Quantitative Macro – Homework 5

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Question 1: Factor Input Misallocation

I will do the entire exercise with only 1,000,000 simulated data points as my RAM gets overloaded when I use 10,000,000. This issue could be resolved by some form of pickling/unpickling. As there are a lot of plots in this Problem Set readability may suffer a bit which is why I exclude some of them.

Simulation of the data

We are given the following pieces of information: ln z_i and ln k_i are jointly normally distributed. Both have unit variance and a correlation of zero which implies the following covariance matrix:

$$COV(lnz_i, lnk_i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

In later exercises the off-diagonal elements can be adjusted to mean different correlations. Moreover, we are given that $E[k] = E[s] = 1$. The variable k comes from a log-normal distribution which means that $E[k] = e^{\mu+0.5\sigma^2} = 1$ which gives $\mu_k = -0.5$. Moreover, we can construct the density function of the random variable s (assuming that $f_{\tilde{z}}(z)$ is the density of a log-normal density which is a bijection).

$$s = z^{\frac{1}{1-\gamma}} = \phi(s)$$
$$z = s^{1-\gamma} = \phi^{-1}(s)$$
$$g(s) = f_{\tilde{z}}(s^{1-\gamma})|(1-\gamma)s^{-\gamma}|$$

Simplifying and then forming expectation of s yields

$$E[s] = \int_0^\infty s\frac{1-\gamma}{\sqrt{2\pi}\sigma^2 s^{1-\gamma}}e^{-\frac{1}{2}\left(\frac{ln(s^{1-\gamma})-\mu_z}{\sigma}\right)^2}(1-\gamma)s^{-\gamma}ds = 1$$

Solving this yields $\mu_z = 1.25$. Simulating the normal random variable ln z with variance 1 and mean -1.25 does indeed produce E[s] = 1 as desired. We now have a full description of the joint normal density for ln k and ln z:

$$\begin{pmatrix} ln(k) \\ ln(z) \end{pmatrix} \sim N \left[\begin{pmatrix} -0.5 \\ -1.25 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

The resulting joint density plots do indeed show that the variables k and z follow the pattern of a log-normal distribution (which is good as then all observations are positive).

Figures 2, 4 and 6 have a lot of blank space due to outliers in the log-normal distribution which is not a centered distribution. In the figures for the normal distribution we can observe the effect of different correlations on the distribution of our data.

Comparing the allocations of the efficient k against the data

Plotting z against k^e and against k yields the Figures 7 to 12. While the efficient capital allocation is along a straight line of roughly 45 degrees, the inefficient allocation, our data, is scattered around a flat line with higher mass around low values and a high variance.

Calculating the gains from reallocation

Using the given formula for calculating gains from reallocation

$$G = 100 \left(\frac{Y^e}{Y^a} - 1 \right) \tag{2}$$

we get the following results under the different correlations:

Table 1: Gains from reallocation for different correlations, $\gamma = 0.6$

Correlation coefficient	0	0.5	-0.5
Gains from reallocation	138.4%	75.89%	221.32 %

Question 2: Higher span of control

We now change the parameter γ from 0.6 to 0.8. I will only report plots for the identity covariance case to not overload the document even more.

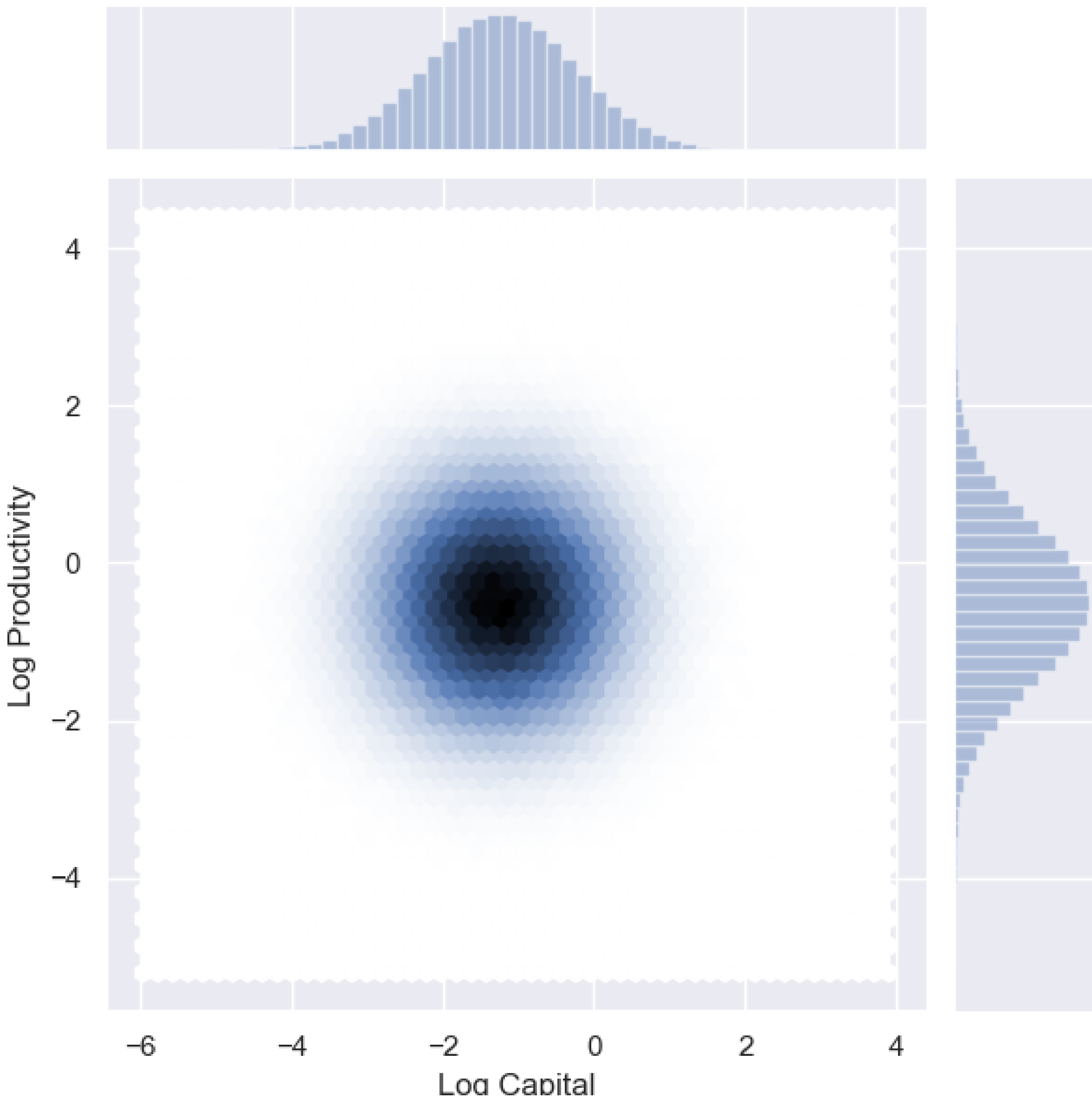

Figure 1: Joint density of ln(z) and ln(k), no correlation

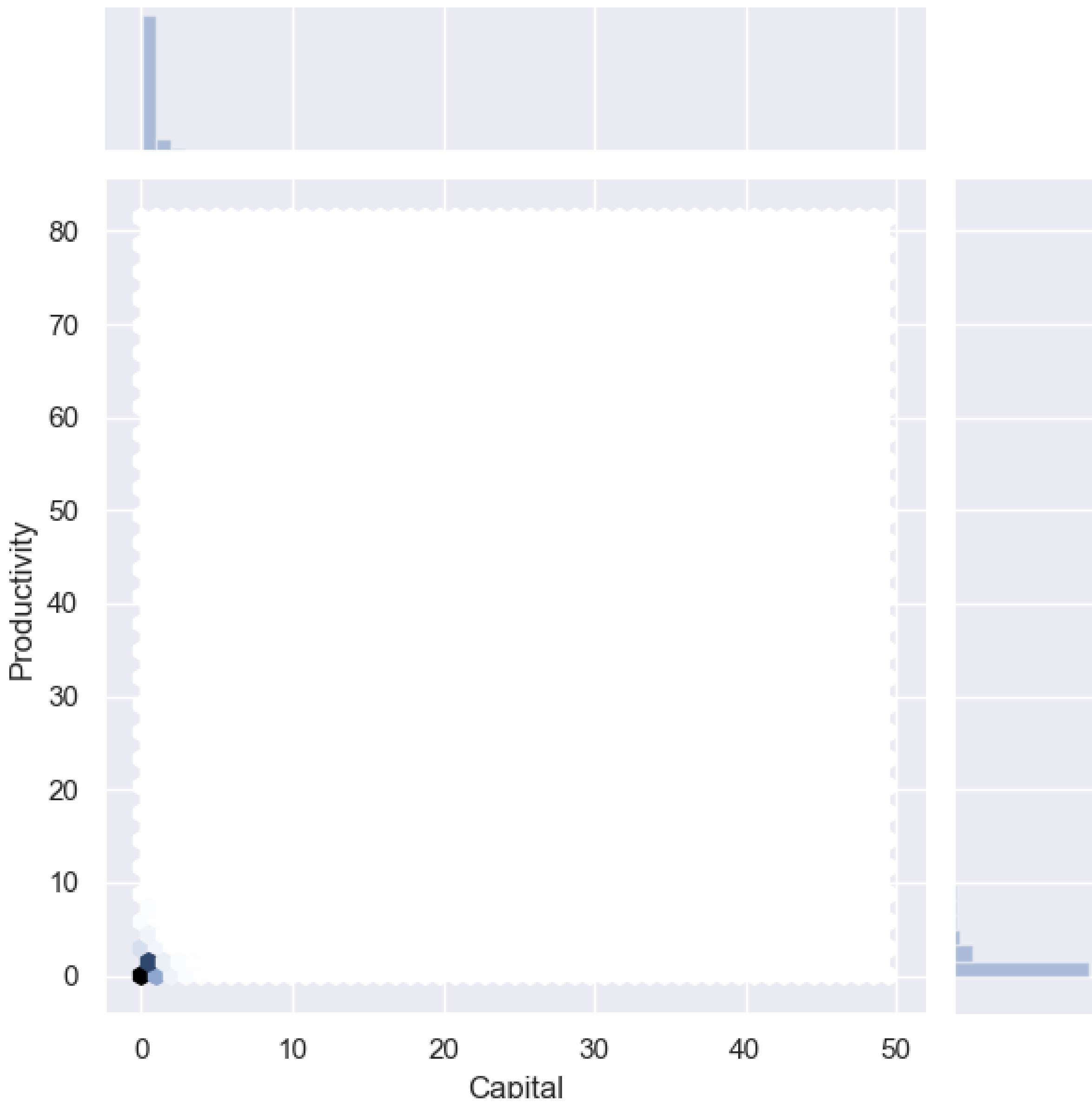

Figure 2: Joint density of z and k, no correlation

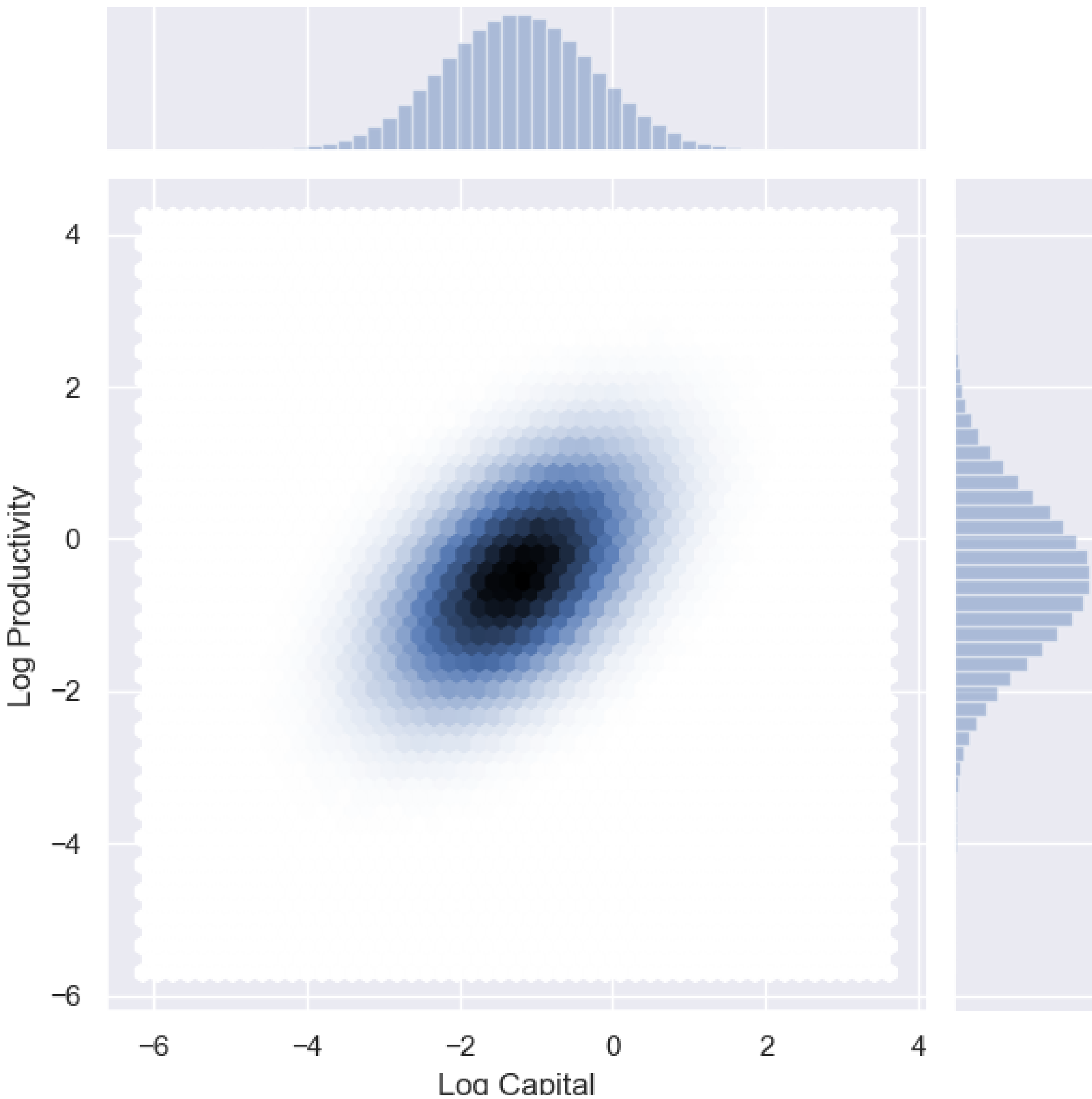

Figure 3: Joint density of ln(z) and ln(k), correlation=0.5

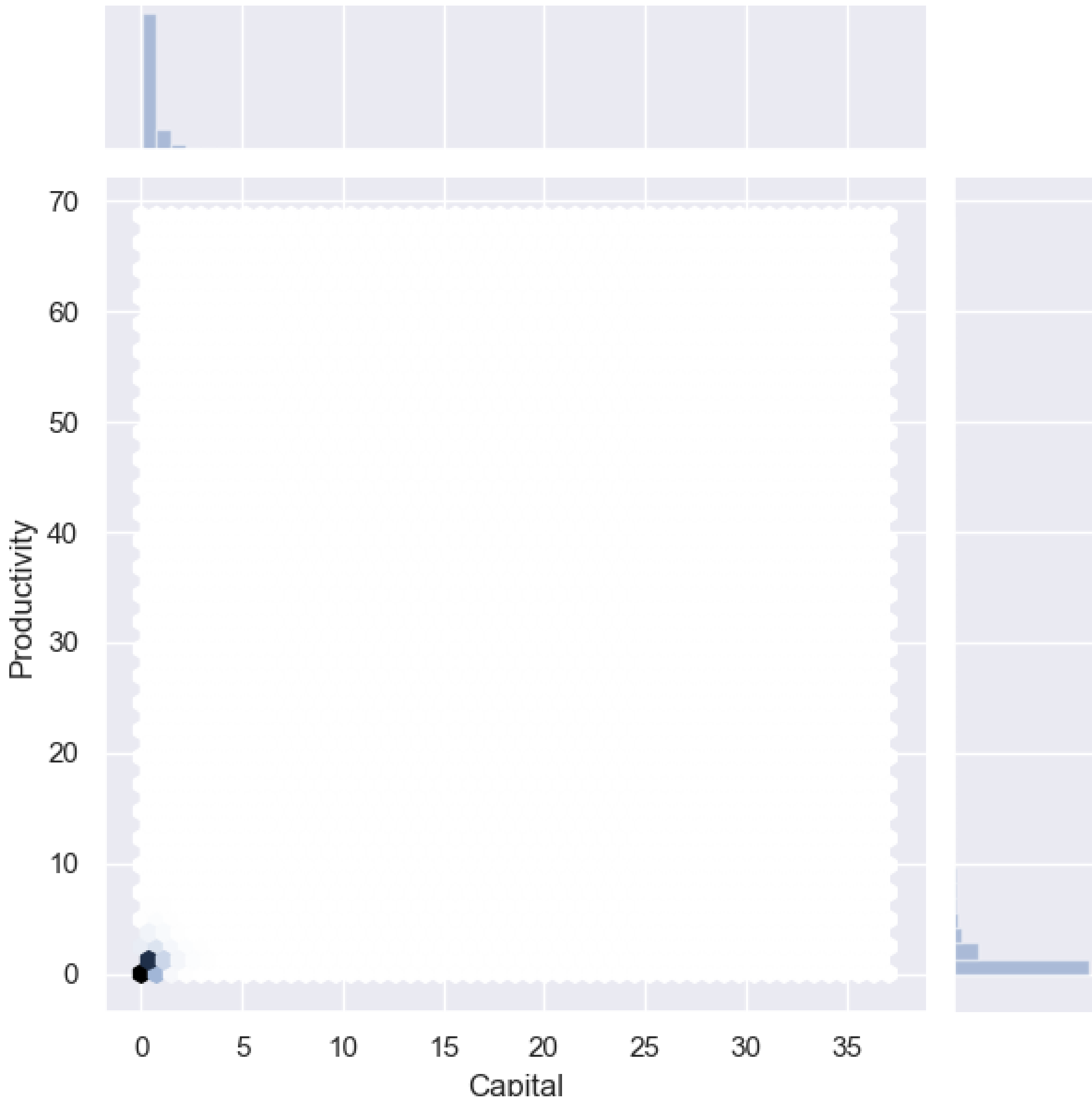

Figure 4: Joint density of z and k, correlation = 0.5

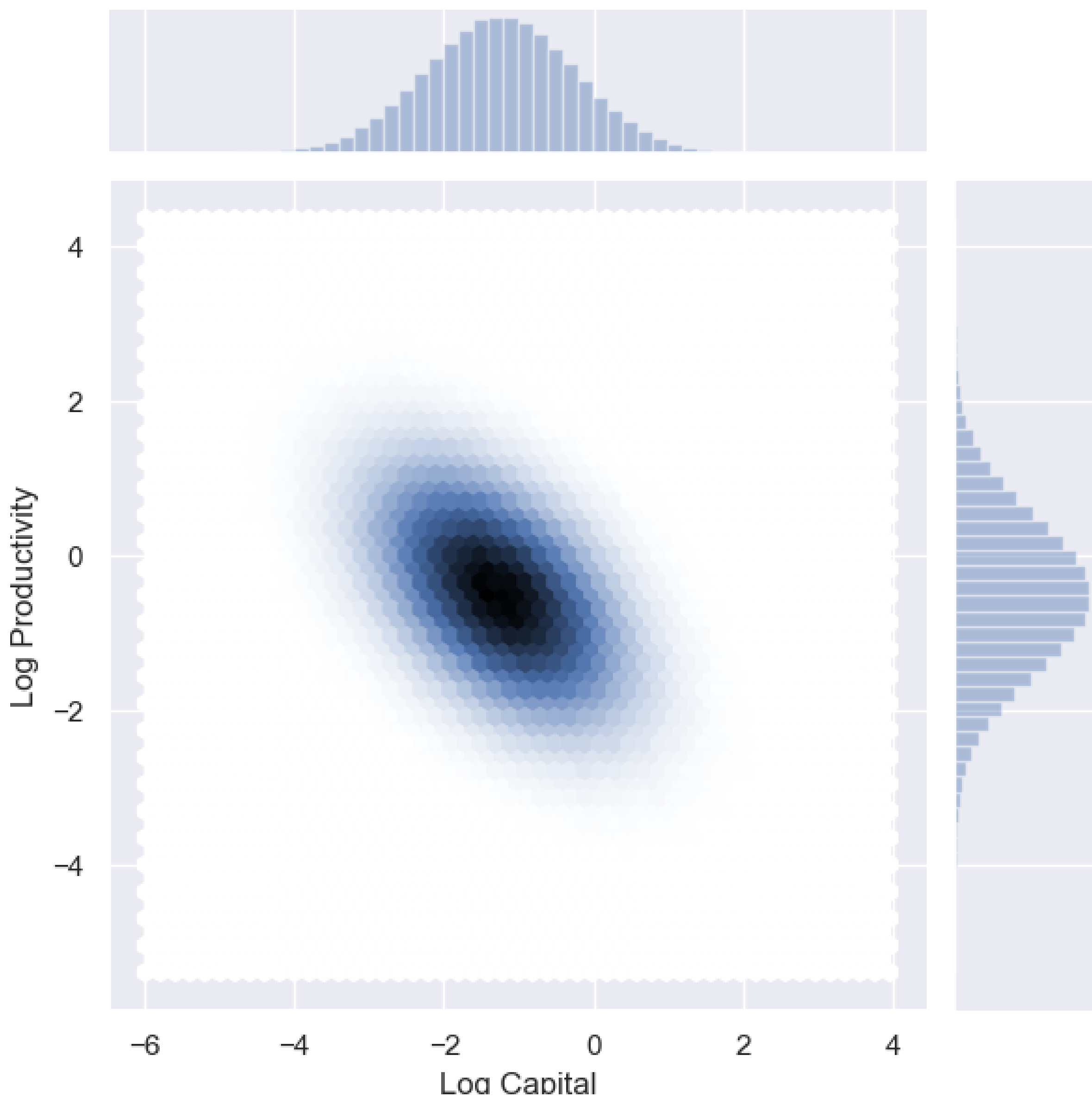

Figure 5: Joint density of ln(z) and ln(k), correlation = -0.5

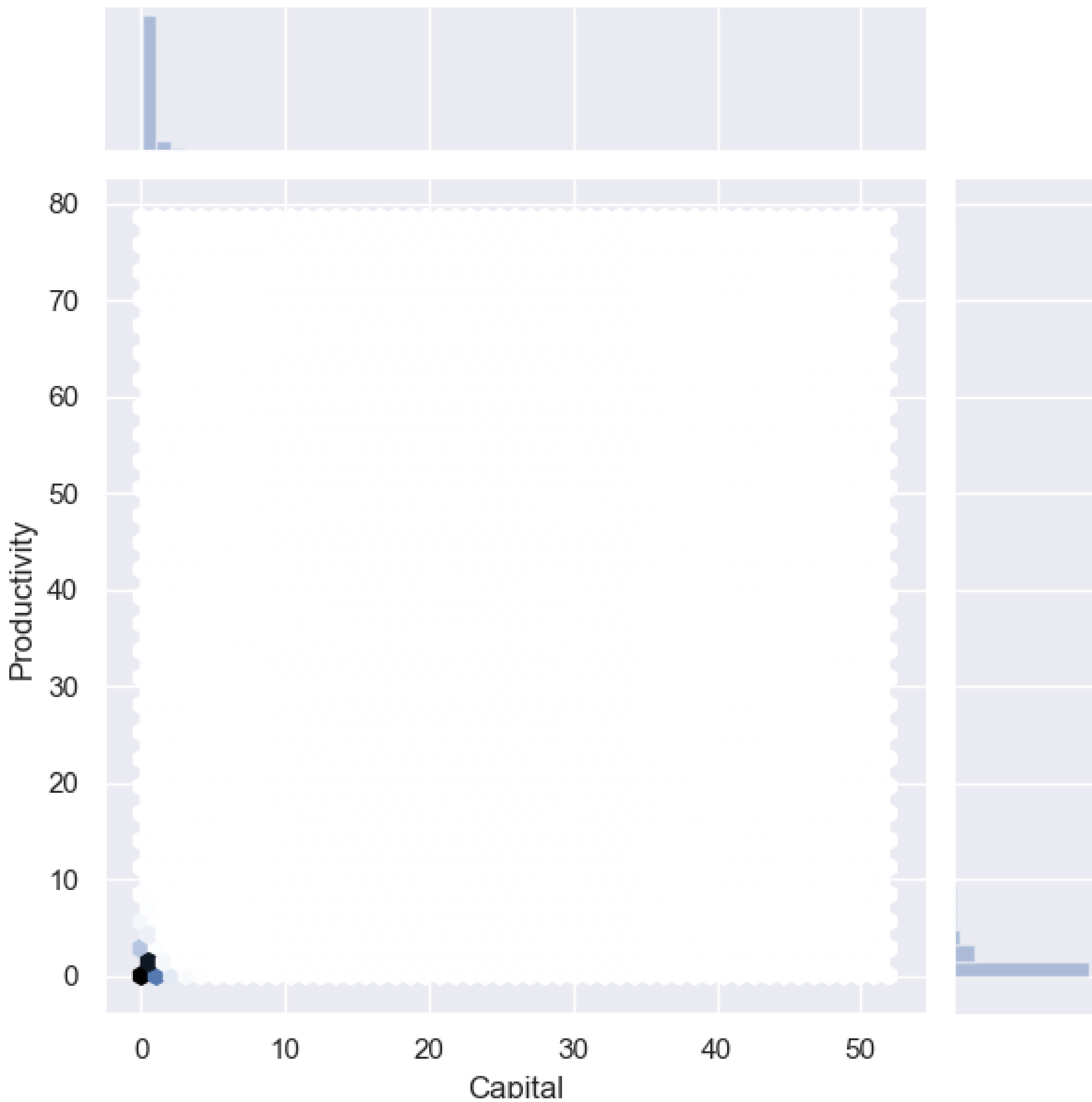

Figure 6: Joint density of z and k, correlation = -0.5

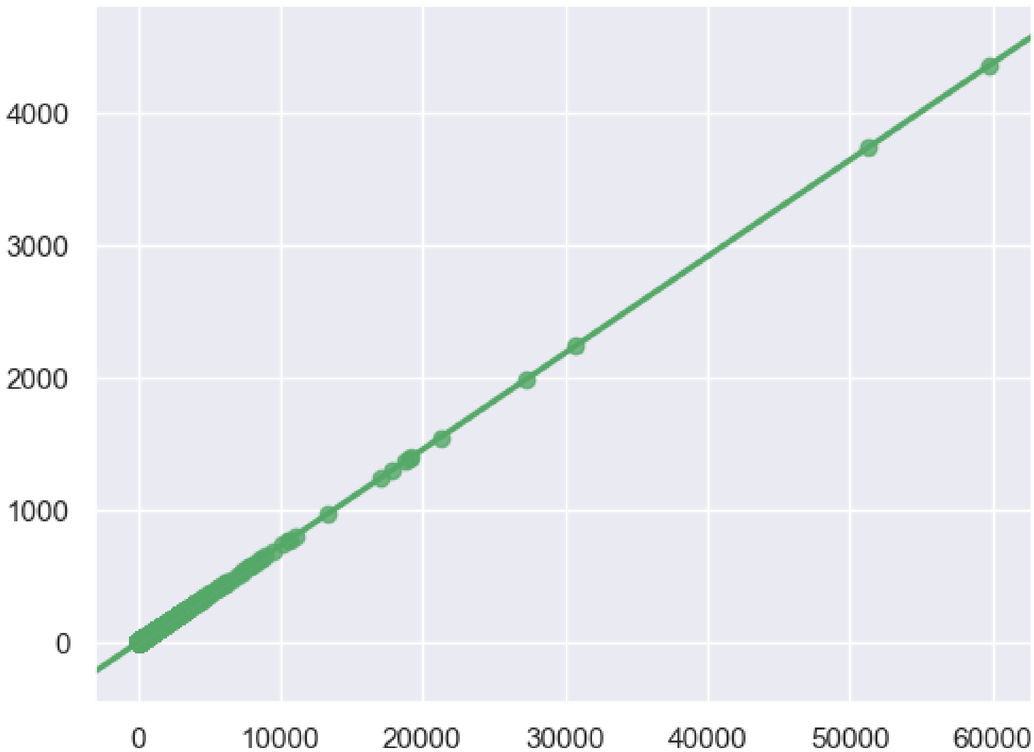

Figure 7: Efficient allocation of capital, no correlation

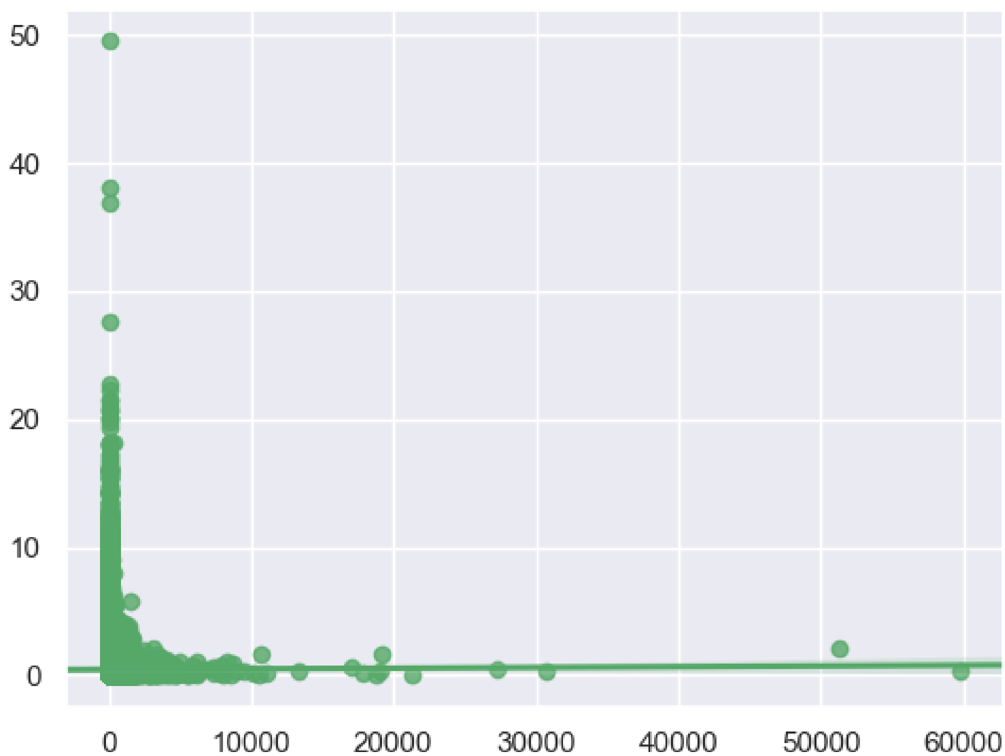

Figure 8: Inefficient allocation of capital in data, no correlation

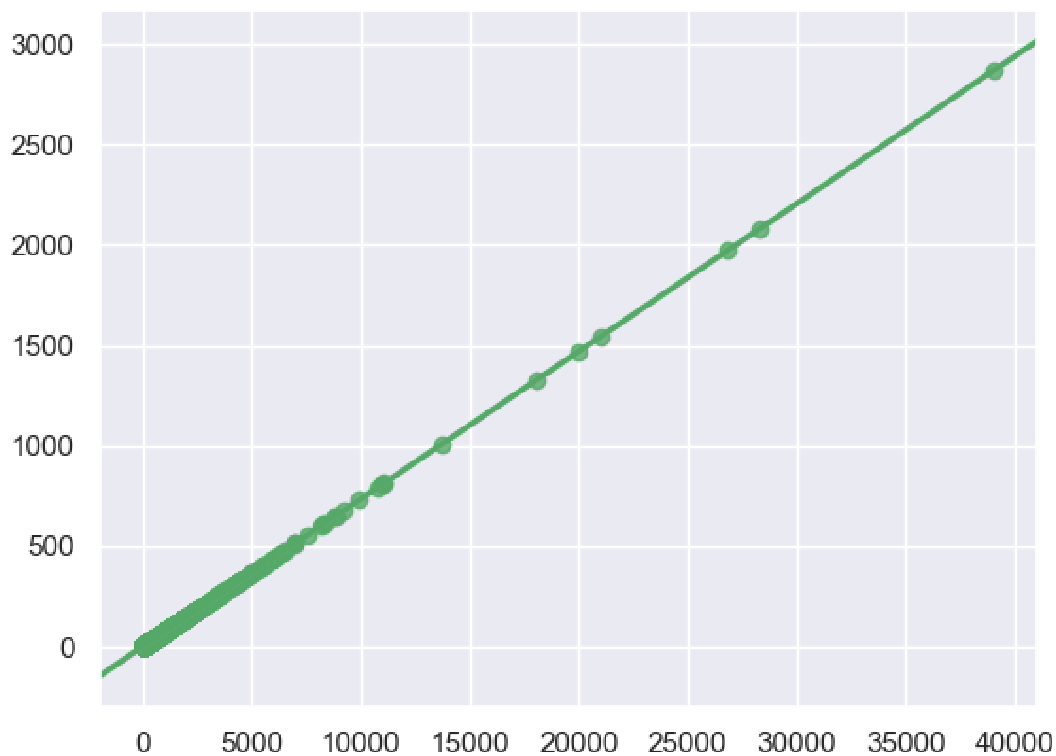

Figure 9: Efficient allocation of capital, correlation = 0.5

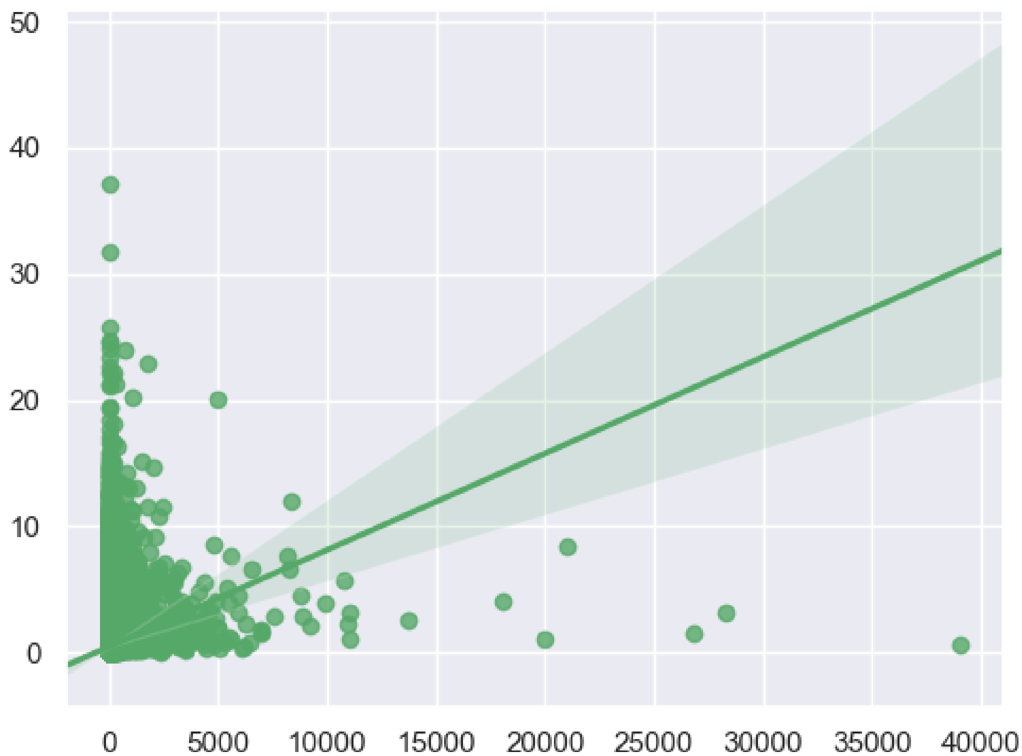

Figure 10: Inefficient allocation of capital in data, correlation = 0.5

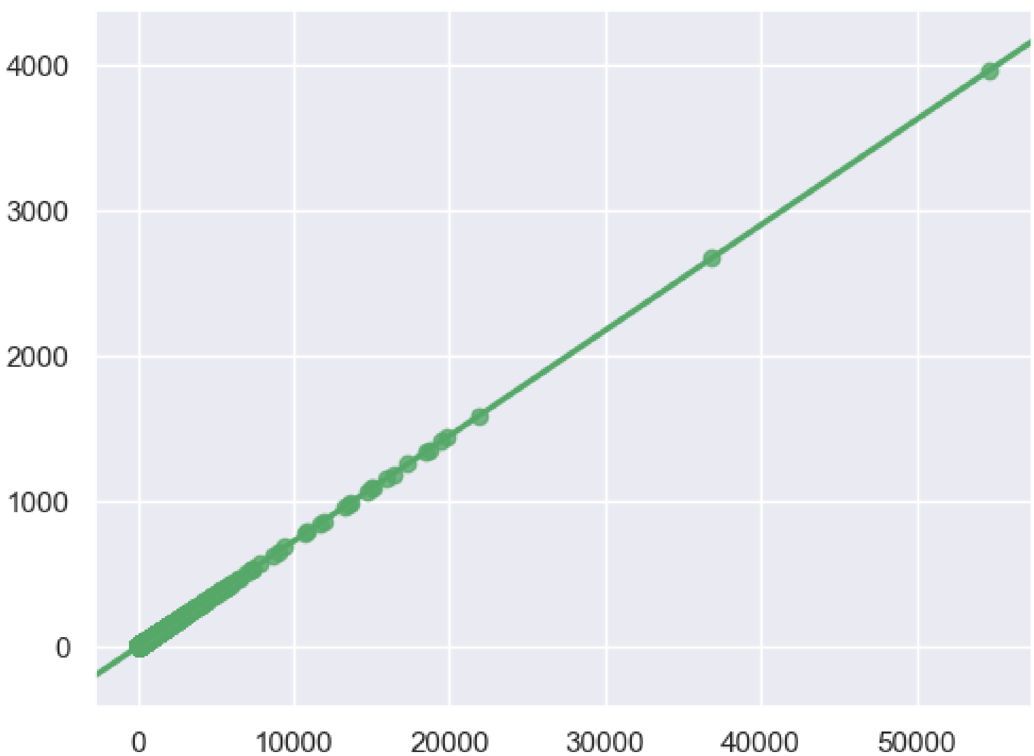

Figure 11: Efficient allocation of capital, correlation = -0.5

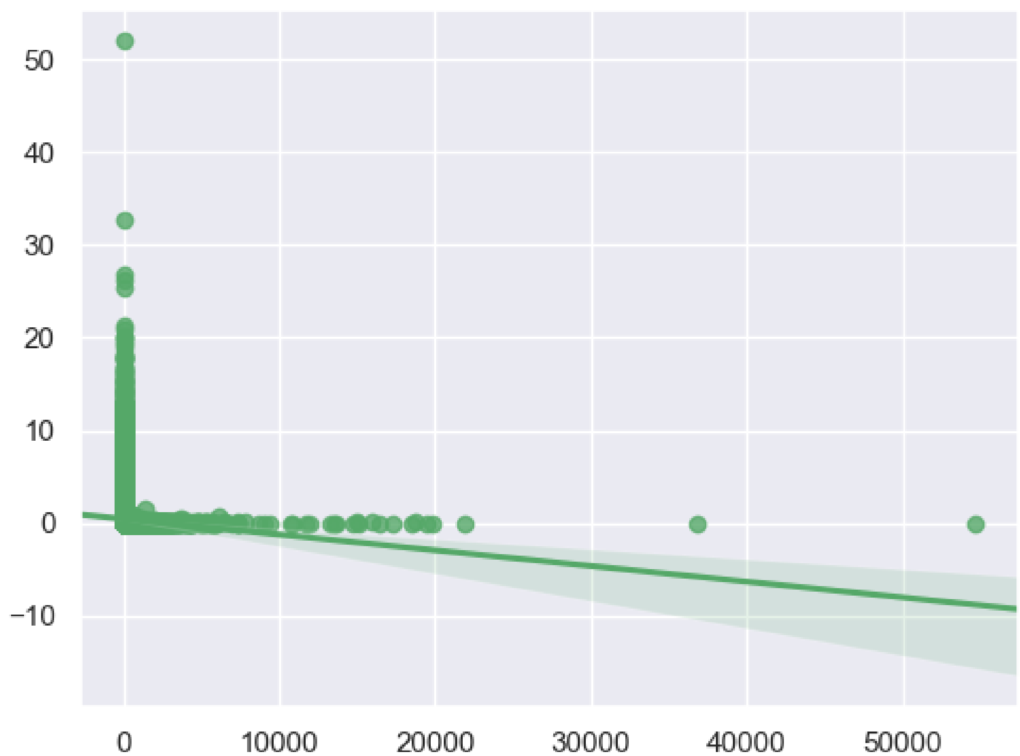

Figure 12: Inefficient allocation of capital in data, correlation = -0.5

Simulating the data

Since we changed γ this will change the means of our distribution as well. I now get

$$\begin{pmatrix} ln(k) \\ ln(z) \end{pmatrix} \sim N \left[\begin{pmatrix} -0.5 \\ -2.5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

although with these values I fail to reproduce $E[s] = 1$ as I get $E[s] = 0.72$ instead. This leads me to believe, that there is a mistake in the derivation of the density of s. I will still carry on with this parameterization.

Comparing allocations

Here I compare the allocations for higher span of control. This is given by Figures 15 and 16.

Calculating the gains from reallocation

The gains from reallocation in the case for $\gamma = 0.8$ are calculated as 641.58 %.

Question 3: From Complete Distributions to Random Samples

As I only have 1,000,000 data points I will have to work with somewhat different sample to population ratios. Moreover, for this exercise I will stick to the identity covariance matrix case and keep $\gamma = 0.6$. Results are reported in the table below:

Table 2: Descriptive statistics for different sample sizes

Sample to population ratio	$V_{ln(k)}$	$V_{ln(z)}$	Pearson corr.	Gains	Prob 10%
1/100	1.01	0.99	0	133.83%	75.7%
1/1000	0.97	1.05	0	128.06%	38.7%
1/10000	0.94	1.08	-0.11	136.3%	14.4%

As we decrease the sample size, the probability to get near the actual gains from the data decreases significantly when compared to a 10% probability band.

Plotting (in)efficient capital allocation in the sample

The results for this comparison are given in Figures 17 to 22.

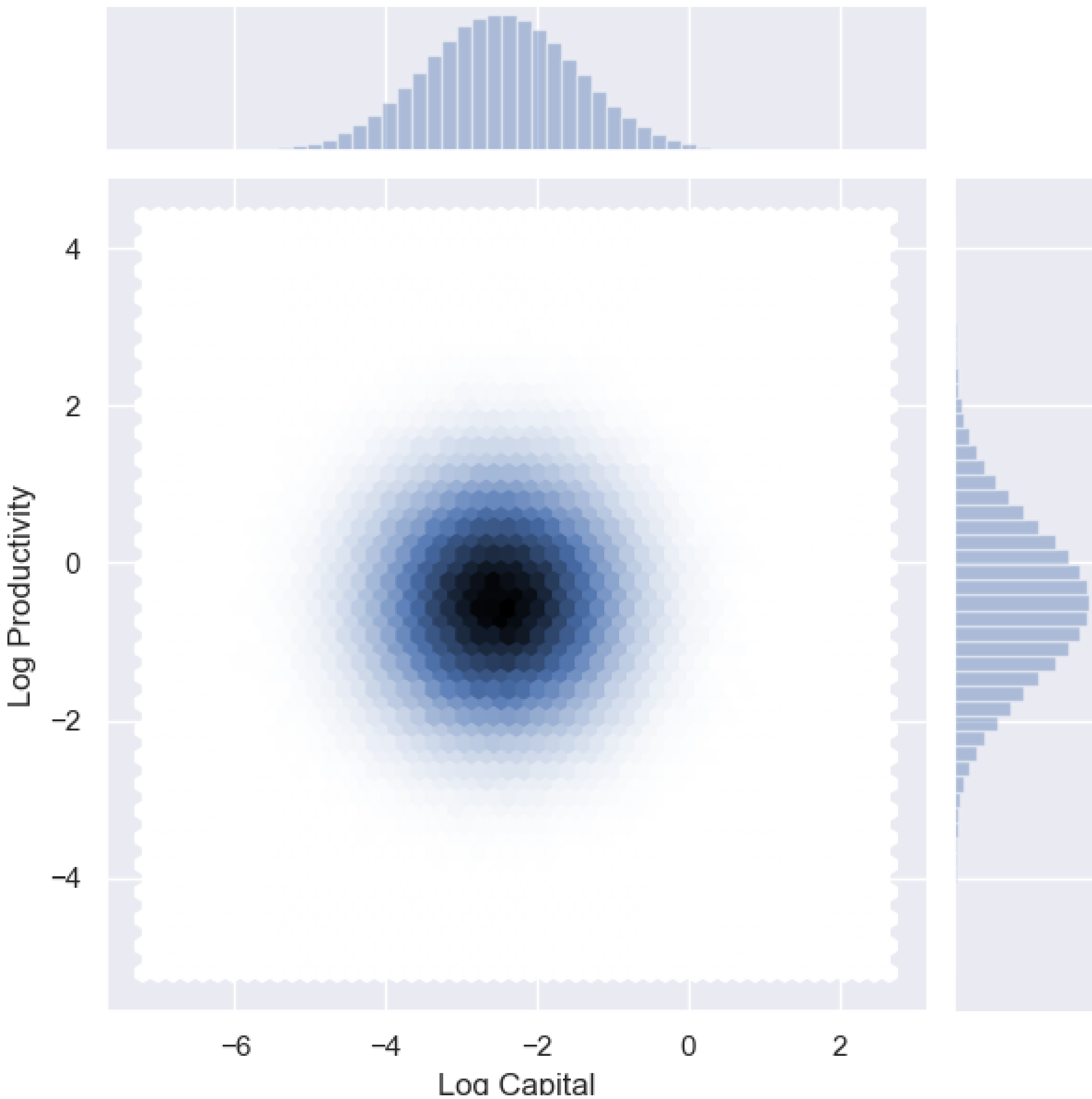

Figure 13: Joint density of ln(z) and ln(k), no correlation, $\gamma = 0.8$

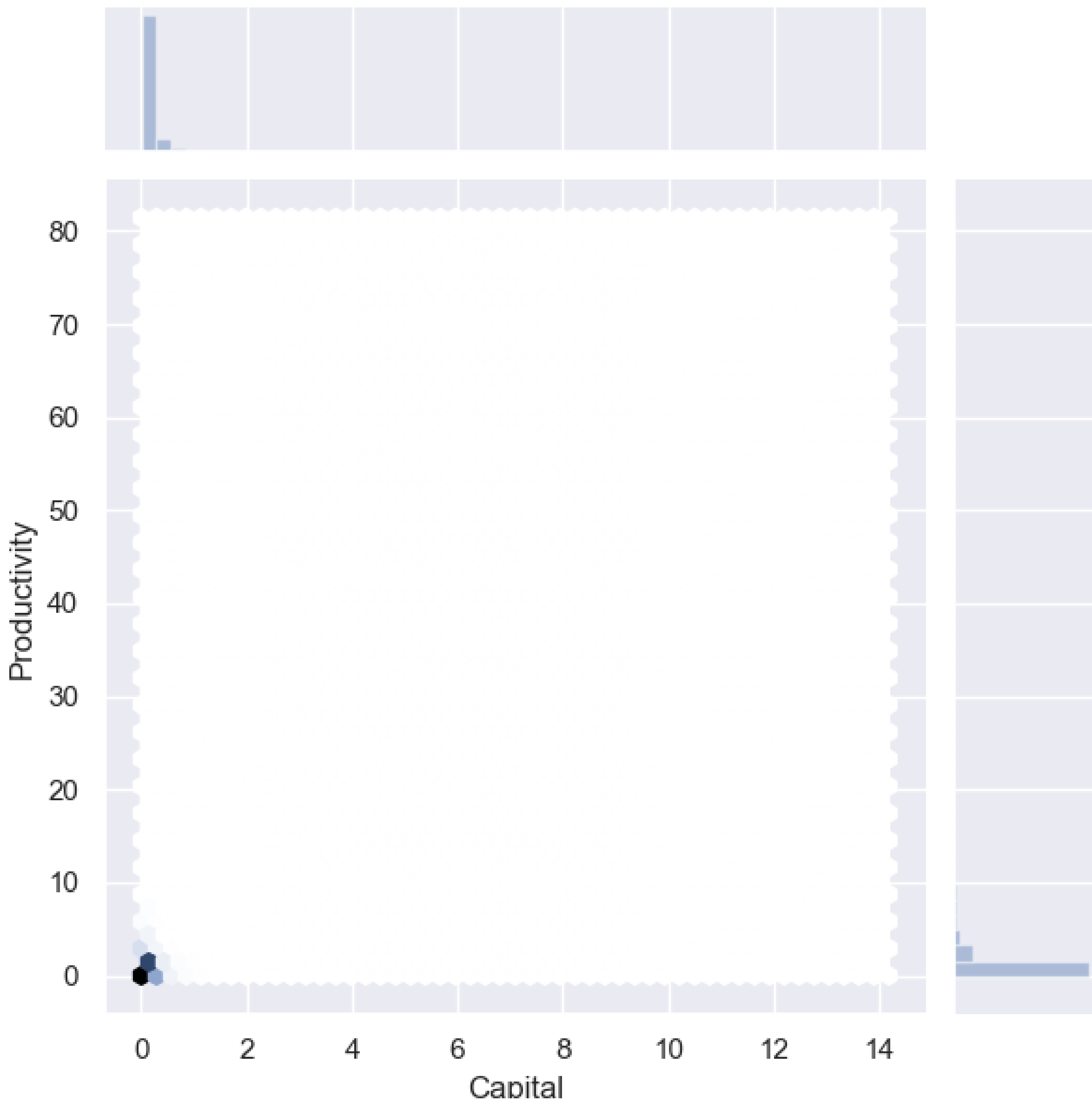

Figure 14: Joint density of z and k, no correlation, $\gamma = 0.8$

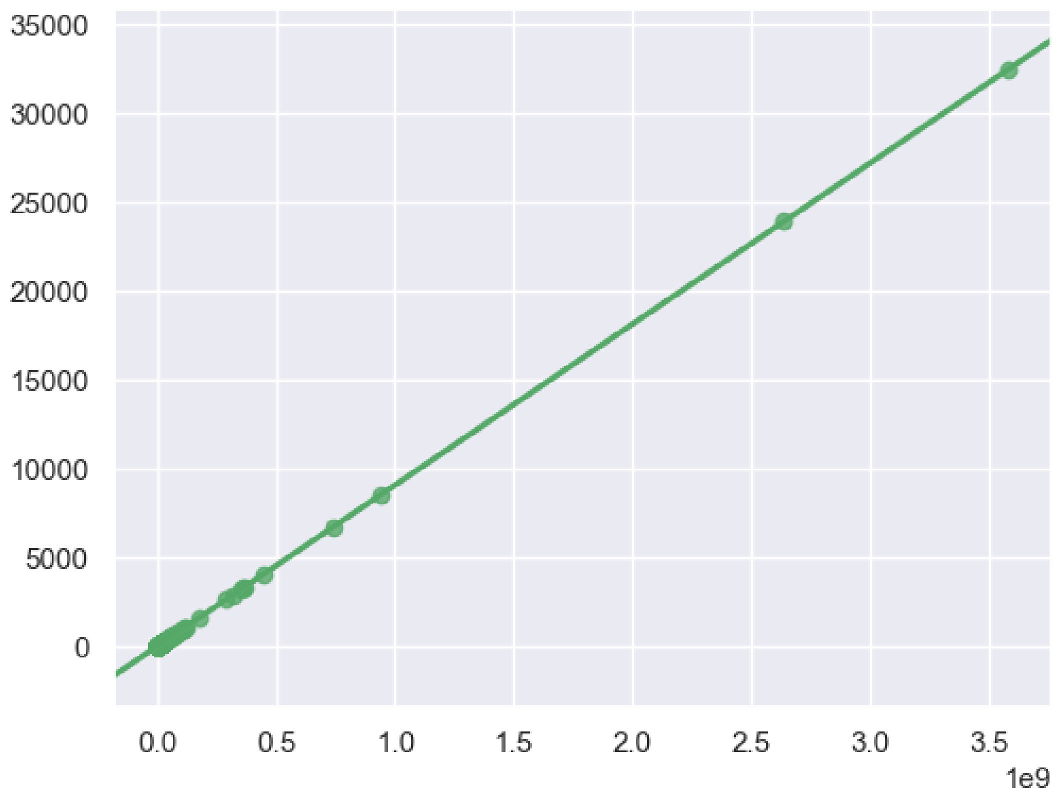

Figure 15: Efficient allocation of capital, no correlation, $\gamma = 0.8$

Plotting histograms for the gains

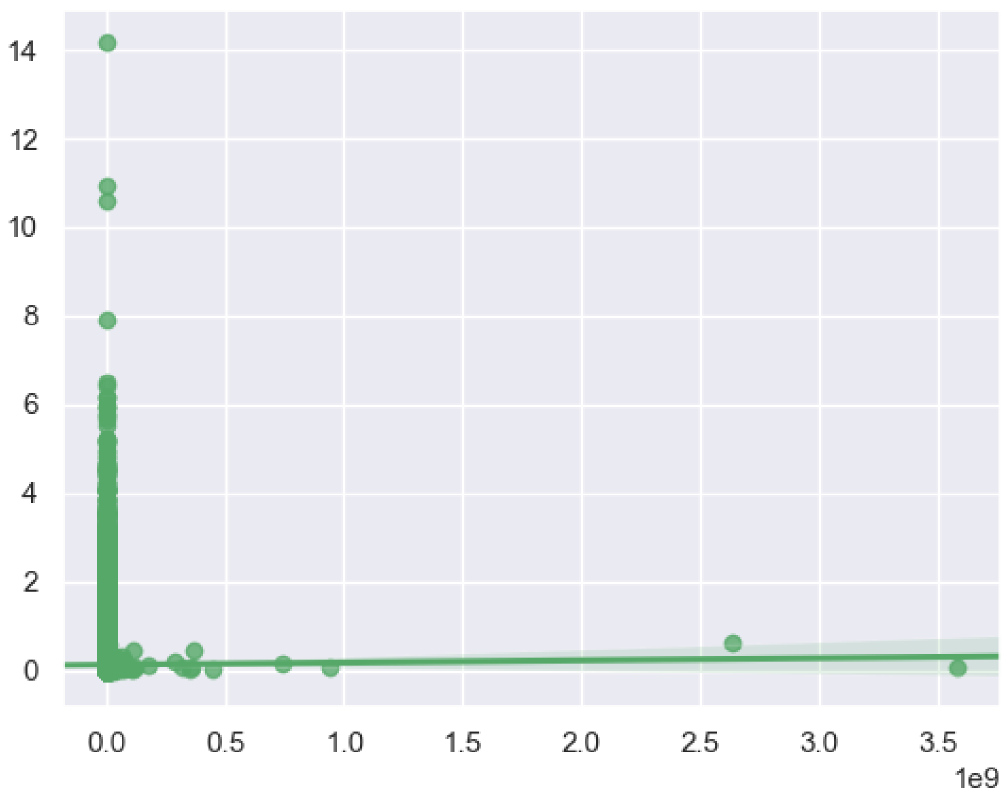

Figure 16: Inefficient allocation of capital in data, no correlation,$\gamma = 0.8$

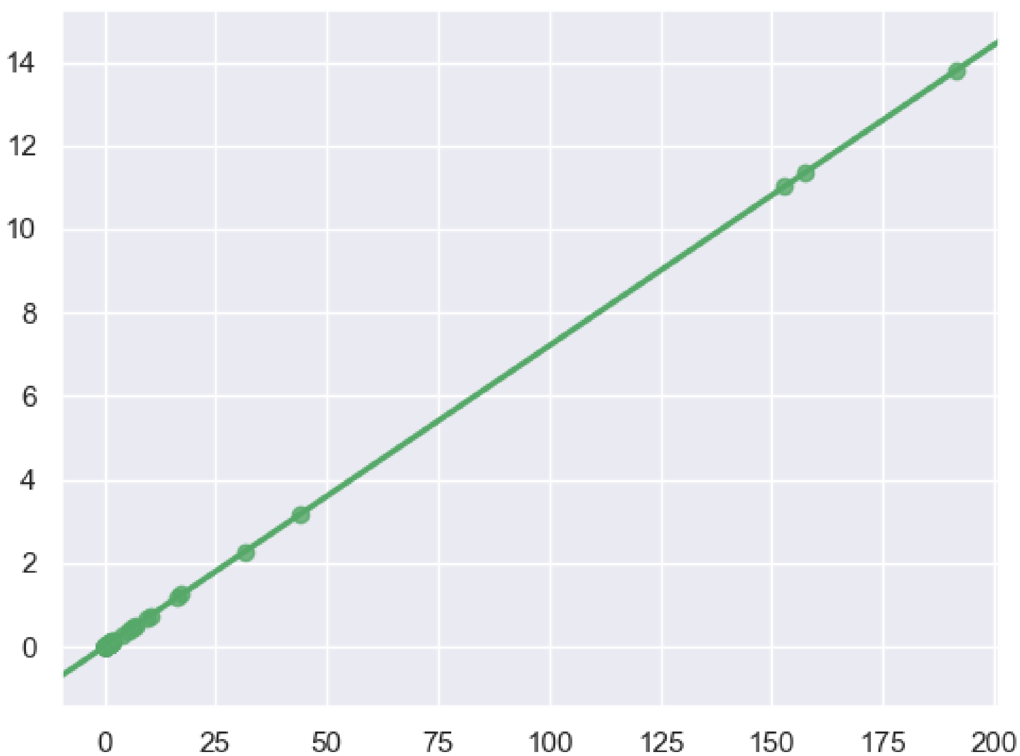

Figure 17: Efficient allocation of capital in sample of size 100

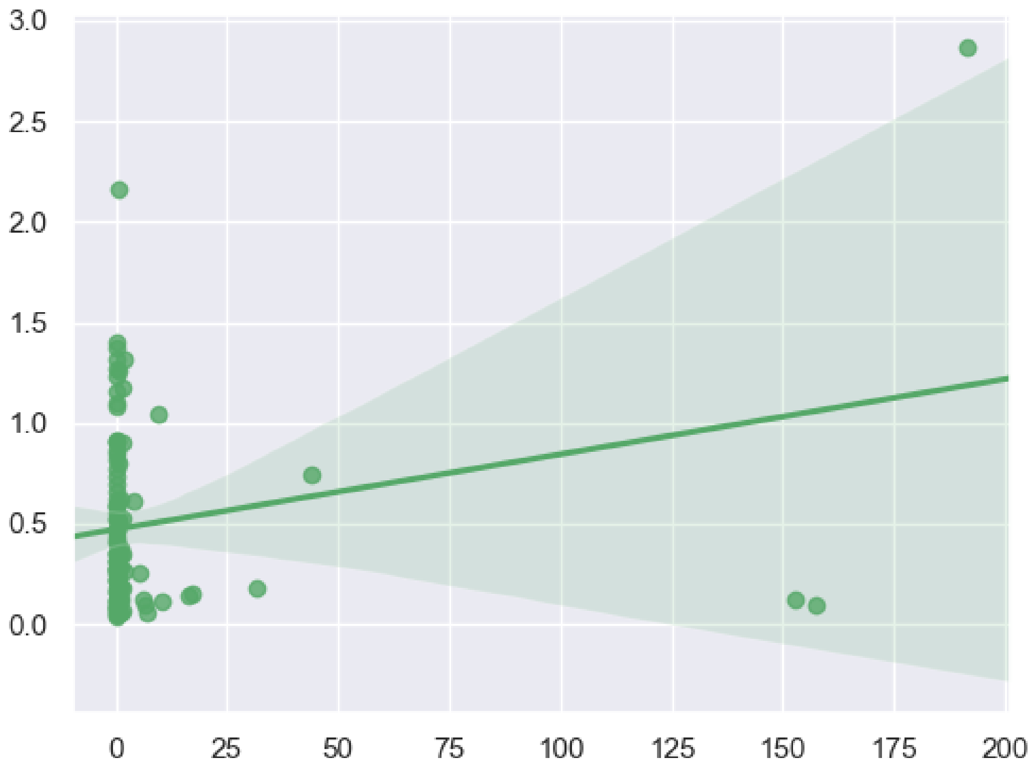

Figure 18: Inefficient allocation of capital in sample of size 100

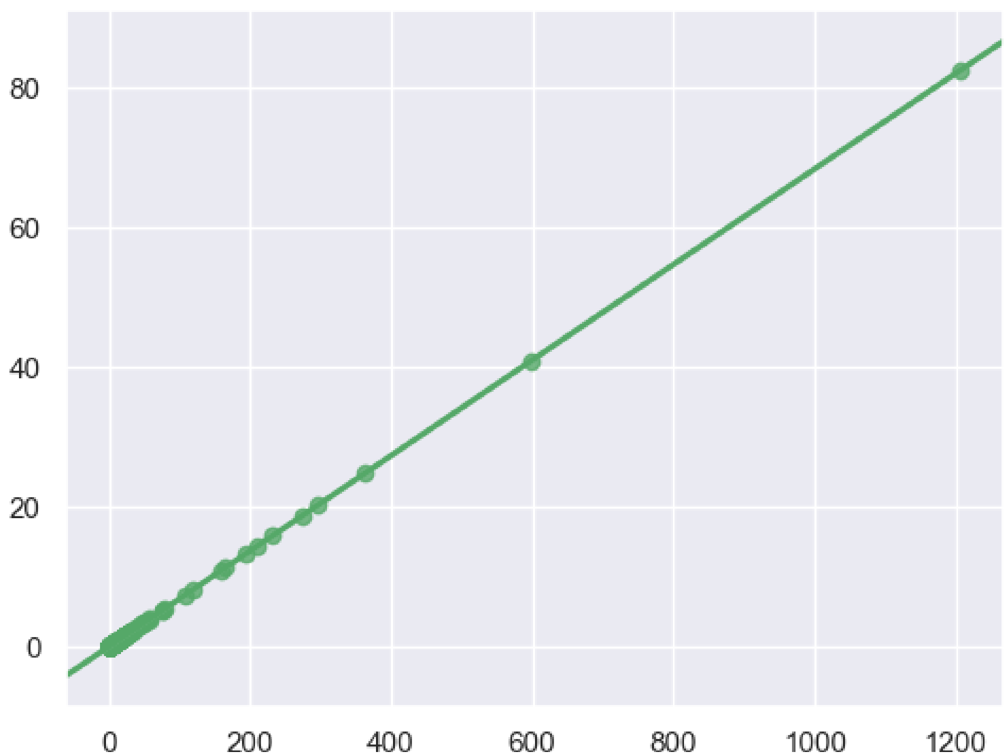

Figure 19: Efficient allocation of capital in sample of size 1000

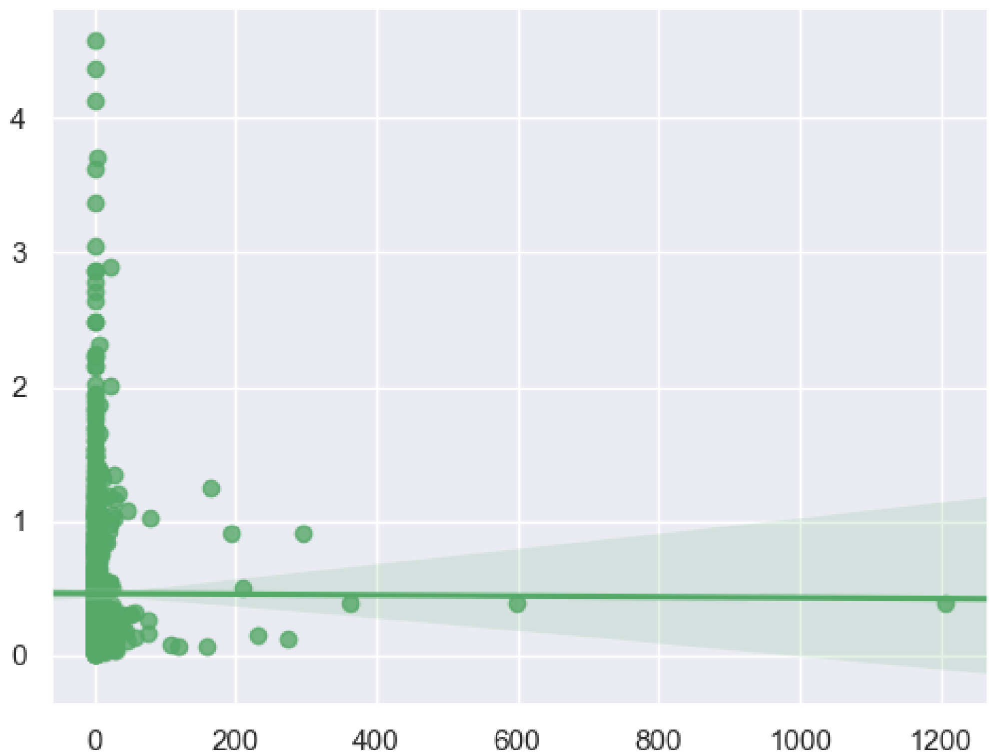

Figure 20: Inefficient allocation of capital in sample of size 1000

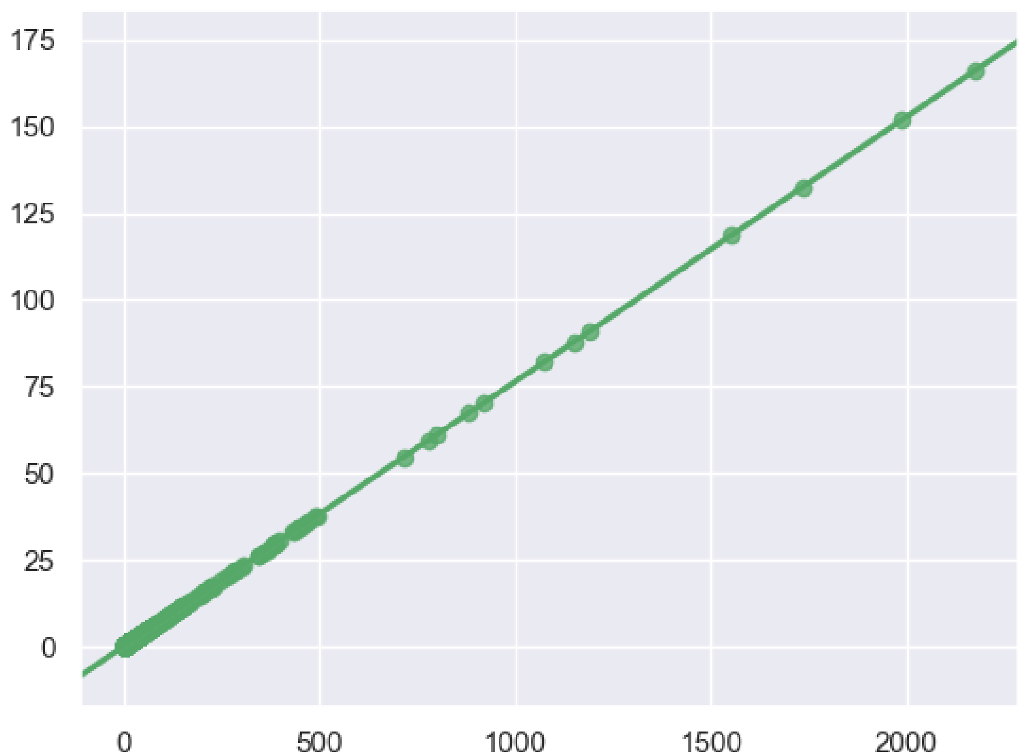

Figure 21: Efficient allocation of capital in sample of size 10,000

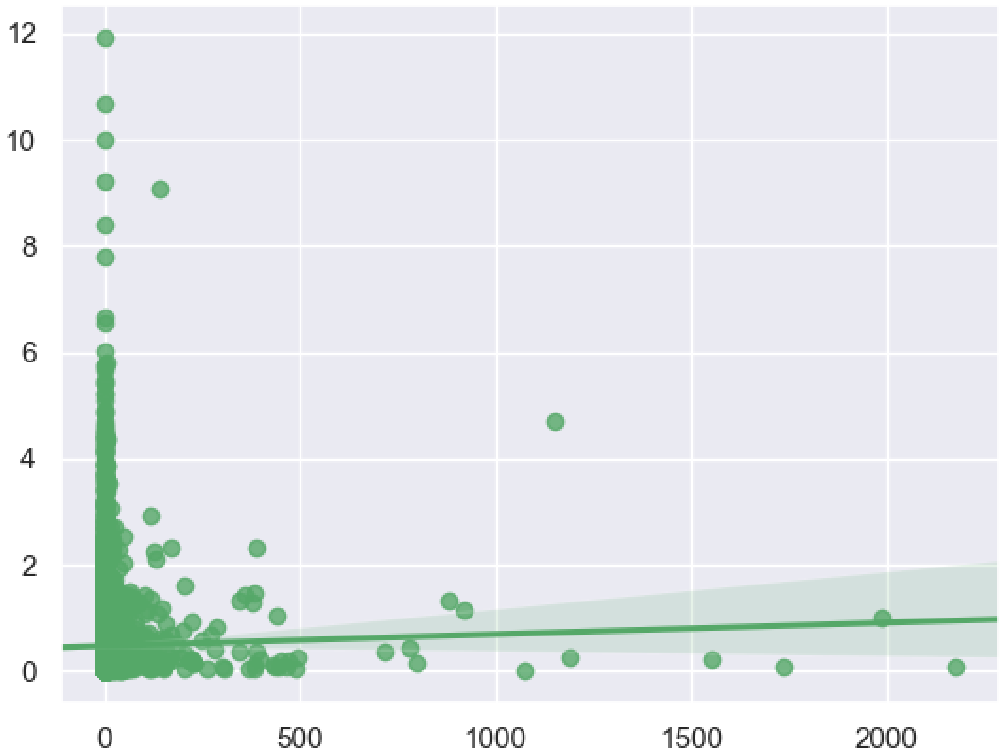

Figure 22: Inefficient allocation of capital in sample of size 10,000

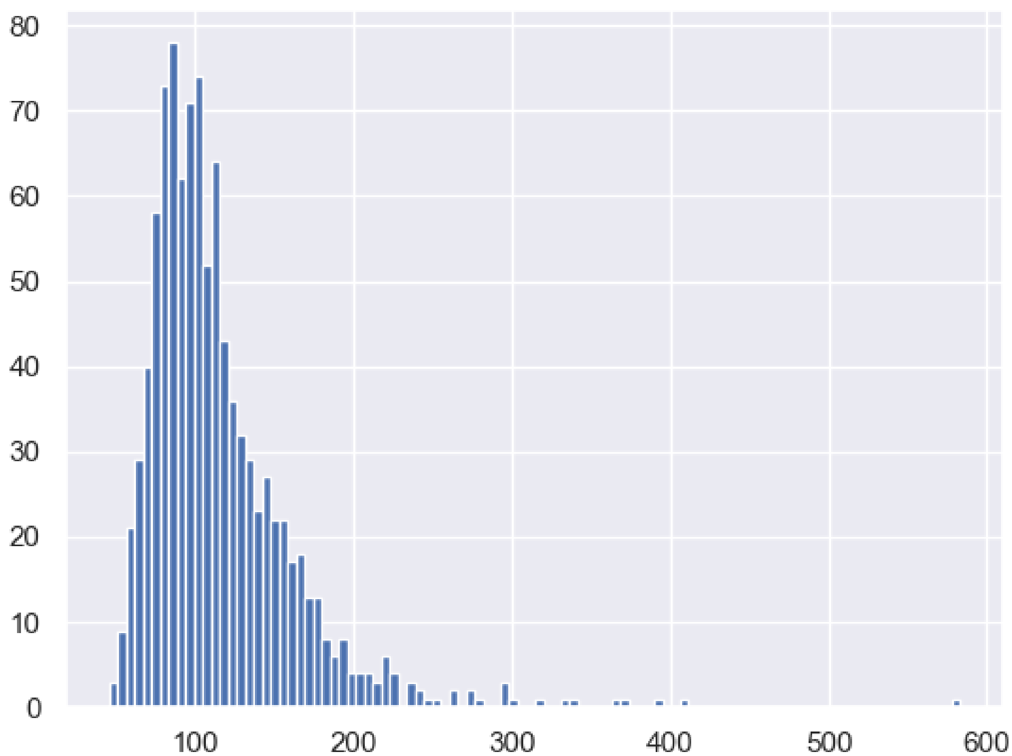

Figure 23: Histogram of gains for 100 sample points and 1,000 repetitions

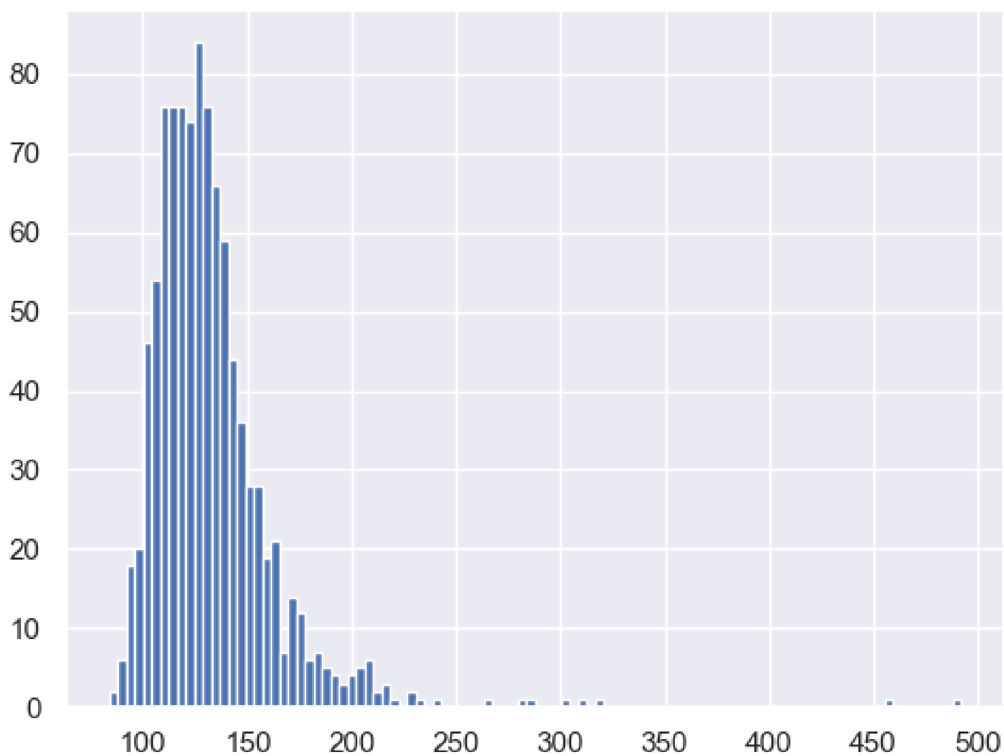

Figure 24: Histogram of gains for 1,000 sample points and 1,000 repetitions

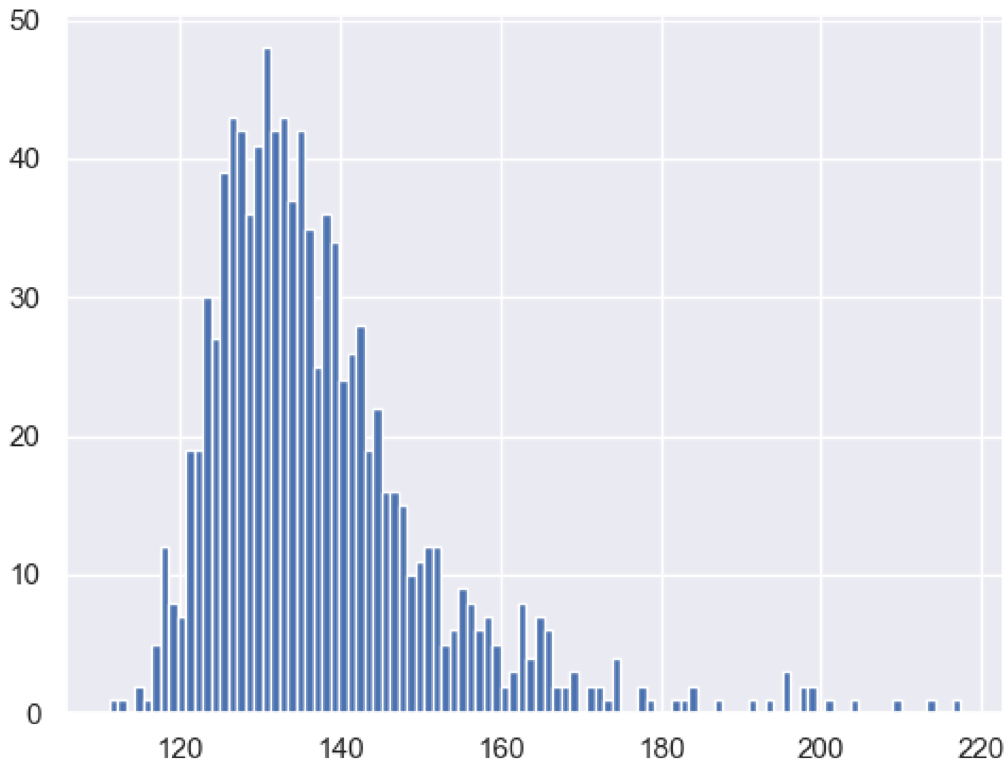

Figure 25: Histogram of gains for 10,000 sample points and 1,000 repetitions