Quantitative Macro

Problem Set 8

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Problem 1:

A three period model with a discrete choice on working in the last period.

- Look at the plot of the comparison between the explicit and the numerical solution for the three different solution methods.

Comparing 1 and 2

1 = global solution

2 = discrete-continuous endogeneous grid method

→ we see that solutions largely overlap
→ numerical solution continues for more values of x
→ discontinuity in global solution is around $x \approx 2.5$

Comparing 1 and 3

3 = exogeneous grid method

→ exogeneous grid method gets "discontinuity" (not really discontinuous) at $x \approx 2$, so below the analytical solution
→ after the drop the trend is picked up correctly and as of $x \approx 3$, the two solutions overlap again

2. Describe in words those 3 solution methods:

Solution method 1: "cloud". This is an implementation of the analytical results in the handout.

- using the perturbation from taste shocks we can get an analytical expression for V_2
- we need a threshold condition when the discrete choice in the last period becomes 1 or when it stays zero

- if savings are apparently high, working may be suboptimal for the individual
- using the condition we distinguish two cases and assign choice probabilities using logit
- the expected utility in the last period can be expressed using the log-sum property
- we can solve the maximization problem in the penultimate period for savings and consumption policies, obtain $V_0(x_0)$ and then solve the maximization in the initial period for optimal savings and consumption
- this gives us all policies for consumption and savings and we can check the discrete choice in the last.
- because we have to loop over a grid of shocks for the value function evaluations

solution method 2: endogenous grid method

- only necessary to solve pre-discrete choice problem with discontinuous/continuous choice
  - -> create a grid of savings
  - -> recursively loop over the savings grid and compute the solution for the household problem at each grid point
  - -> gives you a consumption policy, cash-on-hand policy and value function
  - -> this computation is made for all ~~discrete choice~~ state states given by the income shock (i.e. endogenously calculating cash-on-hand)

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following Ishakov, Jorgensen, Rust, Schjerning (2016)

- -> take a current decision d_t and form exogenous, ascending wealth grid (end of period wealth)
- -> compute from this a "grid" of next period wealth
- -> if $u(\cdot)$ is invertible, calculate the discrete choice d_t

from the inverted Euler equation

→ compute current consumption from the inverted Euler equation

→ this gives the endogeneous grid over current period wealth

→ get the choice specific value function by maximizing plugging in the consumption policy into the value function

(→ approximate corresponding functions by linear interpolation)

→ now we have approximations to all solutions of the Euler equation

→ but: some of these may not be optimal choices what are the suboptimal choices we need to kick out?

→ cleanup step: I think this is skipped in the code ???
⇒ it is, line 359 (checking monotonicity)

· how do the preference shocks appear?

→ Type-I EV shocks allow for log-sum representation of continuation value in workers' problem given a discrete choice

→ this is needed when we recursively get the value-functions in the EGM step.

solution method 1 ("traditional VFI") →

· ~~for every point on the~~ savings grid:
· guess a solution for the consumption policy
· evaluate the value function at your guess
· pick the maximum value of the vector
· verify if local or global
· store consumption policy and value function

Method 3: Exogeneous grid root finder

→ make a grid for cash-on-hand

→ use univariate solver to compute optimal consumption from Euler equation

→ solve value function problem of worker (for all possible states given by shocks and decisions) in T-1

→ here we use again the logsum property created by the taste shocks

→ obtain V_0 as $V + \sum P(\eta_j) \cdot V_j(\eta_j)$

~~expected value~~:

~~for every~~

→ do for other periods

3. Compare the closed form solution in period 1 to the numerical solution.

→ solutions: policy functions and value function

$g_1^c(x)$

$g_1^{a'}(x)$

$V_1(x)$

→ we set param. nt = 1 (meaning the agent solves essentially a 2-period model where in the last period they have a discrete choice)

→ we keep the solution method equal to 2

- with 1 period the global solution for c_1 gives larger values for the same grid than for 2 periods before the discrete choice.

=> there is one period with uncertainty fewer than before

- we also get lower value functions (clearly since we have fewer periods to add up) (i.e. lower value function for t=1 perspective relative to t=0 perspective)

- consumption and savings policies change quite significantly
- in fact, they overlap after the kink in $X=2$

4) Measure the running times of the three alternatives
→ intel i7, 8GB RAM

method 1: 25.6787

method 2: 1.2831

method 3: 18.4754

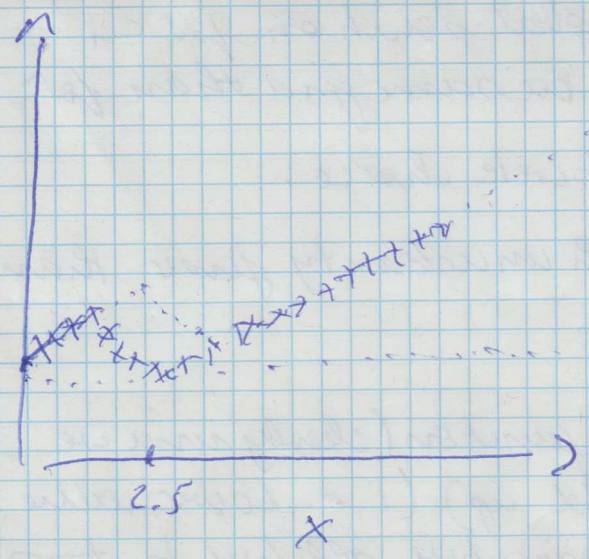
→ endogeneous grid is by far the fastest, method 2 needs to solve non linear problem to find root,

→ method 1 is using a golden search method to check global maximum

→ endogenous grid moves state to one side of Eulr equation

5) Why does method 3 have problems in characterizing the solution?

→ recall, the solution path of method 3 does not coincide well with the path of the global / analytical one



- exogeneous grid method uses Brent's algorithm to find the root of the Euler equation
- the Euler equation ~~terms of c_0 is not continuous~~ and the may have exponential and $1/x$ terms which cangive Brent's algorithm problems to converge
- the method is unable to locate the 1 (analytical) kinks precisely due to linear interpolation.

6. What happens if you increase the variance of the income shocks to

$\sigma_\varepsilon = 0.01$ under the three methods?

method 3: we get a second kink around $x = 1$ for the solution of the problem in terms of c_0

the policies also get a second kink (consumption and savings at $t=0$

method 2: method 2 no longer follows the global solution for higher values of x

also appears to have multiple kink in solution path and policy function

method 1: also two kinks

Why do we get more kinks?

- kinks occur at points where an agent is indifferent between working and not working (i.e. where the value function is not differentiable)
- these so-called primary kinks reverberate back through the system (i.e. through the impact of V_t on V_{t-1} and of the discontinuity in the policies) and can manifest themselves as additional kinks (secondary kinks)
- the number of kinks depends on the wealth threshold attained per period and so depend on the magnitude of the wage shock
- ~~if the precautionary saving (in wealth) of optimal consumption policy carries for $\tau = 3$ periods~~ which ~~calculated are such that choices jump at different~~ realizations of the wage shock ~~we can get~~
- the optimal consumption policy is piecewise linear and may therefore exhibit many kinks depending on the wealth thresholds (and hence the shocks to income).

7.) Elsing param. sigma = 1

method 2: analytical and numerical solution are indistinguishable

smoothing is strong so that kinks appear to be gone

method 3: corresponds closely to analytical and global solution

slight kink in consumption and in savings policies, but close to none

=> numerical solution get closer to analytical solution due to smoothing

→ for $\sigma = 1$ (I assume they mean the scale parameter for the Gumbel shocks) the log sums become very simple and the expected utility can be evaluated well

Problem B

$$V(x, \eta, j) = \max_{c, a'} \left\{ u(c) + \beta \underbrace{E_j}_{\text{survival risk}} \sum_{\eta'} \pi(\eta'|\eta) V(x', \eta', j+1) \right\}$$

s.t.

$$a' = s = \begin{cases} a(1+r) + \eta w - c & \text{for } j < j_r \\ a(1+r) + p - c & \text{for } j \geq j_r \end{cases}$$

$$x = a(1+r) + \eta w$$

$$a' \geq 0$$

$$u(c) = \frac{c^{1-\theta}}{1-\theta} \qquad \theta \neq 1$$

$$\log c \qquad \theta = 1$$

Problem 1: a) Explain all sections of the code

1. lines 1–16 : Intro

→ we have a couple of options to set, whether to include survival risk, stochastic element and we need to set the number of states for the income shock, in this case 2

2. lines 17–33: Solution

→ uses function that are created later to

a) calibrate the parameters of the model to typical values

get invariant distribution starting from 50%/50%

b) solve the household model

c) aggregate onto

d) create the trajectories over the lifecycle of our variables of interest

3. lines 34 - 108: Plots

- Plots paths for variables of interest over the life-cycle + growth rate of consumption, the asset distribution and the value function

4. line 109 - 210: Calibration

- based on input from intro (essentially whether stochastic or deterministic) calibrate model parameters, long run distribution, survival rates, Markov process shocks and transitions, population and pension income

5. line 211 - 343: solve the household problem

- uses backward iteration and an endogeneous grid method to recuperate consumption policy and value function
- first set up grids for cash-on-hand, savings and assets at all times and states
- for the two states compute final period wealth to get final period value, policies
- from there iterate backwards in time using a grid on savings
- for all income shocks (i.e. high or low), for all grid points in the c-o-h grid do the following
  a) compute the value tomorrow under each shock by interpolating
  b) since we have ~~consumption~~ savings only on RHS of Euler equation → evaluate
  c) solve for

grid

e) calculate value function and update

f) go back one period and proceed as before

6. line 344 - 377: function interpolation

- implement linear interpolation of a function between points

7. line 378 - ~~445~~ 511: Aggregation

- this section controls for the measure of individuals at points in the cash-on-hand grid
- for this we need a transfer function which determines the new shape of the distribution
- then we aggregate over the new distribution and store this at each period
- then we check if the measure in total adds to 1
- for the actual aggregation of assets we need these distributions (we also compute total consumption)
- the last bit of this section is a function we use to generate the new distribution, i.e. for the transfer function, from the old distribution

8. line 512 - 544: Life-Cycle Profile

- the function here calculates the asset holdings, consumptions, incomes, value functions at all periods of the population

9. rest: these are auxiliary functions needed in the solution above

b) Compare the solution procedure in func_hh to a standard implementation using the exogeneous grid method

→ What are the differences:

the grid is used for savings instead of cash-on-hand

→ Advantages / disadvantage?

- the method is faster and reduces the number of continuous state variables
- we do not need to use a non-linear root finder but can solve for consumption "analytically" via the Euler equation
- disadvantage: the exogeneous savings grid can lead to an endogeneous assets grid with infeasible grid points

↳ Hence need to figure out a way how to deal with this in interpolation

c) Implement exogeneous grid method

→ running time egm: 6.7431 s

→ running time exogm (flawed): 27.7066 s

d) Have a look at func_aggr.

i) what is the role of the objects TT and Phi?

lecture notes: Ψ_t = cross sectional measure of households

Q_t = Markov transition function

→ TT: how do individuals move within the distribution over the ages

→ Phi: determines the measure over ages, savings and income shocks

- in the lecture notes the survival risk is multiplied onto the transition function rather than the law of motion of the measure
- the integral is not applied in the code, rather we sum over the entire state space (for the law of motion of the measure)

- merging reduces time from 6.0929 to 5.9223

- breaking reduces time to 5.8635 but delivers strange results

e) solve the model under 3 alternative settings

1. opt_ret = true , opt_nosr = true
   - non-stochastic income and the transition function has $\pi = 1$
   - perfect survival until the last period
   - problem: value function is convex
   - consumption constant except 1 period around retirement, policy kinked
2. opt_ret = false , opt_nosr = true
   - income is determined by shocks that follow a Markov process, there is no risk from survival chance
   - consumption becomes flat after retirement
   - consumption policy at age 20 and 40 overlap
   - asset decumulation is more concave
3. (I assume there is a typo in the PS)
   opt_ret = false , opt_nosr = false
   → agent save to smooth consumption and face survival risk

Stock

- simulate the model with varying theta
→ I put both previous options to "false"

θ=2:

- value function from −38 to −2 (relatively straight)
- savings from 0.4 to 0.15 (ages 20 − 55)
  at retirement sharp drop, before slight increase
  after retirement steady around −0.45, then increase to 0
- consumption hump shaped, peak at 6.5 (1.25)
- asset "triangular": peaked at 68 (11)
- income: sharp drop at retirement (from 1.4)
- asset tomorrow dist.: mass at zero, hump at (10, 0.05)
  relatively high variance
- policies for young agents almost overlapping (consumption)

θ=4

- value function far lower at early ages
- savings are similar to above setting
- slightly higher consumption, but also hump-shaped
- higher asset accumulation towards retirement
- income very similar (a bit higher)
- asset distribution (tomorrow) even higher variance
- policies for consumption roughly the same

ρ=0.2, r=0.04 (leads to lower β) → more impatience

- consumption policies overlap completely at young age
- value function decreasing after retirement
- savings similar but somewhat lower
- consumption not really hump shaped but skewed towards age 0
- asset accumulation decelerates before retirement
- lower income, asset distribution less variance, higher mass at zero

Problem 2

a) Embed the code into a general equilibrium model. Assume that accidental bequests are taxed 100% and are not redistributed.

→ the file "geneq" implements this, the equilibrium rate of return is 0.8% after 54 iterations

b) Why can an OLG economy with idiosyncratic risk be dynamically inefficient?

→ it can be dynamically inefficient when the capital stock is too high (when the return rate is to low)

→ in this case, a social security system can be improving for efficiency

→ since there are in theory infinitely many periods and finitely many agents, if the consumption profile is increasing towards later periods, this goes at the burden of the "older" generations in the absence of a social security system

→ the file "geneq" can implement such a system by introducing a tax and benefit system "pay as you go" and/or by having the life-cycle structure (??)

c) Augment the code by a welfare analysis for the life-time utility of new borns by calculating the consumption equivalent variation

$$\Rightarrow \quad g_t = \left(\frac{V_t}{\overline{V}_t}\right)^{\frac{1}{1-\theta}} - 1$$

(VO computed in code geneq - ceqv.m)

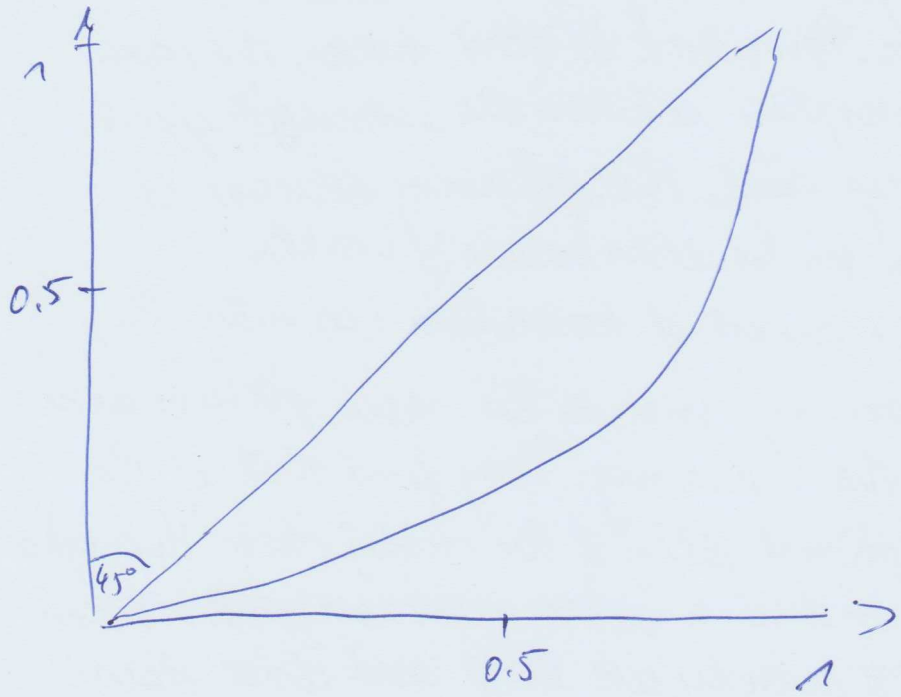
d) Compare pens = 0 vs pens = 0.6
(I compute this by running twice instead of the loop)

pens = 0: 28.1278 pens = 0.6: same → g_0 = damn

e) Compute the Gini-Index of wealth distribution

→ for the default calibration and $\rho_{ins} = 0.6$
I get a Gini coefficient of 0.4540 (assets)

→ the Lorenz curve looks like this



. comparing this to DeNardi (2004):

US data: Gini 0.~~78~~

Sweden: 0.73

⇒ our simulated economy is more equal than the US data or Swedish data permit

Part C: On the optimal Provision of Social Insurance

Ludwig & Krüger (2016)

The authors develop a life-cycle model which endogenizes the decision of young agents to attend college in order to investigate a problem of optimal taxation and college subsidy. They find that it is optimal for government to heavily subsidize college education and implement a progressive income tax code to finance the expenditure. Compared to other studies, the paper considers explicitly the interplay between the subsidy's effect of increasing college take-up and the following decrease of the college wage premium on the distribution of wealth. Moreover, they look at the period of transition (as in the PS) between the state of the economy prior to the new policies and the new steady state of the economy. They find that if the government sticks to the optimal policy of the steady state throughout the transition it could induce a severe recession as the taxes needed to finance the college subsidy are large and drive down consumption. If such transition dynamics are accounted for, however, the authors find that the progressivity degree should be 10%, the marginal tax rate on labor income should be 22.9% and the subsidy of college education 120%. This is associated with a 3% increase in annual consumption compared to the status quo. The model entails idiosyncratic income risk, skill-propagation across households, survival risk, borrowing constraints and optimal policy is conducted by maximizing a Utilitarian SWF as in Conesa et al. (2009). It is calibrated to match US data. In my opinion, the crucial aspect is the consideration of transitional dynamics for optimal policy which can change the outcome from boom to recession. It would be interesting to see how such a model performs in countries where college education is not as expensive as in the US, where college is often a person's first major financial decision.

Problem B.3

Provide a detailed write-up:

1) We first fix a time period T (realistically equal to 10, but convergence can require up to 200) Compute initial and final steady state after (MIT shock)

2) Make a guess on the sequence of interest rates $\{r_t^m\}_{t=1}^{T-1}$ by interpolation between r_0 and r_T

3) Goo to the firms problem and use the guess in 2) to obtain

$\{w_t^m = w_t(1-\tau_m)\}_{t=1}^{T-1}$ where τ_m is the tax rate

Then compute the pension system

$$\tau_m = \frac{rr \sum_{j=jr+1}^{J} N_j}{\sum_{j=0}^{jr} N_j + \overline{rr} \sum_{j=jr}^{J} N_j}$$

4) Calculate now the solution to the household problem given r_m, $w_m(1-\tau_m)$ and $\overline{rr}$

$$\{\{v^m(t,j,\cdot), a_{t+1}^m(t,j,\cdot), c^m(t,j,\cdot)\}_{j=0}^{J-1}\}_{t=1}^{T-1}$$

by iterating backwards in time

5) Use $\{a_{t+1}^m(t,j,a_t,\eta_t)\}_{t=0}^{T-1}$ and $\pi(\eta'|\eta)$ to calculate the transition laws $\{H_t^m(j)\}_{t=1}^{T-1}$

6) Iterate forward on $\Phi_0(j)$ to get the law of motion of Φ: $\Phi_{t+1}^m = H_t^m(\Phi_t^m)$ up to $T-1$

7) Aggregate asets: $A_t^m = \int a_t^m(j-1, a_{t-1}, \eta_{t-1}) \Phi_{t-1}^m$

$(a_j \times d\eta \times da)$

8) Compute the new $\{r_t^m\}_{t=1}^{T-1}$ implied by clearing the capital market $\tilde{K}_t^m = A_t^m \ \forall t$

9) Check if $dist < \varepsilon$ where dist is the sup norm on $\tilde{r}_t^m - r_t$

→ if yes, then stop the iteration, otherwise update $r_t^{m+1} = \omega \tilde{r}_t^m + (1-\omega) r_t^m$ for some small ω and go back to 3)

10) Once convergence is reached, check $\| A_T - K_T \| < \varepsilon$ if not increase T and goto 2)