}, {1, 3}}}, {False, {{5, 4}, {1, 5}}}}
```

Koczkodaj's Iterative Inconsistency Reduction algorithm

Calculate the value of the Koczkodaj inconsistency index

```
N[KoczkodajIdx[M]]
```

```
0.78125
```

Prints the worst Koczkodaj triad in M. As we can see it is $m_{5,3} == \frac{1}{4}$, $m_{3,1} == \frac{7}{4}$, $m_{5,1} == 2$. The value of inconsistency introduced by this triad is $\frac{25}{32}$

```
KoczkodajTheWorstTriad[M]
```

$\left\{\{5, 3, 1\}, \left\{\frac{1}{4}, \frac{7}{4}, 2\right\}, \frac{25}{32}\right\}$

Perform one step of the Koczkodaj inconsistency reduction algorithm. On the output there is a new slightly modified matrix M2 that is expected to be more consistent than M

```
M₂ = N[KoczkodajImproveMatrixStep[M]]
```

$$\begin{pmatrix} 1. & 0.6 & 0.344306 & 0.625 & 0.829827 \\ 1.66667 & 1. & 0.714286 & 2.5 & 3.33333 \\ 2.90439 & 1.4 & 1. & 3.5 & 2.41014 \\ 1.6 & 0.4 & 0.285714 & 1. & 1.33333 \\ 1.20507 & 0.3 & 0.414913 & 0.75 & 1. \end{pmatrix}$$

```
KoczkodajIdx[M₂]
```

```
0.585087
```

Aggregation of Individual Judgments (AIJ)

Let us consider three different PC matrices *X*, *Y* and *Z* that come from three different experts

$$X = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ x_{21} & 1 & x_{23} & x_{24} \\ x_{31} & x_{32} & 1 & x_{34} \\ x_{41} & x_{42} & x_{43} & 1 \end{pmatrix};$$

$$Y = \begin{pmatrix} 1 & y_{12} & y_{13} & y_{14} \\ y_{21} & 1 & y_{23} & y_{24} \\ y_{31} & y_{32} & 1 & y_{34} \\ y_{41} & y_{42} & y_{43} & 1 \end{pmatrix};$$

$$Z = \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ z_{21} & 1 & z_{23} & z_{24} \\ z_{31} & z_{32} & 1 & z_{34} \\ z_{41} & z_{42} & z_{43} & 1 \end{pmatrix};$$

Then it is possible to aggregate the results using appropriate functions.

Aggregate Individual Judgments additively (AIJadd):

```
AIJadd[X, Y, Z] // MatrixForm
```

$$\begin{pmatrix} 1 & \frac{1}{3}(x_{12}+y_{12}+z_{12}) & \frac{1}{3}(x_{13}+y_{13}+z_{13}) & \frac{1}{3}(x_{14}+y_{14}+z_{14}) \\ \frac{1}{3}(x_{21}+y_{21}+z_{21}) & 1 & \frac{1}{3}(x_{23}+y_{23}+z_{23}) & \frac{1}{3}(x_{24}+y_{24}+z_{24}) \\ \frac{1}{3}(x_{31}+y_{31}+z_{31}) & \frac{1}{3}(x_{32}+y_{32}+z_{32}) & 1 & \frac{1}{3}(x_{34}+y_{34}+z_{34}) \\ \frac{1}{3}(x_{41}+y_{41}+z_{41}) & \frac{1}{3}(x_{42}+y_{42}+z_{42}) & \frac{1}{3}(x_{43}+y_{43}+z_{43}) & 1 \end{pmatrix}$$

Aggregate Individual Judgments geometrically (AIJgeom)

```
AIJgeom[X, Y, Z] // MatrixForm
```

$$\begin{pmatrix} 1 & (x_{12}\,y_{12}\,z_{12})^{1/3} & (x_{13}\,y_{13}\,z_{13})^{1/3} & (x_{14}\,y_{14}\,z_{14})^{1/3} \\ (x_{21}\,y_{21}\,z_{21})^{1/3} & 1 & (x_{23}\,y_{23}\,z_{23})^{1/3} & (x_{24}\,y_{24}\,z_{24})^{1/3} \\ (x_{31}\,y_{31}\,z_{31})^{1/3} & (x_{32}\,y_{32}\,z_{32})^{1/3} & 1 & (x_{34}\,y_{34}\,z_{34})^{1/3} \\ (x_{41}\,y_{41}\,z_{41})^{1/3} & (x_{42}\,y_{42}\,z_{42})^{1/3} & (x_{43}\,y_{43}\,z_{43})^{1/3} & 1 \end{pmatrix}$$

Note that these two functions works also for result lists i.e.:

```
AIJgeom[{x₁, x₂, x₃, x₄}, {y₁, y₂, y₃, y₄}]
```

$$\{\sqrt{x_1\,y_1},\ \sqrt{x_2\,y_2},\ \sqrt{x_3\,y_3},\ \sqrt{x_4\,y_4}\}$$

```
AIJadd[{x₁, x₂, x₃, x₄}, {y₁, y₂, y₃, y₄}]
```

$$\left\{\frac{1}{2}(x_1+y_1),\ \frac{1}{2}(x_2+y_2),\ \frac{1}{2}(x_3+y_3),\ \frac{1}{2}(x_4+y_4)\right\}$$

Incomplete Pairwise Comparisons Matrix - Harker Method

The idea comes from P. T. Harker, Alternative Modes of Questioning in The Anaslytic Hiserarchy Process, Math Modeling, 1987.

Let M be an incomplete pairwise comparisons matrix.

$$M = \begin{pmatrix} 1 & 2 & \square \\ 1/2 & 1 & 2 \\ \square & 1/2 & 1 \end{pmatrix};$$

Thus, to compute the ranking based on M we need to compute matrix A (hereinafter referred to as Harker matrix)

```
A = HarkerMatrix[M];
```

```
A // MatrixForm
```

$$\begin{pmatrix} 2 & 2 & 0 \\ \frac{1}{2} & 1 & 2 \\ 0 & \frac{1}{2} & 2 \end{pmatrix}$$

Then compute the eigenvalue based ranking and inconsistency index as usuall:

```
EigenvalueRank[A]
```

```
{0.571429, 0.285714, 0.142857}
```

```
SaatyIdx[A]
```

```
0
```