Chapter 1

Basics of finite-element method

Chapter outline

Various phenomena treated in science and engineering are often described in terms of differential equations formulated by using their continuum mechanics models. Solving differential equations under various conditions such as boundary or initial conditions leads to the understanding of the phenomena and can predict the future of the phenomena (determinism). Exact solutions for differential equations, however, are generally difficult to obtain. Numerical methods are adopted to obtain approximate solutions for differential equations. Among these numerical methods, those which approximate continua with infinite degree of freedom by a discrete body with finite degree of freedom are called "discrete analysis." Popular discrete analyses are the finite difference method, the method of weighted residuals, and the Rayleigh–Ritz method. Via these methods of discrete analysis, differential equations are reduced to simultaneous linear algebraic equations and thus can be solved numerically.

This chapter will explain first the method of weighted residuals and the Rayleigh–Ritz method which furnish a basis for the finite-element method (FEM) by taking examples of one-dimensional boundary-value problems, and then will compare

the results with those by the one-dimensional FEM in order to acquire a deeper understanding of the basis for the FEM.

1.1 Method of weighted residuals

Differential equations are generally formulated so as to be satisfied at any points which belong to regions of interest. The method of weighted residuals determines the approximate solution $\bar{u}$ to a differential equation such that the integral of the weighted error of the differential equation of the approximate function $\bar{u}$ over the region of interest is zero. In other words, this method determines the approximate solution which satisfies the differential equation of interest on average over the region of interest:

$$\begin{cases} L[u(x)] = f(x) \quad (a \le x \le b) \\ \text{BC (Boundary Conditions): } u(a) = u_a, \;\; u(b) = u_b \end{cases} \tag{1.1}$$

where L is a linear differential operator, $f(x)$ a function of x, and u_a and u_b the values of a function $u(x)$ of interest at the endpoints, or the one-dimensional boundaries of the region D. Now, let us suppose an approximate solution to the function u be

$$\bar{u}(x) = \phi_0(x) + \sum_{i=1}^{n} a_i \phi_i(x) \tag{1.2}$$

where ϕ_i are called trial functions $(i = 1, 2, \ldots, n)$ which are chosen arbitrarily as any function $\phi_0(x)$ and a_i some parameters which are computed so as to obtain a good "fit."

The substitution of $\bar{u}$ into Equation (1.1) makes the right-hand side non-zero but gives some error R:

$$L[\bar{u}(x)] - f(x) = R \tag{1.3}$$

The method of weighted residuals determines $\bar{u}$ such that the integral of the error R over the region of interest weighted by arbitrary functions w_i $(i = 1, 2, \ldots, n)$ is zero, i.e., the coefficients a_i in Equation (1.2) are determined so as to satisfy the following equation:

$$\int_D w_i R \, dv = 0 \tag{1.4}$$

where D is the region considered.

1.1.1 Sub-domain method (finite volume method)

The choice of the following weighting function brings about the sub-domain method or finite-volume method.

$$w_i(x) = \begin{cases} 1 & (\text{for } x \in D) \\ 0 & (\text{for } x \notin D) \end{cases} \tag{1.5}$$

EXAMPLE 1.1

Consider a boundary-value problem described by the following one-dimensional differential equation:

$$\begin{cases} \dfrac{d^2u}{dx^2} - u = 0 \quad (0 \le x \le 1) \\ \text{BC: } u(0) = 0 \quad u(1) = 1 \end{cases} \tag{1.6}$$

The linear operator $L[\cdot]$ and the function $f(x)$ in Equation (1.6) are defined as follows:

$$L[\cdot] \equiv \frac{d^2(\,\cdot\,)}{dx^2} \quad f(x) \equiv u(x) \tag{1.7}$$

For simplicity, let us choose the power series of x as the trial functions ϕ_i, i.e.,

$$\bar{u}(x) = \sum_{i=0}^{n+1} c_i x^i \tag{1.8}$$

For satisfying the required boundary conditions,

$$c_0 = 0, \quad \sum_{i=1}^{n+1} c_i = 1 \tag{1.9}$$

so that

$$\bar{u}(x) = x + \sum_{i=1}^{n} A_i (x^{i+1} - x) \tag{1.10}$$

If the second term of the right-hand side of Equation (1.10) is chosen as a first-order approximate solution

$$\bar{u}_1(x) = x + A_1(x^2 - x) \tag{1.11}$$

the error or residual is obtained as

$$R = \frac{d^2\bar{u}}{dx^2} - \bar{u} = -A_1x^2 + (A_1 - 1)x + 2A_1 \neq 0 \tag{1.12}$$

$$\int_0^1 w_i R\, dx = \int_0^1 1 - \left[-A_1x^2 + (A_1 - 1)x + 2A_1\right] dx = \frac{13}{6}A_1 - \frac{1}{2} = 0 \tag{1.13}$$

Consequently, the first-order approximate solution is obtained as

$$\bar{u}_1(x) = x + \frac{3}{13}x(x - 1) \tag{1.14}$$

(Example 1.1 continued)

which agrees well with the exact solution

$$u(x) = \frac{e^x - e^{-x}}{e - e^{-1}} \tag{1.15}$$

as shown by the dotted and the solid lines in Figure 1.1.

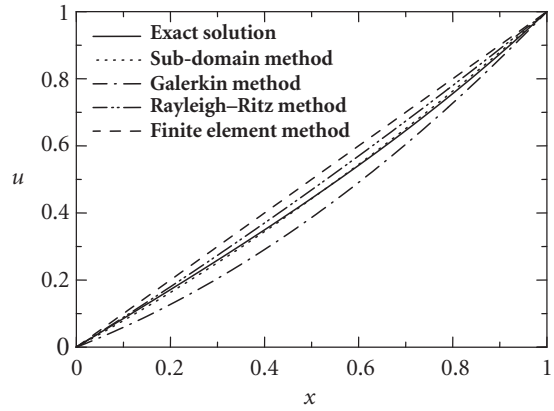

Figure 1.1 Comparison of the results obtained by various kinds of discrete analyses.

1.1.2 GALERKIN METHOD

When the weighting function w_i in Equation (1.4) is chosen equal to the trial function ϕ_i, the method is called the Galerkin method, i.e.,

$$w_i(x) = \phi_i(x) \quad (i = 1, 2, \ldots, n) \tag{1.16}$$

and thus Equation (1.4) is changed to

$$\int_D \phi_i R \, dv = 0 \tag{1.17}$$

This method determines the coefficients a_i by directly using Equation (1.17) or by integrating it by parts.

EXAMPLE 1.2

Let us solve the same boundary-value problem as described by Equation (1.6) in the preceding Section 1.1.1 by the Galerkin method.

The trial function ϕ_i is chosen as the weighting function w_i in order to find the first-order approximate solution:

$$w_1(x) = \phi_1(x) = x(x - 1) \tag{1.18}$$

(Example 1.2 continued)

Integrating Equation (1.4) by parts,

$$\int_0^1 w_i R\,dx = \int_0^1 w_i \left(\frac{d^2\bar{u}}{dx^2} - \bar{u}\right) dx = \left[w_i \frac{d\bar{u}}{dx}\right]_0^1 - \int_0^1 \frac{dw_i}{dx}\frac{d\bar{u}}{dx}dx - \int_0^1 w_i \bar{u}\,dx = 0 \tag{1.19}$$

is obtained. Choosing $\bar{u}_1$ in Equation (1.11) as the approximate solution $\bar{u}$, the substitution of Equation (1.18) into (1.19) gives

$$\int_0^1 \phi_1 R\,dx = -\int_0^1 \frac{d\phi_1}{dx}\frac{d\bar{u}}{dx}dx - \int_0^1 \phi_1 \bar{u}\,dx = -\int_0^1 (2x-1)[1 + A_1(2x-1)]dx$$
$$-\int_0^1 (x^2 - x)[1 + A_1(x^2 - x)]dx = -\frac{A_1}{3} + \frac{1}{12} - \frac{A_1}{30} = 0 \tag{1.20}$$

Thus, the following approximate solution is obtained:

$$\bar{u}_1(x) = x + \frac{5}{22}x(x-1) \tag{1.21}$$

Figure 1.1 shows that the approximate solution obtained by the Galerkin method also agrees well with the exact solution throughout the region of interest.

1.2 Rayleigh–Ritz method

When there exists the functional which is equivalent to a given differential equation, the Rayleigh–Ritz method can be used.

Let us consider the example problem illustrated in Figure 1.2 where a particle having a mass of M slides from point P_0 to lower point P_1 along a curve in a vertical plane under the force of gravity. The time t that the particle needs for sliding from the points P_0 to P_1 varies with the shape of the curve denoted by $y(x)$ which connects the two points. Namely, the time t can be considered as a kind of function $t = F[y]$ which is determined by a function $y(x)$ of an independent variable x. The function

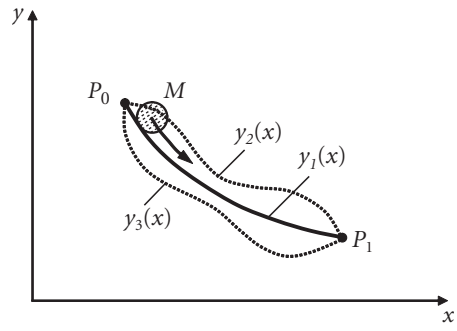

Figure 1.2 Particle M sliding from point P_0 to lower point P_1 under gravitational force.

of a function $F[y]$ is called a functional. The method for determining the maximum or the minimum of a given functional is called the variational method. In the case of Figure 1.2, the method determines the shape of the curve $y(x)$ which gives the possible minimum time $t_{\min}$ in which the particle slides from P_0 to P_1.

The principle of the virtual work or the minimum potential energy in the field of the solid mechanics is one of the variational principles which guarantee the existence of the function which makes the functional minimum or maximum. For unsteady thermal conductivity problems and viscous flow problems, no variational principle can be established; in such a case, the method of weighting residuals can be adopted instead.

Now, let $\Pi[u]$ be the functional which is equivalent to the differential equation in Equation (1.1). The Rayleigh–Ritz method assumes that an approximate solution $\bar{u}(x)$ of $u(x)$ is a linear combination of trial functions ϕ_i as shown in the following equation:

$$\bar{u}(x) = \sum_{i=1}^{n} a_i \phi_i(x) \tag{1.22}$$

where a_i $(i = 1, 2, \ldots, n)$ are arbitrary constants, ϕ_i are C^0-class functions which have continuous first-order derivatives for $a \leq x \leq b$ and are chosen such that the following boundary conditions are satisfied:

$$\sum_{i=1}^{n} a_i \phi_i(a) = u_a \quad \sum_{i=1}^{n} a_i \phi_i(b) = u_b \tag{1.23}$$

The approximate solution $\bar{u}(x)$ in Equation (1.22) is the function which makes the functional $\Pi[u]$ take stationary value and is called the admissible function.

Next, integrating the functional Π after substituting Equation (1.22) into the functional, the constants a_i are determined by the stationary conditions:

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad (i = 1, 2, \ldots, n) \tag{1.24}$$

The Rayleigh–Ritz method determines the approximate solution $\bar{u}(x)$ by substituting the constants a_i into Equation (1.22). It is generally understood to be a method which determines the coefficients a_i so as to make the distance between the approximate solution $\bar{u}(x)$ and the exact one $u(x)$ minimum.

EXAMPLE 1.3

Let us solve again the boundary-value problem described by Equation (1.6) by the Rayleigh–Ritz method. The functional equivalent to the first equation of Equation (1.6) is written as

$$\Pi[u] = \int_0^1 \left[\frac{1}{2} \left(\frac{du}{dx} \right)^2 + \frac{1}{2} u^2 \right] dx \tag{1.25}$$

Equation (1.25) is obtained by intuition, but Equation (1.25) is shown to really give the functional of the first equation of Equation (1.6) as follows: first, let us take the first variation of Equation (1.25) in order to obtain the stationary value of the

equation:

$$\delta\Pi = \int_0^1 \left[\frac{du}{dx} \delta\left(\frac{du}{dx}\right) + u\delta u \right] dx \tag{1.26}$$

Then, integrating the above equation by parts, we have

$$\begin{aligned}\delta\Pi &= \int_0^1 \left(\frac{du}{dx}\frac{d\delta u}{dx} + u\delta u \right) dx = \left[\frac{du}{dx}\delta u \right]_0^1 - \int_0^1 \left[\frac{d}{dx}\left(\frac{du}{dx}\right)\delta u - u\delta u \right] dx \\ &= -\int_0^1 \left(\frac{d^2u}{dx^2} - u \right) \delta u \, dx \end{aligned} \tag{1.27}$$

For satisfying the stationary condition that $\delta\Pi = 0$, the rightmost-hand side of Equation (1.27) should be identically zero over the interval considered ($a \le x \le b$), so that

$$\frac{d^2u}{dx^2} - u = 0 \tag{1.28}$$

This is exactly the same as the first equation of Equation (1.6).

Now, let us consider the following first-order approximate solution $\bar{u}_1$ which satisfies the boundary conditions:

$$\bar{u}_1(x) = x + a_1 x(x-1) \tag{1.29}$$

Substitution of Equation (1.29) into (1.25) and integration of Equation (1.25) lead to

$$\Pi[\bar{u}_1] = \int_0^1 \left\{ \frac{1}{2}[1 + a_1(2x-1)]^2 + \frac{1}{2}\left[x + a_1(x^2 - x)\right]^2 \right\} dx = \frac{2}{3} - \frac{1}{12}a_1 + \frac{1}{3}a_1^2 \tag{1.30}$$

Since the stationary condition for Equation (1.30) is written by

$$\frac{\partial\Pi}{\partial a_1} = -\frac{1}{12} + \frac{2}{3}a_1 = 0 \tag{1.31}$$

the first-order approximate solution can be obtained as follows:

$$\bar{u}_1(x) = x + \frac{1}{8}x(x-1) \tag{1.32}$$

Figure 1.1 shows that the approximate solution obtained by the Rayleigh–Ritz method agrees well with the exact solution throughout the region considered.

1.3 Finite-element method

There are two ways for the formulation of the FEM: one is based on the direct variational method (such as the Rayleigh–Ritz method) and the other on the method of weighted residuals (such as the Galerkin method). In the formulation based on the variational method, the fundamental equations are derived from the stationary conditions of the functional for the boundary-value problems. This formulation has an advantage that the process of deriving functionals is not necessary, so it is

easy to formulate the FEM based on the method of the weighted residuals. In the formulation based on the variational method, however, it is generally difficult to derive the functional except for the case where the variational principles are already established as in the case of the principle of the virtual work or the principle of the minimum potential energy in the field of the solid mechanics.

This section will explain how to derive the fundamental equations for the FEM based on the Galerkin method.

Let us consider again the boundary-value problem stated by Equation (1.1):

$$\begin{cases} L[u(x)] = f(x) \quad (a \le x \le b) \\ \text{BC (Boundary Conditions): } u(a) = u_a \quad u(b) = u_b \end{cases} \tag{1.33}$$

First, divide the region of interest ($a \le x \le b$) into n subregions as illustrated in Figure 1.3. These subregions are called "elements" in the FEM.

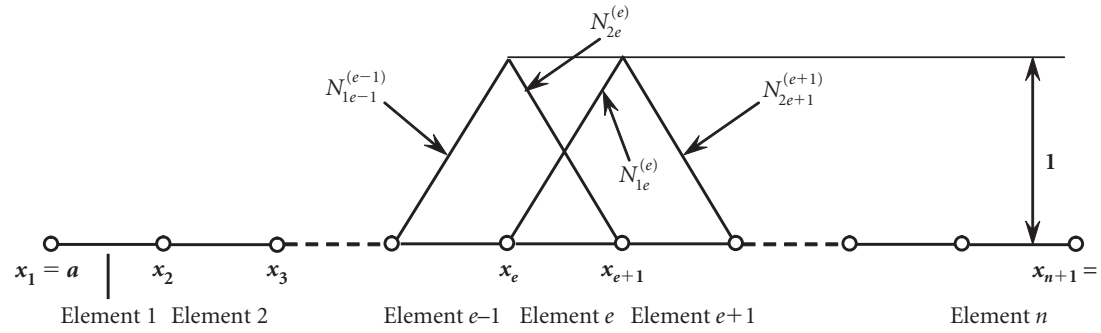

Figure 1.3 Discretization of the domain to analyze by finite elements and their interpolation functions.

Now, let us assume that an approximate solution $\bar{u}$ of u can be expressed by a piecewise linear function which forms a straight line in each subregion, i.e.,

$$\bar{u}(x) = \sum_{i=1}^{n+1} u_i N_i(x) \tag{1.34}$$

where u_i represents the value of u in element "e" at a boundary point, or a nodal point "i" between two one-dimensional elements. Functions $N_i(x)$ are the following piecewise linear functions and are called interpolation or shape functions of the nodal point "i:"

$$\begin{cases} N_{1e}^{(e)} = \dfrac{x_{e+1} - x}{x_{e+1} - x_e} = \dfrac{x_{2e} - x}{x_{2e} - x_{1e}} = \dfrac{h^{(e)} - \xi}{h^{(e)}} \\ N_{2e}^{(e)} = \dfrac{x - x_e}{x_{e+1} - x_e} = \dfrac{x - x_{1e}}{x_{2e} - x_{1e}} = \dfrac{\xi}{h^{(e)}} \end{cases} \tag{1.35}$$

where e ($e = 1, 2, \ldots, n$) denotes the element number, x_i the global coordinate of the nodal point i ($i = 1, \ldots, e-1, e, \ldots n, n+1$), $N_{ie}^{(e)}$ the value of the interpolation function at the nodal point i_e ($i_e = 1_e, 2_e$) which belongs to the eth element, 1_e and 2_e

the number of two nodal points of the eth element. Symbol ξ is the local coordinate of an arbitrary point in the eth element, $\xi = x - x_e = x - x_{1e}$ $(0 \le \xi \le h^{(e)})$, $h^{(e)}$ is the length of the eth element, and $h^{(e)}$ is expressed as $h^{(e)} = x_{e+1} - x_e = x_{2e} - x_{1e}$.

As the interpolation function, the piecewise linear or quadric function is often used. Generally speaking, the quadric interpolation function gives better solutions than the linear one.

The Galerkin method-based FEM adopts the weighting functions $w_i(x)$ equal to the interpolation functions $N_i(x)$, i.e.,

$$w_i(x) = N_i(x) \quad (i = 1, 2, \ldots, n+1) \tag{1.36}$$

Thus, Equation (1.4) becomes

$$\int_D N_i R \, dv = 0 \tag{1.37}$$

In the FEM, a set of simultaneous algebraic equations for unknown variables of $u(x)$ at the ith nodal point u_i and those of its derivatives du/dx, $(du/dx)_i$ are derived by integrating Equation (1.37) by parts and then by taking boundary conditions into consideration. The simultaneous equations can be easily solved by digital computers to determine the unknown variables u_i and $(du/dx)_i$ at all the nodal points in the region considered.

EXAMPLE 1.4

Let us solve the boundary-value problem stated in Equation (1.6) by FEM.
First, the integration of Equation (1.37) by parts gives

$$\int_0^1 w_i R \, dx = \int_0^1 w_i \left(\frac{d^2\bar{u}}{dx^2} - \bar{u} \right) dx = \left[w_i \frac{d\bar{u}}{dx} \right]_0^1 - \int_0^1 \left(\frac{dw_i}{dx} \frac{d\bar{u}}{dx} + w_i \bar{u} \right) dx = 0$$
$$(i = 1, 2, \ldots, n+1) \tag{1.38}$$

Then, the substitution of Equations (1.34) and (1.36) into Equation (1.38) gives

$$\sum_{j=1}^{n+1} \int_0^1 \left(\frac{dN_i}{dx} \frac{dN_j}{dx} + N_i N_j \right) u_j dx - \left[N_i \frac{d\bar{u}}{dx} \right]_0^1 = 0 \quad (i = 1, 2, \ldots, n+1) \tag{1.39}$$

Equation (1.39) is a set of simultaneous linear algebraic equations composed of $(n+1)$ nodal values u_i of the solution u and also $(n+1)$ nodal values $(du/dx)_i$ of its derivative du/dx. The matrix notation of the simultaneous equations above is written in a simpler form as follows:

$$[K_{ij}]\{u_j\} = \{f_i\} \tag{1.40}$$

where $[K_{ij}]$ is a square matrix of $(n+1)$ by $(n+1)$, $\{f_i\}$ is a column vector of $(n+1)$ by 1, and the components of the matrix and the vector K_{ij} and f_i are expressed as

$$\begin{cases} K_{ij} \equiv \displaystyle\int_0^1 \left(\frac{dN_i}{dx} \frac{dN_j}{dx} + N_i N_j \right) dx & (1 \le i, j \le n+1) \\ f_i \equiv \left[N_i \dfrac{d\bar{u}}{dx} \right]_0^1 & (1 \le i \le n+1) \end{cases} \tag{1.41}$$

1.3.1 One-element case

As the first example, let us compute Equation (1.37) by regarding the whole region as one finite element as shown in Examples 1.1 through 1.3. From Equations (1.34) and (1.35), since $x_1 = 0$ and $x_2 = 1$, the approximate solution $\bar{u}$ and the interpolation functions N_i $(i = 1, 2)$ become

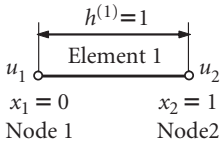

Figure 1.4 One-element model of one-dimensional FEM.

$$\bar{u}(x) = u_1 N_1 + u_2 N_2 \tag{1.42}$$

$$\begin{cases} N_1 = N_{11}^{(1)} = \dfrac{x_2 - x}{x_2 - x_1} = 1 - x \\ N_2 = N_{21}^{(1)} = \dfrac{x - x_1}{x_2 - x_1} = x \end{cases} \tag{1.43}$$

Thus, from Equation (1.41),

$$K_{ij} \equiv K_{ij}^{(1)} \equiv \int_0^1 \left(\frac{dN_i}{dx}\frac{dN_j}{dx} + N_i N_j \right) dx = \begin{cases} 4/3 & (i = j) \\ -5/6 & (i \neq j) \end{cases}$$

$$f_i \equiv \left[N_i \frac{d\bar{u}}{dx} \right]_0^1 = \begin{cases} -\left.\dfrac{d\bar{u}}{dx}\right|_{x=0} \\ \left.\dfrac{d\bar{u}}{dx}\right|_{x=1} \end{cases} \tag{1.44}$$

The global simultaneous equations are obtained as

$$\begin{bmatrix} \frac{4}{3} & -\frac{5}{6} \\ -\frac{5}{6} & \frac{4}{3} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -\left.\dfrac{d\bar{u}}{dx}\right|_{x=0} \\ \left.\dfrac{d\bar{u}}{dx}\right|_{x=1} \end{Bmatrix} \tag{1.45}$$

According to the boundary conditions, $u_1 = 0$ and $u_2 = 1$ in the left-hand side of the above equations are known variables, whereas $(du/dx)_{x=0}$ and $(du/dx)_{x=1}$ in the left-hand side are unknown variables. The substitution of the boundary conditions into Equation (1.45) directly gives the nodal values of the approximate solution, i.e.,

$$\begin{cases} \left.\dfrac{d\bar{u}}{dx}\right|_{x=0} = 0.8333 \\ \left.\dfrac{d\bar{u}}{dx}\right|_{x=1} = 1.3333 \end{cases} \tag{1.46}$$

which agrees well with those of the exact solution

$$\begin{cases} \left.\dfrac{du}{dx}\right|_{x=0} = \dfrac{2}{e-e^{-1}} = 0.8509 \\ \left.\dfrac{du}{dx}\right|_{x=1} = \dfrac{e+e^{-1}}{e-e^{-1}} = 1.3130 \end{cases} \tag{1.47}$$

The approximate solution in this example is determined as

$$\bar{u}(x) = x \tag{1.48}$$

and agrees well with the exact solution throughout the whole region of interest as depicted in Figure 1.1.

1.3.2 THREE-ELEMENT CASE

In this section, let us compute the approximate solution $\bar{u}$ by dividing the whole region considered into three subregions having the same length as shown in Figure 1.5. From Equations (1.34) and (1.35), the approximate solution $\bar{u}$ and the interpolation functions N_i $(i = 1, 2)$ are written as

$$\bar{u}(x) = \sum_{i=1}^{4} u_i N_i \tag{1.49}$$

$$\begin{cases} N_{1e} = \dfrac{x_{2e} - x}{x_{2e} - x_{1e}} = \dfrac{h^{(e)} - \xi}{h^{(e)}} \\ N_{2e} = \dfrac{x - x_{1e}}{x_{2e} - x_{1e}} = \dfrac{\xi}{h^{(e)}} \end{cases} \tag{1.50}$$

where $h^{(e)} = 1/3$ and $0 \le \xi \le 1/3$ $(e = 1, 2, 3)$.

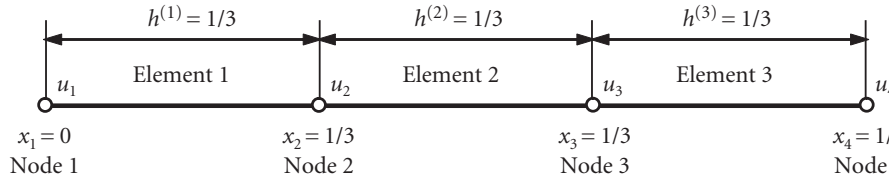

Figure 1.5 Three-element model of a one-dimensional FEM.

By calculating all the components of the K-matrix in Equation (1.41), the following equation is obtained:

$$K_{ij}^{(e)} \equiv \int_0^1 \left(\frac{dN_i^{(e)}}{dx} \frac{dN_j^{(e)}}{dx} + N_i^{(e)} N_j^{(e)} \right) dx$$

$$= \begin{cases} \dfrac{1}{h^{(e)}} + \dfrac{h^{(e)}}{3} = \dfrac{28}{9} & (i = j \text{ and } i, j = 1e, 2e) \\ -\dfrac{1}{h^{(e)}} + \dfrac{h^{(e)}}{6} = -\dfrac{53}{18} & (i \neq j \text{ and } i, j = 1e, 2e) \\ 0 & (i, j \neq 1e, 2e) \end{cases} \tag{1.51a}$$

The components relating to the first derivative of the function u in Equation (1.41) are calculated as follows:

$$f_i \equiv \left[N_i \frac{d\bar{u}}{dx} \right]_0^1 = \begin{cases} -\left. \dfrac{d\bar{u}}{dx} \right|_{x=0} & (i = 1) \\ 0 & (i = 2, 3) \\ \left. \dfrac{d\bar{u}}{dx} \right|_{x=1} & (i = 4) \end{cases} \tag{1.51b}$$

The coefficient matrix in Equation (1.51a) calculated for each element is called "element matrix" and the components of the matrix are obtained as follows:

$$\left[K_{ij}^{(1)} \right] = \begin{bmatrix} \dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{3} & -\dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{6} & 0 & 0 \\ -\dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{6} & \dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.52a}$$

$$\left[K_{ij}^{(2)} \right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{h^{(2)}} + \dfrac{h^{(2)}}{3} & -\dfrac{1}{h^{(2)}} + \dfrac{h^{(2)}}{6} & 0 \\ 0 & -\dfrac{1}{h^{(2)}} + \dfrac{h^{(2)}}{6} & \dfrac{1}{h^{(2)}} + \dfrac{h^{(2)}}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.52b}$$

$$\left[K_{ij}^{(3)} \right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{3} & -\dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{6} \\ 0 & 0 & -\dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{6} & \dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{3} \end{bmatrix} \tag{1.52c}$$

From Equations (1.52a) through (1.52c) above, it is concluded that only the components of the element matrix relating to the nodal points which belong to the corresponding element are non-zero and that the other components are zero. Namely, for example, element 2 is composed of nodal points 2 and 3 and among the components of the element matrix only $K_{22}^{(2)}$, $K_{23}^{(2)}$, $K_{32}^{(2)}$, and $K_{33}^{(2)}$ are non-zero and the other components are zero. The superscript (2) of the element matrix components above indicates that the components are computed in element 2, and the subscripts indicate that the components are calculated for nodal points 2 and 3 in element 2.

A matrix which relates all the known and the unknown variables for the problem concerned is called the global matrix. The global matrix can be obtained simply by summing up Equations (1.52a) through (1.52c) as follows:

$$
\begin{aligned}
\left[K_{ij}\right] &= \left[\sum_{e=1}^{3} K_{ij}^{(e)}\right] \\
&= \begin{bmatrix}
\dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{3} & -\dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{6} & 0 & 0 \\
-\dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{6} & \displaystyle\sum_{e=1}^{2}\left(\frac{1}{h^{(e)}} + \frac{h^{(e)}}{3}\right) & -\dfrac{1}{h^{(2)}} + \dfrac{h^{(2)}}{6} & 0 \\
0 & -\dfrac{1}{h^{(2)}} + \dfrac{h^{(2)}}{6} & \displaystyle\sum_{e=2}^{3}\left(\frac{1}{h^{(e)}} + \frac{h^{(e)}}{3}\right) & -\dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{6} \\
0 & 0 & -\dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{6} & \dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{3}
\end{bmatrix}
\end{aligned}
\tag{1.53}
$$

Consequently, the global simultaneous equation becomes

$$
\begin{aligned}
&\begin{bmatrix}
\dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{3} & -\dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{6} & 0 & 0 \\
-\dfrac{1}{h^{(1)}} + \dfrac{h^{(1)}}{6} & \displaystyle\sum_{e=1}^{2}\left(\frac{1}{h^{(e)}} + \frac{h^{(e)}}{3}\right) & -\dfrac{1}{h^{(2)}} + \dfrac{h^{(2)}}{6} & 0 \\
0 & -\dfrac{1}{h^{(2)}} + \dfrac{h^{(2)}}{6} & \displaystyle\sum_{e=2}^{3}\left(\frac{1}{h^{(e)}} + \frac{h^{(e)}}{3}\right) & -\dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{6} \\
0 & 0 & -\dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{6} & \dfrac{1}{h^{(3)}} + \dfrac{h^{(3)}}{3}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \\
&= \begin{Bmatrix}
-\left.\dfrac{d\bar{u}}{dx}\right|_{x=0} \\ 0 \\ 0 \\ \left.\dfrac{d\bar{u}}{dx}\right|_{x=1}
\end{Bmatrix}
\end{aligned}
\tag{1.54}
$$

Note that the coefficient matrix $[K_{ij}]$ in the left-hand side of Equation (1.54) is symmetric with respect to the non-diagonal components ($i \neq j$), i.e., $K_{ij} = K_{ji}$. Only the components in the band region around the diagonal of the matrix are non-zero and the others are zero. Due to this nature, the coefficient matrix is called the sparse or band matrix.

From the boundary conditions, the values of u_1 and u_4 in the left-hand side of Equation (1.54) are known, i.e., $u_1 = 0$ and $u_4 = 1$ and, from Equation (1.51b), the values of f_2 and f_3 in the right-hand side are also known, i.e., $f_2 = 0$ and $f_3 = 0$. On the other hand, u_2 and u_3 in the left-hand side and $\left.\frac{d\bar{u}}{dx}\right|_{x=0}$ and $\left.\frac{d\bar{u}}{dx}\right|_{x=1}$ in the right-hand side are unknown variables. By changing unknown variables $\left.\frac{d\bar{u}}{dx}\right|_{x=0}$ and $\left.\frac{d\bar{u}}{dx}\right|_{x=1}$ with the first and the fourth components of the vector in the left-hand side of Equation (1.54) and by substituting $h^{(1)} = h^{(2)} = h^{(3)} = 1/3$ into Equation (1.54), after rearrangement of the equation, the global simultaneous equation is rewritten as follows:

$$\begin{bmatrix} -1 & -\frac{53}{18} & 0 & 0 \\ 0 & \frac{56}{9} & -\frac{53}{18} & 0 \\ 0 & -\frac{53}{18} & \frac{56}{9} & 0 \\ 0 & 0 & -\frac{53}{18} & -1 \end{bmatrix} \begin{Bmatrix} \left.\frac{d\bar{u}}{dx}\right|_{x=0} \\ u_2 \\ u_3 \\ \left.\frac{d\bar{u}}{dx}\right|_{x=1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{53}{18} \\ -\frac{28}{9} \end{Bmatrix} \tag{1.55}$$

where the new vector in the left-hand side of the equation is an unknown vector and the one in the right-hand side is a known vector.

After solving Equation (1.55), it is found that $u_2 = 0.2885$, $u_3 = 0.6098$, $\left.\frac{d\bar{u}}{dx}\right|_{x=0} = 0.8496$, and $\left.\frac{d\bar{u}}{dx}\right|_{x=1} = 1.3157$. The exact solutions for u_2 and u_3 can be calculated as $u_2 = 0.2889$ and $u_3 = 0.6102$ from Equation (1.55). The relative errors for u_2 and u_3 are as small as 0.1% and 0.06%, respectively. The calculated values of the derivative $\left.\frac{d\bar{u}}{dx}\right|_{x=0}$ and $\left.\frac{d\bar{u}}{dx}\right|_{x=1}$ are improved when compared to those by the one-element FEM described in Section 1.3.1.

In this section, only one-dimensional FEM was described. The FEM can be applied to two- and three-dimensional continuum problems of various kinds which are described in terms of ordinary and partial differential equations. There is no essential difference between the formulation for one-dimensional problems and the formulations for higher dimensions except for the intricacy of formulation.

1.4 FEM IN TWO-DIMENSIONAL ELASTOSTATIC PROBLEMS

Generally speaking, elasticity problems are reduced to solving the partial differential equations known as the equilibrium equations together with the stress–strain relations or the constitutive equations, the strain–displacement relations, and the

compatibility equation under given boundary conditions. The exact solutions can be obtained in quite limited cases only and in general cannot be solved in closed forms. In order to overcome these difficulties, the FEM has been developed as one of the powerful numerical methods to obtain approximate solutions for various kinds of elasticity problems. The FEM assumes an object of analysis as an aggregate of elements having arbitrary shapes and finite sizes (called finite element), approximates partial differential equations by simultaneous algebraic equations, and numerically solves various elasticity problems. Finite elements take the form of line segment in one-dimensional problems as shown in the preceding section, triangle or rectangle in two-dimensional problems, and tetrahedron, cuboid, or prism in three-dimensional problems. Since the procedure of the FEM is mathematically based on the variational method, it can be applied not only to elasticity problems of structures but also to various problems related to thermodynamics, fluid dynamics, and vibrations which are described by partial differential equations.

1.4.1 Elements of finite-element procedures in the analysis of plane elastostatic problems

Limited to static (without time variation) elasticity problems, the procedure described in the preceding section is essentially the same as that of the stress analyses by the FEM.

The procedure is summarized as follows:

- Procedure 1: *Discretization* Divide the object of analysis into a finite number of finite elements.
- Procedure 2: *Selection of the interpolation function* Select the element type or the interpolation function which approximates displacements and strains in each finite element.
- Procedure 3: *Derivation of element stiffness matrices* Determine the element stiffness matrix which relates forces and displacements in each element.
- Procedure 4: *Assembly of stiffness matrices into the global stiffness matrix* Assemble the element stiffness matrices into the global stiffness matrix which relates forces and displacements in the whole elastic body to be analyzed.
- Procedure 5: *Rearrangement of the global stiffness matrix* Substitute prescribed applied forces (mechanical boundary conditions) and displacements (geometrical boundary conditions) into the global stiffness matrix, and rearrange the matrix by collecting unknown variables for forces and displacements, say in the left-hand side, and known values of the forces and displacements in the right-hand side in order to set up simultaneous equations.
- Procedure 6: *Derivation of unknown forces and displacements* Solve the simultaneous equations set up in Procedure 5 above to solve the unknown variables for forces and displacements. The solutions for unknown forces are reaction forces and those for unknown displacements are deformations of the elastic body of interest for given geometrical and mechanical boundary conditions, respectively.
- Procedure 7: *Computation of strains and stresses* Compute the strains and stresses from the displacements obtained in Procedure 6 by using the strain–displacement relations and the stress–strain relations explained later.

1.4.2 Fundamental formulae in plane elastostatic problems

1.4.2.1 Equations of equilibrium

Consider the static equilibrium state of an infinitesimally small rectangle with its sides parallel to the coordinate axes in a two-dimensional elastic body as shown in Figure 1.6. If the body forces F_x and F_y act in the directions of the x- and the y-axes, respectively, the equations of equilibrium in the elastic body can be derived as follows:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + F_x = 0 \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + F_y = 0 \end{cases} \tag{1.56}$$

where σ_x and σ_y are normal stresses in the x- and the y-axes, respectively, with τ_{xy} and τ_{yx} shear stresses acting in the x–y plane. The shear stresses τ_{xy} and τ_{yx} are generally equal to each other due to the rotational equilibrium of the two-dimensional elastic body around its center of gravity.

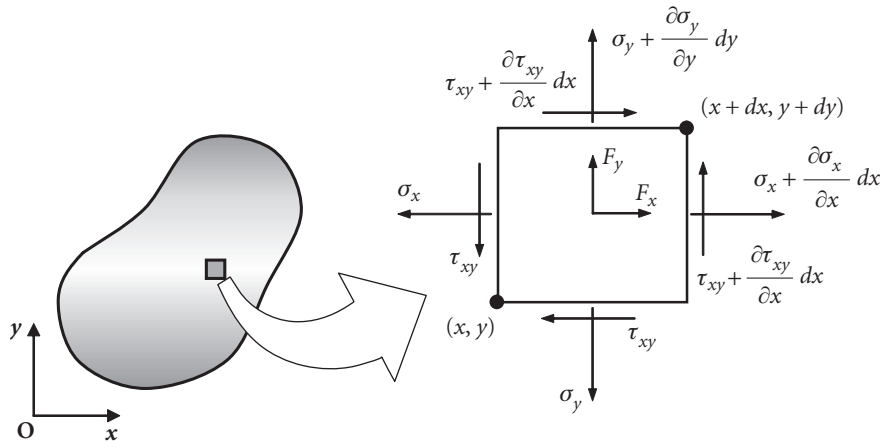

Figure 1.6 Stress states in an infinitesimal element of a two-dimensional elastic body.

1.4.2.2 Strain–displacement relations

If the deformation of a two-dimensional elastic body is infinitesimally small under the applied load, the normal strains ε_x and ε_y in the directions of the x- and the y-axes, respectively, and the engineering shearing strain γ_{xy} in the x–y plane are expressed by

the following equations:

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{cases} \tag{1.57}$$

where u and v are infinitesimal displacements in the directions of the x- and the y-axes, respectively.

1.4.2.3 STRESS–STRAIN RELATIONS (CONSTITUTIVE EQUATIONS)

The stress–strain relations describe states of deformation, strains induced by the internal forces, or stresses resisting against applied loads. Unlike the other fundamental equations shown in Equations (1.56) and (1.57) which can be determined mechanistically or geometrically, these relations depend on the properties of the material, and they are determined experimentally and often called constitutive relations or constitutive equations. One of the most popular relations is the generalized Hooke's law which relates six components of the three-dimensional stress tensor with those of strain tensor through the following simple linear expressions:

$$\begin{cases} \sigma_x = \dfrac{\nu E}{(1+\nu)(1-2\nu)} e_v + 2G\varepsilon_x \\ \sigma_y = \dfrac{\nu E}{(1+\nu)(1-2\nu)} e_v + 2G\varepsilon_y \\ \sigma_y = \dfrac{\nu E}{(1+\nu)(1-2\nu)} e_v + 2G\varepsilon_z \\ \tau_{xy} = G\gamma_{xy} = \dfrac{E}{2(1+\nu)}\gamma_{xy} \\ \tau_{yz} = G\gamma_{yz} = \dfrac{E}{2(1+\nu)}\gamma_{yz} \\ \tau_{zx} = G\gamma_{zx} = \dfrac{E}{2(1+\nu)}\gamma_{zx} \end{cases} \tag{1.58a}$$

or inversely

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \\ \varepsilon_z = \dfrac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G} \\ \gamma_{yz} = \dfrac{\tau_{yz}}{G} \\ \gamma_{zx} = \dfrac{\tau_{zx}}{G} \end{cases} \tag{1.58b}$$

where E is Young's modulus, ν Poisson's ratio, G the shear modulus, and e_v the volumetric strain expressed by the sum of the three normal components of strain, i.e., $e_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$. The volumetric strain e_v can be written in other words as $e_v = \Delta V/V$, where V is the initial volume of the elastic body of interest in an undeformed state and ΔV the change of the volume after deformation.

In the two-dimensional elasticity theory, the three-dimensional Hooke's law is converted into two-dimensional form by using the following two types of approximations:

(1) Plane stress approximation: For thin plates, for example, one can assume the plane stress approximation that all the stress components in the direction perpendicular to the plate surface vanish, i.e., $\sigma_z = \tau_{zx} = \tau_{yz} = 0$. The stress–strain relations in this approximation are written by the following two-dimensional Hooke's law:

$$\begin{cases} \sigma_x = \dfrac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \\ \sigma_y = \dfrac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \\ \tau_{xy} = G\gamma_{xy} = \dfrac{E}{2\,(1+\nu)}\gamma_{xy} \end{cases} \tag{1.59a}$$

or

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \\ \varepsilon_y = \dfrac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G} = \dfrac{2\,(1+\nu)}{E}\tau_{xy} \end{cases} \tag{1.59b}$$

The normal strain component ε_z in the thickness direction, however, is not zero, but $\varepsilon_z = -\nu(\sigma_x + \sigma_y)/E$.

The plane stress approximation satisfies the equations of equilibrium (1.56); nevertheless, the normal strain in the direction of the z-axis ε_z must take a special form, i.e., ε_z must be a linear function of coordinate variables x and y in order to satisfy the compatibility condition which ensures the single-valuedness and continuity conditions of strains. Since this approximation imposes a special requirement for the form of the strain ε_z and thus the forms of the normal stresses σ_x and σ_y, this approximation cannot be considered as a general rule. Strictly speaking, the plane stress state does not exist in reality.

(2) Plane strain approximation: In cases where plate thickness (in the direction of the z-axis) is large, displacement is subjected to large constraints in the direction of the z-axis such that $\varepsilon_z = \gamma_{zx} = \gamma_{yz} = 0$. This case is called the plane stress approximation. The generalized Hooke's law can be written as follows:

$$\begin{cases} \sigma_x = \dfrac{E}{(1+\nu)\,(1-2\nu)}\left[(1-\nu)\,\varepsilon_x + \nu\varepsilon_y\right] \\ \sigma_y = \dfrac{E}{(1+\nu)\,(1-2\nu)}\left[\nu\varepsilon_x + (1-\nu)\,\varepsilon_y\right] \\ \tau_{xy} = G\gamma_{xy} = \dfrac{E}{2\,(1+\nu)}\gamma_{xy} \end{cases} \tag{1.60a}$$

or

$$\begin{cases} \varepsilon_x = \dfrac{1+\nu}{E}\left[(1-\nu)\,\sigma_x - \nu\sigma_y\right] \\ \varepsilon_y = \dfrac{1+\nu}{E}\left[-\nu\sigma_x + (1-\nu)\,\sigma_y\right] \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G} = \dfrac{2\,(1+\nu)}{E}\tau_{xy} \end{cases} \tag{1.60b}$$

The normal stress component σ_z in the thickness direction is not zero, but $\sigma_z = \nu E(\sigma_x + \sigma_y)/[(1+\nu)\,(1-2\nu)]$. Since the plane strain state satisfies the equations of equilibrium (1.56) and the compatibility condition, this state can exist in reality.

If we redefine Young's modulus and Poisson's ratio by the following formulae:

$$E' = \begin{cases} E & \text{(plane stress)} \\ \dfrac{E}{1-\nu} & \text{(plane strain)} \end{cases} \tag{1.61a}$$

$$\nu' = \begin{cases} \nu & \text{(plane stress)} \\ \dfrac{\nu}{1-\nu} & \text{(plane strain)} \end{cases} \tag{1.61b}$$

the two-dimensional Hooke's law can be expressed in a unified form:

$$\begin{cases} \sigma_x = \dfrac{E'}{1-\nu'^2}\left(\varepsilon_x + \nu'\varepsilon_y\right) \\ \sigma_y = \dfrac{E'}{1-\nu'^2}\left(\varepsilon_y + \nu'\varepsilon_x\right) \\ \tau_{xy} = G\gamma_{xy} = \dfrac{E'}{2\,(1+\nu')}\gamma_{xy} \end{cases} \tag{1.62a}$$

or

$$\begin{cases} \varepsilon_x = \dfrac{1}{E'}\left(\sigma_x - \nu'\sigma_y\right) \\ \varepsilon_y = \dfrac{1}{E'}\left(\sigma_y - \nu'\sigma_x\right) \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G} = \dfrac{2\left(1+\nu'\right)}{E'}\tau_{xy} \end{cases} \tag{1.62b}$$

The shear modulus G is invariant under the transformations as shown in Equations (1.61a) and (1.61b), i.e.,

$$G = \frac{E}{2(1+\nu)} = \frac{E'}{2(1+\nu')} = G'$$

1.4.2.4 BOUNDARY CONDITIONS

When solving the partial differential equation (1.56), there remains indefiniteness in the form of integral constants. In order to eliminate this indefiniteness, prescribed

conditions on stress and/or displacements must be imposed on the bounding surface of the elastic body. These conditions are called boundary conditions. There are two types of boundary conditions, i.e. (1) mechanical boundary conditions prescribing stresses or surface tractions and (2) geometrical boundary conditions prescribing displacements.

Let us denote a portion of the surface of the elastic body where stresses are prescribed by S_σ and the remaining surface where displacements are prescribed by S_u. The whole surface of the elastic body is denoted by $S = S_\sigma + S_u$. Note that it is not possible to prescribe both stresses and displacements on a portion of the surface of the elastic body.

The mechanical boundary conditions on S_σ are given by the following equations:

$$\begin{cases} t_x^* = \bar{t}_x^* \\ t_y^* = \bar{t}_y^* \end{cases} \tag{1.63}$$

where t_x^* and t_y^* are the x- and the y-components of the traction force $\mathbf{t}^\star$, respectively, while the bar over t_x^* and t_y^* indicates that those quantities are prescribed on that portion of the surface. Taking $\boldsymbol{n} = [\cos\alpha, \sin\alpha]$ as the outward unit normal vector at a point of a small element of the surface portion S_σ, the Cauchy relations which represent the equilibrium conditions for surface traction forces and internal stresses are given by the following equations:

$$\begin{cases} \bar{t}_x^* = \sigma_x \cos\alpha + \tau_{xy} \sin\alpha \\ \bar{t}_y^* = \tau_{xy} \cos\alpha + \sigma_y \sin\alpha \end{cases} \tag{1.64}$$

where α is the angle between the normal vector $\boldsymbol{n}$ and the x-axis. For free surfaces where no forces are applied, $t_x^* = 0$ and $t_y^* = 0$.

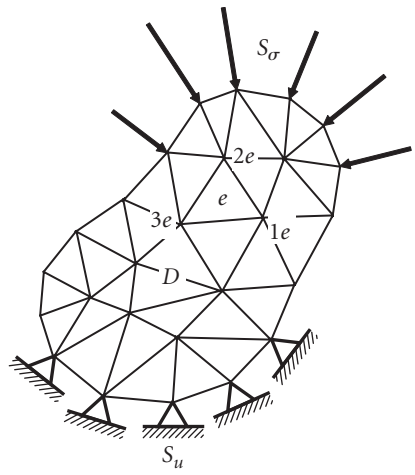

Figure 1.7 Finite-element discretization of a two-dimensional elastic body by triangular elements.

The geometrical boundary conditions on S_u are given by the following equations:

$$\begin{cases} u = \overline{u} \\ v = \overline{v} \end{cases} \tag{1.65}$$

where $\bar{u}$ and $\bar{v}$ are the x- and the y-components of prescribed displacements $\boldsymbol{u}$ on S_u. One of the most popular geometrical boundary conditions, i.e., clamped end condition, is denoted by $u = 0$ and/or $v = 0$ as shown in Figure 1.7.

1.4.3 VARIATIONAL FORMULAE IN ELASTOSTATIC PROBLEMS: THE PRINCIPLE OF VIRTUAL WORK

The variational principle used in two-dimensional elasticity problems is the principle of virtual work which is expressed by the following integral equation:

$$\iint_D \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy} \delta\gamma_{xy}\right) t \, dx \, dy - \iint_D \left(F_x \delta u + F_y \delta v\right) t \, dx \, dy$$
$$- \int_{S_\sigma} \left(\overline{t}_x^* \delta u + \overline{t}_y^* \delta v\right) t \, ds = 0 \tag{1.66}$$

where D denotes the whole region of a two-dimensional elastic body of interest, S_σ the whole portion of the surface of the elastic body $S(=S_\sigma \cup S_u)$, where the mechanical boundary conditions are prescribed and t the thickness.

The first term in the left-hand side of Equation (1.66) represents the increment of the strain energy of the elastic body, the second term the increment of the work done by the body forces, and the third term the increment of the work done by the surface traction forces. Therefore, Equation (1.66) claims that the increment of the strain energy of the elastic body is equal to the work done by the forces applied.

The fact that the integrand in each integral in the left-hand side of Equation (1.66) is identically equal to zero brings about the equations of equilibrium (1.56) and the boundary conditions (1.63) and/or (1.65). Therefore, instead of solving the partial differential equations (1.56) under the boundary conditions of Equations (1.63) and/or (1.65), two-dimensional elasticity problems can be solved by using the integral equation (1.66).

1.4.4 FORMULATION OF THE FUNDAMENTAL FINITE-ELEMENT EQUATIONS IN PLANE ELASTOSTATIC PROBLEMS

1.4.4.1 STRAIN–DISPLACEMENT MATRIX OR [B] MATRIX

Let us use the constant-strain triangular element (see Figure 1.8(a)) to derive the fundamental finite-element equations in plane elastostatic problems. The constant-strain triangular element assumes the displacements within the element to be expressed by

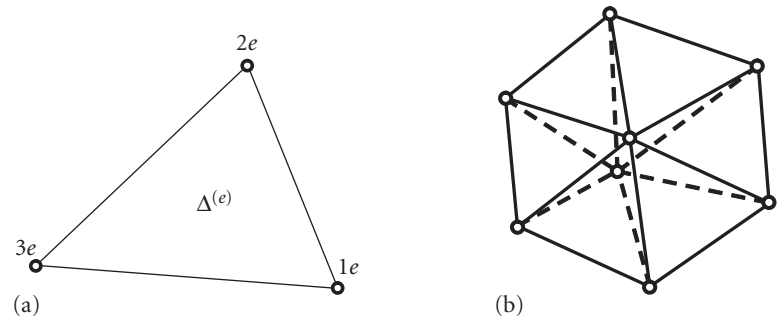

Figure 1.8 (a) Triangular constant strain element and (b) the continuity of displacements.

the following linear functions of the coordinate variables (x, y):

$$\begin{cases} u = \alpha_0 + \alpha_1 x + \alpha_2 y \\ v = \beta_0 + \beta_1 x + \beta_2 y \end{cases} \tag{1.67}$$

The above interpolation functions for displacements convert straight lines joining arbitrarily two points in the element into straight lines after deformation. Since the boundaries between neighboring elements are straight lines joining the apices or nodal points of triangular elements, incompatibility does not occur along the boundaries between adjacent elements and displacements are continuous everywhere in the domain to be analyzed as shown in Figure 1.8(b). For the eth triangular element consisting of three apices or nodal points $(1_e, 2_e, 3_e)$ having the coordinates (x_{1e}, y_{1e}), (x_{2e}, y_{2e}), and (x_{3e}, y_{3e}) and the nodal displacements (u_{1e}, v_{1e}), (u_{2e}, v_{2e}), and (u_{3e}, v_{3e}), the coefficients $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$, and β_2 in Equations (1.67) are obtained by the following equations:

$$\begin{cases} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{pmatrix} a_{1e} & a_{2e} & a_{3e} \\ b_{1e} & b_{2e} & b_{3e} \\ c_{1e} & c_{2e} & c_{3e} \end{pmatrix} \begin{Bmatrix} u_{1e} \\ u_{2e} \\ u_{3e} \end{Bmatrix} \\ \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{Bmatrix} = \begin{pmatrix} a_{1e} & a_{2e} & a_{3e} \\ b_{1e} & b_{2e} & b_{3e} \\ c_{1e} & c_{2e} & c_{3e} \end{pmatrix} \begin{Bmatrix} v_{1e} \\ v_{2e} \\ v_{3e} \end{Bmatrix} \end{cases} \tag{1.68}$$

where

$$\begin{cases} a_{1e} = \dfrac{1}{2\Delta^{(e)}} \left(x_{2e} y_{3e} - x_{3e} y_{2e} \right) \\ b_{1e} = \dfrac{1}{2\Delta^{(e)}} \left(y_{2e} - y_{3e} \right) \\ c_{1e} = \dfrac{1}{2\Delta^{(e)}} \left(x_{3e} - x_{2e} \right) \end{cases} \tag{1.69a}$$

$$\begin{cases} a_{2e} = \dfrac{1}{2\Delta^{(e)}}\left(x_{3e}y_{1e} - x_{1e}y_{3e}\right) \\ b_{2e} = \dfrac{1}{2\Delta^{(e)}}\left(y_{3e} - y_{1e}\right) \\ c_{2e} = \dfrac{1}{2\Delta^{(e)}}\left(x_{1e} - x_{3e}\right) \end{cases} \tag{1.69b}$$

$$\begin{cases} a_{3e} = \dfrac{1}{2\Delta^{(e)}}\left(x_{1e}y_{2e} - x_{2e}y_{1e}\right) \\ b_{3e} = \dfrac{1}{2\Delta^{(e)}}\left(y_{1e} - y_{2e}\right) \\ c_{3e} = \dfrac{1}{2\Delta^{(e)}}\left(x_{2e} - x_{1e}\right) \end{cases} \tag{1.69c}$$

The numbers with subscript "e", 1_e, 2_e, and 3_e, in the above equations are called element nodal numbers and denote the numbers of three nodal points of the eth element. Nodal points should be numbered counterclockwise. These three numbers are used only in the eth element. Nodal numbers of the other type called global nodal numbers are also assigned to the three nodal points of the eth element, being numbered throughout the whole model of the elastic body. The symbol $\Delta^{(e)}$ represents the area of the eth element and can be expressed only by the coordinates of the nodal points of the element, i.e.,

$$\Delta^{(e)} = \frac{1}{2}\left[\left(x_{1e} - x_{3e}\right)\left(y_{2e} - y_{3e}\right) - \left(y_{3e} - y_{1e}\right)\left(x_{3e} - x_{2e}\right)\right] = \frac{1}{2}\begin{vmatrix} 1 & x_{1e} & y_{1e} \\ 1 & x_{2e} & y_{2e} \\ 1 & x_{3e} & y_{3e} \end{vmatrix} \tag{1.69d}$$

Consequently, the components of the displacement vector $[u, v]$ can be expressed by the components of the nodal displacement vectors $[u_{1e}, v_{1e}]$, $[u_{2e}, v_{2e}]$, and $[u_{3e}, v_{3e}]$ as follows:

$$\begin{cases} u = \left(a_{1e} + b_{1e}x + c_{1e}y\right)u_{1e} + \left(a_{2e} + b_{2e}x + c_{2e}y\right)u_{2e} + \left(a_{3e} + b_{3e}x + c_{3e}y\right)u_{3e} \\ v = \left(a_{1e} + b_{1e}x + c_{1e}y\right)v_{1e} + \left(a_{2e} + b_{2e}x + c_{2e}y\right)v_{2e} + \left(a_{3e} + b_{3e}x + c_{3e}y\right)v_{3e} \end{cases} \tag{1.70}$$

Matrix notation of Equation (1.70) is

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_{1e}^{(e)} & 0 & N_{2e}^{(e)} & 0 & N_{3e}^{(e)} & 0 \\ 0 & N_{1e}^{(e)} & 0 & N_{2e}^{(e)} & 0 & N_{3e}^{(e)} \end{bmatrix} \begin{Bmatrix} u_{1e} \\ v_{1e} \\ u_{2e} \\ v_{2e} \\ u_{3e} \\ v_{3e} \end{Bmatrix} = [\mathbf{N}]\{\boldsymbol{\delta}\}^{(e)} \tag{1.71}$$

where

$$\begin{cases} N_{1e}^{(e)} = a_{1e} + b_{1e}x + c_{1e}y \\ N_{2e}^{(e)} = a_{2e} + b_{2e}x + c_{2e}y \\ N_{3e}^{(e)} = a_{3e} + b_{3e}x + c_{3e}y \end{cases} \tag{1.72}$$

and the superscript of $\{\boldsymbol{\delta}\}^{(e)}$, (e), indicates that $\{\boldsymbol{\delta}\}^{(e)}$ is the displacement vector determined by the three displacement vectors at the three nodal points of the eth triangular element. Equation (1.72) formulates the definitions of the interpolation functions or shape functions $N_{ie}^{(e)}(i = 1, 2, 3)$ for the triangular constant-strain element.

Now, let us consider strains derived from the displacements given by Equation (1.71). Substitution of Equation (1.71) into (1.57) gives

$$\{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_{1e}^{(e)}}{\partial x} & 0 & \dfrac{\partial N_{2e}^{(e)}}{\partial x} & 0 & \dfrac{\partial N_{3e}^{(e)}}{\partial x} & 0 \\ 0 & \dfrac{\partial N_{1e}^{(e)}}{\partial y} & 0 & \dfrac{\partial N_{2e}^{(e)}}{\partial y} & 0 & \dfrac{\partial N_{3e}^{(e)}}{\partial y} \\ \dfrac{\partial N_{1e}^{(e)}}{\partial x} & \dfrac{\partial N_{1e}^{(e)}}{\partial y} & \dfrac{\partial N_{2e}^{(e)}}{\partial x} & \dfrac{\partial N_{2e}^{(e)}}{\partial y} & \dfrac{\partial N_{3e}^{(e)}}{\partial x} & \dfrac{\partial N_{3e}^{(e)}}{\partial y} \end{bmatrix} \begin{Bmatrix} u_{1e} \\ v_{1e} \\ u_{2e} \\ v_{2e} \\ u_{3e} \\ v_{3e} \end{Bmatrix}$$

$$= \begin{bmatrix} b_{1e} & 0 & b_{2e} & 0 & b_{3e} & 0 \\ 0 & c_{1e} & 0 & c_{2e} & 0 & c_{3e} \\ c_{1e} & b_{1e} & c_{2e} & b_{2e} & c_{3e} & b_{3e} \end{bmatrix} \begin{Bmatrix} u_{1e} \\ v_{1e} \\ u_{2e} \\ v_{2e} \\ u_{3e} \\ v_{3e} \end{Bmatrix} = [\mathbf{B}]\{\boldsymbol{\delta}\}^{(e)} \tag{1.73}$$

where $[\mathbf{B}]$ establishes the relationship between the nodal displacement vector $\{\boldsymbol{\delta}\}^{(e)}$ and the element strain vector $\{\boldsymbol{\varepsilon}\}$, and is called the strain–displacement matrix or $[\mathbf{B}]$ matrix. All the components of the $[\mathbf{B}]$ matrix are expressed only by the coordinate values of the three nodal points consisting of the element.

From the above discussion, it can be concluded that strains are constant throughout a three-node triangular element, since its interpolation functions are linear functions of the coordinate variables within the element. For this reason, a triangular element with three nodal points is called a "constant-strain" element. Three-node triangular elements cannot satisfy the compatibility condition in the strict sense, since strains are discontinuous among elements. It is demonstrated, however, that the results obtained by elements of this type converge to exact solutions as the size of the elements becomes smaller.

It is known that elements must fulfill the following three criteria for the finite-element solutions to converge to the exact solutions as the subdivision into even-smaller elements is attempted. Namely, the elements must

(1) represent rigid body displacements,

(2) represent constant strains, and

(3) ensure the continuity of displacements among elements.

1.4.4.2 STRESS–STRAIN MATRIX OR [D] MATRIX

Substitution of Equation (1.73) into (1.62a) gives

$$\{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E'}{1-\nu'^2} \begin{bmatrix} 1 & \nu' & 0 \\ \nu' & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu'}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\mathbf{D}^e]\{\boldsymbol{\varepsilon}\} = [\mathbf{D}^e][\mathbf{B}]\{\boldsymbol{\delta}\}^{(e)} \tag{1.74}$$

where $[\mathbf{D}^e]$ establishes the relationship between stresses and strains, or the constitutive relations. The matrix $[\mathbf{D}^e]$ is for elastic bodies and thus is called the elastic stress–strain matrix or just $[\mathbf{D}]$ matrix. In the case where initial strains $\{\boldsymbol{\varepsilon}_0\}$ such as plastic strains, thermal strains, and residual strains exist, $\{\boldsymbol{\varepsilon}\} - \{\boldsymbol{\varepsilon}_0\}$ is used instead of $\{\boldsymbol{\varepsilon}\}$.

1.4.4.3 ELEMENT STIFFNESS EQUATIONS

First, let $\{\mathbf{P}\}^{(e)}$ define the equivalent nodal forces which are statically equivalent to the traction forces $\mathbf{t}^* = [t_x^*, t_y^*]$ on the element boundaries and the body forces $\{\mathbf{F}\}^{(e)}$ in the element:

$$\{\mathbf{F}\}^{(e)T} = [F_x, F_y] \tag{1.75}$$

$$\{\mathbf{P}\}^{(e)T} = [X_{1e}, Y_{1e}, X_{2e}, Y_{2e}, X_{3e}, Y_{3e}] \tag{1.76}$$

In the above equations, $\{\mathbf{F}\}$ represents a column vector, $[\mathbf{P}]$ a row vector, and superscript T the transpose of a vector or a matrix.

In order to make differentiations shown in Equation (1.57), displacements assumed by Equation (1.71) must be continuous everywhere in an elastic body of interest. The remaining conditions to be satisfied are the equations of equilibrium (1.56) and the mechanical boundary conditions (1.63); nevertheless these equations generally cannot be satisfied in the strict sense. Hence, the equivalent nodal forces, for instance (X_{1e}, Y_{1e}), (X_{2e}, Y_{2e}), and (X_{3e}, Y_{3e}), are defined on the three nodal points of the eth element via determining these forces by the principle of the virtual work in order to satisfy the equilibrium and boundary conditions element by element. Namely, the principle of the virtual work to be satisfied for arbitrary virtual displacements $\{\boldsymbol{\delta}^*\}^{(e)}$ of the eth element is derived from Equation (1.66) as

$$\{\boldsymbol{\delta}^*\}^{(e)T}\{\mathbf{P}\}^{(e)} = \iint_D (\{\boldsymbol{\varepsilon}^*\}^T\{\boldsymbol{\sigma}\} - \{\mathbf{f}^*\}^T\{\mathbf{F}\}^{(e)})t\,dx\,dy \tag{1.77}$$

where

$$\{\boldsymbol{\varepsilon}^*\} = [\mathbf{B}]\{\boldsymbol{\delta}^*\}^{(e)} \tag{1.78}$$

$$\{\mathbf{f}^*\} = [\mathbf{N}]\{\boldsymbol{\delta}^*\}^{(e)} \tag{1.79}$$

Substitution of Equations (1.78) and (1.79) into (1.77) gives

$$\{\boldsymbol{\delta}^*\}^{(e)T}\{\mathbf{P}\}^{(e)} = \{\boldsymbol{\delta}^*\}^{(e)T}\left(\iint_D [\mathbf{B}]^T\{\boldsymbol{\sigma}\}t\,dx\,dy - \iint_D [\mathbf{N}]^T\{\mathbf{F}\}^{(e)}t\,dx\,dy\right) \tag{1.80}$$

Since Equation (1.80) holds true for any virtual displacements $\{\boldsymbol{\delta}^*\}^{(e)}$, the equivalent nodal forces can be obtained by the following equation:

$$\{\mathbf{P}\}^{(e)} = \iint_D [\mathbf{B}]^T\{\boldsymbol{\sigma}\}t\,dx\,dy - \iint_D [\mathbf{N}]^T\{\mathbf{F}\}^{(e)}t\,dx\,dy \tag{1.81}$$

From Equations (1.73) and (1.74),

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}^e]\,(\{\boldsymbol{\varepsilon}\} - \{\boldsymbol{\varepsilon}_0\}) = [\mathbf{D}^e][\mathbf{B}]\{\boldsymbol{\delta}\}^{(e)} - [\mathbf{D}^e]\{\boldsymbol{\varepsilon}_0\} \tag{1.82}$$

Substitution of Equation (1.82) into (1.81) gives

$$\{\mathbf{P}\}^{(e)} = \left(\iint_D [\mathbf{B}]^T[\mathbf{D}^e][\mathbf{B}]t\,dx\,dy\right)\{\boldsymbol{\delta}\}^{(e)} - \iint_D [\mathbf{B}]^T[\mathbf{D}^e]\{\boldsymbol{\varepsilon}_0\}t\,dx\,dy - \iint_D [\mathbf{N}]^T\{\mathbf{F}\}^{(e)}t\,dx\,dy \tag{1.83}$$

Equation (1.83) is rewritten in the form

$$\{\mathbf{P}\}^{(e)} = [\mathbf{k}^{(e)}]\{\boldsymbol{\delta}\}^{(e)} + \{\mathbf{F}_{\varepsilon_0}\}^{(e)} + \{\mathbf{F}_F\}^{(e)} \tag{1.84}$$

where

$$[\mathbf{k}^{(e)}] \equiv \iint_D [\mathbf{B}]^T[\mathbf{D}^e][\mathbf{B}]t\,dx\,dy = \Delta^{(e)}[\mathbf{B}]^T[\mathbf{D}^e][\mathbf{B}]t \tag{1.85}$$

$$\{\mathbf{F}_{\varepsilon_0}\}^{(e)} \equiv -\iint_D [\mathbf{B}]^T[\mathbf{D}^e]\{\boldsymbol{\varepsilon}_0\}t\,dx\,dy \tag{1.86}$$

$$\{\mathbf{F}_F\}^{(e)} \equiv -\iint_D [\mathbf{N}]^T\{\mathbf{F}\}^{(e)}t\,dx\,dy \tag{1.87}$$

Equation (1.84) is called the element stiffness equation for the eth triangular finite element and $[\mathbf{k}^{(e)}]$ defined by Equation (1.85) the element stiffness matrix. Since the matrices $[\mathbf{B}]$ and $[\mathbf{D}^e]$ are constant throughout the element, they can be taken out of the integral and the integral is simply equal to the area of the element $\Delta^{(e)}$ so that the rightmost side of Equation (1.85) is obtained. The forces $\{\mathbf{F}_{\varepsilon_0}\}^{(e)}$ and $\{\mathbf{F}_F\}^{(e)}$ are the equivalent nodal forces due to initial strains and body forces, respectively. Since the integrand in Equation (1.85) is generally a function of the coordinate variables x and y except for the case of three-node triangular elements, the integrals appearing in Equation (1.85) are often evaluated by a numerical integration scheme such as the Gaussian quadrature.

The element stiffness matrix $[\mathbf{k}^{(e)}]$ in Equation (1.85) is a 6 by 6 square matrix which can be decomposed into nine 2 by 2 submatrices as shown in the following equation:

$$[\mathbf{k}^{(e)}_{ieje}] = \begin{bmatrix} \mathbf{k}^{(e)}_{1e1e} & \mathbf{k}^{(e)}_{1e2e} & \mathbf{k}^{(e)}_{1e3e} \\ \mathbf{k}^{(e)}_{2e1e} & \mathbf{k}^{(e)}_{2e2e} & \mathbf{k}^{(e)}_{2e3e} \\ \mathbf{k}^{(e)}_{3e1e} & \mathbf{k}^{(e)}_{3e2e} & \mathbf{k}^{(e)}_{3e3e} \end{bmatrix} \tag{1.88}$$

$$\mathbf{k}^{(e)}_{ieje} = \mathbf{k}^{(e)T}_{jeie} \tag{1.89}$$

$$\mathbf{k}^{(e)}_{ieje} = \int_{\Delta^{(e)}} [\mathbf{B}_{ie}]^T [\mathbf{D}^e][\mathbf{B}_{je}] t \, dx \, dy \quad \begin{cases} (2 \times 2) \text{ asymmetric matrix} (i_e \neq j_e) \\ (2 \times 2) \text{ symmetric matrix} (i_e = j_e) \end{cases} \tag{1.90}$$

where

$$[\mathbf{B}_{ie}] \equiv \frac{1}{2\Delta^{(e)}} \begin{bmatrix} b_{ie} & 0 \\ 0 & c_{ie} \\ c_{ie} & b_{ie} \end{bmatrix} \quad (i_e = 1, 2, 3) \tag{1.91}$$

and the subscripts i_e and j_e of $\mathbf{k}^{(e)}_{ieje}$ refer to element nodal numbers and

$$\mathbf{k}^{(e)}_{ieje} = [\mathbf{B}_{ie}]^T [\mathbf{D}^e][\mathbf{B}_{je}] t \Delta^{(e)} \tag{1.92}$$

In the above discussion, the formulae have been obtained for just one triangular element, but are available for any elements, if necessary, with some modifications.

1.4.4.4 GLOBAL STIFFNESS EQUATIONS

Element stiffness equations as shown in Equation (1.84) are determined for element by element, and then they are assembled into the global stiffness equations for the whole elastic body of interest. Since nodal points which belong to different elements but have the same coordinates are the same points, the following items during the assembly procedure of the global stiffness equations are to be noted:

(1) The displacement components u and v of the same nodal points which belong to different elements are the same; i.e., there exist no incompatibilities such as cracks between elements.

(2) For nodal points on the bounding surfaces and for those in the interior of the elastic body to which no forces are applied, the sums of the nodal forces are to be zero.

(3) Similarly, for nodal points to which forces are applied, the sums of the nodal forces are equal to the sums of the forces applied to those nodal points.

The same global nodal numbers are to be assigned to the nodal points which have the same coordinates. Taking the items described above into consideration, let us rewrite the element stiffness matrix $[\mathbf{k}^{(e)}]$ in Equation (1.88) by using the global

nodal numbers I, J, and K ($I, J, K = 1, 2, \ldots, 2n$) instead of the element nodal numbers i_e, j_e, and k_e ($i_e, j_e, k_e = 1, 2, 3$); i.e.,

$$\left[\mathbf{k}^{(e)}\right] = \begin{bmatrix} \mathbf{k}_{II}^{(e)} & \mathbf{k}_{IJ}^{(e)} & \mathbf{k}_{IK}^{(e)} \\ \mathbf{k}_{JI}^{(e)} & \mathbf{k}_{JJ}^{(e)} & \mathbf{k}_{JK}^{(e)} \\ \mathbf{k}_{KI}^{(e)} & \mathbf{k}_{KJ}^{(e)} & \mathbf{k}_{KK}^{(e)} \end{bmatrix} \tag{1.93}$$

Then, let us embed the element stiffness matrix in a square matrix having the same size as the global stiffness matrix of $2n$ by $2n$ as shown in Equation (1.94):

$$[\mathbf{K}^{(e)}]\{\boldsymbol{\delta}\} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ 2I-1 \\ 2I \\ \vdots \\ 2J-1 \\ 2J \\ \vdots \\ 2K-1 \\ 2K \\ \vdots \\ 2n-1 \\ 2n \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & \cdots & 2I-1 & 2I & \cdots & 2J-1 & 2J & \cdots & 2K-1 & 2K & \cdots \end{array} \quad 2n-1 \quad 2n \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \multirow{2}{*}{$\mathbf{k}_{II}^{(e)}$} & \cdots & & \multirow{2}{*}{$\mathbf{k}_{IJ}^{(e)}$} & \cdots & & \multirow{2}{*}{$\mathbf{k}_{IK}^{(e)}$} & \cdots & & 0 & 0 \\ 0 & 0 & \cdots & & \cdots & & & \cdots & & & \cdots & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \multirow{2}{*}{$\mathbf{k}_{JI}^{(e)}$} & \cdots & & \multirow{2}{*}{$\mathbf{k}_{JJ}^{(e)}$} & \cdots & & \multirow{2}{*}{$\mathbf{k}_{JK}^{(e)}$} & \cdots & & 0 & 0 \\ 0 & 0 & \cdots & & \cdots & & & \cdots & & & \cdots & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \multirow{2}{*}{$\mathbf{k}_{KI}^{(e)}$} & \cdots & & \multirow{2}{*}{$\mathbf{k}_{KJ}^{(e)}$} & \cdots & & \multirow{2}{*}{$\mathbf{k}_{KK}^{(e)}$} & \cdots & & 0 & 0 \\ 0 & 0 & \cdots & & \cdots & & & \cdots & & & \cdots & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & & \cdots & & & \cdots & & & \cdots & & 0 & 0 \\ 0 & 0 & \cdots & & \cdots & & & \cdots & & & \cdots & & 0 & 0 \end{bmatrix} \end{array}$$

$$\times \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_I \\ v_I \\ \vdots \\ u_j \\ v_j \\ \vdots \\ u_k \\ v_k \\ \vdots \\ u_n \\ v_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ X_I^{(e)} \\ Y_I^{(e)} \\ \vdots \\ X_J^{(e)} \\ Y_J^{(e)} \\ \vdots \\ X_K^{(e)} \\ Y_k^{(e)} \\ \vdots \\ 0 \\ 0 \end{Bmatrix} \tag{1.94}$$

where n denotes the number of nodal points. This procedure is called the method of extended matrix. The number of degrees of freedom here means the number of unknown variables. In two-dimensional elasticity problems, since two of displacements and forces in the x- and the y-directions are unknown variables for one nodal point, every nodal point has two degrees of freedom. Hence, the number of degrees of freedom for a finite-element model consisting of n nodal points is $2n$.

By summing up the element stiffness matrices for all the n_e elements in the finite-element model, the global stiffness matrix $[\mathbf{K}]$ is obtained as shown in the following equation:

$$[\mathbf{K}] \equiv [K_{ij}] = \sum_{e=1}^{ne} [\mathbf{K}^{(e)}] \quad (i, j = 1, 2, \ldots, 2n \quad \text{and} \quad e = 1, 2, \ldots, n_e) \tag{1.95}$$

Since the components of the global nodal displacement vector $\{\boldsymbol{\delta}\}$ are common for all the elements, they remain unchanged during the assembly of the global stiffness equations. By rewriting the components of $\{\boldsymbol{\delta}\}$, u_1, u_2, …, u_n as u_1, u_3, …, u_{2i-1}, …, u_{2n-1} and v_1, v_2, …, v_n as u_2, u_4, …, u_{2i}, …, u_{2n}, the following expression for the global nodal displacement vector $\{\boldsymbol{\delta}\}$ is obtained:

$$\{\boldsymbol{\delta}\} = \{u_1, u_2, \ldots, u_{2I-1}, u_{2I}, \ldots, u_{2J-1}, u_{2J}, \ldots, u_{2K-1}, u_{2K}, \ldots, u_{2n-1}, u_{2n}\}^T \tag{1.96}$$

The global nodal force of a node is the sum of the nodal forces for all the elements to which the node belongs. Hence, the global nodal force vector $\{\mathbf{P}\}$ can be written as

$$\{\mathbf{P}\} = \{X_1, Y_1, \ldots, X_I, Y_I, \ldots, X_J, V_J, \ldots, X_K, Y_K, \ldots, X_n, Y_n\}^T \tag{1.97}$$

where

$$X_I = \sum X_I^{(e)} \quad Y_I = \sum Y_I^{(e)} \quad (I = 1, 2, \ldots, n) \tag{1.98}$$

By rewriting the global nodal force vector $\{\mathbf{P}\}$ in a similar way to $\{\boldsymbol{\delta}\}$ in Equation (1.96) as

$$\{\mathbf{P}\} = \{X_1, X_2, \cdots, X_{2I-1}, X_{2I}, \ldots, X_{2J-1}, X_{2J}, \ldots, X_{2K-1}, X_{2K}, \ldots, X_{2n-1}, X_{2n}\}^T \tag{1.99}$$

where

$$X_I = \sum X_I^{(e)} \quad (I = 1, 2, \ldots, 2n) \tag{1.100}$$

The symbol Σ in Equations (1.98) and (1.100) indicates that the summation is taken over all the elements that possess the node in common. The values of X_I in Equation (1.100), however, are zero for the nodes inside of the elastic body and for those on the bounding surfaces which are subjected to no applied loads.

Consequently, the following formula is obtained as the governing global stiffness equation:

$$[\mathbf{K}]\{\boldsymbol{\delta}\} = \{\mathbf{P}\} \tag{1.101}$$

which is the $2n$th degree simultaneous linear equations for $2n$ unknown variables of nodal displacements and/or forces.

1.4.4.5 EXAMPLE: FINITE-ELEMENT CALCULATIONS FOR A SQUARE PLATE SUBJECTED TO UNIAXIAL UNIFORM TENSION

Procedures 5 through 7 described in Section 1.4.1 will be explained by taking an example of the finite-element calculations for a square plate subjected to uniaxial uniform tension as illustrated in Figure 1.9. The square plate model has a side of unit length 1 and a thickness of unit length 1, and consists of two constant-strain triangular elements, i.e., the model has four nodes and thus eight degrees of freedom.

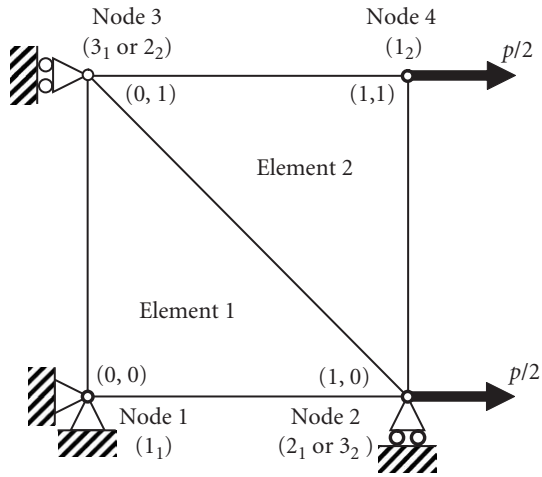

Figure 1.9 Finite element model of a square plate subjected to uniaxial uniform tension.

Let us determine the element stiffness matrix for Element 1. From Equations (1.73) and (1.69a) through (1.69c), the [**B**] matrix of Element 1 is calculated as

$$[\mathbf{B}] = \begin{bmatrix} b_{1_1} & 0 & b_{2_1} & 0 & b_{3_1} & 0 \\ 0 & c_{1_1} & 0 & c_{2_1} & 0 & c_{3_1} \\ c_{1_1} & b_{1_1} & c_{2_1} & b_{2_1} & c_{3_1} & b_{3_1} \end{bmatrix}$$

$$= \frac{1}{2\Delta^{(1)}} \begin{bmatrix} y_{2_1} - y_{3_1} & 0 & y_{3_1} - y_{1_1} & 0 & y_{1_1} - y_{2_1} & 0 \\ 0 & x_{3_1} - x_{2_1} & 0 & x_{1_1} - x_{3_1} & 0 & x_{2_1} - x_{1_1} \\ x_{3_1} - x_{2_1} & y_{2_1} - y_{3_1} & x_{1_1} - x_{3_1} & y_{3_1} - y_{1_1} & x_{2_1} - x_{1_1} & y_{1_1} - y_{2_1} \end{bmatrix} \tag{1.102}$$

Since the area of Element 1 $\Delta^{(e)}$ in the above equation can be easily obtained as 1/2 without using Equation (1.69d),

$$[\mathbf{B}] = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{1.103}$$

Hence, from Equation (1.85), the element stiffness matrix for Element 1 $[\mathbf{k}^{(e)}]$ is calculated as

$$[\mathbf{k}^{(1)}] = \frac{t}{2}\,\frac{E'}{1-\nu'^2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \nu' & 0 \\ \nu' & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{1.104}$$

where $\alpha = (1-\nu')/2$. After multiplication of the matrices in Equation (1.104), the element stiffness equation is obtained from Equation (1.84) as

$$\begin{Bmatrix} X_{1_1}^{(1)} \\ Y_{1_1}^{(1)} \\ X_{2_1}^{(1)} \\ Y_{2_1}^{(1)} \\ X_{3_1}^{(1)} \\ Y_{3_1}^{(1)} \end{Bmatrix} = \frac{t}{2}\,\frac{E'}{1-\nu'^2}\begin{bmatrix} 1+\alpha & \nu'+\alpha & -1 & -\alpha & -\alpha & -\nu' \\ \nu'+\alpha & 1+\alpha & -\nu' & -\alpha & -\alpha & -1 \\ -1 & -\nu' & 1 & 0 & 0 & \nu' \\ -\alpha & -\alpha & 0 & \alpha & \alpha & 0 \\ -\alpha & -\alpha & 0 & \alpha & \alpha & 0 \\ -\nu' & -1 & \nu' & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{1_1} \\ v_{1_1} \\ u_{2_1} \\ v_{2_1} \\ u_{3_1} \\ v_{3_1} \end{Bmatrix} \tag{1.105}$$

since the equivalent nodal forces due to initial strains ε_0 and body forces F_x and F_y, $\{\mathbf{F}_{\varepsilon 0}\}^{(1)}$ and $\{\mathbf{F}_F\}^{(1)}$ are zero. The components of the nodal displacement and force vectors are written by the element nodal numbers. By rewriting these components by the global nodal numbers as shown in Figure 1.9, Equation (1.105) is rewritten as

$$\begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{Bmatrix} = \frac{t}{2}\,\frac{E'}{1-\nu'^2}\begin{bmatrix} 1+\alpha & \nu'+\alpha & -1 & -\alpha & -\alpha & -\nu' \\ \nu'+\alpha & 1+\alpha & -\nu' & -\alpha & -\alpha & -1 \\ -1 & -\nu' & 1 & 0 & 0 & \nu' \\ -\alpha & -\alpha & 0 & \alpha & \alpha & 0 \\ -\alpha & -\alpha & 0 & \alpha & \alpha & 0 \\ -\nu' & -1 & \nu' & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{1.106}$$

In a similar way, the element stiffness equation for Element 2 is obtained as

$$\begin{Bmatrix} X_{1_2}^{(2)} \\ Y_{1_2}^{(2)} \\ X_{2_2}^{(2)} \\ Y_{2_2}^{(2)} \\ X_{3_2}^{(2)} \\ Y_{3_2}^{(2)} \end{Bmatrix} = \frac{t}{2}\,\frac{E'}{1-\nu'^2}\begin{bmatrix} 1+\alpha & \nu'+\alpha & -1 & -\alpha & -\alpha & -\nu' \\ \nu'+\alpha & 1+\alpha & -\nu' & -\alpha & -\alpha & -1 \\ -1 & -\nu' & 1 & 0 & 0 & \nu' \\ -\alpha & -\alpha & 0 & \alpha & \alpha & 0 \\ -\alpha & -\alpha & 0 & \alpha & \alpha & 0 \\ -\nu' & -1 & \nu' & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{1_2} \\ v_{1_2} \\ u_{2_2} \\ v_{2_2} \\ u_{3_2} \\ v_{3_2} \end{Bmatrix} \tag{1.107}$$

and is rewritten by using the global nodal numbers as

$$\begin{Bmatrix} X_4 \\ Y_4 \\ X_3 \\ Y_3 \\ X_2 \\ Y_2 \end{Bmatrix} = \frac{t}{2}\ \frac{E'}{1-\nu'^2} \begin{bmatrix} 1+\alpha & \nu'+\alpha & -1 & -\alpha & -\alpha & -\nu' \\ \nu'+\alpha & 1+\alpha & -\nu' & -\alpha & -\alpha & -1 \\ -1 & -\nu' & 1 & 0 & 0 & \nu' \\ -\alpha & -\alpha & 0 & \alpha & \alpha & 0 \\ -\alpha & -\alpha & 0 & \alpha & \alpha & 0 \\ -\nu' & -1 & \nu' & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ v_4 \\ u_3 \\ v_3 \\ u_2 \\ v_2 \end{Bmatrix} \tag{1.108}$$

By assembling the two-element stiffness matrices, the following global stiffness equation for the square plate subjected to uniform tension is obtained (Procedure 4):

$$\frac{t}{2}\ \frac{E'}{1-\nu'^2} \begin{bmatrix} 1+\alpha & \nu'+\alpha & -1 & -\alpha & -\alpha & -\nu' & 0 & 0 \\ \nu'+\alpha & 1+\alpha & -\nu' & -\alpha & -\alpha & -1 & 0 & 0 \\ -1 & -\nu' & 1+\alpha & 0 & 0 & \nu'+\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & 0 & 1+\alpha & \nu'+\alpha & 0 & -\nu' & -1 \\ -\alpha & -\alpha & 0 & \nu'+\alpha & 1+\alpha & 0 & -1 & -\nu' \\ -\nu' & -1 & \nu'+\alpha & 0 & 0 & 1+\alpha & -\alpha & -\alpha \\ 0 & 0 & -\alpha & -\nu' & -1 & -\alpha & 1+\alpha & \nu'+\alpha \\ 0 & 0 & -\alpha & -1 & -\nu' & -\alpha & \nu'+\alpha & 1+\alpha \end{bmatrix}$$
$$\times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{Bmatrix} \tag{1.109}$$

where the left- and right-hand sides of the equation are replaced with each other.

Let us now impose boundary conditions on the nodes. Namely, the node 1 is clamped in both the x- and the y-directions, the node 2 is clamped only in the x-direction, and the nodes 2 and 4 are subjected to equal nodal forces $X_2 = X_4 = (p \times 1 \times 1)/2 = p/2$, respectively, in the x-direction. A pair of the equal nodal forces $p/2$ applied to the nodes 2 and 4 in the x-direction is the finite-element model of a uniformly distributed tension force p per unit area exerted on the side $\overline{24}$ in the x-direction as illustrated in Figure 1.9. The geometrical boundary and mechanical conditions for the present case are

$$u_1 = v_1 = v_2 = u_3 = 0 \tag{1.110}$$

and

$$X_2 = X_4 = (pt)/2 = p/2 \quad Y_3 = Y_4 = 0 \tag{1.111}$$

respectively. Substitution of Equations (1.110) and (1.111) into Equation (1.109) gives the following global stiffness equation:

$$\frac{t}{2}\frac{E'}{1-\nu'^2}\begin{bmatrix} 1+\alpha & \nu'+\alpha & -1 & -\alpha & -\alpha & -\nu' & 0 & 0 \\ \nu'+\alpha & 1+\alpha & -\nu' & -\alpha & -\alpha & -1 & 0 & 0 \\ -1 & -\nu' & 1+\alpha & 0 & 0 & \nu'+\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & 0 & 1+\alpha & \nu'+\alpha & 0 & -\nu' & -1 \\ -\alpha & -\alpha & 0 & \nu'+\alpha & 1+\alpha & 0 & -1 & -\nu' \\ -\nu' & -1 & \nu'+\alpha & 0 & 0 & 1+\alpha & -\alpha & -\alpha \\ 0 & 0 & -\alpha & -\nu' & -1 & -\alpha & 1+\alpha & \nu'+\alpha \\ 0 & 0 & -\alpha & -1 & -\nu' & -\alpha & \nu'+\alpha & 1+\alpha \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ 0 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ p/2 \\ Y_2 \\ X_3 \\ 0 \\ p/2 \\ 0 \end{Bmatrix} \tag{1.112}$$

Rearrangement of Equation (1.112) by collecting the unknown variables for forces and displacements in the left-hand side and the known values of the forces and displacements in the right-hand side brings about the following simultaneous equations (Procedure 5):

$$\frac{t}{2}\frac{E'}{1-\nu'^2}\begin{bmatrix} -\frac{2}{t}\frac{E'}{1-\nu'^2} & 0 & -1 & 0 & 0 & -\nu' & 0 & 0 \\ 0 & -\frac{2}{t}\frac{E'}{1-\nu'^2} & -\nu' & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1+\alpha & 0 & 0 & \nu'+\alpha & -\alpha & -\alpha \\ 0 & 0 & 0 & -\frac{2}{t}\frac{E'}{1-\nu'^2} & 0 & 0 & -\nu' & -1 \\ 0 & 0 & 0 & 0 & -\frac{2}{t}\frac{E'}{1-\nu'^2} & 0 & -1 & -\nu' \\ 0 & 0 & \nu'+\alpha & 0 & 0 & 1+\alpha & -\alpha & -\alpha \\ 0 & 0 & -\alpha & 0 & 0 & -\alpha & 1+\alpha & \nu'+\alpha \\ 0 & 0 & -\alpha & 0 & 0 & -\alpha & \nu'+\alpha & 1+\alpha \end{bmatrix} \times \begin{Bmatrix} X_1 \\ Y_1 \\ u_2 \\ Y_2 \\ X_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ p/2 \\ 0 \\ 0 \\ 0 \\ p/2 \\ 0 \end{Bmatrix} \tag{1.113}$$

Equation (1.113) can be solved numerically by, for instance, the Gauss elimination procedure. The solutions for Equation (1.113) are

$$u_2 = u_4 = p/E', \quad v_3 = v_4 = -\nu' p/E', \quad X_1 = X_3 = -p/2, \quad Y_1 = Y_2 = 0 \tag{1.114}$$

(Procedure 6).

The strains and stresses in the square plate can be calculated by substituting the solutions (1.114) into Equations (1.73) and (1.74), respectively (Procedure 7). The resultant strains and stresses are given by the following equations:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\mathbf{B}] \begin{Bmatrix} u_4 \\ v_4 \\ u_3 \\ v_3 \\ u_2 \\ v_2 \end{Bmatrix} = \frac{p}{E'} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ -\nu' \\ 0 \\ \nu' \\ 1 \\ 0 \end{Bmatrix}$$

$$= \frac{p}{E'} \begin{Bmatrix} 1 \\ -\nu' \\ 1 - \nu' + \nu' - 1 \end{Bmatrix} = \frac{p}{E'} \begin{Bmatrix} 1 \\ -\nu' \\ 0 \end{Bmatrix} \tag{1.115}$$

and

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E'}{1 - \nu'^2} \begin{bmatrix} 1 & \nu' & 0 \\ \nu' & 1 & 0 \\ 0 & 0 & \dfrac{1 - \nu'}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{p}{1 - \nu'^2} \begin{bmatrix} 1 & \nu' & 0 \\ \nu' & 1 & 0 \\ 0 & 0 & \dfrac{1 - \nu'}{2} \end{bmatrix} \begin{Bmatrix} 1 \\ -\nu' \\ 0 \end{Bmatrix}$$

$$= \frac{p}{1 - \nu'^2} \begin{Bmatrix} 1 - \nu'^2 \\ 0 \\ 0 \end{Bmatrix} = p \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{1.116}$$

The results obtained by the present finite-element calculations imply that a square plate subjected to uniaxial uniform tension in the x-direction is elongated by a uniform strain of p/E' in the loading direction, whereas it is contacted by a uniform strain of $-\nu' p/E'$ in the direction perpendicular to the loading direction and that only a uniform normal stress of $\sigma_x = p$ is induced in the plate. The result that the nodal reaction forces at the nodes 1 and 3 are equal to $-p/2$, i.e., $X_1 = X_3 = -p/2$ implies that a uniform reaction force of $-p$ is produced along the side $\overline{13}$. It is concluded that the above results obtained by the FEM agree well with the physical interpretations of the present problem.

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