Natural Deduction Proof Verifier

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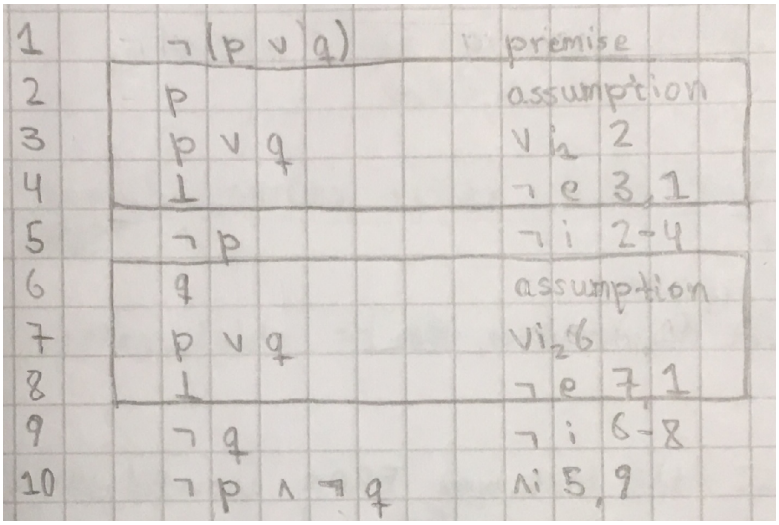
Valid Proof of de Morgan's Law

The following is a valid proof of one of de Morgan's laws. Specifically, it proves the sequent $\neg(p \vee q) \vdash \neg p \wedge \neg q$.



In Prolog notation, this proof is written:

```
[1,   neg(or(p, q)),          premise],
[
   [2,   p,                   assumption],
   [3,   or(p, q),            orint1(2)],
   [4,   cont,                negel(3, 1)]
],
```

```
[5,   neg(p),                  negint(2, 4)],
[
   [6,   q,                    assumption],
   [7,   or(p, q),             orint2(6)],
   [8,   cont,                 negel(7, 1)]
],
[9,   neg(q),                  negint(6, 8)],
[10,  and(neg(p), neg(q)),     andint(5, 9)]
```

The proof verification algorithm (view appendix) successfully identifies this as a valid proof. To do this, the algorithm does the following:

1. `verify(InputFileName)` is called and reads in the proof from the specified file. The premises are stored in an array called `Premises`, the sequent's conclusion is stored in `Conclusion`, and the array containing the proof is stored in `Proof`.

2. When Prolog has parsed the file, the rule `valid_proof(Premises, Conclusion, Proof)` is invoked, which in turn calls `verify_end(Conclusion, Proof)` and `verify_proof(Premises, Proof, [])`, in the given order.

3. `verify_end(Conclusion, Proof)` ensures that the final line of the proof is the same as the sequent's conclusion. This is achieved by first applying Prolog's built-in `last/3` predicate to extract the last element (line) in `Proof`, and then utilizing `nth0/3` to select the formula present on the last line. If the proof is valid, this formula is equal to the sequent's conclusion. Accordingly, `verify_end` concludes with an equality check that ensures that the described equality holds.

4. `verify_proof(Premises, Proof, Verified)` recursively validates the provided proof line-by-line, from top to bottom. It validates each line (element in `Proof`) by invoking `rule/3` wite the premises, line, and previously verified lines as arguments. `rule/3` then returns `true` whenever the line follows by natural deduction from the premises and previously

verified lines, and false otherwise. If a line is valid, it gets appended to the `Verified` array and `verify_proof` is called again with the yet unverified lines and the new array of verified lines.

5. `rule/3` takes in the array of premises, a line or assumption box, and the, as of then, verified lines of a proof. It then ensures that the line or assumption box follows by natural deduction from the premises and previous lines of the proof. It does this by inspecting the last element of each line, which specifies the deduction rule that was applied to deduce the formula of that line, and then checking that the prerequisites for using the rule are fulfilled. To exemplify, in the above proof the first line invokes the premise verification rule

```
rule(Premises, [_, Formula, premise], _) :-
   member(Formula, Premises).
```

which checks that `Formula` (the formula on the given line) is a member of `Premises`. If `Formula` is a member, the line is valid and `rule` returns `true`. Line two is an assumption and thus opens a box. It is handled by the assumption rule

```
rule(Premises, [[R, Formula, assumption] | T], Verified) :-
   append(Verified, [[R, Formula, assumption]], VerifiedNew),
   verify_proof(Premises, T, VerifiedNew).
```

which returns `true` if the sub-proof of the assumption box is valid. To verify that the assumption box contains a valid proof the assumption line is appended to the array of verified lines, and `verify_proof/3` is called to validate the sub-proof. Within the box, line three is verified by ensuring that its application of the first disjunction introduction rule is valid. The below code does this

```
rule(_, [_, or(X, _), orint1(R)], Verified) :-
   member([R, X, _], Verified).
```

by confirming that the formula p has been deduced earlier in the proof. Line four is deduced by administering the rule of negation elimination. The below code verifies that the rule can be employed

```
rule(_, [_, cont, negel(R1, R2)], Verified) :-
   member([R1, X, _], Verified),
   member([R2, neg(X), _], Verified).
```

by checking that both $p \vee q$ and $\neg(p \vee q)$ occur previously in the proof (are members of `Verified`). When the sub-proof of the assumption box has been validated, the entire box is appended to `Verified` before continuing to verify the next line or assumption box. This ensures that no single line that is predicated on the assumption can be referenced without also referencing the assumption. In the above proof, the fifth line, which also is the first line following the upper assumption box, is deduced by applying the rule of negation introduction. The subsequent code verifies that the rule can be applied

```
rule(_, [_, neg(X), negint(R1, R2)], Verified) :-
   member([[R1, X, assumption] | T], Verified),
   last(T, BoxConclusion),
   [R2, cont, _] = BoxConclusion.
```

by establishing that the assumption box starts with p, and that it ends with a contradiction.

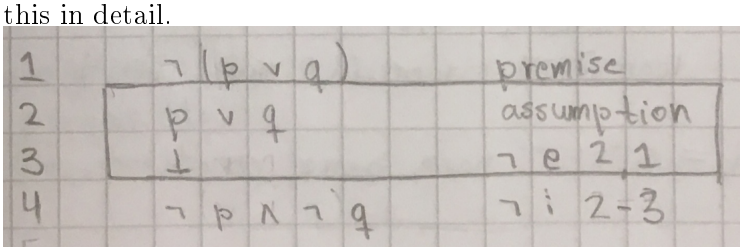
Lines six through nine repeat the same deductive pattern as lines two through five. The concluding line of the proof is arrived at by utilizing the rule of conjunction introduction. The employment of this rule is validated by the below code

```
rule(_, [_, and(X, Y), andint(R1, R2)], Verified) :-
   member([R1, X, _], Verified),
   member([R2, Y, _], Verified).
```

which states that the rule can be administered if, and only if, both $\neg p$ and $\neg q$ occur previously in the proof (exist in `Verified`).

Invalid Proof of de Morgan's Law

The following is an invalid proof of the same sequent as above. Lines one through three of the proposed lines are deduced by rules that we have already discussed, and hence they will not be discussed any further. Nevertheless, the fourth and final line incorrectly applies the rule of negation introduction. The subsequent text examines how the proof verification algorithm handles this in detail.



Recall that the negation introduction rule can be employed to conclude $\neg\phi$, for some formula ϕ, if, and only if, there exists a box starts with an assumption of ϕ and ends in a contradiction. The proof passes the latter requirement, which is enforced by the code

```
last(T, BoxConclusion),
[R2, cont, _] = BoxConclusion.
```

but fails the former, as the assumption $\phi = p \vee q$ combined with the contradiction and negation introduction rule, can only be used to conclude $\neg(p \vee q)$, and not $\neg p \vee \neg q$.