Natural Deduction Proof Verifier

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Lab 2 – DD1351

November 22, 2019

1 Introduction

Natural deduction is a proof calculus comprised of a set of inference rules that are used to infer formulas from other formulas. The rules can be categorized into introduction and elimination rules. The former combine or negate formulas by introducing connectives; the latter decompose them by eliminating connectives. These rules are defined in terms of formula patterns that must hold true for a connective to be introduced or eliminated. For instance, to claim $\phi \land \psi$, it must be known that both of the formulas ϕ and ψ are true. Prolog is a logic programming language that enables easy expression of logical statements. To exemplify, the Prolog code

```
rule(_, [_, and(X, Y), andint(R1, R2)], Verified) :-
    member([R1, X, _], Verified),
    member([R2, Y, _], Verified).
```

states that $\phi \land \psi$ can be concluded, by the and introduction rule, if both ϕ and ψ are known to be true.

This report presents a Prolog-based algorithm that can verify the validity of any proof in propositional logical that has been written exclusively using the rules of natural deduction listed in the below table.

Prolog	Logisk	Prolog	Logisk
`premise`	premise	`impel(x,y)`	$\rightarrow$e x,y
`assumption`	assumption	`negint(x,y)`	$\neg$i x-y
`copy(x)`	copy x	`negel(x,y)`	$\neg$e x,y
`andint(x,y)`	$\wedge$i x, y	`contel(x)`	$\bot$e x
`andel1(x)`	$\wedge$e$_1$ x	`negnegint(x)`	$\neg\neg$i x
`andel2(x)`	$\wedge$e$_2$ x	`negnegel(x)`	$\neg\neg$e x
`orint1(x)`	$\vee$i$_1$ x	`mt(x,y)`	MT x,y
`orint2(x)`	$\vee$i$_2$ x	`pbc(x,y)`	PBC x-y
`orel(x,y,u,v,w)`	$\vee$e x,y–u,v–w	`lem`	LEM
`impint(x,y)`	$\rightarrow$i x–y		

Figure 1: Algorithm-Supported Inference Rules

2 Method

A proof is valid if all of its line are valid. A line is valid if it can be inferred by applying the rules of natural deduction to the proof's premises and previously inferred formulas. Accordingly, the validity of a proof can be determined if, and only if, the premises and previously derived formulas are known, and the required conditions for the employment of all used rules are defined. To this end, 19 instances of the `rule/3` predicate were defined, one for each rule from TABLE 1, that given the premises, a line, and all previous lines, returns `true` if the passed in line can be inferred from the premises and previous lines, and `false` otherwise. The set of all `rule/3` predicates achieve this behavior by, first, matching the rule used to deduce the passed in line (which is specified at the end of the line) with the predicate handling that rule; and secondly, verifying that the premises and derived formulas that were used to invoke the rule satisfy the conditions for its invocation. The premises of a proof are specified in the input file to the algorithm and stored in a list called `Premises`. Verified lines are appended to a list, usually named `Verified`, immediately after a `rule/3` predicate has declared them as `true`.

2.1 Premise Handling

2.2 Handling of Inference Rules

2.3 Assumption Handling

The most outstanding `rule/3` predicate is the one that handles assumptions. There are two reasons for this. Firstly, rather than being matched with lines that declare themselves as "derived by assumption", it is matched to boxes (implemented as a list of lines) that start with an assumption. Secondly, unlike the other predicates, it does not check for the fulfillment of a set of predefined conditions; instead, it treats the lines of the box as a proof in its own right, where the assumption becomes a premise, and returns `true` only if the proof within the box is valid. In this way, the algorithm handles assumptions by regarding their associated boxes as sub-proofs that can be verified recursively.

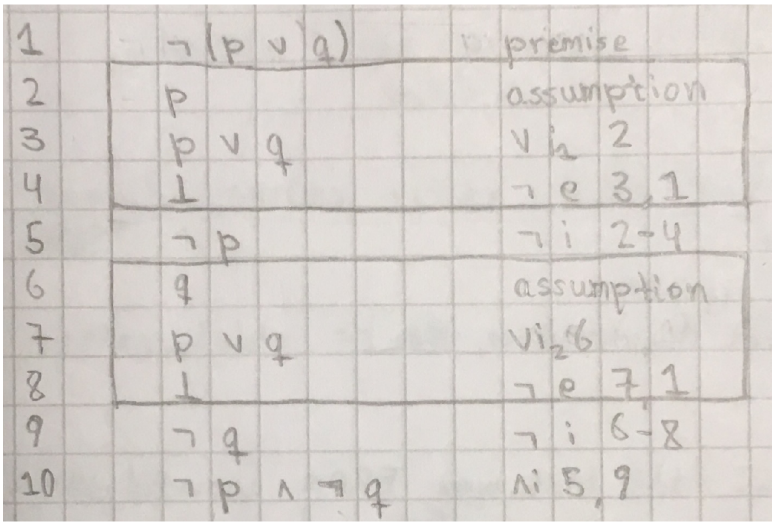
The subsequent code handles assumptions in the described manner.

```
rule(Premises, [[R, Formula, assumption] | T], Verified) :-
   append(Verified, [[R, Formula, assumption]], VerifiedNew),
   verify_proof(Premises, T, VerifiedNew).
```

Preimises is this. [[da...]] is proof box. Verified is this. The append predicate appends the assumption to the list of verified lines. Then, verify_proof is recursively called to verify the sub-proof within the box.

Valid Proof of de Morgan's Law

The following is a valid proof of one of de Morgan's laws. Specifically, it proves the sequent $\neg(p \vee q) \vdash \neg p \wedge \neg q$.



In Prolog notation, this proof is written:

```
[1,   neg(or(p, q)),           premise],
[
   [2,   p,                    assumption],
   [3,   or(p, q),             orint1(2)],
   [4,   cont,                 negel(3, 1)]
],
[5,   neg(p),                  negint(2, 4)],
[
   [6,   q,                    assumption],
   [7,   or(p, q),             orint2(6)],
   [8,   cont,                 negel(7, 1)]
],
[9,   neg(q),                  negint(6, 8)],
[10,  and(neg(p), neg(q)),     andint(5, 9)]
```

The proof verification algorithm (view appendix) successfully identifies this as a valid proof. To do this, the algorithm does the following:

1. `verify(InputFileName)` is called and reads in the proof from the specified file. The premises are stored in an array called `Premises`, the

sequent's conclusion is stored in `Conclusion`, and the array containing the proof is stored in `Proof`.

2. When Prolog has parsed the file, the rule `valid_proof(Premises, Conclusion, Proof)` is invoked, which in turn calls `verify_end(Conclusion, Proof)` and `verify_proof(Premises, Proof, [])`, in the given order.

3. `verify_end(Conclusion, Proof)` ensures that the final line of the proof is the same as the sequent's conclusion. This is achieved by first applying Prolog's built-in `last/3` predicate to extract the last element (line) in `Proof`, and then utilizing `nth0/3` to select the formula present on the last line. If the proof is valid, this formula is equal to the sequent's conclusion. Accordingly, `verify_end` concludes with an equality check that ensures that the described equality holds.

4. `verify_proof(Premises, Proof, Verified)` recursively validates the provided proof line-by-line, from top to bottom. It validates each line (element in `Proof`) by invoking `rule/3` wite the premises, line, and previously verified lines as arguments. `rule/3` then returns `true` whenever the line follows by natural deduction from the premises and previously verified lines, and false otherwise. If a line is valid, it gets appended to the `Verified` array and `verify_proof` is called again with the yet unverified lines and the new array of verified lines.

5. `rule/3` takes in the array of premises, a line or assumption box, and the, as of then, verified lines of a proof. It then ensures that the line or assumption box follows by natural deduction from the premises and previous lines of the proof. It does this by inspecting the last element of each line, which specifies the deduction rule that was applied to deduce the formula of that line, and then checking that the prerequisites for using the rule are fulfilled. To exemplify, in the above proof the first line invokes the premise verification rule

```
rule(Premises, [_, Formula, premise], _) :-
    member(Formula, Premises).
```

which checks that Formula (the formula on the given line) is a member of Premises. If Formula is a member, the line is valid and rule returns true. Line two is an assumption and thus opens a box. It is handled by the assumption rule

```
rule(Premises, [[R, Formula, assumption] | T], Verified) :-
   append(Verified, [[R, Formula, assumption]], VerifiedNew),
   verify_proof(Premises, T, VerifiedNew).
```

which returns true if the sub-proof of the assumption box is valid. To verify that the assumption box contains a valid proof the assumption line is appended to the array of verified lines, and verify_proof/3 is called to validate the sub-proof. Within the box, line three is verified by ensuring that its application of the first disjunction introduction rule is valid. The below code does this

```
rule(_, [_, or(X, _), orint1(R)], Verified) :-
   member([R, X, _], Verified).
```

by confirming that the formula p has been deduced earlier in the proof. Line four is deduced by administering the rule of negation elimination. The below code verifies that the rule can be employed

```
rule(_, [_, cont, negel(R1, R2)], Verified) :-
   member([R1, X, _], Verified),
   member([R2, neg(X), _], Verified).
```

by checking that both $p \vee q$ and $\neg(p \vee q)$ occur previously in the proof (are members of Verified). When the sub-proof of the assumption box has been validated, the entire box is appended to Verified before continuing to verify the next line or assumption box. This ensures that no single line that is predicated on the assumption can be referenced without also referencing the assumption. In the above proof, the fifth line, which also is the first line following the upper assumption box, is deduced by applying the rule of negation introduction. The subsequent code verifies that the rule can be applied

```
rule(_, [_, neg(X), negint(R1, R2)], Verified) :-
    member([[R1, X, assumption] | T], Verified),
    last(T, BoxConclusion),
    [R2, cont, _] = BoxConclusion.
```

by establishing that the assumption box starts with p, and that it ends with a contradiction.

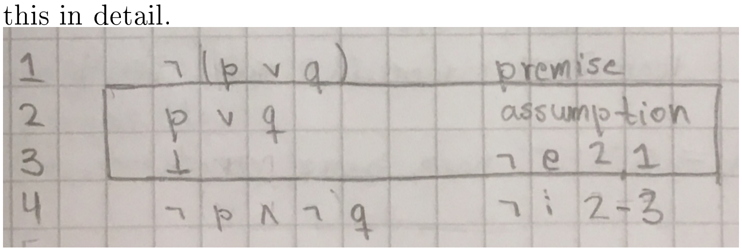
Lines six through nine repeat the same deductive pattern as lines two through five. The concluding line of the proof is arrived at by utilizing the rule of conjunction introduction. The employment of this rule is validated by the below code

```
rule(_, [_, and(X, Y), andint(R1, R2)], Verified) :-
    member([R1, X, _], Verified),
    member([R2, Y, _], Verified).
```

which states that the rule can be administered if, and only if, both $\neg p$ and $\neg q$ occur previously in the proof (exist in `Verified`).

Invalid Proof of de Morgan's Law

The following is an invalid proof of the same sequent as above. Lines one through three of the proposed lines are deduced by rules that we have already discussed, and hence they will not be discussed any further. Nevertheless, the fourth and final line incorrectly applies the rule of negation introduction. The subsequent text examines how the proof verification algorithm handles this in detail.



Recall that the negation introduction rule can be employed to conclude $\neg\phi$, for some formula ϕ, if, and only if, there exists a box starts with an assumption of ϕ and ends in a contradiction. The proof passes the latter requirement, which is enforced by the code

```
last(T, BoxConclusion),
[R2, cont, _] = BoxConclusion.
```

but fails the former, as the assumption $\phi = p \vee q$ combined with the contradiction and negation introduction rule, can only be used to conclude $\neg(p \vee q)$, and not $\neg p \vee \neg q$.

Predicate	**Parameters**	**Truth Conditions**
`verify`	`InputFileName`	`valid_proof`

`valid_proof`	`Premise`
	`Conclusion`
	`Proof`
		`verify_end`
		`verify_proof`

`verify_end`	`Conclusion` `Proof`	Proof ends with sequent's conclusion.

`verify_proof`	`Premises`
	`Proof`
	`Verified`
		The first line, and recursively all subsequent lines, are provable by natural deduction.

`rule`	`Premises`
	`Line/Box`
	`Verified`
		If the second argument is an assumption box, `rule` is true if the sub-proof of the box is valid (i.e. if `verify_proof` returns true when invoked with the sub-proof as its `Proof` argument).
		
		If the second argument is a line, the conditions determining the return value of `rule` differ based on what deduction rule was used to arrive at the formula on the line.
		
		For lines that are *premises*, the `Formula` on the line must exist in the `Premises` of the proof.
		
		For lines that are deduced by the *copy* rule, the copied formula must appear previously in the proof (i.e. be a member of `Verified`).
		
		For lines with a formula $\neg\neg\phi$, that is deduced by *double negation, introduction*, the formula ϕ must appear previously in the proof (i.e. exist in `Verified`).
		
		For lines with a formula ϕ that is deduced by *double negation elimination*, the formula $\neg\neg\phi$ must appear previously in the proof (i.e exist in `Verified`).

Predicate	Parameters	Truth Conditions
		For a line with a formula $\phi \wedge \psi$ that is deduced by *and introduction*, the formulas ϕ and ψ must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula ϕ, that is deduced by *first and elimination*, a formula of the form $\phi \wedge \psi$ must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula ϕ, that is deduced by *second and elimination*, a formula of the form $\psi \wedge \phi$ must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula $\phi \vee \psi$, that is deduced by *first or introduction*, a formula ϕ must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula $\phi \vee \psi$, that is deduced by *second or introduction*, a formula ψ must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula χ, that is deduced by *or elimination*, a formula $\phi \vee \psi$, a box assuming ϕ and ending in χ, and a box assuming ψ and ending in χ, must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula $\phi \rightarrow \psi$, that is deduced by *implication introduction*, a box assuming ϕ and concluding in ψ must appear previously in the proof (i.e. exist in `Verified`)

Predicate	Parameters	Truth Conditions
		For a line with a formula ψ that is deduced by *implication elimination* a formula ϕ, and a box assuming ϕ and ending in ψ, must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula $\neg\phi$ that is deduced by *negation introduction* a box assuming ϕ and concluding in $\perp$ must appear previously in the proof (i.e. exist in `Verified`). For a line with a contradiction $\perp$ that is deduced by *negation elimination* two formulas ϕ and $\neg\phi$ must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula ϕ that is deduced by *contradiction elimination* a line with a contradiction $\perp$ must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula $\neg\phi$ that is deduced by *modus tollens* a formula $\phi \to \psi$, and a formula $\neg\psi$, must appear previously in the proof (i.e. exist in `Verified`). For a line with a formula ϕ that is derived by a *proof by contradiction* a box assuming $\neg\phi$ and ending in a contradiction $\perp$ must appear previously in the proof (i.e. exist in `Verified`). By the *law of the excluded middle* a line with a formula of the form $\phi \vee \neg\phi$ is always valid.