Quantitative Macro – Homework 1

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October 1, 2019

Question 1.1: Taylor series

The objective is to use Taylor series approximations of orders 1,2,5 and 20 of the function $f(x) = x^{0.321}$ around a point $\bar{x} = 1$ on the domain (0,4).

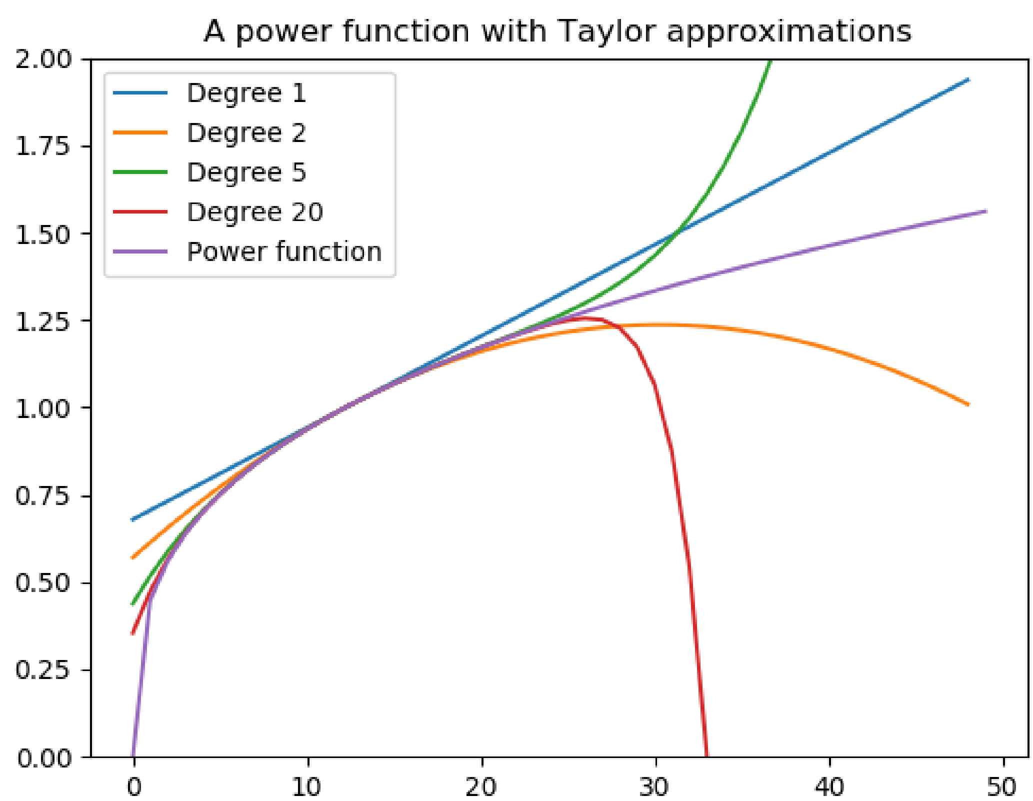

Figure 1: Taylor series on a power function

Figure 1 shows that the order one approximation does pretty well in approximating the function whereas orders 5 and 20 digress very sharply. The x-axis labels come from the fact that I used a grid of size 50 to approximate the function.

Question 1.2: Taylor series Ramp function

The objective is to use Taylor series approximations of orders 1,2,5 and 20 of the function $f(x) = \frac{x+|x|}{2}$ around a point $\bar{x} = 2$ on the domain (-2,6).

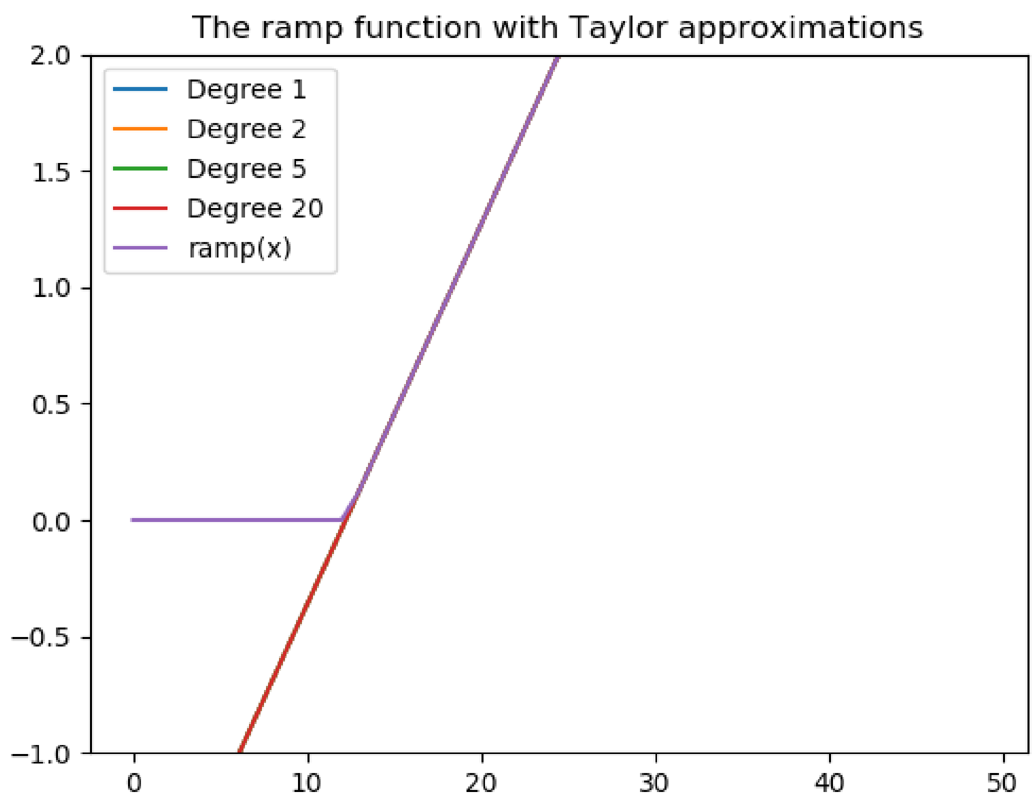

Figure 2: Taylor series on the Ramp function

Figure 2 shows the kink of the ramp function Is missed by the approximations. Since derivatives higher than degree 1 all produce a zero in this approximation we just get straight lines through the $\bar{x} = 2$ we set.

Question 2: Interpolation nodes and approximation using polynomials

For the entire exercise I will leave out the graphs and interpretations for the exponential example as there is a discontinuity in the function which I could not plot properly in Python.

Evenly spaced interpolation nodes and a cubic polynomial

The Ramp function

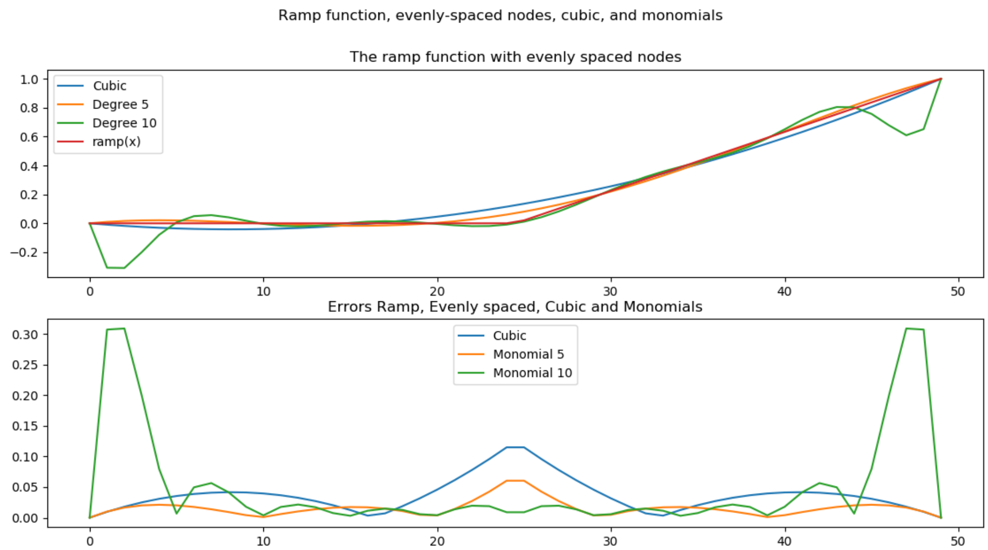

Figure 3: Ramp: Approximation using evenly-spaced nodes and monomials

For the ramp function approximation using evenly-spaced nodes and different polynomials we observe that the higher degree approximation tends to leave bigger errors at the interval ends whereas the lower degrees capture the overall structure quite well, the functions can be under or over the original.

The Runge function

Figure 4 shows that the approximation error again is larger at the ends of the interval. However, the higher monomial does a better job in capturing the original function in the center than do the other approximations. Outside the edges of the interval, the 10-degree monomial leads to low errors.

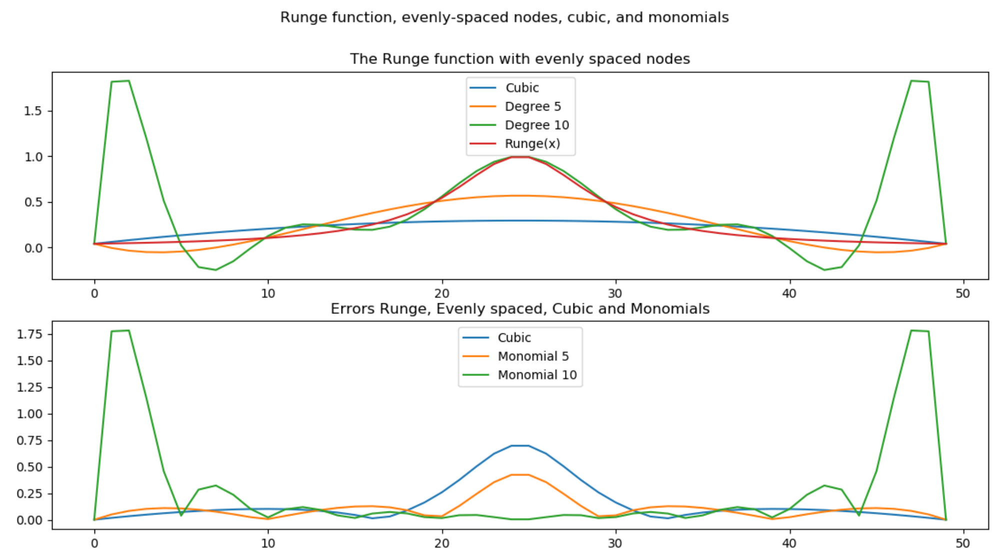

Figure 4: Runge: Approximation using evenly-spaced nodes and monomials

Chebyshev interpolation nodes and a cubic polynomial

The Ramp function

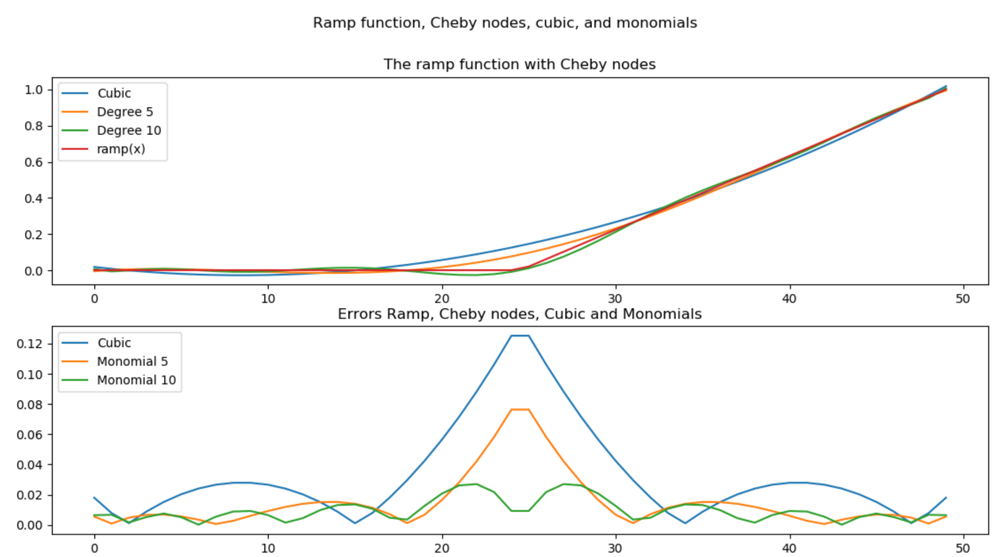

Figure 5: Ramp: Approximation using Cheby nodes and monomials

We observe that the approximation using Chebyshev nodes is much better at the ends of the interval but struggles to capture the kink of the ramp function. The higher the degree of the polynomial, however, the lower the variance of the errors becomes.

The Runge function

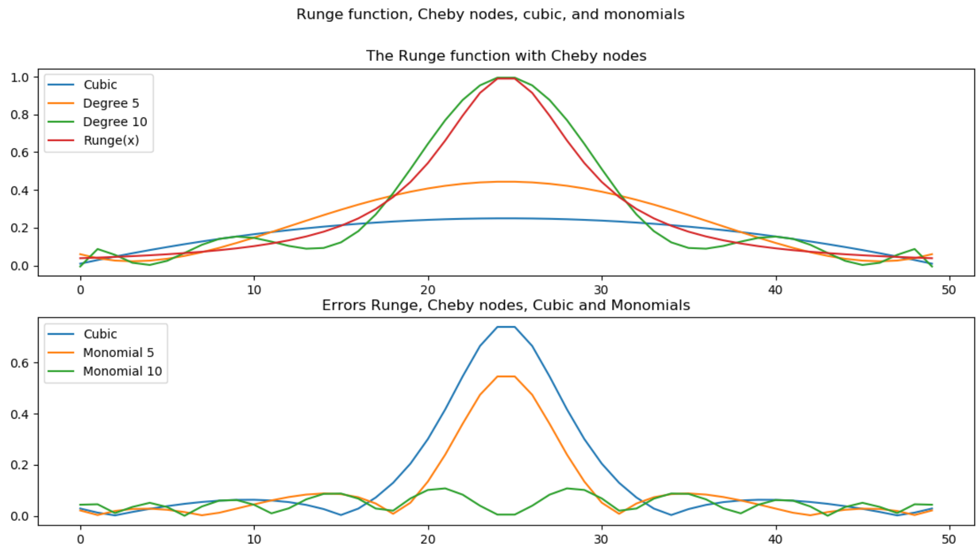

Figure 6: Runge: Approximation using Cheby nodes and monomials

The results for the Runge function are very similar in quality to those of the ramp function using Chebyshev nodes. Yet, the only polynomial that comes close to capturing the hump in the middle of the graph is the highest order polynomial. Again, we observe the way pattern of errors for the order-10 approximation.

Chebyshev interpolation nodes and Chebyshev polynomials

The Ramp function

The approximation using the Chebyshev nodes and polynomials are quite similar to those of Chebyshev nodes and monomials in quality. It appears that the nodes are more of an improvement over evenly-spaced ones than Chebyshev polynomials are over the monomials.

The Runge function

The same as above applies

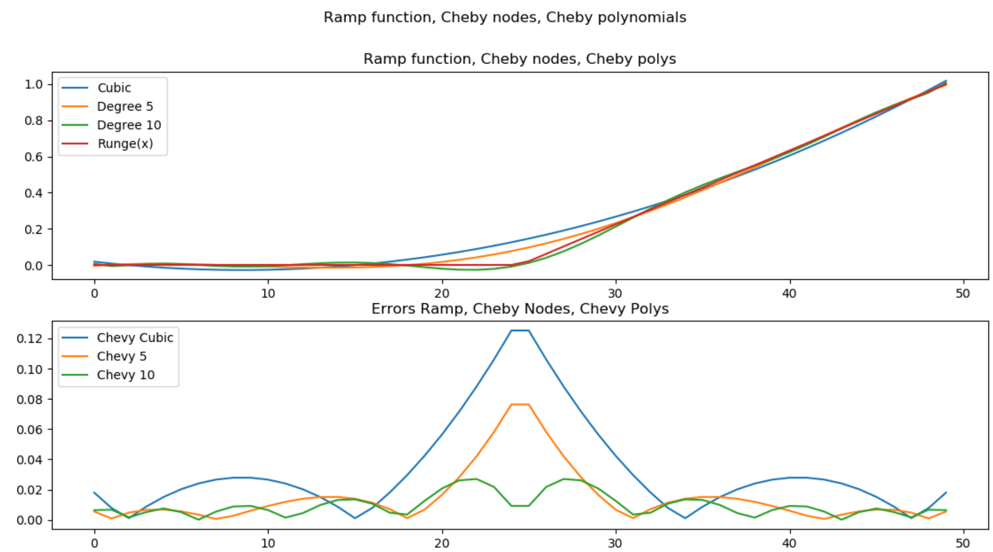

Figure 7: Ramp: Approximation using Cheby nodes and Cheby polynomials

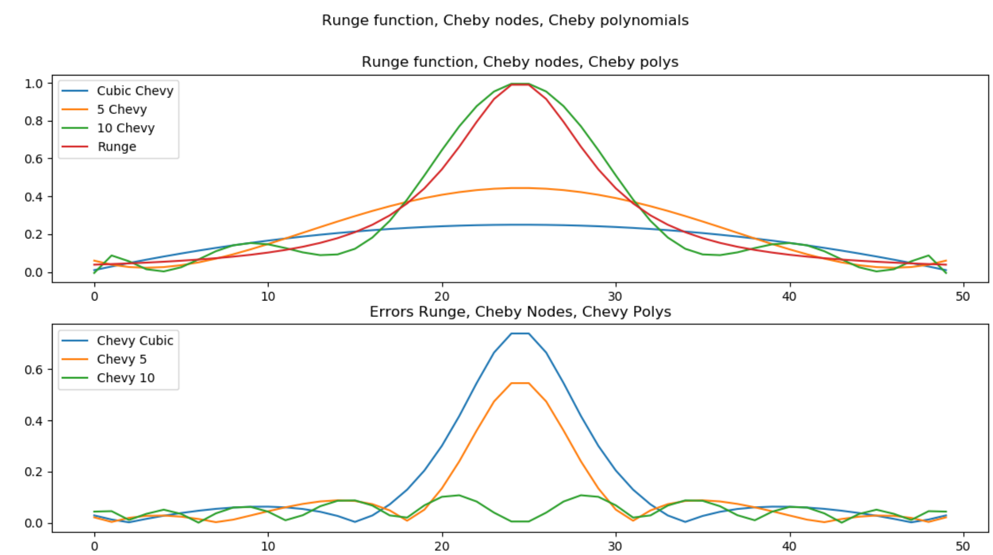

Figure 8: Runge: Approximation using Cheby nodes and Cheby polynomials

Question 2: Function Approximation CES

Derivation of ES

In order to obtain the ES of the given production function I use the following formula:

$$\sigma_{k,h} = \frac{dln(\frac{k}{h})}{dln\theta} \tag{1}$$

where

$$MRTS \equiv \theta = \frac{f'_h(h,k)}{f'_k(h,k)}$$

Straightforward derivation of the CES gives

$$\theta = \frac{\alpha}{1-\alpha}\left(\frac{k}{h}\right)^{\frac{1}{\sigma}}$$

Rearranging then give

$$\frac{k}{h} = \left(\theta\frac{1-\alpha}{\alpha}\right)^{\sigma}$$

Plugging into the above formula and applying the rules for differentiation of the natural logarithm yields

$$\sigma_{k,h} = \sigma\frac{\frac{1-\alpha}{\alpha}}{\frac{1-\alpha}{\alpha}\theta}\frac{\theta}{1}$$

which then reduces to the desired result.

Derivation of labor ratio

I use the standard definition of the labor share as

$$LS = \frac{wh}{pY} \tag{2}$$

Normalizing p = 1 and exploiting competitive markets means that

$$w = f'_h(h,k) = \alpha h^{\frac{-1}{\sigma}}\left[(1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}}$$

Applying the formula we obtain

$$\frac{\alpha h^{\frac{-1}{\sigma}} y^{\frac{1}{\sigma}} h}{y} = \alpha h^{\frac{\sigma-1}{\sigma}} y^{\frac{1-\sigma}{\sigma}} = \alpha\left(\frac{h}{y}\right)^{\frac{\sigma-1}{\sigma}}$$

Approximation using Chebyshev nodes and polynomials

In this section I will only compare approximations of degree 3 and 15 to the actual graph as otherwise the readability would suffer (even more). More degrees are readily available if needed.

The actual CES is depicted in the figure below.

We can plot isoquants for the given percentiles of the total production in the xy plane according to Figure 10

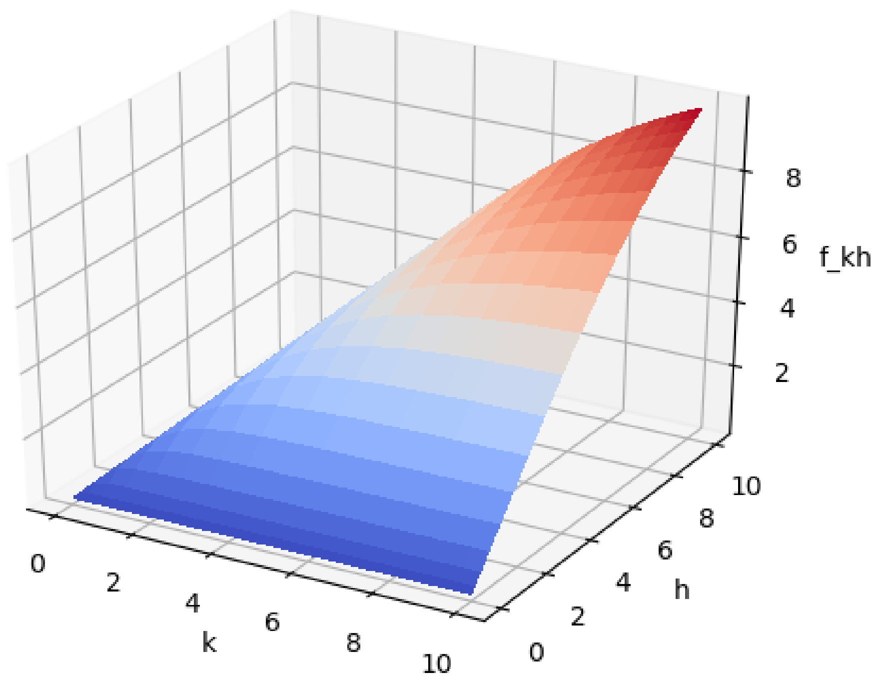

Figure 9: Actual CES production function

Approximation of degree 3

The Chebyshev approximation of degree 3 leads to Figure 11

The errors (as absolute values of the difference between the actual and the approximated function) are plotted in Figure 12.

For the given percentiles we plot the Isoquants of the approximation and errors to the actual percentiles in Figures 13 and 14:

Approximation of degree 15

The Chebyshev approximation of degree 15 leads to Figure 15

The errors (as absolute values of the difference between the actual and the approximated function) are plotted in Figure 16.

For the given percentiles we plot the Isoquants of the approximation and errors to the actual percentiles in Figures 17 and 18:

We can observe from the figures that the higher order polynomial leads to more accurate approximations of the CES function compared to the cubic approximation. We

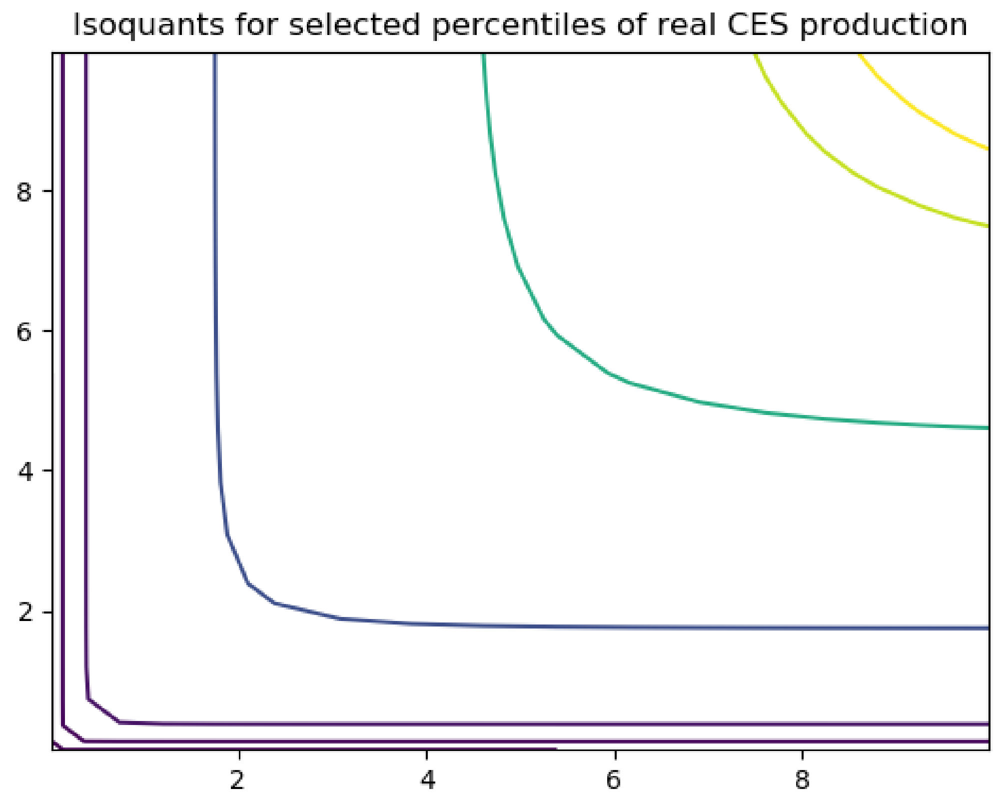

Figure 10: Actual CES Isoquants

notice that there appears to be relatively high error in the approximation for the lowest quantiles under consideration whereas higher quantiles do not appear problematic. I do not repeat the exercise for higher values of ES, but from analyzing my code and the corresponding graphs it becomes obvious that increasing ES leads to higher approximation errors throughout and clearly visible problems around the lower quantiles.

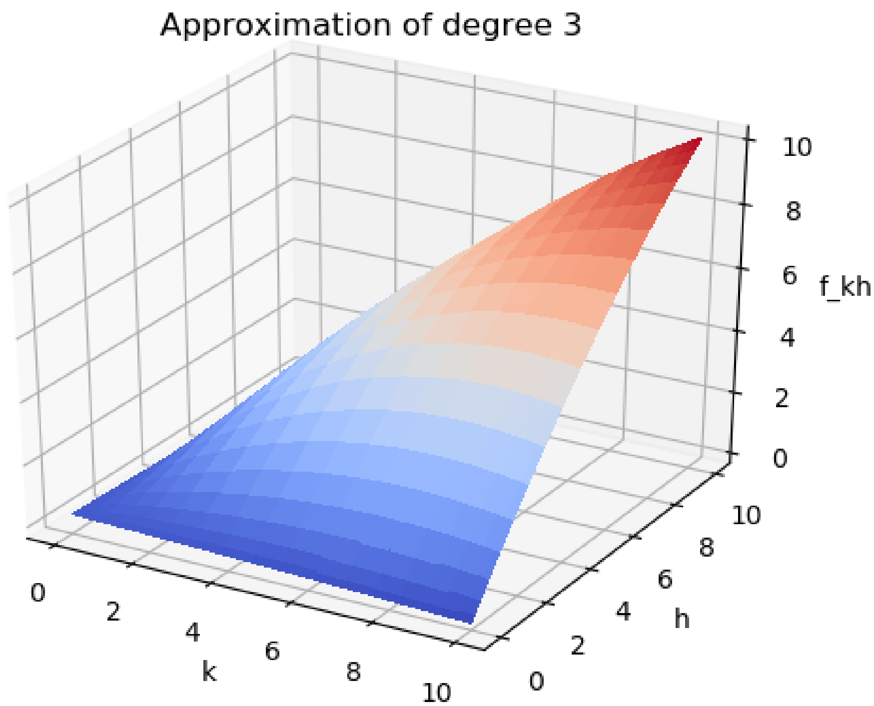

Figure 11: Approximation using 20 Cheby nodes and Cheby polynomial of degree 3

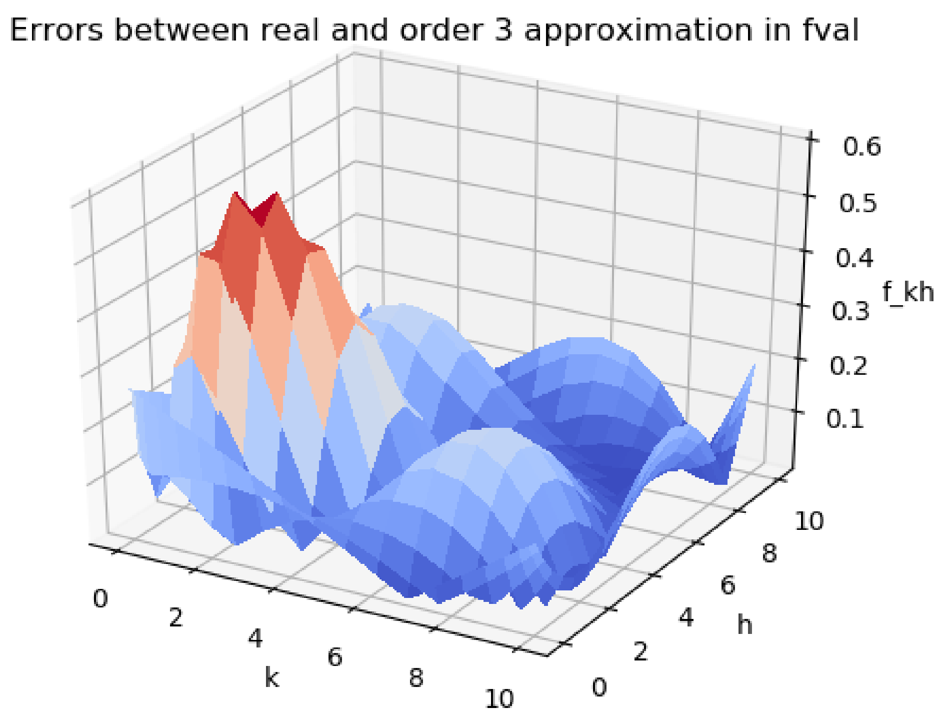

Figure 12: Approximation errors using Cheby polynomial of degree 3

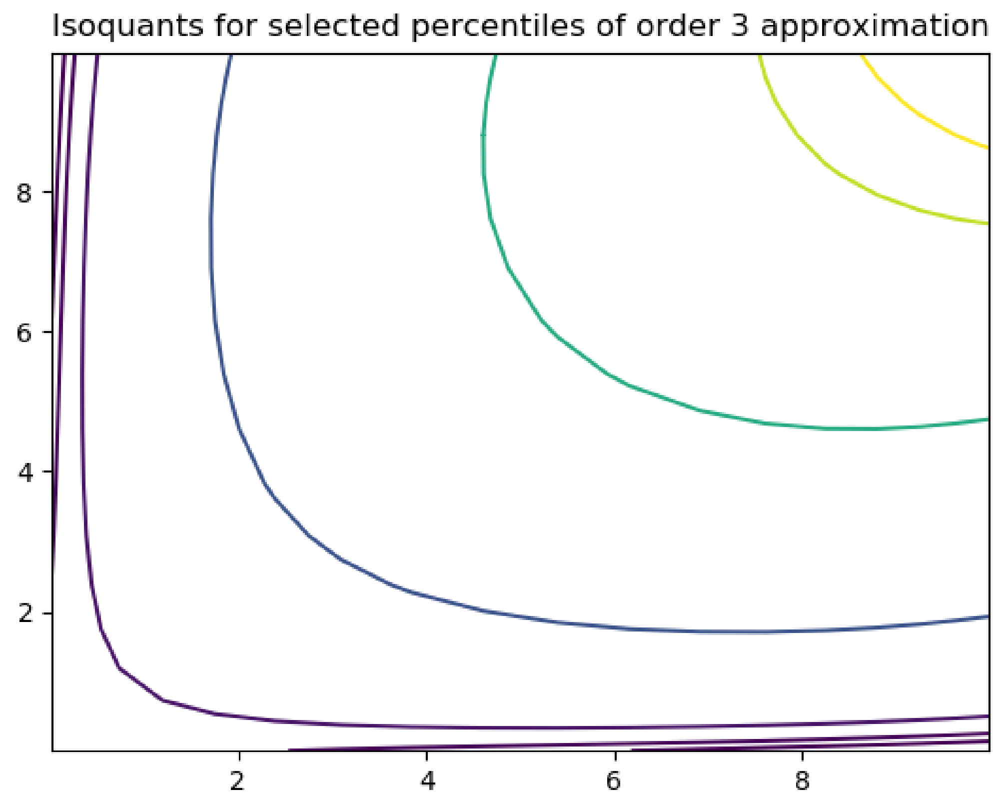

Figure 13: Isoquants for approximation using Cheby polynomial of order 3

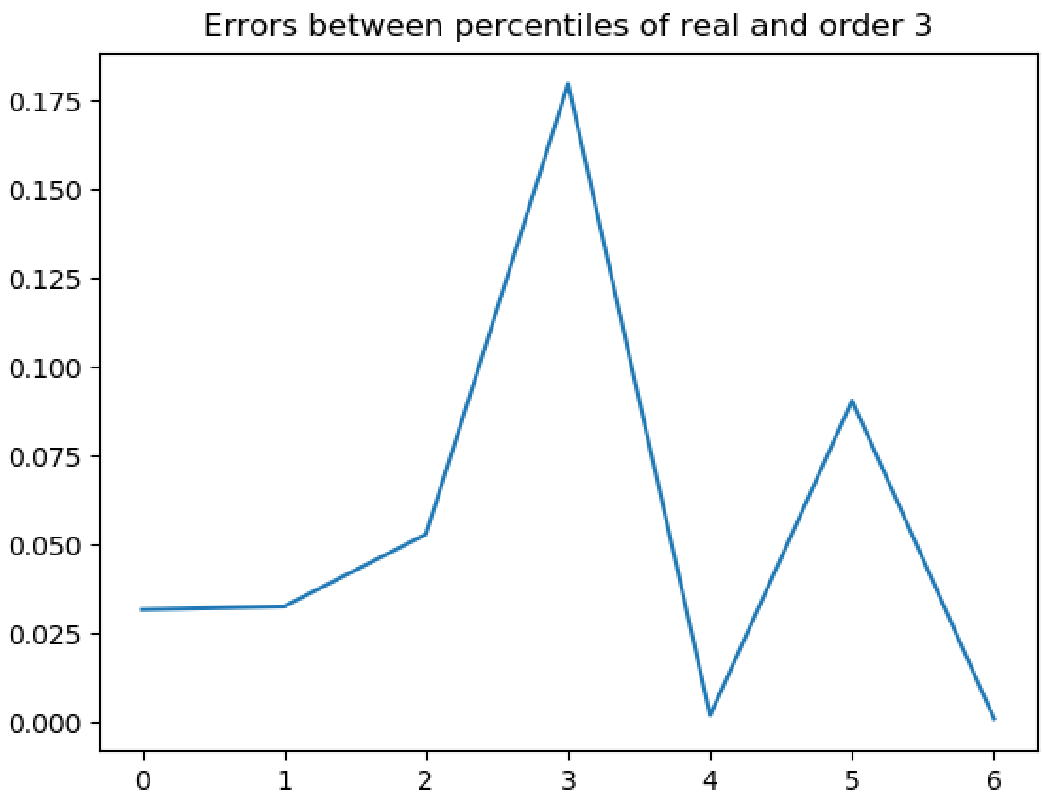

Figure 14: Errors between isoquants for Cheby polynomial of order 3

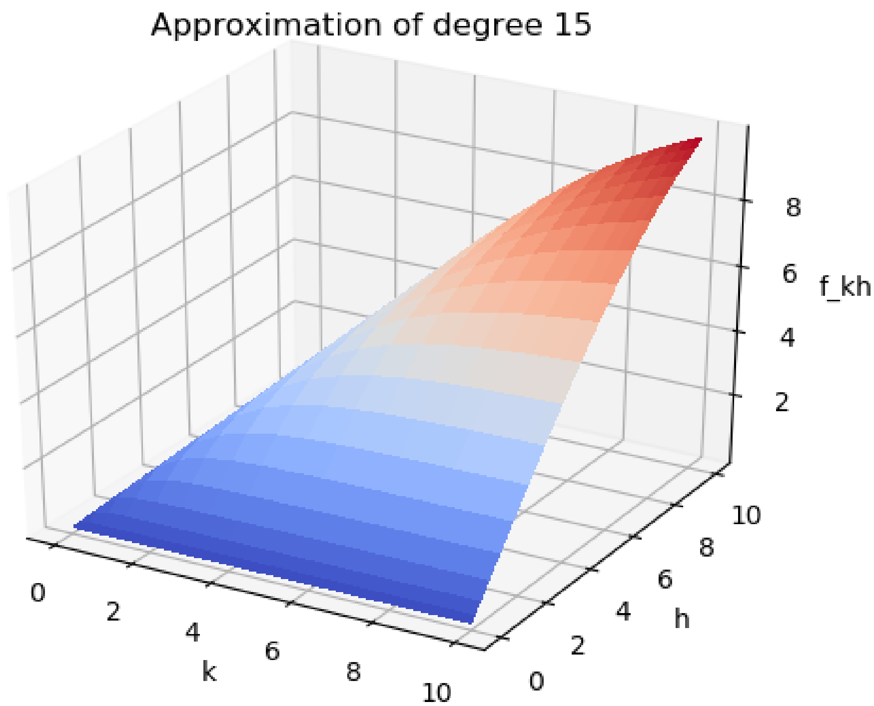

Figure 15: Approximation using 20 Cheby nodes and Cheby polynomial of degree 15

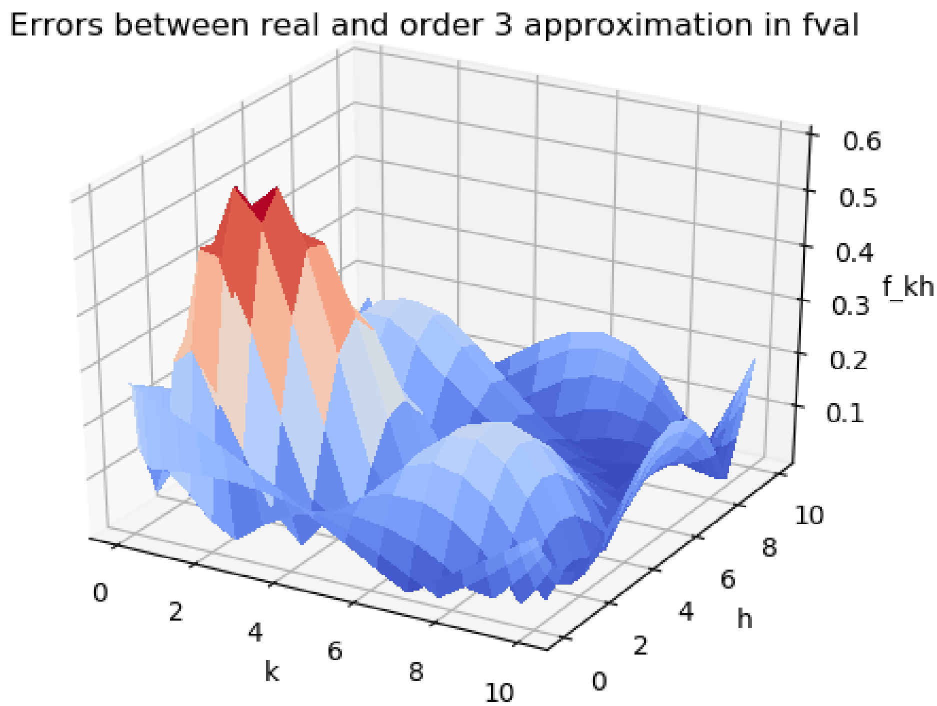

Figure 16: Approximation errors using Cheby polynomial of degree 15

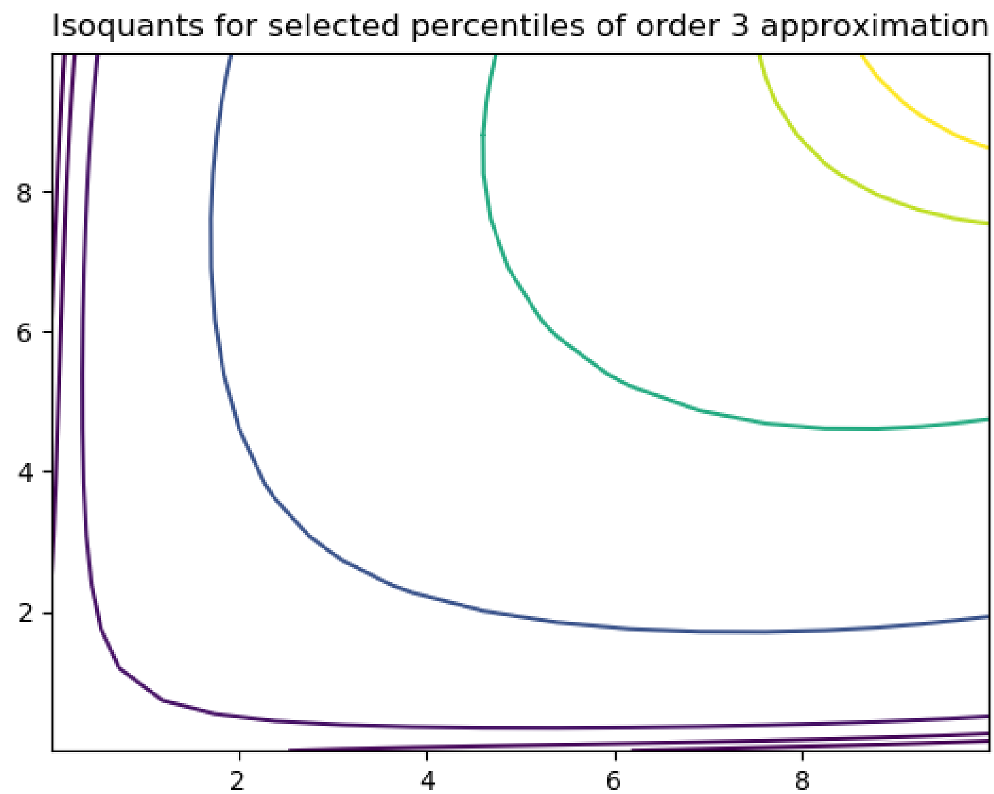

Figure 17: Isoquants for approximation using Cheby polynomial of order 15

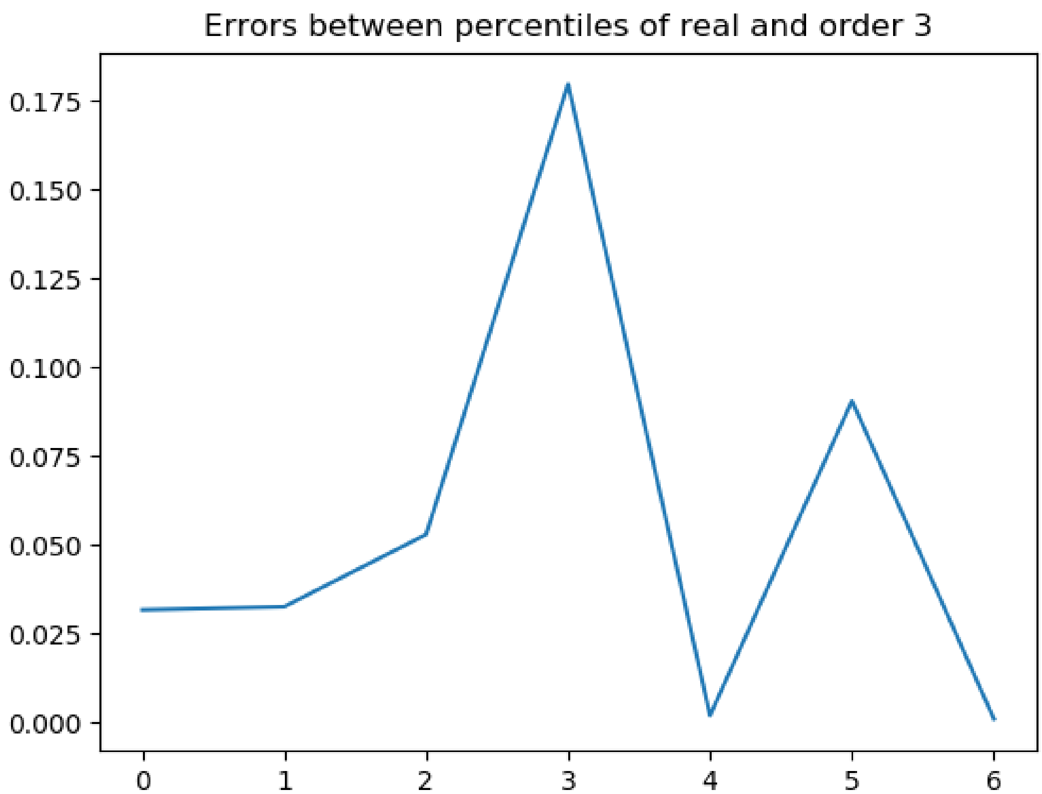

Figure 18: Errors between isoquants for Cheby polynomial of order 15