Assignment 1: Linear Regression (based on Normal Distribution) within GLM Framework

- Assignment 1.1: Membership of Normal Distribution in the Exponential Family
- Assignment 1.2: Deriving $E[Y]$ and $Var[Y]$ for Normal Distribution from $a(\theta)$

1.1

Normal:

$$f(y \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

Common form:

$$f(y \mid \theta, \phi) = e^{\left(\frac{y\theta - a(\theta)}{\phi} + b(y,\phi)\right)}$$

$$X = e^{\ln X}$$

rewrite it using this prop

$$f(y \mid \mu, \sigma^2) =$$

$$= \exp\left(\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\right)\right) =$$

$$= \exp\left(\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{(y-\mu)^2}{2\sigma^2}\right) =$$

$$= \exp\left(\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{y^2}{2\sigma^2} + \frac{2y\mu}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}\right) =$$

$$= \frac{\cancel{2}y\mu - (\mu^2/2)}{\cancel{2}\sigma^2} + \left(\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{y^2}{2\sigma^2}\right)$$

compare it with:

$$f(y \mid \theta, \phi) = \exp\left(\frac{y\theta - a(\theta)}{\phi} + b(y,\phi)\right)$$

So: $\mu = \theta \quad \phi = \sigma^2$

$$a(\theta) = \frac{\theta^2}{2}$$

$$b(y,\phi) = \ln\left(\frac{1}{\sqrt{2\pi\phi}}\right) - \frac{y^2}{2\phi}$$

1.2

$$E[Y] = a(\theta)' = \left(\frac{\mu^2}{2}\right)' = \mu$$

$$Var[Y] = \phi \cdot a(\theta)'' = \phi = \sigma^2 \quad \theta = \eta$$

Linear regr.:

$$\eta = \langle X, w \rangle \quad g(\mu) = \theta = \eta = \mu$$

$$\mu = E[Y] \Rightarrow \eta = \langle X, w \rangle = E[Y]$$

Assignment 2: Logistic Regression (based on Bernoulli Distribution) within GLM Framework

- Assignment 2.1: Membership of Bernoulli Distribution in the Exponential Family
- Assignment 2.2: Deriving $E[Y]$ and $Var[Y]$ for Bernoulli from $a(\theta)$

2.1

$$p(y \mid p) = p^y \cdot (1-p)^y, \; y \in \{0,1\}$$

common form:

$$f(y \mid \theta, \phi) = \exp\left(\frac{y\theta - a(\theta)}{\phi} + b(y,\phi)\right)$$

$$\exp(\ln(p^y \cdot (1-p)^{1-y})) =$$

$$\exp(y\ln p + (1-y)\ln(1-p)) =$$

$$= \exp(y\ln p + \ln(1-p) - y\ln(1-p)) =$$

$$= \exp(y\ln p - y\ln(1-p) + \ln(1-p)) =$$

$$= \exp(y(\ln p - \ln(1-p)) + \ln(1-p)) =$$

$$= \exp\left(y\ln\frac{p}{1-p} + \ln(1-p)\right)$$

$$\theta = \ln\frac{p}{1-p} \qquad a(\theta) = -\ln(1-p)$$

$$\phi = 1 \qquad b(\phi, y) = 0$$

Express a in terms of θ:

$$e^\theta = \frac{p}{1-p}$$

for this we need to replace p with θ

$$e^\theta(1-p) = p$$

$$e^\theta - e^\theta p = p$$

$$e^\theta = p + e^\theta p = p(1+e^\theta)$$

$$p = \frac{e^\theta}{1+e^\theta}$$

$$a(\theta) = -\ln\left(1 - \frac{e^\theta}{1+e^\theta}\right) = -\ln\left(\frac{1+e^\theta-e^\theta}{1+e^\theta}\right)$$

$$= -\ln\left(\frac{1}{1+e^\theta}\right) = \ln(1+e^\theta)$$

2.2

$$E[Y] = a'(\theta) = \frac{e^\theta}{1+e^\theta} = p$$

$$Var[Y] = \phi \cdot a''(\theta) =$$

$$= 1 \cdot \left(\frac{e^\theta}{1+e^\theta}\right)' = \frac{e^\theta(1+e^\theta) - e^{2\theta}}{(1+e^\theta)^2}$$

$$= \frac{e^\theta(1+e^\theta-e^\theta)}{(1+e^\theta)^2} = \underbrace{\frac{e^\theta}{1+e^\theta}}_{p} \cdot \underbrace{\frac{1}{1+e^\theta}}_{?} = p(1-p)$$

1-p

$$= \frac{1}{1+e^\theta} = \frac{\frac{e^\theta}{1+e^\theta}}{e^\theta} = \frac{p}{e^\theta} =$$

{ remember that $e^\theta = \frac{p}{1-p}$ }

$$= \underline{1-p}$$

Logistic regression:

$$\theta = \ln\frac{p}{1-p} \qquad \eta = \langle X, w \rangle \qquad \mu = E[Y] = p$$

$g(\mu)$ must equals η. We know that $g(\mu) = \theta = \ln\frac{p}{1-p}$

$\ln\frac{p}{1-p} = \langle X, w \rangle \leftarrow$ that's logit!

$$p = \sigma(\eta)$$

Assignment 3: Poisson Regression (based on Poisson Distribution) within GLM Framework

- Assignment 3.1: Membership of Poisson Distribution in the Exponential Family
- Assignment 3.2: Deriving $E[Y]$ and $Var[Y]$ for Poisson from $a(\theta)$